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Non-adiabatic Pulsations of a Stellar Model

By

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EDDINGTON, SCHWARZSCHILD, WOLTJER and others have discussed the Non-adiabatic Pulsations of a Star in connection with the problem of phase lag and maintenance and destruction of pulsations in the case of cepheid variable stars, but so far no general method for the solution of the non-adiabatic equation has been given. ROSSELAND, in connection with the problem of secular stability of variable stars, has suggested a systematic method for the solution of the non-adiabatic equation. We have applied this method and obtained the solution of the non-adiabatic pulsation equation in terms of a series of characteristic functions of the corresponding adiabatic equation. We have worked up to higher approximation than the previous workers. In the second part of the paper we have applied the solution obtained to the study of a homogeneous star, and have made a numerical estimate of the period of pulsation in the fundamental mode and the time of relaxation in case of the cepheid variable having mass equal to 5.02 solar masses and radius equal to 16.382 solar radii. These numerical estimates suggest that the departure from the adiabatic conditions must be taken into account more precisely than what has been done in the present paper.

1. Introduction

A number of authors have worked out the non-adiabatic pulsations of special stellar models in connection with the problem of phase lag and the generation and destruction of pulsations in cepheid variables. EDDINGTON [1], assuming radiative equilibrium and a constant ratio of the specific heats, tried to work out the problem of non-adiabatic pulsations with a view to explain the phenomenon of phase lag but he found that small deviations from adiabatic conditions produce too small an effect. SCHWARZSCHILD [5] later considered two stellar models for which the atmospheres, assumed in radiative equilibrium, extended to 0.8 and 2.6% of the stellar radius. He assumed the pulsations in the main body of the star to be adiabatic while in the atmosphere to be non-adiabatic. For simplicity in mathematical analysis he took the various physical quantities as varying in a single frequency. He also tried to work out the case when radiation pressure is taken into account and ε_0, the equilibrium value of the rate of energy generation per unit time per unit mass, is taken to be zero. He found that on account of the singularity of the differential equation at the centre the various physical quantities could not be uniquely specified there. Later, EDDINGTON and WOLTJER pointed out that a solution of the pulsation equation for a star with perfect adiabatic conditions prevailing inside and non-adiabatic outside cannot be worked out if it is required to satisfy the boundary conditions at the centre and also at the surface. They suggested that to get a well-fitted solution one has to take slight deviations from the adiabatic conditions even in the interior of the star.

WOLTJER [8] tried to develop a theory of non-adiabatic pulsations by introducing canonical variables. He used adiabatic Hamiltonian equations introduced by him [7] and Miss KLUYVER [3] and worked out departure from these. He succeeded in obtaining a solution of these modified equations in the form of a series which converged rapidly. This solution, however, satisfied the boundary condition only at the surface, namely the temperature vanishes there; moreover, WOLTJER did not apply his results to any specific variable star. SCHWARZSCHILD [6] taking the equantions of motion from his previous work [5] obtained a solution in case of a stellar model which satisfied the boundary conditions: (i) the temperature vanishes at the surface, (ii) the equilibrium values of density, temperature and the energy generation at the centre are constant and greater than zero, (iii) the radiation flux F and $\frac{\partial \varepsilon}{\partial r}$ vanish at the centre. His calculations yielded a negative result for the phase lag. He, then, tried a solution of the progressive wave type throughout the interior of the star for the pulsation equation (correct to first order approximation) and also a solution in which the standing wave type of pulsations were assumed in the interior and progressive wave type of pulsations in the outer region. His studies gave a phase lag much smaller than the observed one. EDDINGTON [2] suggested that inside the core and the radiative equilibrium zone, the pulsations are adiabatic while in the outermost part where a critical layer of hydrogen is being ionized and deionized, they are non-adiabatic. According to his views a sharp decrease in dissipation supplies the mechanical energy for pulsations and the condition of minimum dissipation determines the phase lag. EDDINGTON's hypothesis, for want of a general solution of the non-adiabatic equation of pulsation could not be tested quantitatively.

ROSSELAND [4] in connection with the problem of secular stability of the variable stars suggested a systematic method for the solution of the non-adiabatic equation. He expressed the displacement in terms of the characteristic functions of the corresponding adiabatic equation and adopted for the energy generation and the absorption coefficients expressions of the form $\varepsilon \propto \varrho^{\mu} T^{\nu}$ and $k \propto \varrho^{n} T^{-s}$. In part A of the present paper, following ROSSELAND, we have taken $\varepsilon \propto \varrho^{\mu} T^{\nu}$ and $k \propto \varrho^{n} T^{-s}$ throughout the stellar model and obtained the formal solution of the non-adiabatic equation correct to second order using the iteration process and taking γ, the ratio of specific heats, as constant. In part B we have applied the solution to a homogenous star. In a subsequent paper the problem of dissipation of energy and the phase lag will be considered.

2. *Non-adiabatic Pulsation Equation*

The equation of motion for radial oscillations, energy and the continuity of mass, in the Lagrangian form, are:

$$\ddot{r} = -\frac{1}{\varrho r'}\frac{\partial p}{\partial a} - g_0\frac{a^2}{r^2}\,; \quad f' = \frac{\partial f}{\partial a}\,, \quad \dot{f} = \frac{\partial f}{\partial t} \tag{2.1}$$

$$\dot{p} = \gamma p \frac{\dot{\varrho}}{\varrho} + \varrho(\gamma - 1)\left[\varepsilon - \frac{\partial H}{\partial m}\right] \tag{2.2}$$

and

$$\varrho r^2 r' = \varrho_0 a^2 \tag{2.3}$$

where r, ϱ, p, g and m denote the radius vector, density, pressure, gravity of an element and mass within r; a, ϱ_0, p, g_0, and m denote the corresponding equilibrium values of these quantities. Further, ε is the rate of energy generation per unit mass per unit time and H, the total radiant flux across the spherical surface of radius r.

