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Flow induced in a cylindrical column by a uniformly rotating magnetic field

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Under laboratory conditions, the magnetic Reynolds number is quite small in a conductor, but can be made appreciable if a high frequency rotating field is applied. Moffatt investigated this problem for high magnetic Reynolds numbers and concluded that there existed a magnetic boundary layer due to spiralling of field lines. Applying Fourier transforms and solving the corrected equations, we find that at low magnetic Reynolds numbers the field lines uniformly penetrate the cylindrical column and do not exhibit any appreciable spiralling. The rotation opposes the drift due to conductivity which is evened out as one proceeds from the centre to the surface. This uniform behaviour persists for small magnetic Reynolds number inside and outside. When the magnetic Reynolds number becomes large, of the order of 100 (say), the field lines passing through the axis of the cylinder exhibit spiralling as suggested by Moffatt since the diffusion is unable to counterbalance the rotational effects.

1. Introduction

In astronomical and geophysical problems, due to large length scales involved, the effect of the induced electromotive force predominates the small diffusion of magnetic lines of force across the material. In laboratories, the working fluids like mercury or liquid sodium have very small conductivity of the order of $10^{-5}\,\Omega/\mathrm{s}$ and relatively small length scales. Consequently, unless the conducting fluid moves across the magnetic lines of force with considerable velocity, the magnetic Reynolds number ($4\pi\mu\sigma LV$) will be small. This approximation of 'weakly conducting fluids' has been initiated by Lundquist (1952) and studied in detail by Lehnert & Sjogren (1960), Braginskii (1960), Murty (1963) and others. Recently, in an interesting paper, Moffatt (1965) has given a practical method of obtaining high magnetic Reynolds numbers by imposing a high frequency rotating magnetic field. He has concentrated on high values of magnetic Reynolds number of the orders of 10, 10^2, etc., and has found that at those situations the field lines have a tendency to spiral and crowd near the boundary of the cylindrical column, suggestive of skin effect. The present note may be regarded as a complementary investigation to that of Moffatt. We have confined ourselves to comparatively low magnetic Reynolds numbers (R_M), namely of the order of 0·01, 0·25, 1, 4 and 100. At low R_M, the diffusion of the magnetic fields can counterbalance the high frequency rotation and hence it is possible to avoid such spiralling and skin effects. We find that for $R_M \ll 100$, the field lines do not show the above effects. However, for $R_M = 100$, the field line, corresponding to the magnetic potential $\Phi = 0$, does show spiralling and hence introduces a sort of funnelling of lines from $\Phi = 0$ to $\Phi = 0{\cdot}1$ near the region of entry of these lines but not at other places. Effectively, for large values of R_M, the material behaves as if the electrical conductivity of the matter is enhanced

Perhaps, this fact accounts for this type of behaviour of the field lines at large R_M.

In passing, we note that we have rectified a certain mathematical error in Moffatt's paper in writing the equations governing the magnetic stream function and their solutions, though in the asymptotic limit, this error does not affect Moffatt's qualitative picture at large magnetic Reynolds number.

2. Formulation of the problem and basic equations

Consider the motion induced in a column of conducting fluid contained in a cylinder of insulating material by an alternating magnetic field acting perpendicular to the axis of the cylinder. Neglecting the end effects, the induced flow is purely two dimensional in the plane transverse to the axis. Accordingly, the basic equations of the problem are:

inside the cylinder $(r < a)$

$$\operatorname{div} \mathbf{v} = 0, \tag{2·1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + [(\mathbf{v} . \nabla)\,\mathbf{v}] = -\frac{1}{\rho}\operatorname{grad} p + \frac{1}{\rho}\mathbf{J} \times \mathbf{B}, \tag{2·2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = [\operatorname{curl}(\mathbf{v} \times \mathbf{B})] + \frac{1}{4\pi\mu\sigma}\nabla^2 \mathbf{B}, \tag{2·3}$$

