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On the zeros of $\zeta^{(l)}(s) - a$ (on the zeros of a class of a generalized Dirichlet series – XVII)*

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Abstract. Some very precise results (see Theorems 4 and 5) are proved about the a-values of the lth derivative of a class of generalized Dirichlet series, for $l \geqslant l_0 = l_0(a)$ (l_0 being a large constant). In particular for the precise results on the zeros of $\zeta^{(l)}(s) - a$ (a any complex constant and $l \geqslant l_0$) see Theorems 1 and 2 of the introduction.

Keywords. Riemann zeta function; generalized Dirichlet series; derivatives; distribution of zeros.

1. Introduction

The object of this paper is to prove the following two theorems.

Theorem 1. *Let* $\delta = \left(\log\left(\frac{\log 3}{\log 2}\right)\right)\left(\log\frac{3}{2}\right)^{-1}$. *There exists an effective constant* $\varepsilon_0 > 0$ *such that if* ε *is any constant satisfying* $0 < \varepsilon \leqslant \varepsilon_0$, *then the rectangle*

$$\left\{\sigma \geqslant l(\delta - \varepsilon), 2k\pi\left(\log\frac{3}{2}\right)^{-1} \leqslant t \leqslant (2k+2)\pi\left(\log\frac{3}{2}\right)^{-1}\right\}$$

contains precisely one zero of $\zeta^{(l)}(s)$, *provided* l *exceeds a constant* $l_0 = l_0(\varepsilon)$ *depending only on* ε. *This zero is a simple zero. Moreover this zero does not lie on the boundary of this rectangle and further lies in*

$$\sigma \leqslant l(\delta + \varepsilon).$$

Here as usual $s = \sigma + it$ *and* k *is any integer, positive negative or zero.*

Theorem 2. *Let* $\delta = (\log\log 15)(\log 15)^{-1}$ *and* a *any non-zero complex constant. There exists an effective constant* $\varepsilon_0 > 0$ *such that if* ε *is any constant satisfying* $0 < \varepsilon \leqslant \varepsilon_0$, *then the rectangle*

$$\{\sigma \geqslant l(\delta - \varepsilon), T_0 - \pi(\log 15)^{-1} \leqslant t \leqslant T_0 + \pi(\log 15)^{-1}\}$$

where $T_0 = (\mathrm{Im}\ \log\frac{1}{a} + \pi l + 2k\pi)(\log 15)^{-1}$, *contains precisely one zero of* $\zeta^{(l)}(s) - a$, *provided* l *exceeds an effective constant* $l_0 = l_0(a, \varepsilon)$ *depending only on* a *and* ε. *This zero is a simple zero. Moreover this zero does not lie on the boundary of this rectangle and further lies in*

$$\sigma \leqslant l(\delta + \varepsilon).$$

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*Dedicated to Prof. Paul Erdös on his eighty-first birthday

Here k is any integer, positive negative or zero.

Remark. In [1] we dealt with slightly different questions on the zeros in $\sigma > \frac{1}{2}$ of $\zeta^{(l)}(s) - a$ where a is any complex constant and l is any fixed positive integer. Interested reader may consult this paper. However the results of the present paper deal with large l and are more precise.

The main ingredient of the proof of Theorems 1 and 2 (and the more general results to be stated and proved in § 3 and § 4) is the following theorem (see Theorem 3.42 on page 116 on [2]).

Theorem 3. (Rouché's Theorem). *If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C, and $|g(z)| < |f(z)|$ on C then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.*

Remark 1. In what follows we use s in place of z.

Remark 2. It is somewhat surprising that we can prove (with the help of Theorem 3) Theorems 4 and 5, which are much more general than Theorems 1 and 2. These will be stated in § 3 and § 4 respectively.

Remark 3. Theorems 4 and 5 can be generalized to include derivatives of ζ and L functions and also of ζ function of ray classes of any algebraic number field and so on. But we have not done so.

2. Notation

$\{\lambda_n\}(n = 1, 2, 3, \ldots)$ will denote any sequence of real numbers with $\lambda_1 = 1$ and $\frac{1}{A} \leqslant \lambda_{n+1} - \lambda_n \leqslant A$ where $A(\geqslant 1)$ is any fixed constant. $\{a_n\}(n = 1, 2, 3, \ldots)$ will denote any sequence of complex numbers with $a_1 = 1$ and $|a_n| \leqslant n^A$. k will be any integer, positive negative or zero. $\delta_n(n \geqslant 2)$ will denote $(\log\log \lambda_n)(\log \lambda_n)^{-1}$