If we assume the star to be in radiative equilibrium

$$H = -\frac{16\pi^2 c \bar{a} r^4}{3\varkappa}\frac{\partial T^4}{\partial m} \tag{2.4}$$

where c, $\bar{a}$, $\varkappa$ are velocity of light, the Stefan-Boltzmann constant and the absorption coefficient.

In the following we shall assume that the ratio of specific heats is constant and the gas obeys the perfect gas equation

$$p = \mathrm{k}\,\varrho\,T \quad (\mathrm{k} = \text{Boltzmann's constant})\,. \tag{2.5}$$

We adopt for mathematical simplicity the following laws for ε and k:

$$\varepsilon \propto \varrho^{\mu} \cdot T^{\nu} \tag{2.6}$$

$$\mathrm{k} \propto \varrho^{n} T^{-s}\,. \tag{2.7}$$

Writing

$$r = a(1 + \zeta) \tag{2.8}$$

and retaining only first order terms in ζ, the continuity equation (2.3) gives

$$\frac{\varrho}{\varrho_0} = 1 - 3\zeta - a\zeta'\,. \tag{2.9}$$

We shall now solve the energy equation by the method of successive approximations. For this we take the oscillations to be adiabatic to the zeroth order approximation and hence

$$\dot{p} = \gamma p \frac{\dot{\varrho}}{\varrho} \tag{2.10}$$

or

$$\frac{p}{p_0} = \left(\frac{\varrho}{\varrho_0}\right)^{\gamma} = 1 - 3\,\gamma\,\zeta - a\,\gamma\,\zeta' \tag{2.11}$$

and from the gas equation (2.5)

$$\frac{T}{T_0} = \left(\frac{\varrho}{\varrho_0}\right)^{\gamma-1} = 1 - 3(\gamma - 1)\,\zeta - a(\gamma - 1)\,\zeta'\,. \tag{2.12}$$

Using (2.11) and (2.12) in (2.6), (2.4) and (2.7), we get

$$\varepsilon = \varepsilon_0[1 - 3(\mu + \nu\,\gamma - \nu) - (\mu + \nu\,\gamma - \nu)\,a\,\zeta'] \tag{2.13}$$

and

$$H = H_0\Big[1 + \{16 + 3(n+s) - 3\,\gamma(s+4)\}\,\zeta - \{-(n+s+4) + \gamma(s+4)\}\,a\,\zeta' - \\ - \frac{T_0}{T_0'}(\gamma - 1)\{4\,\zeta' + a\,\zeta''\}\Big] \tag{2.14}$$

so that

$$\varepsilon - \frac{\partial H}{\partial m} = -\,\varepsilon_0\Big[\{16 + 3\,(n + s + \mu - \nu) - 3\,\gamma\,(s + 4 - \nu)\}\,\zeta + \\ + \{4 + (n+s) - \gamma\,(s+4) + \mu + \gamma\,\nu - \nu\}\,a\,\zeta' - (\gamma - 1)\,\frac{T_0}{T_0'}\,(4\,\zeta' + a\,\zeta'')\Big] - \\ - \frac{H_0}{4\,\pi\,a^2\,\varrho_0}\Big[\{20 + 4\,(n+s) - 4\,\gamma(s+4)\}\,\zeta' + \{4 + (n+s) - \gamma\,(s+4)\}\,a\,\zeta'' - \\ - (\gamma - 1)\Big\{\frac{T_0}{T_0'}\,(5\,\zeta'' + a\,\zeta''') + (4\,\zeta' + a\,\zeta'')\,\frac{\partial}{\partial a}\Big(\frac{T_0}{T_0'}\Big)\Big\}\Big] \tag{2.15}$$

where ε_0 and H_0 are the equilibrium values of ε and H satisfying the equation

$$\varepsilon_0 = \frac{\partial H_0}{\partial m}\,. \tag{2.16}$$

Substituting $\varepsilon - \frac{\partial H}{\partial m}$ from (2.15) in (2.2) we have

$$\dot{p} - \gamma\,p\,\frac{-3\,\dot{\zeta} - a\,\dot{\zeta}'}{1 - 3\,\zeta - a\,\zeta'} = -\,(\gamma - 1)\,P \tag{2.17}$$

where

$$P = \varrho_0\,\varepsilon_0\Big[\{16 + 3\,(n + s + \mu - \nu) + 3\,\gamma(\nu - s - 4)\}\,\zeta + \\ + \{4 + (n + s + \mu - \nu) + \gamma\,(\nu - s - 4)\}\,s\,\zeta' - \\ - \frac{T_0}{T_0'}\,(\gamma - 1)\,(4\,\zeta' + a\,\zeta'')\Big] + \\ + \frac{H_0}{4\,\pi\,a^2}\Big[\{20 + 4\,(n+s) - 4\,\gamma\,(s+4)\}\,\zeta' + \\ + \{4 + (n+s) - \gamma\,(s+4)\}\,a\,\zeta'' - (\gamma - 1)\,\frac{T_0}{T_0'}\,(5\,\zeta'' + a\,\zeta''') - \\ - (\gamma - 1)\,(4\,\zeta' + a\,\zeta'')\,\frac{\partial}{\partial a}\Big(\frac{T_0}{T_0'}\Big)\Big]\,. \tag{2.18}$$

The solution of the equation (2.17) is

$$\frac{p}{(1-3\zeta-a\zeta')^{\gamma}}-p_0=-(\gamma-1)\int_0^t P\,dt \tag{2.19}$$

where we have measured time from the instant when the star is passing through the equilibrium configuration, so that when

$$t=0$$
$$p=p_0 \text{ and } \zeta=\zeta'=0\,.$$

(2.19) gives p correct to first order approximation in the non-adiabatic term.