$$\operatorname{div} \mathbf{B} = 0; \tag{2·4}$$

outside the cylinder $(r > a)$

$$\nabla^2 \mathbf{B} = 0, \tag{2·5}$$

$$\operatorname{div} \mathbf{B} = 0. \tag{2·6}$$

Since the flow is induced by an oscillating magnetic field of fairly high frequency, we have the following asymptotic conditions and boundary conditions

$$\left.\begin{aligned} B_r &\sim B_0 \cos(\theta - \omega t) \\ B_\theta &\sim -B_0 \sin(\theta - \omega t) \end{aligned}\right\} \quad (r \to \infty), \tag{2·7}$$

and

$$\mathbf{B}_{\text{outside}}(a, t) = \mathbf{B}_{\text{inside}}(a, t). \tag{2·8}$$

In consequence of our assumptions, the terms in the square bracket in (2·2) and (2·3) are neglected in comparison with the remaining terms.

For convenience, we introduce the 'magnetic potential' by

$$B_r = \frac{1}{r}\frac{\partial \psi_B}{\partial \theta}, \quad B_\theta = -\frac{\partial \psi_B}{\partial r}, \tag{2·9}$$

so that (2·4) and (2·6) are automatically satisfied. Further, we introduce the following nondimensional quantities:

$$\xi = \frac{r}{a}, \quad \tau = \frac{t}{4\pi\mu\sigma a^2} \equiv \frac{\lambda t}{a^2},$$

$$\Psi = \frac{\psi_{B\,(\text{exterior})}}{B_0 a}, \quad \Phi = \frac{\psi_{B\,(\text{interior})}}{B_0 a},$$

and

$$\beta = 4\pi\mu\sigma a^2 \omega.$$

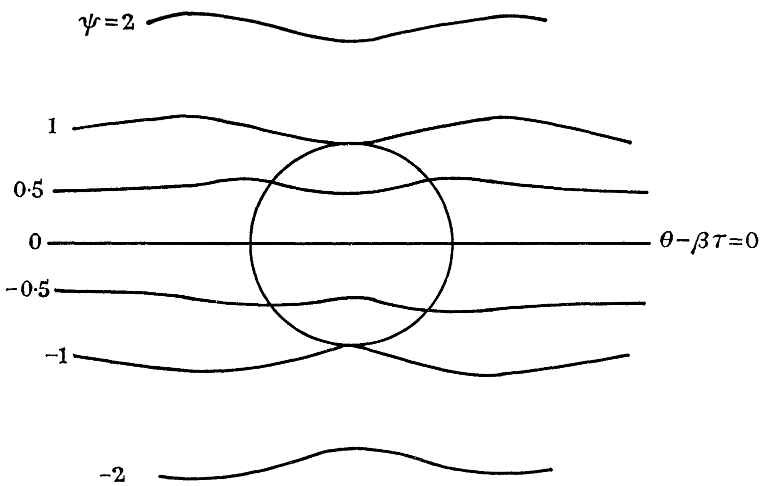

FIGURE 1. Magnetic field lines for $\sqrt{\beta} = 0{\cdot}1$. The circle represents the cross section of the cylinder.

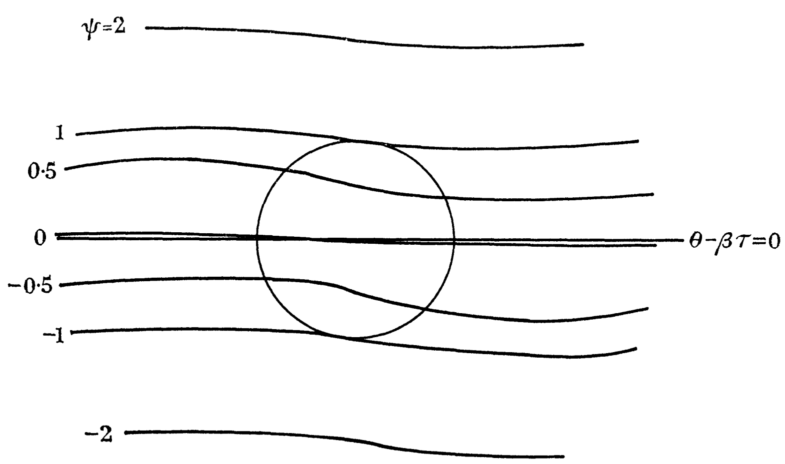

FIGURE 2. Magnetic field lines for $\sqrt{\beta} = 0{\cdot}5$.

