ON THE SUMMABILITY OF THE CONJUGATE SERIES OF FOURIER SERIES.

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Received December 13, 1937.
(Communicated by Dr. S. Chowla.)

1. Let $f(x)$ be a function integrable in the sense of Lebesgue and periodic with period 2π. Let

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \tag{1·1}$$

be the conjugate series to the Fourier series corresponding to $f(x)$.

It was shown[1] by Paley that the series (1·1) is summable (c, δ) for $\delta > 1$ to the conjugate function

$$\frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot \frac{t}{2}\, dt, \tag{1·2}$$

provided the above integral exists, where

$$\psi(t) = f(x+t) - f(x-t).$$

Later on, it was shown[2] by Dr. B. N. Prasad that the condition of the existence of the integral (1·2) can be removed off and he proved that the series (1·1) is summable (c, δ) for $\delta > 1$ to the generalised integral

$$\frac{1}{4\pi} \int_0^{\pi} \Psi(t) \operatorname{cosec}^2 \frac{t}{2}\, dt, \tag{1·3}$$

which integral may exist even if the integral (1·2) is divergent, provided

$$\frac{\Psi(t)}{t} = O(1) \tag{1·4}$$

or

$$\int_0^t \left| \frac{\Psi(t)}{t} \right| dt = o(t), \tag{1·5}$$

where

$$\Psi(t) = \int_0^t \psi(t)\, dt.$$

[1] Paley (2).
[2] B. N. Prasad (4).

The object of this paper is to prove that for the summability (c, δ) for $\delta > 1$, of the series (1·1) even the conditions (1·4) and (1·5) can be relaxed, and the theorem can be stated as follows :

THEOREM : *The conjugate series*

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx)$$

is summable (c, δ) *for* $\delta > 1$ *to the sum*

$$\frac{1}{4\pi} \int_0^{\pi} \Psi(t) \operatorname{cosec}^2 \frac{t}{2}\, dt,$$

at a point x, where this integral exists at least as a non-absolutely convergent integral.

My thanks are due to Dr. B. N. Prasad for his keen interest in the preparation of this paper.

2. The Rieszian means being equivalent to Cesàro means, we shall use the former in the present investigation and shall employ Young's function[3] and, in particular, its following properties :

If $C_p(t)$ denote the Young's function, then

$$C_p(t) = \frac{t^p}{\Gamma(1+p)} \left[1 - \frac{t^2}{(p+1)(p+2)} + \frac{t^4}{(p+1)(p+2)(p+3)(p+4)} - \cdots \right], \quad p \geqslant 0 \qquad (2\cdot1)$$

$$C_p(u) = \frac{u^p}{\Gamma p} \int_0^1 (1-t)^{p-1} \cos ut\, dt, \qquad (2\cdot2)$$

$$\frac{d}{du} C_p(u) = C_{p-1}(u), \quad (p \geqslant 1), \qquad (2\cdot3)$$

$$C_{p+2}(u) = \frac{u^p}{\Gamma(1+p)} - C_p(u) \qquad (2\cdot4)$$

and

for large values of u,

$$C_p(u) = O(1), \qquad (p \leqslant 2) \qquad (2\cdot5)$$

$$C_p(u) = O(u^{p-2}), \quad (p > 2) \qquad (2\cdot6)$$

3. In what follows we shall require the following Lemmas :

Lemma 1. *If* $p > 1$, *the function* $u^{-p} C_p(u)$ *is of bounded variation in* $(0, \infty)$ *and tends to zero as* u *tends to infinity.*[4]

[3] E. W. Hobson (1).

[4] Young (5).

