FERMI-DIRAC AND BOSE-EINSTEIN GAS IN A GRAVITATIONAL FIELD.

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ABSTRACT.

The distribution with height of pressure and concentration for a Fermi-Dirac and a Bose-Einstein gas subject to a uniform gravitational field is considered. The non-relativistic non-degenerate and degenerate, and the relativistic non-degenerate and degenerate cases are discussed. Because of its connection with radiation, the relativistic case of Bose-Einstein statistics is particularly interesting.

§1. In a recent paper (Kothari and Auluck, 1942) the problem of density distribution for a Fermi-Dirac and a Bose-Einstein gas, both for the relativistic and non-relativistic cases, in the presence of a uniform field of force has been discussed. The force on a particle of the gas due to the field is assumed to be independent of the energy of the particle. This assumption, however, is in general untenable when the field of force is due to gravitation, for in this case the force on a particle is proportional to its mass and this depends upon the energy of the particle in accordance with relativity. If ϵ be the kinetic energy of a particle whose rest mass is m, then the force on the particle due to a gravitational field of intensity g will be $(\epsilon + mc^2)\frac{g}{c^2}$ (where c is the velocity of light), and this in the relativistic case, when the kinetic energy is very large compared to the rest mass energy, becomes $\frac{g\epsilon}{c^2}$. In the present paper we shall treat the problem of the distribution of pressure and concentration for a Fermi-Dirac and a Bose-Einstein gas subject to a gravitational field, taking into account the effect of relativistic mechanics. As usual, we shall consider four limiting cases, viz. non-relativistic non-degeneracy and degeneracy, and relativistic non-degeneracy and degeneracy.

§2. Let us consider an ideal Fermi-Dirac or Bose-Einstein gas placed in a uniform gravitational field which we assume to be directed along the negative direction of the x-axis and extending from $x = 0$ to $x = \infty$. Then the equation of hydrostatic equilibrium is

$$dp = -n\,(E + mc^2)\,\frac{g}{c^2}\,dx, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

where p is the pressure, E the *average* kinetic energy per particle and n the number of particles per unit volume, all at height x.

The pressure and ζ, the Gibb's free energy per particle, are connected by the thermodynamic relation

$$\left(\frac{\partial p}{\partial \zeta}\right)_T = n,$$

where T is the temperature. This relation is well known, but for the sake of completeness the proof for the relevant special case is indicated below.

The pressure is given by*

$$p = \frac{4\pi q}{3c^3h^3} \int_0^\infty \frac{(\epsilon^2+2mc^2\epsilon)^{\frac{3}{2}}}{e^{\frac{\epsilon-\zeta}{kT}}+\beta}\, d\epsilon, \qquad (2)$$

and hence

$$\left(\frac{\partial p}{\partial \zeta}\right)_T = \frac{4\pi q}{3c^3h^3} \int_0^\infty \frac{d\epsilon}{kT} \frac{(\epsilon^2+2mc^2\epsilon)^{\frac{3}{2}} e^{\frac{\epsilon-\zeta}{kT}}}{\left[e^{\frac{\epsilon-\zeta}{kT}}+\beta\right]^2}, \qquad (3)$$

where q is the weight factor of the particle and k and h have their usual meanings. For classical statistics we have $\beta = 0$, for Bose-Einstein statistics $\beta = -1$ and for Fermi-Dirac statistics $\beta = +1$. Integrating (3) by parts we obtain

$$\left(\frac{\partial p}{\partial \zeta}\right)_T = \frac{4\pi q}{c^3h^3} \int_0^\infty \frac{(\epsilon^2+2mc^2\epsilon)^{\frac{1}{2}}(\epsilon+mc^2)\, d\epsilon}{e^{\frac{\epsilon-\zeta}{kT}}+\beta},$$

or

$$\left(\frac{\partial p}{\partial \zeta}\right)_T = n. \qquad (4)$$

Substituting (4) in (1) we have

$$\left(\frac{d\zeta}{dx}\right)_T = -\left(E+mc^2\right)\frac{g}{c^2}. \qquad (5)$$

We shall now proceed to discuss separately the non-relativistic $\left(\frac{E}{mc^2}\to 0\right)$ and relativistic $\left(\frac{E}{mc^2}\to\infty\right)$ cases. Consider first the non-relativistic case. We have from (5)

$$\zeta-\bar{\zeta} = -mgx, \qquad (6)$$

where a *bar* over a quantity denotes its value at the level $x = 0$.

* The expressions for p, ζ, etc., quoted here will be found collected in a paper by Kothari and Singh (1941).