Substituting this value of p in $\varepsilon-\frac{\partial H}{\partial m}$ we get

$$\begin{aligned}\varepsilon-\frac{\partial H}{\partial m}=&-\frac{P}{\varrho_0}+(s+4-\nu)(\gamma-1)\frac{\varepsilon_0}{p_0}\int P\,dt+\\&+(\gamma-1)\,\varepsilon_0\frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{1}{p_0}\int P\,dt\right)\\&+\frac{H_0}{4\pi a^2\varrho_0}\left[(s+4)(\gamma-1)\frac{\partial}{\partial a}\left(\frac{1}{p_0}\int P\,dt\right)+\right.\\&\left.+(\gamma-1)\frac{\partial}{\partial a}\left\{\frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{1}{p_0}\int P\,dt\right)\right\}\right]\end{aligned} \tag{2.20}$$

and then (2.2) gives

$$\begin{aligned}\dot{p}=&-\gamma p_0(3\dot{\zeta}+a\dot{\zeta}')-(\gamma-1)P+\\&+(\gamma-1)^2\varepsilon_0\varrho_0\left[(s+4-\nu)\frac{1}{p_0}\int P\,dt+\frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{1}{p_0}\int P\,dt\right)\right]+\\&+(\gamma-1)^2\frac{H_0}{4\pi a^2}\left[(s+4)\frac{\partial}{\partial a}\left(\frac{1}{p_0}\int P\,dt\right)+\frac{\partial}{\partial a}\left\{\frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{1}{p_0}\int P\,dt\right)\right\}\right].\end{aligned} \tag{2.21}$$

The solution of (2.21) will give the expression for p to the second order approximation in the non-adiabatic term.

Differentiating (2.1) partially with respect to t, we have

$$\begin{aligned}-\frac{\partial}{\partial a}\dot{p}&=\frac{\partial}{\partial t}\{(\ddot{r}+g)\,\varrho\,r'\}\\&=\frac{\partial}{\partial t}\left[\left\{a\ddot{\zeta}+\frac{M(a)}{a^2}G(1-2\zeta)\right\}\varrho_0(1-3\zeta-a\zeta')(1+\zeta+a\zeta')\right]\\&=\varrho_0\dddot{\zeta}-4\varrho_0 g_0\dot{\zeta}\,.\end{aligned} \tag{2.22}$$

Differentiating now (2.21) partially with respect to a and eliminating $\frac{\partial\dot{p}}{\partial a}$ with the help of (2.22), we get

$$\begin{aligned}&\varrho_0 a\dddot{\zeta}+g_0\varrho_0\{(3\gamma-4)\dot{\zeta}+\gamma a\dot{\zeta}'\}-\gamma p_0(4\dot{\zeta}'+a\dot{\zeta}'')\\&=(\gamma-1)\frac{\partial}{\partial a}\left[P-(\gamma-1)\varrho_0\varepsilon_0\left\{(s+4-\nu)\frac{1}{p_0}\int P\,dt+\frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{1}{p_0}\int P\,dt\right)\right\}-\right.\\&\left.-(\gamma-1)\frac{H_0}{4\pi a^2}\left\{(s+4)\frac{\partial}{\partial a}\left(\frac{1}{p_0}\int P\,dt\right)+\frac{\partial}{\partial a}\left(\frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{1}{p_0}\int P\,dt\right)\right)\right\}\right]\end{aligned} \tag{2.23}$$

which on partial differentiation with respect to t gives

$$\varrho_0 a \ddddot{\zeta} + \varrho_0 g_0 \{(3\gamma - 4)\ddot{\zeta} + \gamma a \ddot{\zeta}'\} - 4 p_0 (4\ddot{\zeta}' + a\ddot{\zeta}'')$$
$$= (\gamma - 1)\frac{\partial}{\partial a}\left[\dot{P} - (\gamma - 1)\varrho_0 \varepsilon_0 \left\{(s + 4 - \nu)\frac{P}{p_0} + \frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{P}{p_0}\right)\right\} - \right.$$
$$\left. - (\gamma - 1)\frac{H_0}{4\pi a^2}\left\{(s+4)\frac{\partial}{\partial a}\left(\frac{P}{p_0}\right) + \frac{\partial}{\partial a}\left(\frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{P}{p_0}\right)\right)\right\}\right]. \quad (2.24)$$

Following ROSSELAND we now set

$$\zeta = \sum_k \xi_k q_k(t) \quad (2.25)$$

where ξ_k are the time-independent orthogonal eigen-functions of the adiabatic equation

$$\xi_k'' + \frac{1}{a}\left[4 - \frac{a\varrho_0 g_0}{p_0}\right]\xi_k' + \left[\frac{\varrho_0}{\gamma p_0}\sigma_k^2 - \left(3 - \frac{4}{\gamma}\right)\frac{\varrho_0 g_0}{a p_0}\right]\xi_k = 0 \quad (2.26)$$

normalized according to

$$I_s = \int_0^M (a\,\xi_s)^2\, dm = 1 \quad (2.27)$$

and $q_k(t)$ are functions of time only.

