A NOTE ON THE PULSATION THEORY OF CEPHEID VARIABLES

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Summary

The problem of the anharmonic pulsations for the homogeneous model (and also the Roche model) pulsating in the fundamental mode is discussed. The ratio of specific heats is assumed to be 5/3. The results are compared with those of Rosseland.

In his George Darwin Lecture * on the Pulsation Theory of Cepheid Variables Professor Rosseland has developed the general theory of anharmonic pulsations and has applied it to a homogeneous star pulsating in the fundamental mode, which case is amenable to a simple mathematical treatment. In this case Rosseland finds that for an appreciable skewness in the velocity-time curve, the semi-amplitude of pulsation nearly approaches one-fourth the stellar radius (the observed amplitude is about one-tenth the radius). However, this and other results obtained by Rosseland are not inherent in the model, but arise on account of an approximation introduced in the investigation. The work discussed in the present note is free from this approximation (the value of the ratio of the specific heats is assumed to be 5/3). As it happens, the exact treatment is simpler than that of Rosseland. In the following, as far as practicable, we use his notation.

1. Let r, p, ρ, T and g denote radius vector, pressure, density, temperature and gravity of an element in the Lagrangian sense and a, p_0, ρ_0, T_0 and g_0 denote these quantities at a given initial time t_0. For a homogeneous star pulsating in the fundamental mode we write for the displacement at any point a

$$r-a=a\eta q, \tag{1}$$

where η is a constant and q is a function of time only. In the sequel we shall identify t_0 with the instant when the pulsation velocity is maximum, it being directed outwards.

We note that for the homogeneous model

$$p_0=\tfrac{1}{2}\rho_0\frac{GM}{R}\left(1-\frac{a^2}{R^2}\right) \tag{2}$$

and

$$g_0=GMa/R^3, \tag{3}$$

where M is the mass of the star and R its radius at time t_0. We shall call R the equilibrium radius.

The kinetic energy $W_1(t)$ of radial oscillation is

$$W_1=\tfrac{1}{2}\int_0^R 4\pi a^2\rho_0 da\ a^2\eta^2\dot{q}^2=\tfrac{3}{10}MR^2\eta^2\dot{q}^2 \tag{4}$$

and the work $W_2(t)$ done against gravitation, measured from the state at time t_0, is

$$W_2=-\frac{3}{5}\frac{GM^2}{R}\left[\frac{1}{1+q\eta}-1\right]=\frac{3}{5}\frac{GM^2}{R}\frac{q\eta}{1+q\eta}. \tag{5}$$

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Further, the increase in the thermal energy $W_3(t)$, measured again from the state at time t_0, is

$$W_3=\int_0^R C_v(T-T_0)4\pi a^2\rho_0 da,$$

where C_v is the constant volume specific heat per unit mass. Substituting the perfect gas equation

$$p=C_v(\gamma-1)\rho T$$

and the adiabatic relation

$$p/p_0=(\rho/\rho_0)^\gamma=\frac{1}{(1+q\eta)^{3\gamma}},$$

where γ is the ratio of the specific heats, we obtain using (2)

$$W_3=\frac{1}{5(\gamma-1)}\frac{GM^2}{R}\left[\frac{1}{(1+q\eta)^{3\gamma-3}}-1\right]. \tag{6}$$

Adding (4), (5) and (6) we have for the total energy W, which does not vary with time,

$$W=\frac{3}{5}MR^2\left[\tfrac{1}{2}\eta^2\dot{q}^2+\frac{MG}{R^3}-\frac{q\eta}{1+q\eta}+\frac{1}{3(\gamma-1)}\frac{MG}{R^3}\left\{\frac{1}{(1+q\eta)^{3\gamma-3}}-1\right\}\right]. \tag{7}$$

Expanding to the third power of η we get

$$W=\tfrac{3}{5}MR^2\eta^2[\tfrac{1}{2}\dot{q}^2+\tfrac{1}{2}\sigma_1{}^2q^2-\tfrac{1}{2}\sigma_1{}^2(\gamma+\tfrac{1}{3})\eta q^3], \tag{8}$$

and hence we get

$$\ddot{q}+\sigma_1{}^2q=\tfrac{1}{2}(3\gamma+1)\sigma_1{}^2\eta q^2; \qquad \sigma_1{}^2=(3\gamma-4)\frac{MG}{R^3}, \tag{9}$$

which is the equation of motion to the first power of η and has been given by Rosseland. His discussion of the homogeneous model is based on it.