Substituting for ζ the expression

$$\zeta = kT\,[\log A_0 + 2\beta b_2 A_0 - \frac{3}{2} b_3 A_0^3 + \ldots],$$

where *

$$A_0 = \frac{nh^3}{q\,(2\pi mkT)^{\frac{3}{2}}}$$

$$b_2 = \frac{1}{2^{\frac{5}{2}}} = 0{\cdot}1768$$

$$b_3 = \frac{2}{3^{\frac{5}{2}}} - \frac{1}{4^{\frac{3}{2}}} = 0{\cdot}00330,$$

we have [retaining terms up to the first order in A_0]

$$(\log A_0 + 2\beta b_2 A_0) - (\log \bar{A}_0 + 2\beta b_2 \bar{A}_0) = -\frac{mgx}{kT}, \quad \ldots \quad (7)$$

and hence

$$\log \frac{A_0}{\bar{A}_0} = \log \frac{n}{\bar{n}} = -\frac{x}{x_0} + 2\,\beta b_2\,\bar{A}_0 \left(1 - e^{-\frac{x}{x_0}}\right),$$

or

$$n = \bar{n}\left\{1 + 2\beta b_2\,\bar{A}_0\left(1 - e^{-\frac{x}{x_0}}\right)\right\} e^{-\frac{x}{x_0}}, \quad \ldots \quad \ldots \quad (8)$$

where x_0 is a quantity of the dimension of length defined by

$$x_0 = \frac{kT}{mg}.$$

Equation (8) shows that *for a given* $\bar{n}$, the decrease of concentration with height is *smaller* for a Fermi-Dirac gas than what it would be for a classical gas ($\beta = 0$). In the Bose-Einstein case the decrease is greater than the classical value. These results are what would be expected from the physical properties of the Fermi-Dirac and Bose-Einstein statistics.

Let N denote the total number of particles in a cylinder of unit cross-section and extending from $x = 0$ to $x = \infty$, then we have

$$N = \int_0^\infty n\,dx = \bar{n}x_0 + \bar{n}x_0\beta b_2\bar{A}_0,$$

or

$$\bar{n} = \frac{N}{x_0}(1 - \beta b_2 \bar{A}_0). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Substituting (9) in (8) we obtain

$$n = \frac{N}{x_0}\left\{1 + \beta b_2\,\bar{A}_0\left(1 - 2e^{-\frac{x}{x_0}}\right)\right\} e^{-\frac{x}{x_0}}. \quad \ldots \quad \ldots \quad (10)$$

* $A_0 << 1$ for non-degeneracy.

Thus we see that *for a given N*, the total number of particles, and for any height $x < x_0 \log 2$, the concentration n in the Fermi-Dirac case is less than, and in the Bose-Einstein case more than, what it would be for the classical case ($\beta = 0$). At the height $x = x_0 \log 2$ the difference between the classical, Fermi-Dirac, and Bose-Einstein cases vanishes. For $x > x_0 \log 2$ the concentration for the Fermi-Dirac statistics is greater and for Bose-Einstein statistics less than what it would be for the classical statistics.

Let us define a quantity D, called for brevity the 'statistical variation' by the relation

$$D = \frac{n - n^*}{n^*},$$

where n^* is the concentration in the classical case ($\beta = 0$), n, n^* both referring to the same height x. From (8) and (10) we have

$$(a) \quad D = 2\beta b_2 A_0 \left(1 - e^{-\frac{x}{x_0}}\right) \text{ for fixed } \bar{n},$$

and

$$(b) \quad D = \beta b_2 A_0 \left(1 - 2e^{-\frac{x}{x_0}}\right) \text{ for fixed } N.$$

In the case of fixed $\bar{n}$ (the concentration at $x = 0$), D is practically zero for $x << x_0$, and increases to a value $2\beta b_2 A_0$ when $x >> x_0$. On the other hand in the case of a fixed N (total number of particles), D is practically equal to $-\beta b_2 A_0$ for $x << x_0$, vanishes for $x = x_0 \log 2$, and for $x >> x_0$ approaches the value $\beta b_2 A_0$.

§3. We shall now consider the *relativistic* non-degenerate case.

In the relativistic case the rest-mass energy of a particle is negligible compared to its kinetic energy. Equation (1) therefore becomes

$$dp = -\frac{ngE}{c^2} dx,$$

and as

$$p = \frac{1}{3} nE$$

we have

$$p = \bar{p} e^{-\frac{x}{x_0}}, \qquad \qquad (11)$$

where for the relativistic case x_0 is defined by

$$x_0 = \frac{c^2}{3g}. \qquad \qquad (12)$$

Substituting for p the expression

$$p = nkT [1 + \beta b_2 A_0 - b_3 A_0^2 + \dots],$$

where *

$$A_0 = \frac{n}{8\pi q}\left(\frac{ch}{kT}\right)^3$$

$$b_2 = \frac{1}{2^4} = 0{\cdot}06250$$

$$b_3 = \frac{2}{3^4} - \frac{1}{4^3} = 0{\cdot}009066\,,$$

we obtain [up to first powers in A_0]

$$n = \bar{n}\left[1 + \beta b_2 \bar{A}_0\left(1 - e^{-\frac{x}{x_0}}\right)\right] e^{-\frac{x}{x_0}}. \qquad \ldots \qquad \ldots \quad (13)$$

Denoting as before by N the total number of particles in a column of unit cross-section we have

$$N = \bar{n}x_0\left\{1 + \beta b_2 \frac{A_0}{2}\right\},$$

and hence

$$n = \frac{N}{x_0}\left\{1 + \frac{\beta b_2 A_0}{2}\left(1 - 2e^{-\frac{x}{x_0}}\right)\right\} e^{-\frac{x}{x_0}}. \qquad \ldots \qquad \ldots \quad (14)$$