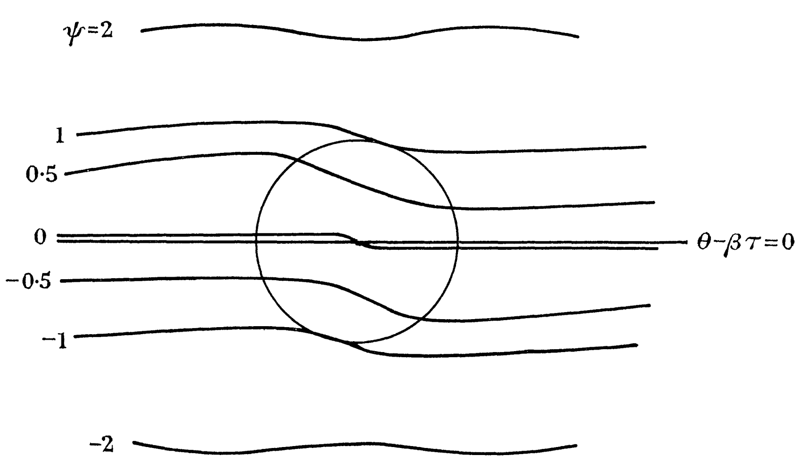

FIGURE 3. Magnetic field lines for $\sqrt{\beta} = 1$.

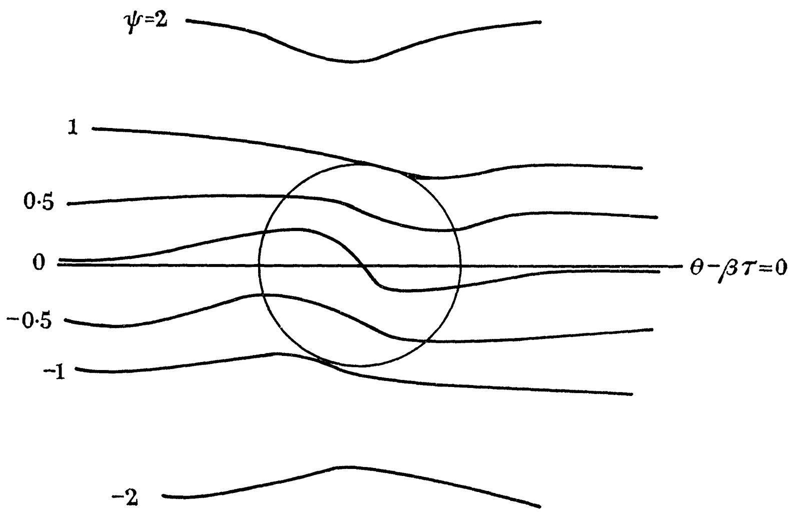

FIGURE 4. Magnetic field lines for $\sqrt{\beta} = 2$.