It is convenient to introduce a new variable $x(t)$, where Rx represents the displacement at the surface of the star at time t. We have from (1)

$$x=\eta q. \tag{10}$$

Introducing this variable and substituting $\gamma=\frac{5}{3}$, we have for W the expression

$$W=\frac{3}{5}MR^2\left[\frac{1}{2}\dot{x}^2+\frac{MG}{R^3}\frac{x}{1+x}+\frac{1}{2}\frac{MG}{R^3}\left\{\frac{1}{(1+x)^2}-1\right\}\right]. \tag{11}$$

2. Introducing x and substituting $\gamma=\frac{5}{3}$ in Rosseland's equation we have

$$\ddot{x}=-\sigma_1{}^2x+3\sigma_1{}^2x^2; \qquad \sigma_1{}^2=\frac{MG}{R^3}, \tag{12}$$

which on integration gives

$$\dot{x}^2=-\sigma_1{}^2x^2+2\sigma_1{}^2x^3+c, \tag{13}$$

c being a constant of integration.

Let x_1 and x_2 represent respectively the "outside" and the "inside" amplitudes of oscillation, *i.e.* the stellar radius oscillates between the limits $R(1+x_1)$ and $R(1-x_2)$. The semi-amplitude K of oscillation is defined by

$$K=\frac{x_1+x_2}{2}.$$

It follows from (12) that for oscillatory motion x_1 cannot exceed $\frac{1}{3}$, for otherwise $\ddot{x}$

would become positive. For $x_1 = \frac{1}{3}$, equation (13) gives $x_2 = \frac{1}{6}$, and hence the limiting value for the semi-amplitude K is

$$K = \tfrac{1}{4}.$$

It is to be observed that the existence of these upper limits for x_1, x_2 and K have no physical basis; they arise merely because in the equation of motion terms higher than the second power of x have been ignored. According to the exact equation (11) x_1 can be infinitely large.

Equation (12) can be solved, as has been done by Rosseland, in terms of elliptic functions in the usual way. Let P denote the period of oscillation, t_1 the part of the period during which the radius of the star exceeds the equilibrium value R (expansion phase) and t_2 the part of the period for which the radius is less than R (compression phase),

$$P = t_1 + t_2.$$

We shall call $\delta = t_1/t_2$ the skewness of oscillation. In the limit of vanishing amplitude $\delta \to 1$ and $P \to P_0$, where

$$P_0 = \frac{2\pi}{\sigma_1} = 2\pi\left(\frac{R^3}{MG}\right)^{\frac{1}{2}}.$$

As $K \to \frac{1}{4}$, $x_1 \to \frac{1}{3}$, $x_2 \to \frac{1}{6}$ and P/P_0 and δ both tend to infinity.

Table I(*a*) shows values for x_1, x_2, K, P/P_0 and δ obtained by solving (12) with the help of tables of the elliptic functions. From the well-known properties of the elliptic functions the asymptotic expression for P/P_0 is readily obtained,

$$\frac{P}{P_0} \sim \frac{1}{\pi} \log_e \frac{1}{2 - 8K} \quad \text{as} \quad K \to \tfrac{1}{4}. \tag{14}$$

TABLE I(*a*)

x_1	x_2	K	P/P_0	δ
0	0	0	1	1
0·0025	0·00249	0·0025	1·00	1·01
0·027	0·026	0·0265	1·00	1·07
0·074	0·064	0·069	1·02	1·20
0·163	0·120	0·142	1·09	1·50
0·220	0·145	0·183	1·19	1·78
0·279	0·161	0·220	1·39	2·30
0·318	0·166	0·242	1·74	3·15
0·330	0·167	0·248	2·00	3·77
$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$	∞	∞

TABLE I(*b*)

x_1	x_2	K	P/P_0	δ
0	0	0	1	1
0·0025	0·00249	0·0025	1·00	1·01
0·027	0·026	0·0265	1·00	1·07
0·074	0·064	0·069	1·01	1·19
0·163	0·123	0·143	1·03	1·43
0·220	0·153	0·187	1·05	1·60
0·279	0·179	0·229	1·08	1·76
0·318	0·194	0·256	1·095	1·84
0·330	0·199	0·265	1·10	1·91
0·333	0·200	0·267	1·10	1·92
∞	0·500	∞	∞	∞