Lemma 2. *If* $p > 0$, *the function* $u^{-p} C_p(u)$ *is of bounded variation in every finite interval.*[5]

Lemma 3. *When* $f(x)$ *is periodic with period* 2π *and integrable in the sense of Lebesgue, then*

$$\Psi(t) = \int_0^t \psi(t)\, dt = \int_0^t \{f(x+t) - f(x-t)\}\, dt$$

is a periodic function with period 2π *and bounded.*[6]

Lemma 4. *If*

$$\frac{1}{\pi} \int_0^\infty \frac{\Psi(t)}{t^2}\, dt$$

is convergent, then it is equivalent to the integral

$$\frac{1}{4\pi} \int_0^\pi \Psi(t) \operatorname{cosec}^2 \frac{t}{2}\, dt,$$

provided

$\Psi(t)$ *is periodic.*[7]

4. *Proof of the Theorem*: Working in Rieszian means, in order to prove that the conjugate series (1·1) be summable (c, δ) for $\delta > 1$, we have to show[8] that

$$\sigma_\delta(\omega) = \frac{\Gamma(1+\delta)}{\pi} \int_0^\infty \psi\left(\frac{t}{\omega}\right) t^{-(1+\delta)} C_{2+\delta}(t)\, dt \tag{4·1}$$

tends to the integral

$$\frac{1}{\pi} \int_0^\infty \frac{\Psi(t)}{t^2}\, dt,$$

as $\omega \to \infty$.

Since $\delta > 1$, put $\delta = 1 + \eta$, where $\eta > 0$ in (4·1). Then

$$\sigma_{1+\eta}(\omega) = \frac{\Gamma(2+\eta)}{\pi} \int_0^\infty \psi\left(\frac{t}{\omega}\right) t^{-(2+\eta)} C_{3+\eta}(t)\, dt$$

$$= \frac{1}{\pi} \int_0^\infty \frac{\psi\left(\frac{t}{\omega}\right)}{t}\, dt - \frac{\Gamma(2+\eta)}{\pi} \int_0^\infty \frac{\psi\left(\frac{t}{\omega}\right)}{t} t^{-(1+\eta)} C_{1+\eta}(t)\, dt,$$

by virtue of (2·4).

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[5] B. N. Prasad (3).

[6] B. N. Prasad (4).

[7] B. N. Prasad (4).

[8] B. N. Prasad (3).

Integrating by parts, we get

$$\sigma_{1+\eta}(\omega) = \frac{1}{\pi}\left[\frac{\omega}{t}\Psi\left(\frac{t}{\omega}\right)\left\{1 - \Gamma(2+\eta)\, t^{-(1+\eta)}\, C_{1+\eta}(t)\right\}\right]_0^\infty + \frac{1}{\pi}\int_0^\infty \frac{\Psi(t)}{t^2}\,dt$$

$$+ \frac{\Gamma(2+\eta)}{\pi}\int_0^\infty \omega\,\frac{\Psi\left(\frac{t}{\omega}\right)}{t^2}\, t^{-\eta}\, C_\eta(t)\,dt$$

$$- \frac{\Gamma(3+\eta)}{\pi}\int_0^\infty \omega\,\frac{\Psi\left(\frac{t}{\omega}\right)}{t^2}\, t^{-(1+\eta)}\, C_{1+\eta}(t)\,dt.$$

From Lemma 3, $\Psi(t)$ being bounded in $(0, \infty)$,

$$\lim_{t\to\infty} \frac{\Psi(t)}{t} = 0.$$

Also by virtue of Lemma 1, $t^{-(1+\eta)}\, C_{1+\eta}(t)$ tends to 0 as $t \to \infty$.

Therefore the expression within the square brackets vanishes when $t = \infty$.

Substituting the expansion of $t^{-(1+\eta)}\, C_{1+\eta}(t)$ from (2·1) and taking into consideration that

$$\Psi(0) = 0,$$

we find that the expression within the square brackets also vanishes when $t = 0$.