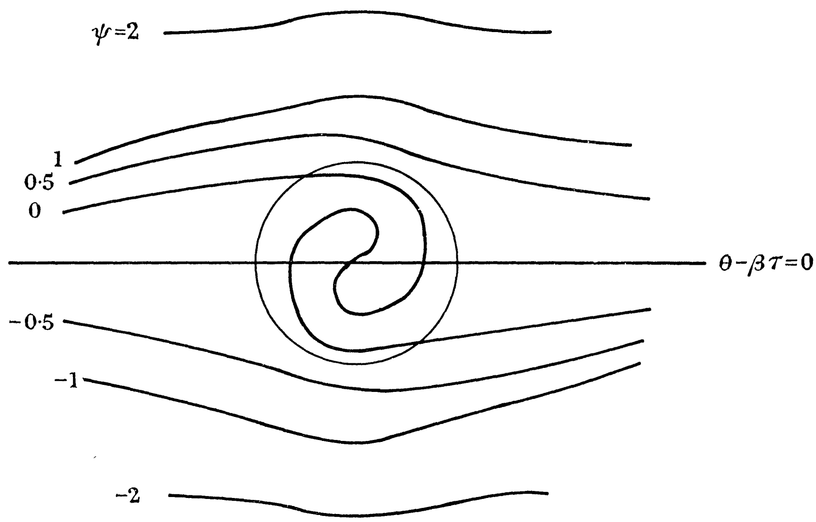

FIGURE 5. Magnetic field lines for $\sqrt{\beta} = 10$.

In terms of these dimensionless quantities equations (2·5) and (2·3) for the magnetic fields reduce to

$$\xi^2 \frac{\partial^2 \Psi}{\partial \xi^2} + \xi \frac{\partial \Psi}{\partial \xi} + \frac{\partial^2 \Psi}{\partial \theta^2} = 0, \tag{2·10}$$

and

$$\frac{\partial \Phi}{\partial \tau} = \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \Phi}{\partial \xi} + \frac{1}{\xi^2}\frac{\partial^2 \Phi}{\partial \theta^2}, \tag{2·11}^*$$

while the conditions (2·7) and (2·8) reduce to

$$\Psi \sim \frac{\xi}{2\mathrm{i}}\left[\mathrm{e}^{\mathrm{i}(\theta-\beta\tau)} - \mathrm{e}^{-\mathrm{i}(\theta-\beta\tau)}\right] \quad (\xi \to \infty), \tag{2·12}$$

* In writing equation (2·11) the variation of the unit vectors i_r and i_θ have been neglected in Moffatt (1965).

and
$$\frac{\partial\Psi}{\partial\xi} = \frac{\partial\Phi}{\partial\xi}, \quad \frac{\partial\Psi}{\partial\theta} = \frac{\partial\Phi}{\partial\theta} \quad \text{at} \quad \xi = 1. \tag{2·13}$$

The most general solution of (2·10) satisfying the condition (2·12) is

$$\Psi = \xi\sin(\theta-\beta\tau) + \sum_{n=1}^{\infty}\frac{D_n(\tau)}{\xi^n}\sin[n\theta+\alpha_n(\tau)], \tag{2·14}$$

where the amplitudes $D_n(\tau)$ and the phase lag $\alpha_n(\tau)$ are to be determined from the boundary conditions.

If we introduce the Fourier transform

$$G(\xi,\theta,\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\xi,\theta,\tau)\,e^{i\mu\tau}\,d\tau, \tag{2·15}$$

the equation (2·11) reduces to

$$\frac{\partial^2 G}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial G}{\partial\xi}+\frac{1}{\xi^2}\frac{\partial^2 G}{\partial\theta^2}+i\mu G = 0. \tag{2·16}$$

The general solution of this equation, which is finite as $\xi \to 0$, is

$$G = \sum_{n=1}^{\infty}[B_n(\mu)\cos n\theta + C_n(\mu)\sin n\theta]\,J_n(\sqrt{(i\mu)}\,\xi).$$

On inversion, we get

$$\Phi = \int_{-\infty}^{\infty}\sum_{n=1}^{\infty}[B_n(\mu)\cos n\theta + C_n(\mu)\sin n\theta]\,J_n(\sqrt{(i\mu)}\,\xi)\,e^{-i\mu\tau}\,d\mu. \tag{2·17}$$