Inserting this expression of ζ in (2.24), multiplying by $4\pi a^3 \xi_s\, da$ and integrating over a from o to R (the Stellar radius), we get

$$\dddot{q}_s + \sigma_s^2 \ddot{q}_s = \sum_k A_{sk}\dot{q}_k + \sum_k B_{sk} q_k \quad (2.28)$$

where

$$A_{sk} = (\gamma - 1)\int_0^R 4\pi a^3 \xi_s\, da \frac{\partial}{\partial a}\{Q_k\} \quad (2.29)$$

and

$$B_{sk} = (\gamma - 1)^2 \int_0^R 4\pi a^3 \xi_s da \frac{\partial}{\partial a}\{R_k\} \quad (2.30)$$

with

$$Q_k = \varrho_0 \varepsilon_0 \left[\{16 + 3(n + s + \mu - \nu) + 3\gamma(\nu - s - 4)\}\xi_k + \right.$$
$$+ \{4 + (n + s + \mu - \nu) - \gamma(s + 4 - \nu)\} a\,\xi_k' -$$
$$\left. - \frac{T_0}{T_0'}(\gamma - 1)(4\xi_k' + a\xi_k'')\right] +$$
$$+ \frac{H_0}{4\pi a^2}\left[\{20 + 4(n+s) - 4\gamma(s+4)\}\xi_k' + \right.$$
$$+ \{4 + (n+s) - \gamma(s+4)\} a\,\xi_k'' - (\gamma - 1)\frac{T_0}{T_0'}(5\xi_k'' + a\xi_k''') -$$
$$\left. - (\gamma - 1)(4\xi_k' + a\xi_k'')\frac{\partial}{\partial a}\left(\frac{T_0}{T_0'}\right)\right] \quad (2.31)$$

and
$$R_k = -\frac{\varepsilon_0 \varrho_0}{p_0}(s+4-\nu)\,Q_k - \varrho_0 \varepsilon_0 \frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{Q_k}{p_0}\right) - \\ -\frac{H_0}{4\pi a^2}\left[(s+4)\frac{\partial}{\partial a}\left(\frac{Q_k}{p_0}\right) + \frac{\partial}{\partial a}\left\{\frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{Q_k}{p_0}\right)\right\}\right]. \tag{2.32}$$

Taking
$$\varepsilon_0 = \bar{\varepsilon}\,\varrho_0^{\mu}\,T_0^{\nu} \tag{2.33}$$
and
$$\varkappa_0 = \bar{\varkappa}\,\varrho_0^{n}\,T_0^{-s} \tag{2.34}$$
we get the following expressions for Q_k and R_k:
$$Q_k = \bar{\varepsilon}\,\varrho_0^{\mu+1}\,T_0^{\nu}\Big[\{16 + 3(n+s+\mu) + 3\,\gamma(\nu - s - 4)\}\,\xi_k + \\ + \{4 + (n+s+\mu-\nu) + \gamma(\nu - s - 4)\}\,a\,\xi_k' - \frac{T_0}{T_0'}(\gamma-1)(4\,\xi_k' + a\,\xi_k'')\Big] - \\ - \frac{4\,c\,\bar{a}}{3\,\bar{\varkappa}}\frac{T_0^{s+3}}{\varrho_0^{n+1}}\,T_0'\Big[\{20 + 4(n+s) - 4\,\gamma\,(s+4)\}\,\xi_k' + \\ + \{4 + (n+s) - \gamma(s+4)\}\,a\,\xi_k'' - \\ - (\gamma-1)\left\{\frac{T_0}{T_0'}(5\,\xi_k'' + a\,\xi_k''') + (4\,\xi_k' + a\,\xi_k'')\frac{\partial}{\partial a}\left(\frac{T_0}{T_0'}\right)\right\}\Big] \tag{2.35}$$
and
$$R_k = -\,\bar{\varepsilon}\,\varrho_0^{\mu+1}\,T_0^{\nu}\left[(s+4-\nu)\frac{Q_k}{p_0} + \frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{Q_k}{p_0}\right)\right] + \\ + \frac{4\,c\,\bar{a}}{3\,\bar{\varkappa}}\frac{T_0^{s+3}}{\varrho_0^{n+1}}\,T_0'\left[(s+4)\frac{\partial}{\partial a}\left(\frac{Q_k}{p_0}\right) + \frac{\partial}{\partial a}\left\{\frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{Q_k}{p_0}\right)\right\}\right], \tag{2.36}$$
where $\bar{\varepsilon}$ and $\bar{\varkappa}$ are related by the condition (2.16) which now becomes
$$\bar{\varepsilon}\,\varrho_0^{\mu}\,T_0^{\nu} = -\frac{4\,c\,\bar{a}}{3\,\bar{\varkappa}}\frac{1}{a^2\,\varrho_0^{n+2}}\left[(s+3)\,T_0^{s+2}a^2\,T_0'^{\,2} + 2\,T_0^{s+3}\,a\,T_0' + \\ + T_0^{s+3}\,a^2\,T_0''\right]. \tag{2.37}$$

Equation (2.28) is the fundamental equation of this paper and applies to all stars for which (2.4), (2.5), (2.6) and (2.7) hold.