Equations (13) and (14), except for numerical coefficients, are similar to the corresponding non-relativistic equations (8) and (10), and, therefore, their discussion need not be repeated here. We may note that in this case D will be given by

$$(a) \quad D = \beta b_2 A_0\left(1 - e^{-\frac{x}{x_0}}\right) \text{ for fixed } \bar{n}\,,$$

and

$$(b) \quad D = \frac{\beta b_2 A_0}{2}\left(1 - 2e^{-\frac{x}{x_0}}\right) \text{ for fixed } N\,.$$

It should be particularly noticed that in the relativistic case equation (11) for pressure distribution is independent of the statistics obeyed by the gas and also of temperature. However, the dependence on statistics (shown by the appearance of terms containing β as a factor) comes in when we consider the distribution of concentration instead of pressure. In fact equation (11) being merely an expression of hydrostatic equilibrium holds not only for the non-degenerate case that we are considering but also for degeneracy. In the relativistic case, *whatever statistics a gas may follow, and both for degeneracy and non-degeneracy, the pressure distribution is given by equation* (11). In the non-relativistic case, however, the expressions are different for the different cases.

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* $A_0 << 1$ for non-degeneracy.

§4. We shall now consider the degenerate case. In the non-relativistic case, for a particle in a uniform gravitational field the force acting on it, being independent of the kinetic energy, becomes constant, and thus the problem is identical with that treated by Kothari and Auluck (*loc. cit.*). We shall, therefore, here consider only the case of relativistic degeneracy. Taking first the Fermi-Dirac case, and substituting in (11) the expression for pressure

$$p = \frac{1}{4} n \zeta_0 \left[1 + \frac{2\pi^2}{3} \left(\frac{kT}{\zeta_0} \right)^2 \right], \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

where

$$\zeta_0 = ch \left(\frac{3n}{4\pi g} \right)^{\frac{1}{3}},$$

we have

$$n = \bar{n} \left[1 - \frac{\pi^2}{2} \left(\frac{kT}{\zeta_0} \right)^2 \left\{ e^{\frac{x}{2x_0}} - 1 \right\} \right] e^{-\frac{3x}{4x_0}}, \quad \ldots \quad \ldots \quad (16)$$

or for $T = 0$,

$$n = \bar{n} e^{-\frac{3x}{4x_0}}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

Equation (17) may be compared with the relativistic non-degenerate equation (13) which for $\bar{A}_0 \rightarrow 0$ is

$$n = \bar{n} e^{-\frac{x}{x_0}}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17')$$

The two expressions (17) and (17′) differ only by a numerical factor in the power of the exponential*. While, therefore, there is no great difference between the expressions for concentration distribution for the relativistic degeneracy and non-degeneracy, there is a fundamental difference between relativistic degeneracy and non-relativistic degeneracy. In non-relativistic degeneracy, as is well known (see Kothari and Auluck, *loc. cit.*), the distribution effectively extends from $x = 0$ to a certain height $x = l$. Above $x = l$ the number of particles is negligibly small and it vanishes altogether when the degeneracy is complete (i.e. $T = 0$). For *complete degeneracy* ($T = 0$) n is exactly proportional to $(l-x)^{\frac{3}{2}}$ within the interval $x = 0$ to $x = l$, where the length l is given by

$$l = \frac{h^2}{2m^2 g} \left(\frac{3\bar{n}}{4\pi^3} \right)^{\frac{2}{3}}. \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

In the relativistic case even for complete degeneracy the distribution is not confined within any finite interval but continues to vary exponentially with height.

So far we have considered the relativistic degenerate case of Fermi-Dirac statistics. We shall now discuss the Bose-Einstein case. For a Bose-

* The definitions of x_0 in (17) and (17′) are different.

Einstein gas, the Gibb's free energy is always less than zero for non-degeneracy, and equal to zero in degeneracy. According to the fundamental equation (5) ζ decreases with increasing x, its maximum value occurring at $x = 0$. Hence it follows that for a *Bose-Einstein gas the degeneracy can occur only at the level* $x = 0$ *and above this level the gas must necessarily be non-degenerate.*

For a degenerate Bose-Einstein gas the pressure is independent of concentration and for the relativistic case that we are considering p is given by

$$p = \frac{8}{45}\frac{\pi^5}{(ch)^3}(kT)^4 . \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

As in a degenerate Bose-Einstein gas placed in a field of force degeneracy occurs only at the ground level, we have by combining (19) with (11)

$$p = \frac{8\pi^5}{45(ch)^3}(kT)^4 e^{-\frac{x}{x_0}}, \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

where $$x_0 = \frac{c^2}{3g}.$$

Since black-body radiation constitutes a relativistic degenerate Bose-Einstein gas, equation (20) shows the possibility of the occurrence of non-degenerate radiation (at $x \neq 0$) when the gravitational field is extremely large. The astrophysical implications of this result will be considered elsewhere.

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References.

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