3. A generalization of Theorem 1

Theorem 4. *Let $n_0 > 1$ be any integer, $|a_{n_0}| > A^{-1}$, $|a_{n_0+1}| > A^{-1}$ and $\delta = \left(\log\left(\frac{\log \lambda_{n_0+1}}{\log \lambda_{n_0}}\right)\right) \times \left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1}$. Also let $\lambda_{n+1} < \lambda_n^2$ for all $n > 1$. There exists an effective constant ε_0 such that if ε is any constant satisfying $0 < \varepsilon \leqslant \varepsilon_0$, then the rectangle*

$$\left\{\sigma \geqslant l(\delta - \varepsilon),\ T_0 + 2k\pi\left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1} \leqslant t \leqslant T_0 + (2k+2)\pi\left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1}\right\}$$

where $T_0 = \left(\operatorname{Im}\log\left(\frac{a_{n_0+1}}{a_{n_0}}\right)\right)\left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1}$, contains precisely one zero of the analytic function

$$\sum_{n \geqslant n_0} a_n (\log \lambda_n)^l \lambda_n^{-s}$$

provided l exceeds an effective positive constant $l_0 = l_0(A, \varepsilon, n_0)$ depending only on the parameters indicated. This zero is a simple zero. Moreover this zero does not lie on the boundary of this rectangle and further lies in

$$\sigma \leqslant l(\delta + \varepsilon).$$

Remark. Theorem 1 follows by taking $n_0 = 2, \lambda_n = n$ and $a_n = 1$ for all n.

The following lemma will be used in this section and also while applying Theorem 5 of § 4 to deduce Theorem 2.

Lemma 1. *For any $\delta > 0$ the function $(\log x)x^{-\delta}$ (of x in $x \geqslant 1$) is increasing for $1 \leqslant x \leqslant \exp(\delta^{-1})$ and decreasing for $x \geqslant \exp(\delta^{-1})$. It has precisely one maximum at $x = \exp(\delta^{-1})$.*

Remark. The maximum value is $(e\delta)^{-1}$. The proof of this lemma is trivial and will be left as an exercise.

To prove Theorem 4 we apply Theorem 3 to

$$f(s) = 1 + \left(\frac{a_{n_0+1}}{a_{n_0}}\right)\left(\frac{\log \lambda_{n_0+1}}{\log \lambda_{n_0}}\right)^l \left(\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-s}$$

and

$$g(s) = \sum_{n \geqslant n_0+2} a'_n \left(\frac{\log \lambda_n}{\log \lambda_{n_0}}\right)^l \left(\frac{\lambda_n}{\lambda_{n_0}}\right)^{-s}$$

where $a'_n = a_n(a_{n_0})^{-1}$. It suffices to prove that $f(s) + g(s)$ has its zeros as claimed in Theorem 4.

Lemma 2. *The zeros of $f(s)$ are all simple and are given by $s = s_0$ where*

$$s_0 = \left(\log(-a'_{n_0+1}) + l \log\left(\frac{\log \lambda_{n_0+1}}{\log \lambda_{n_0}}\right)\right)\left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1},$$

for all possible values of $\log(-a'_{n_0+1})$. If $s_0 = \sigma_0 + it_0$ then

$$\sigma_0 = \left(\log|a'_{n_0+1}| + l \log\left(\frac{\log \lambda_{n_0+1}}{\log \lambda_{n_0}}\right)\right)\left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1},$$

and

$$t_0 = (\operatorname{Im} \log(-a'_{n_0+1}))\left(\log \frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1}.$$

Also

$$f(s) = 1 - \left(\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-s+s_0}.$$

Proof. The proof is trivial.

Lemma 3. *For $\sigma \geqslant 200A$, we have*

$$|g(s)| \leqslant \left(\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-\sigma+\sigma_0} S$$

where

$$S=\sum_{n\geqslant n_0+2}|a_n||a_{n_0+1}|^{-1}\left(\frac{\log\lambda_{n_0}}{\log\lambda_{n_0+1}}\right)^l\left(\frac{\lambda_n}{\lambda_{n_0+1}}\right)^{-\sigma}.$$

Proof. The proof follows from

$$|g(s)|\leqslant\sum_{n\geqslant n_0+2}|a_n'|\left(\frac{\log\lambda_n}{\log\lambda_{n_0}}\right)^l\left(\frac{\lambda_n}{\lambda_{n_0}}\right)^{-\sigma}$$

$$=\sum_{n\geqslant n_0+2}|a_n'|\left(\frac{\log\lambda_n}{\log\lambda_{n_0}}\right)^l\left(\frac{\lambda_n}{\lambda_{n_0+1}}\right)^{-\sigma}\left(\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-\sigma}$$

and the fact that

$$\left(\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{\sigma_0}=|a_{n_0+1}'|\left(\frac{\log\lambda_{n_0+1}}{\log\lambda_{n_0}}\right)^l.$$

Remark. Hereafter we write $\sigma_0=\delta_0 l$ and

$$\delta_0=l^{-1}(\log|a_{n_0+1}'|)\left(\log\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1}+\delta.$$

Also we remark that the condition $\sigma\geqslant l(\delta_0-\varepsilon)$ is the same as $\sigma\geqslant l(\delta-\varepsilon)$ with a change of ε.