Taking energy generation by the $C\,N$ cycle, KRAMERS' law for the opacity and the gas to be monoatomic,
$$\mu = 1,\ \nu = 16;\ n = 1,\ s = 3.5 \text{ and } \gamma = \tfrac{5}{3} \tag{2.38}$$
so that
$$Q_k = \bar{\varepsilon}\,\varrho_0^{2}\,T_0^{16}\left[27\,\xi_k + \frac{23}{3}\,a\,\xi_k' - \frac{2}{3}\frac{T_0}{T_0'}(4\,\xi_k + a\,\xi_k')\right] + \\ + \frac{4\,c\,\bar{a}}{3\,\bar{\varkappa}}\frac{T_0^{6.5}}{\varrho_0^{2}}\,T_0'\Big[12\,\xi_k' + 4\,a\,\xi_k'' + \frac{2}{3}\frac{T_0}{T_0'}(5\,\xi_k'' + a\,\xi_k''') + \\ + \frac{2}{3}(4\,\xi_k' + a\,\xi_k'')\frac{\partial}{\partial a}\left(\frac{T_0}{T_0'}\right)\Big] \tag{2.39}$$
and
$$R_k = -\,\bar{\varepsilon}\,\varrho_0^{2}\,T_0^{16}\left[-\,8.5\frac{Q_k}{p_0} + \frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{Q_k}{p_0}\right)\right] + \\ + \frac{4\,c\,\bar{a}}{3\,\bar{\varkappa}}\frac{T_0^{6.5}}{\varrho_0^{2}}\,T_0'\left[7.5\frac{\partial}{\partial a}\left(\frac{Q_k}{p_0}\right) + \frac{\partial}{\partial a}\left\{\frac{T_0}{T_0'}\frac{\partial}{\partial a}\left(\frac{Q_k}{p_0}\right)\right\}\right]. \tag{2.40}$$

3. Part B

Homogeneous Model

In this part we shall discuss a homogeneous model i.e. we shall take

$$\varrho_0 = a \text{ constant} \tag{3.1}$$

$$g_0 = \frac{MG}{R^2}\, x \tag{3.2}$$

$$p_0 = \frac{1}{2}\, \varrho_0 \frac{MG}{R} (1 - x^2) \tag{3.3}$$

and

$$T_0 = \frac{1}{2\,\mathrm{k}} \frac{MG}{R} (1 - x^2) \tag{3.4}$$

where we have set

$$x = \frac{a}{R} \tag{3.5}$$

With the help of (3.2), (3.3) and (3.4), the equation (2.37) and hence (2.35) and (2.36) reduce to

$$\begin{aligned} \bar{\varepsilon}\, \varrho_0^4 \left(\frac{1}{2\,\mathrm{k}} \frac{MG}{R}\right)^{16} & (1 - x^2)^{16} \\ &= \frac{8\, c\, \bar{a}}{3\, \varkappa\, R^2} \left(\frac{1}{2\,\mathrm{k}} \frac{MG}{R}\right)^{7.5} (1 - x^2)^{5.5} (3 - 16\, x^2) \end{aligned} \tag{3.6}$$

$$\begin{aligned} Q_k = {} & \frac{8\, c\, \bar{a}}{3\, \varkappa\, \varrho_0^2\, R^2} \left(\frac{1}{2\,\mathrm{k}} \frac{MG}{R}\right)^{7.5} (1 - x^2)^{5.5} \\ & \times \Big[(3 - 16\, x^2) \Big\{27\, \xi_k + \frac{23}{3}\, x\, \xi_k' + \frac{1}{3}\left(\frac{1}{x} - x\right)(4\, \xi_k' + x\, \xi_k'') - \\ & - x(1 - x^2) \Big\{12\, \xi_k' + 4\, x\, \xi_k'' - \frac{1}{3}\left(\frac{1}{x} - x\right)(5\, \xi_k'' + a\, \xi_k''') + \\ & + \frac{1}{3}(4\, \xi_k' + a\, \xi_k'')\left(\frac{1}{x^2} + 1\right)\Big\}\Big] \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} R_k = {} & \frac{8\, c\, \bar{a}}{3\, \varkappa\, \varrho_0^2\, R^2} \left(\frac{1}{2\,\mathrm{k}} \frac{MG}{R}\right)^{7.5} \left(\frac{2}{\varrho_0} \frac{R}{MG}\right) (1 - x^2) \times \\ & \times \Big[(3 - 16\, x^2) \Big\{8.5 \frac{Q_k}{1 - x^2} + \frac{1 - x^2}{2\, x} \frac{\partial}{\partial x}\left(\frac{Q_k}{1 - x^2}\right)\Big\} - \\ & - x(1 - x^2) \Big\{7.5 \frac{\partial}{\partial x}\left(\frac{Q_k}{1 - x^2}\right) - \frac{\partial}{\partial x}\left(\frac{1 - x^2}{2\, x} \frac{\partial}{\partial x}\left(\frac{Q_k}{1 - x^2}\right)\right)\Big\}\Big], \end{aligned} \tag{3.8}$$

where dash now denotes differentiation with respect to x.

Since Q_k and R_k vanish at the surface of the star $x = 1$, we have, on integrating (2.29) and (2.30) by parts once,

$$A_{sk} = -\, 4\, \pi (\gamma - 1)\, R^3 \int_0^1 Q_k \frac{\partial}{\partial x} (x^3\, \xi_s)\, dx \tag{3.9}$$

and

$$B_{sk} = -\, 4\, \pi (\gamma - 1)^2 R^3 \int_0^1 R_k \frac{\partial}{\partial x} (x^3\, \xi_s)\, dx\,. \tag{3.10}$$