Applying the boundary conditions (2·13), we obtain

$$D_1(\tau)\cos\alpha_1(\tau) = \left\{\frac{J_1(\sqrt{(i\beta)})}{\sqrt{(i\beta)}\,J_0(\sqrt{(i\beta)})}+\text{c.c.}\right\}-\cos\beta\tau,$$

$$D_1(\tau)\sin\alpha_1(\tau) = \frac{1}{i}\left\{\frac{J_1(\sqrt{(i\beta)})}{\sqrt{(i\beta)}\,J_0(\sqrt{(i\beta)})}-\text{c.c.}\right\}+\sin\beta\tau,$$

$$D_n(\tau) = 0 \quad (n \geqslant 2),$$

$$C_1(\mu) = \frac{1}{\sqrt{(i\mu)}\,J_0(\sqrt{(i\mu)})}\{\delta(\mu+\beta)+\delta(\mu-\beta)\},$$

$$B_1(\mu) = -\frac{1}{\sqrt{(i\mu)}\,J_0(\sqrt{(i\mu)})}\{\delta(\mu+\beta)-\delta(\mu-\beta)\},$$

$$C_n(\mu) = B_n(\mu) = 0 \quad (n \geqslant 2),$$

where c.c. denotes the complex conjugates and $\delta(x)$ is the Dirac delta function. Substituting these values, we finally have

$$\Phi(\xi,\theta,\tau) = \frac{1}{i}\left[\frac{J_1(\sqrt{(i\beta)}\,\xi)}{\sqrt{(i\beta)}\,J_0(\sqrt{(i\beta)})}\,e^{i(\theta-\beta\tau)}-\text{c.c.}\right], \tag{2·18}$$

and
$$\Psi(\xi,\theta,\tau) = (\xi-1/\xi)\sin(\theta-\beta\tau)+\frac{1}{\xi i}\left[\frac{J_1(\sqrt{(i\beta)})}{\sqrt{(i\beta)}\,J_0(\sqrt{(i\beta)})}\,e^{i(\theta-\beta\tau)}-\text{c.c.}\right]. \tag{2·19}$$

From (2·2), the rate of vorticity production is given by

$$\frac{\partial\omega_z}{\partial\tau} = -\frac{B_0^2 a}{4\pi\rho\lambda\xi}\left[\frac{\partial\Phi}{\partial\tau}\left(\frac{\partial^2\Phi}{\partial\xi^2}+\frac{\partial^2\Phi}{\partial\xi\,\partial\theta}\right)+\frac{\partial\Phi}{\partial\xi}\left(\frac{\partial^2\Phi}{\partial\tau\,\partial\xi}+\frac{\partial^2\Phi}{\partial\tau\,\partial\theta}\right)\right], \tag{2·20}$$

which on integration gives vorticity field. Knowing the divergence and the curl of of the velocity field, by applying Jacob's (1948) theorem, the velocity field can be obtained.

3. Discussion

From (2·18) it is evident that the magnetic potential is of the form

$$\Phi(\xi, \theta, \tau) = D(\xi) \sin [\theta - \beta\tau + r(\xi)]. \tag{3·1}$$

Consequently, for any particular ξ, Φ attains all values from $-|D(\xi)|$ to $+|D(\xi)|$, which for small or moderate values of β are fairly independent of β. The field line $\Phi = 0$ passes through the centre. Other field lines have a constant phase shift owing to the drift of the magnetic field lines resulting from the finite conductivity of the material opposing the rotation of the field and it diminishes as the surface is reached. As ξ approaches unity, the phase lag becomes small. The outside field exhibits very little shift. For $\sqrt{\beta} = 0{\cdot}1, 0{\cdot}5, 1, 2,$ and 10 we have drawn the field lines and the above conclusions are also evident from figures 1 to 5. In the last case, namely $R_M = 100$, the field line $\Phi = 0$ exhibits spiralling tendency. In this case the field lines show crowding only near the entry region and not everywhere near the cylindrical surface. In a subsequent paper, we shall study the stability of the above solution.

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