Lemma 4. *Let $S=S(\sigma)$. Then for $\sigma\geqslant l(\delta-\varepsilon)$ we have,*

$$S(\sigma)<\frac{1}{1000},$$

provided $l\geqslant l_0=l_0(A,\varepsilon,n_0)$, which is effective.

To prove this lemma it suffices to prove that

$$S(l(\delta-\varepsilon))<\frac{1}{1000}.$$

This will be done in two stages. We have (by Lemma 3)

$$S(l(\delta-\varepsilon))=\sum_{n\geqslant n_0+2}|a_n||a_{n_0+1}|^{-1}\left\{\left(\frac{\log\lambda_n}{\log\lambda_{n_0+1}}\right)\left(\frac{\lambda_n}{\lambda_{n_0+1}}\right)^{-\delta+\varepsilon}\right\}^l.$$

In Lemma 5 we prove that $\exp(\delta^{-1})<\lambda_{n_0+1}$ and so by Lemma 1 it follows that $(\log\lambda_n)\lambda_n^{-\delta}$ is decreasing for $n\geqslant n_0+2$. Hence it suffices to prove that

$$\left(\frac{\log\lambda_{n_0+2}}{\log\lambda_{n_0+1}}\right)\left(\frac{\lambda_{n_0+2}}{\lambda_{n_0+1}}\right)^{-\delta+\varepsilon}<1.$$

This will be done in Lemma 6. This would complete the proof of Lemma 4 since for all large n

$$\left(\frac{\log\lambda_n}{\log\lambda_{n_0+1}}\right)\left(\frac{\lambda_n}{\lambda_{n_0+1}}\right)^{-\delta+\varepsilon}$$

is less than a negative constant power of λ_n.

Lemma 5. *We have*

$$\exp(\delta^{-1}) < \lambda_{n_0+1}.$$

Proof. Since for $0 < x < 1$ we have $-\log(1-x) > x$, it follows that

$$\delta = \left(-\log\left(1 - \left(1 - \frac{\log \lambda_{n_0}}{\log \lambda_{n_0+1}}\right)\right)\right)\left(\log\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1}$$
$$> \left(1 - \frac{\log \lambda_{n_0}}{\log \lambda_{n_0+1}}\right)\left(\log\frac{\lambda_{n_0+1}}{\lambda_{n_0}}\right)^{-1}$$
$$= (\log \lambda_{n_0+1})^{-1}.$$

This proves the lemma.

Lemma 6. *We have*

$$\left(\frac{\log \lambda_{n_0+2}}{\log \lambda_{n_0+1}}\right)\left(\frac{\lambda_{n_0+1}}{\lambda_{n_0+2}}\right)^{\delta} < 1.$$

Proof. We have $\lambda_{n_0+2} < \lambda_{n_0+1}^2$ and also for $0 < x < 1$ we have $\log(1+x) < x$. Using these we obtain

$$\left(1 + \left(\log\frac{\lambda_{n_0+2}}{\lambda_{n_0+1}}\right)(\log \lambda_{n_0+1})^{-1}\right)^{\log \lambda_{n_0+1}} < \frac{\lambda_{n_0+2}}{\lambda_{n_0+1}}$$

and so

$$\left(\frac{\log \lambda_{n_0+2}}{\log \lambda_{n_0+1}}\right)\left(\frac{\lambda_{n_0+1}}{\lambda_{n_0+2}}\right)^{(\log \lambda_{n_0+1})^{-1}} < 1$$

and since $(\log \lambda_{n_0+1})^{-1} < \delta$, we obtain Lemma 6. Lemmas 2 and 4 complete the proof of Theorem 4.

4. A generalization of Theorem 2

Theorem 5. *Let* δ_{n_1} *be the maximum of* δ_n *taken over all* n *for which* $a_n \neq 0$ *and* $n > 1$. *Suppose that for all* $n \neq 1, n_1$ *we have* $\delta_{n_1} - \delta_n \geqslant A^{-1}$ *and also* $\lambda_{n_1} - e \geqslant A^{-1}$. *We further suppose that* $|a_{n_1}| \geqslant A^{-1}$ *and put* $\delta_{n_1} = \delta$. *There exists an effective constant* ε_0 *such that for all* ε *satisfying* $0 < \varepsilon \leqslant \varepsilon_0$, *the rectangle*