Setting the dimensionless variable x for a and using (3.2) and (3.3) the equation (2.26) determining ξ_k reduces to

$$\xi_k''(1-x^2)+\frac{4-6x^2}{x}\,\xi_k'+c_k^2\,\xi_k=0 \tag{3.11}$$

with

$$c_k^2=\frac{6}{5}\left(\frac{\sigma_k^2 R^3}{MG}-1\right). \tag{3.12}$$

The eigenfunctions ξ_k of (3.11) normalized according to (2.27) and the corresponding eigenvalues σ_k are given below:

$$\xi_0=\sqrt{\frac{5}{4\pi\varrho_0 R^5}}$$

$$\sigma_0^2=\frac{MG}{R^3}$$

$$\xi_1=3.354\sqrt{\frac{5}{4\pi\varrho_0 R^5}}\left(1-\frac{7}{5}x^2\right)$$

$$\sigma_1^2=\frac{38}{3}\,\frac{MG}{R^3}$$

$$\xi_2=7.544\sqrt{\frac{5}{4\pi\varrho_0 R^5}}\left(1-\frac{18}{5}x^2+\frac{99}{35}x^4\right)$$

$$\sigma_2^2=31\,\frac{MG}{R^3}$$

$$\xi_3=12.107\sqrt{\frac{5}{4\pi\varrho_0 R^5}}\left(1-\frac{33}{5}x^2+\frac{429}{35}x^4-\frac{143}{21}x^6\right)$$

$$\sigma_3^2=56\,\frac{MG}{R^3}$$

$$\xi_4=18.498\sqrt{\frac{5}{4\pi\varrho_0 R^5}}\left(1-\frac{52}{5}x^2+\frac{234}{7}x^4-\frac{884}{21}x^6+\frac{4199}{231}x^8\right)$$

$$\sigma_4^2=71\,\frac{MG}{R^3}$$

$$\xi_5=26.233\sqrt{\frac{5}{4\pi\varrho_0 R^5}}\left(1-15x^2+\frac{510}{7}x^4-\frac{3230}{21}x^6+\right.$$
$$\left.+\frac{1615}{11}x^8-\frac{7429}{143}x^{10}\right)$$

$$\sigma_5^2=\frac{263}{3}\,\frac{MG}{R^3}$$

$$\xi_6=36.887\sqrt{\frac{5}{4\pi\varrho_0 R^5}}\left(1-\frac{102}{5}x^2+\frac{969}{7}x^4-\frac{3876}{9}x^6+\right.$$
$$\left.+\frac{7429}{11}x^8-\frac{74290}{143}x^{10}+\frac{22287}{143}x^{12}\right)$$

$$\sigma_6^2=106\,\frac{MG}{R^3}. \tag{3.13}$$

Using these values of ξ_k we can evaluate Q_k and R_k as functions of x and then, with their help, the coefficients A_{sk} and B_{sk}.
On substituting

$$q_s = A'_s e^{iwt} \tag{3.14}$$

in (2.28), we have

$$\omega^4 A'_s - \omega^2 \sigma_s^2 A'_s - i\,\omega \sum_k A_{sk} A'_k - \sum_k B_{sk} A'_k . \tag{3.15}$$

We now replace ω by a dimensionless quantity w: as

$$w = \frac{\omega}{\left(\frac{MG}{R^3}\right)^{1/2}} \tag{3.16}$$

and write

$$A'_{sk} = \frac{A_{sk}}{\left(\frac{MG}{R^3}\right)^{3/2}} \tag{3.17}$$

$$B'_{sk} = \frac{B_{sk}}{\left(\frac{MG}{R^3}\right)^{2}} \tag{3.18}$$

$$\sigma'_s = \frac{\sigma_s}{\left(\frac{MG}{R^3}\right)^{1/2}} \tag{3.19}$$

so that (3.15) takes the form

$$(w^4 - w^2 \sigma_s'^2)\, A'_s - i\,w \sum_k A'_{sk} A'_k - \sum_k B'_{sk} A'_k \tag{3.20}$$

in which all the symbols are dimensionless.

4. The Characteristic Equation

For numerical computations we shall take the star computed by EDDINGTON (1925)

$M = 5.02$ solar masses

$R = 16.382$ solar radii

Mean Molecular weight $\mu = 2.2$

Opacity coefficient $\quad \bar{\varkappa} = 3.467 \times 10^{25}\,\mathrm{gm}^{-1}\,\mathrm{cm}^2$

To obtain a sufficiently accurate solution of the simultaneous equations (3.20) we found it sufficient to calculate A'_{sk} and B_{sk} for $s = 0, 1, \ldots, 6$ and $k = 0, 1, \ldots 6$ and these are collected in tables I and II.

For the solution of (3.20) be possible, the determinant of all the coefficients of the amplitudes A'_s must vanish. This condition gives

Table 1. A'_{sk}

k \ s	0	1	2	3	4	5	6
0	0	-2.37133×10^{-2}	-6.17188×10^{-2}	-5.77789×10^{-2}	-2.89836×10^{-2}	-6.93635×10^{-3}	-4.67205×10^{-4}
1	-1.89012×10^{-2}	-5.28717×10^{-2}	-1.13086×10^{-1}	-1.48419×10^{-1}	-1.93823×10^{-1}	-6.74059×10^{-2}	-1.55026×10^{-2}
2	-5.51445×10^{-2}	-8.27412×10^{-2}	-9.8100×10^{-2}	-1.62755×10^{-1}	-2.35986×10^{-1}	-2.13483×10^{-1}	-1.10635×10^{-1}
3	-4.45631×10^{-2}	-9.40367×10^{-2}	-6.73047×10^{-2}	-1.10700×10^{-2}	-9.31324×10^{-2}	-2.45415×10^{-1}	-2.67772×10^{-1}
4	-2.23589×10^{-2}	-1.38343×10^{-1}	-8.45855×10^{-2}	8.49984×10^{-2}	2.54354×10^{-1}	1.33588×10^{-1}	-1.710645×10^{-1}
5	-4.96466×10^{-3}	-3.88320×10^{-2}	-8.68874×10^{-2}	4.54538×10^{-2}	4.67898×10^{-1}	8.23156×10^{-1}	6.68937×10^{-1}
6	-4.90063×10^{-4}	-9.82990×10^{-3}	-5.08094×10^{-2}	-2.71854×10^{-2}	3.63570×10^{-1}	1.17726	2.03260