Therefore

$$\sigma_{1+\eta}(\omega) = \frac{1}{\pi}\int_0^\infty \frac{\Psi(t)}{t^2}\,dt + \frac{\Gamma(2+\eta)}{\pi}\int_0^\infty \omega\,\frac{\Psi\left(\frac{t}{\omega}\right)}{t^2}\, t^{-\eta}\, C_\eta(t)\,dt$$

$$- \frac{\Gamma(3+\eta)}{\pi}\int_0^\infty \omega\,\frac{\Psi\left(\frac{t}{\omega}\right)}{t^2}\, t^{-(1+\eta)}\, C_{1+\eta}(t)\,dt$$

$$\equiv I_1 + \frac{\Gamma(2+\eta)}{\pi} I_2 - \frac{\Gamma(3+\eta)}{\pi} I_3 \qquad (4\cdot 2)$$

Let us now evaluate I_2.

$$I_2 = \int_0^\infty \frac{\Psi(t)}{t^2}\,(\omega t)^{-\eta}\, C_\eta(\omega t)\,dt.$$

Now, since

$$\int_0^\infty \frac{\Psi(t)}{t^2}\,dt$$

is convergent, we may put

$$\chi(t) = \int_0^t \frac{\Psi(t)}{t^2}\,dt,$$

where

$$\operatorname*{Lim}_{t \to 0} \chi(t) = 0.$$

Hence corresponding to an arbitrarily chosen small positive number ϵ, another positive number k can be found out such that

$$|\chi(t)| < \epsilon \quad \text{for } 0 \leq t \leq k.$$

Let us now divide the infinite interval $(0, \infty)$ into two parts $(0, k)$ and (k, ∞).

$$I_2 = I_2^{(1)} + I_2^{(2)} = \int_0^k + \int_k^\infty$$

Since $\Psi(t)$ and $C_\eta(t)$ are bounded in (o, ∞) by virtue of $(2\cdot 5)$, we have

$$|I_2^{(2)}| \leqslant \frac{M}{\omega^\eta} \int_k^\infty \frac{1}{t^{2+\eta}}\, dt$$

$$= 0\,(1), \text{ as } \omega \to \infty,$$

where M is the maximum value of $|\Psi(t) \cdot C_\eta(\omega t)|$ in (k, ∞), since k is independent of ω.

Now, since $\dfrac{\Psi(t)}{t^2}$ is integrable and $(t\omega)^{-\eta}\, C_\eta(\omega t)$ is of bounded variation in $(0, k)$ by virtue of Lemma 2, we have on integration by parts

$$I_2^{(1)} = \Big[\chi(t) \cdot (\omega t)^{-\eta}\, C_\eta(\omega t)\Big]_0^k - \int_0^k \chi(t) \cdot \frac{d}{dt}\Big\{(\omega t)^{-\eta}\, C_\eta(\omega t)\Big\}\, dt$$

$$= 0\,(1) - \int_0^k \chi(t)\, \frac{d}{dt}\Big\{(\omega t)^{-\eta}\, C_\eta(\omega t)\Big\}\, dt.$$

$$|I_2^{(1)}| < 0\,(1) + \epsilon \int_0^k \left| \frac{d}{dt}\Big\{(\omega t)^{-\eta}\, C_\eta(\omega t)\Big\}\right| dt$$

$$= 0\,(1) + \epsilon\, V_0^k \Big\{(\omega t)^{-\eta}\, C_\eta(\omega t)\Big\}$$

$$= 0\,(1), \text{ as } \omega \to \infty,$$

since ϵ is arbitrarily small.

Therefore

$$\operatorname*{Lim}_{\omega \to \infty} I_2 = 0. \tag{4·3}$$

Also, since I_3 is obtained from I_2 by putting $1 + \eta$ for η, the relation

$$\operatorname*{Lim}_{\omega \to \infty} I_3 = 0 \tag{4·4}$$

is *a fortiori* satisfied.

Hence, from (4·2), (4·3) and (4·4), we have

$$\lim_{\omega \to \infty} \sigma_{1+\eta}(\omega) = \frac{1}{\pi} \int_0^\infty \frac{\Psi(t)}{t^2}\, dt$$

$$= \frac{1}{4\pi} \int_0^\pi \Psi(t) \operatorname{cosec}^2 \frac{t}{2}\, dt,$$

by virtue of Lemma 4.

This completes the proof of the Theorem.

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