$$\{\sigma \geqslant l(\delta - \varepsilon),\ T_0 - \pi(\log \lambda_{n_1})^{-1} \leqslant t \leqslant T_0 + \pi(\log \lambda_{n_1})^{-1}\}$$

where $T_0 = (\operatorname{Im} \log(-a_{n_1}) + 2k\pi)(\log \lambda_{n_1})^{-1}$, *contains precisely one zero of the analytic function*

$$1 + \sum_{n=2}^{\infty} a_n (\log \lambda_n)^l \lambda_n^{-s}$$

provided l *exceeds an effective constant* $l_0 = l_0(A, \varepsilon, n_1)$ *depending only on the parameters indicated. This zero is a simple zero. Moreover this zero does not lie on the boundary of* *this rectangle and further lies in*

$$\sigma \leqslant l(\delta + \varepsilon).$$

Remark. Theorem 2 follows by taking $\lambda_n = n$ and $a_n = (-1)^{l+1} a^{-1}$ for all $n \geqslant 2$. Note that the maximum of δ_n occurs when $n = 15$. It is necessary to check that $\delta_{15} > \delta_{16}$. In fact we have

$$e^e = 15{\cdot}21\ldots, \log_{10}\delta_{15}^{-1} = 0{\cdot}434357\ldots \text{ and } \log_{10}\delta_{16}^{-1} = 0{\cdot}434455\ldots,$$

by using tables.

To prove Theorem 5 we apply Theorem 3 to

$$f(s) = 1 + a_{n_1}(\log \lambda_{n_1})^l \lambda_{n_1}^{-s}$$

and

$$g(s) = \sum^{*} a_n (\log \lambda_n)^l \lambda_n^{-s}$$

where the asterisk denotes the restrictions $n \neq 1, n_1$.

Lemma 1. *The zeros of* $f(s)$ *are all simple and are given by* $s = s_0$ *where*

$$s_0 = (\log(-a_{n_1}) + l \log\log \lambda_{n_1})(\log \lambda_{n_1})^{-1}$$

for all possible values of $\log(-a_{n_1})$. *If* $s = \sigma_0 + it_0$, *then*

$$\sigma_0 = (\log|a_{n_1}| + l \log\log \lambda_{n_1})(\log \lambda_{n_1})^{-1}$$

and

$$t_0 = (\operatorname{Im} \log(-a_{n_1}))(\log \lambda_{n_1})^{-1}.$$

Also

$$f(s) = 1 - \lambda_{n_1}^{-s+s_0}.$$

Remark. We write $\sigma_0 = \delta_0 l$ and $\delta_0 = l^{-1}(\log|a_{n_1}|)(\log \lambda_{n_1})^{-1} + \delta$. The condition $\sigma \geqslant l(\delta_0 - \varepsilon)$ is the same as $\sigma \geqslant l(\delta - \varepsilon)$ with a change of ε.

Proof. The proof is trivial.

Lemma 2. *For* $\sigma \geqslant l(\delta - \varepsilon)$, *we have*

$$|g(s)| \leqslant \sum^{*} |a_n| (\log \lambda_n)^l \lambda_n^{-l\delta + l\varepsilon}.$$

Proof. LHS is trivially not more than

$$\sum^{*} |a_n| (\log \lambda_n)^l \lambda_n^{-\sigma}$$

for all $\sigma \geqslant 200A$. This proves the lemma.

Lemma 3. *We have for* $\sigma \geqslant l(\delta - \varepsilon)$,

$$|g(s)| \leqslant \frac{1}{1000}.$$

Proof. Using $\log\lambda_n = (\lambda_n)^{\delta_n}$ we obtain, by Lemma 2,

$$|g(s)| \leqslant \sum_n^{*} |a_n| (\lambda_n^{-(\delta-\delta_n)+\varepsilon})^l.$$

By the hypothesis of Theorem 5 we see that $\delta - \delta_n \geqslant A^{-1}$ (note also that $\lambda_{n_1} - e \geqslant A^{-1}$ so that $\delta \geqslant \dfrac{\log\log(e+A^{-1})}{\log(e+A^{-1})}$ if $\lambda_{n_1} \leqslant e^e$) and so Lemma 3 is proved.

Lemmas 1 and 3 complete the proof of Theorem 5.

Open questions

1) How much can one generalize Theorems 1 and 2?
2) Whatever the integer constant $l \geqslant 1$ and whatever the complex constant a, prove that $\zeta^{(l)}(s) - a$ has infinity of simple zeros in $\sigma > \frac{1}{2}$, (more precisely $\gg T$ simple zeros in $(\sigma \geqslant \frac{1}{2} + \delta,\ T \leqslant t \leqslant 2T)$ for some absolute constant $\delta > 0$).

References

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