3. We have so far discussed Rosseland's approximate equation of motion. We now take up the exact equation (11) which we rewrite as

$$\dot{\xi}^2 = \sigma_1^2\left[-A + \frac{2}{\xi} - \frac{1}{\xi^2}\right] \tag{15}$$

$$\xi = 1 + x$$

and

$$A = 1 - \frac{10}{3}\frac{R}{GM^2}W = 1 - \frac{W}{W_0},$$

where $(-W_0)$ represents the total energy for the statical configuration of radius R. The parameter A determines the amplitude of oscillation. If as before $x_1 = \xi_1 - 1$ and

$x_2 = 1 - \xi_2$ denote respectively the "outside" and the "inside" amplitude of oscillation, then, putting $\dot{\xi} = 0$ in (15), we have

$$\xi_1 = \frac{1}{A}\{1 + (1 - A)^{\frac{1}{2}}\}, \qquad \xi_2 = \frac{1}{A}\{1 - (1 - A)^{\frac{1}{2}}\},$$

$$x_2 = x_1/(1 + 2x_1), \tag{16}$$

and solving for A in terms of x_1 and x_2,

$$A = \frac{1 + 2x_1}{(1 + x_1)^2} = \frac{1 - 2x_2}{(1 - x_2)^2}.$$

The period of oscillation P, and t_1 and t_2 (the parts of the period for which the radius of the star is greater and less than R respectively) are immediately obtained from (15). We have

$$\frac{P}{P_0} = \frac{1}{\pi}\int_{\xi_2}^{\xi_1} f(\xi)\,d\xi, \qquad t_1 = \frac{2}{\sigma_1}\int_{1}^{\xi_1} f(\xi)\,d\xi, \qquad t_2 = \frac{2}{\sigma_1}\int_{\xi_2}^{1} f(\xi)\,d\xi, \qquad f(\xi) = \frac{\xi}{(-A\xi^2 + 2\xi - 1)^{\frac{1}{2}}},$$

and after integration we get

$$\frac{P}{P_0} = \frac{(1 + x_1)^3}{(1 + 2x_1)^{3/2}} = \frac{(1 - x_2)^3}{(1 - 2x_2)^{3/2}};$$

$$t_1 = \frac{2}{\sigma_1}\left[\frac{x_1(1 + x_1)}{1 + 2x_1} + \frac{(1 + x_1)^3}{(1 + 2x_1)^{3/2}}\left\{\frac{\pi}{2} + \sin^{-1}\frac{x_1}{1 + x_1}\right\}\right],$$

$$t_2 = \frac{2}{\sigma_1}\left[\frac{x_2(1 - x_2)}{1 - 2x_2} + \frac{(1 - x_2)^3}{(1 - 2x_2)^{3/2}}\left\{\frac{\pi}{2} + \sin^{-1}\frac{x_2}{1 - x_2}\right\}\right],$$

$$t_1 + t_2 = P,$$

x_1 and x_2 being related according to (16).

It is to be noted that $\delta = t_1/t_2$ depends only on x_1 or x_2 and is independent of M and R. Unlike the case of Rosseland, we have obtained here exact expressions for the quantities of physical interest. To facilitate comparison between his results and those obtained here Table I(*b*) has been added.

It will be noticed that for a given skewness of the oscillation the semi-amplitude K must be much larger than that given by Rosseland's approximate treatment. Thus, for example, for $\delta \sim 5$, $K \sim 0{\cdot}8$, which is more than three times the Rosseland value.

4. It may be remarked that the foregoing discussion is equally applicable to the Roche model, which consists of a central point-mass surrounded by a massless atmosphere and, as is easily verified, it is only necessary to write $(5/2)M$ in place of M in the equations given above—M is the mass of the star which in the Roche model is concentrated at the centre. Thus it follows that for the Roche and the homogeneous models, both having the same mean density, for a given skewness of oscillation, *i.e.* value of δ, the semi-amplitude K and so also x_1 and x_2 have the same values, but period for the Roche model is $(2/5)^{\frac{1}{2}}$ times that for the homogeneous model. This smallness of the period for the Roche model compared with that for the homogeneous model for all values of the amplitude provides an illustration of what appears to be a general result *, that, for the stars having the same mean density and the same ratio of specific heats, the period is largest for the homogeneous model and decreases as we pass to models in which the mass is more concentrated towards the centre.

5. From the preceding discussion the conclusion is obvious that the theory of

* T. E. Sterne, *M.N.*, **97**, 582, 1937. (Also P. L. Bhatnagar, *Proc. Nat. Inst.*, in press.)