Table 2. B'_{sk}

k \ s	0	1	2	3	4	5	6
0	-3.42066×10^{-4}	-7.73451×10^{-4}	-9.33362×10^{-4}	-9.85294×10^{-4}	-1.21327×10^{-3}	-1.30415×10^{-3}	-1.05815×10^{-3}
1	-7.10489×10^{-4}	-1.84486×10^{-3}	-2.06845×10^{-3}	-1.84633×10^{-3}	-2.40467×10^{-1}	-1.66390×10^{-3}	-1.66642×10^{-3}
2	-6.18024×10^{-4}	-2.10564×10^{-3}	-1.59693×10^{-3}	-1.71305×10^{-3}	-4.35085×10^{-3}	-2.80963×10^{-3}	2.95132×10^{-3}
3	3.56609×10^{-5}	-5.70823×10^{-4}	2.66366×10^{-3}	-1.41827×10^{-3}	-1.39059×10^{-3}	8.21519×10^{-3}	1.27628×10^{-2}
4	6.31159×10^{-4}	1.57408×10^{-3}	1.17002×10^{-2}	-2.22060×10^{-3}	-4.36305×10^{-2}	4.59047×10^{-4}	1.09840×10^{-2}
5	1.15602×10^{-3}	3.01090×10^{-3}	2.61373×10^{-2}	-2.65408×10^{-3}	-9.96959×10^{-2}	-3.36264×10^{-2}	-3.82502×10^{-2}
6	6.36841×10^{-4}	3.13011×10^{-3}	4.84369×10^{-2}	-9.98777×10^{-4}	-1.88982×10^{-1}	-8.87606×10^{-2}	-1.66731×10^{-1}

us the characteristic equation for evaluating w:

$$D = |d_{sk}| = 0 \,. \tag{4.2}$$

For the model under consideration the values of d_{sk} are given below:

$$
\begin{aligned}
d_{00} &= w^2(w^2 - 1) + 3.42066 \times 10^{-4}\\
d_{01} &= 2.3713 \times 10^{-2} iw + 7.7345 \times 10^{-4}\\
d_{02} &= 6.1719 \times 10^{-2} iw + 9.3336 \times 10^{-4}\\
d_{03} &= 5.7779 \times 10^{-2} iw + 9.8529 \times 10^{-4}\\
d_{04} &= 2.8984 \times 10^{-2} iw + 1.2133 \times 10^{-3}\\
d_{05} &= 6.9363 \times 10^{-3} iw + 1.3415 \times 10^{-3}\\
d_{06} &= 4.6721 \times 10^{-4} iw + 1.0582 \times 10^{-3}
\end{aligned}
$$

$$
\begin{aligned}
d_{10} &= 1.8901 \times 10^{-2} iw \times 7.1049 \times 10^{-4}\\
d_{11} &= w^2(w^2 - 38/3) + 5.2872 \times 10^{-2} iw + 1.8449 \times 10^{-3}\\
d_{12} &= 1.1309 \times 10^{-1} iw + 2.0685 \times 10^{-3}\\
d_{13} &= 1.4842 \times 10^{-1} iw + 1.8463 \times 10^{-3}\\
d_{14} &= 1.9382 \times 10^{-1} iw + 2.4047 \times 10^{-3}\\
d_{15} &= 6.7406 \times 10^{-2} iw + 1.6639 \times 10^{-3}\\
d_{16} &= 1.5503 \times 10^{-2} iw + 1.6664 \times 10^{-3}
\end{aligned}
$$

$$
\begin{aligned}
d_{20} &= 5.5144 \times 10^{-2} iw + 6.1802 \times 10^{-4}\\
d_{21} &= 8.2741 \times 10^{-2} iw + 2.1056 \times 10^{-3}\\
d_{22} &= w^2(w^2 - 31) + 9.8110 \times 10^{-2} iw + 1.5969 \times 10^{-3}\\
d_{23} &= 1.6276 \times 10^{-1} iw + 1.7131 \times 10^{-3}\\
d_{24} &= 2.3599 \times 10^{-1} iw + 4.3509 \times 10^{-3}\\
d_{25} &= 2.1348 \times 10^{-1} iw + 2.8096 \times 10^{-3}\\
d_{26} &= 1.1063 \times 10^{-1} iw - 2.9513 \times 10^{-3}
\end{aligned}
$$

$$
\begin{aligned}
d_{30} &= 4.4563 \times 10^{-2} iw - 3.5661 \times 10^{-5}\\
d_{31} &= 9.4037 \times 10^{-2} iw + 5.7082 \times 10^{-4}\\
d_{32} &= 6.7305 \times 10^{-2} iw - 2.6637 \times 10^{-3}\\
d_{33} &= w^2(w^2 - 56) + 1.1070 \times 10^{-2} iw + 1.4183 \times 10^{-3}\\
d_{34} &= 9.3132 \times 10^{-2} iw + 1.3906 \times 10^{-3}\\
d_{35} &= 2.4542 \times 10^{-1} iw - 8.2152 \times 10^{-3}\\
d_{36} &= 2.6777 \times 10^{-1} iw - 1.2763 \times 10^{-2}
\end{aligned}
$$

$$
\begin{aligned}
d_{40} &= 2.2359 \times 10^{-2} iw - 6.3116 \times 10^{-4}\\
d_{41} &= 1.3834 \times 10^{-1} iw - 1.5741 \times 10^{-3}\\
d_{42} &= 8.4586 \times 10^{-2} iw - 1.1700 \times 10^{-2}\\
d_{43} &= -\,8.4998 \times 10^{-2} iw + 2.2206 \times 10^{-3}\\
d_{44} &= w^2(w^2 - 71) - 2.5435 \times 10^{-1} iw + 4.3630 \times 10^{-2}\\
d_{45} &= -\,1.3359 \times 10^{-1} iw - 4.5905 \times 10^{-4}\\
d_{46} &= 1.7106 \times 10^{-1} iw - 1.0984 \times 10^{-2}
\end{aligned}
$$

$$d_{50} = 4.9647 \times 10^{-3} iw - 1.1560 \times 10^{-3}$$
$$d_{51} = 3.8832 \times 10^{-2} iw - 3.0109 \times 10^{-3}$$
$$d_{52} = 8.6887 \times 10^{-2} iw - 2.6137 \times 10^{-2}$$
$$d_{53} = -4.5454 \times 10^{-2} iw + 2.6541 \times 10^{-3}$$
$$d_{54} = -4.6790 \times 10^{-1} iw + 9.9696 \times 10^{-2}$$
$$d_{55} = w^2 (w^2 - 263/3) - 8.2316 \times 10^{-1} iw + 3.3626 \times 10^{-2}$$
$$d_{56} = -6.6894 \times 10^{-1} iw + 3.8250 \times 10^{-2}$$

$$d_{60} = 4.9006 \times 10^{-4} iw - 6.3684 \times 10^{-4}$$
$$d_{61} = 9.8299 \times 10^{-3} iw - 3.1301 \times 10^{-3}$$
$$d_{62} = 5.0809 \times 10^{-2} iw - 4.8437 \times 10^{-2}$$
$$d_{63} = 2.7185 \times 10^{-2} iw + 9.9878 \times 10^{-4}$$
$$d_{64} = -3.6357 \times 10^{-1} iw \times 1.8898 \times 10^{-1}$$
$$d_{65} = -1.1773 iw + 8.8761 \times 10^{-2}$$
$$d_{66} = w^2 (w^2 - 106) - 2.0326 iw + 1.6673 \times 10^{-1}$$

In what follows we shall determine the roots of the characteristic equation (4.2) corresponding to the fundamental mode of pulsation by the method of successive approximations. For this we start with the zeroth order approximation obtained as the root of

$$d_{00} = 0 . \tag{4.3}$$

The successive approximations are given below

$$w_0 = 1 - 1.71033 \times 10^{-4}$$
$$w_1 = 1 - 1.48020 \times 10^{-4} - 1.7198 \times 10^{-6} i$$
$$w_2 = 1 - 9.4808 \times 10^{-5} - 5.3890 \times 10^{-6} i$$
$$w_3 = 1 - 7.1607 \times 10^{-5} - 2.8625 \times 10^{-6} i$$
$$w_4 = 1 - 6.7144 \times 10^{-5} - 2.7504 \times 10^{-6} i$$
$$w_5 = 1 - 6.6940 \times 10^{-5} - 2.7205 \times 10^{-6} i \tag{4.4}$$

If, as usual, we would have stopped at the first order approximation given by (2.17) in the solution of the energy equation we would have obtained the following approximations to w:

$$w'_0 = 1$$
$$w'_1 = 1 + 1.921 \times 10^{-5} + 8.674 \times 10^{-8} i$$
$$w'_2 = 1 + 7.152 \times 10^{-5} + 5.5827 \times 10^{-7} i$$
$$w'_3 = 1 + 9.7487 \times 10^{-5} + 8.3924 \times 10^{-7} i$$
$$w'_4 = 1 + 1.0396 \times 10^{-4} + 9.3187 \times 10^{-7} i$$
$$w'_5 = 1 + 1.0416 \times 10^{-4} + 9.3630 \times 10^{-7} i \tag{4.5}$$

5. Conclusion

We see from (4.4) that after the third order approximation the solution for ω converges rapidly. Hence we have taken as an approximation.

$$\omega = 1 - 6.6940 \times 10^{-5} - 2.7205 \times 10^{-6} i$$

The period P_0 of the fundamental mode of adiabatic oscillation is given by

$$P_0 = \frac{2\pi}{\sigma_0} = 2.96 \times 10^5 \text{ secs.}$$
$$= 3.43 \text{ days.}$$

The effect of taking the departure from adiabatic conditions on the period is to increase it by 0.007%; also it produces an exponential rise in the amplitude. The time in which the amplitude increases by e comes out to be of the order 1000 years. On the other hand if we confine ourselves to the first order approximation for the solution of the energy equation and accept the value of w as approximately equal to w'_5 given by (4.5) the time period P_0 becomes less by 0.01% and the amplitude of the oscillations decreases becoming $1/e$ of its value in a time of interval of about 3000 years.

These considerations suggest that on the problem concerning the stability of pulsations the departure from the adiabatic conditions must be taken into account more precisely than what we have done in the present note.

Literature

[1] EDDINGTON, A. S.: M. N. R. Astr. Soc., 87, 539 (1927). — [2] EDDINGTON, A. S.: M. N. R. Astr. Soc. **101**, 182 (1941). — [3] KLUYVER: Bull. Astr. Inst. Netherl. **7**, 313 (1936). — [4] ROSSELAND, S.: The Pulsation Theory of veriable stars, S. 60, 1949. — [5] SCHWARZSCHILD, M.: Z. Astrophysik **11**, 152 (1935). — [6] SCHWARZSCHILD, M.: Z. Astrophysik **15**, 14 (1938). — [7] WOLTJER, J.: M. N. R. Astr. Soc **95**, 260 (1935). — [8] WOLTJER, J.: Bull. Astr. Inst. Netherl. 8, 17 (1936).

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