ON TWO-DIMENSIONAL BOUNDARY LAYER IN NON-NEWTONIAN FLUIDS WITH CONSTANT COEFFICIENTS OF VISCOSITY AND CROSS-VISCOSITY

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RECENTLY some papers[1] have come out using the concept of boundary layer in the non-Newtonian flows without actually justifying the boundary layer approximations. In the present note we consider the boundary layer approximations in the two-dimensional flows of the incompressible non-Newtonian fluids with constant coefficient of viscosity μ and constant coefficient of cross-viscosity μ_c. We find that the boundary layer equation reduces to the one of the Newtonian fluid if we suitably replace the pressure p by the modified pressure P defined in the text. With this alterations, the results on two-dimensional boundary layer in Newtonian fluids can be carried over to the non-Newtonian fluids.

2. The equations determining the two-dimensional flow of an incompressible non-Newtonian fluid with constant μ and μ_c are[2]:

Equation of Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1}$$

Momentum Equations:

$$\frac{\partial u}{dt} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \Delta u + \frac{1}{2}\nu_c\left[\frac{\partial}{\partial x}\left\{4\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right\}\right] \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \Delta v + \frac{1}{2}\nu_c\left[\frac{\partial}{\partial y}\left\{4\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right\}\right]. \tag{3}$$

We shall now measure x along the length of a flat plate which is also the direction of the flow at infinity, and y along the normal to the plate.

As usual we assume

$$u, \quad \frac{\partial u}{\partial x}, \quad \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial p}{\partial x}$$

of the standard order and y of the order δ. Then from the equation of continuity, we find

$$v = 0(\delta). \tag{4}$$

Also, if the viscosity and cross-viscosity have to play a role in shaping the flow, we find from (2) that

$$\nu = 0(\delta^2) \text{ and } \nu_c = 0(\delta^2) \tag{5}$$

and then the equation (2) may be written as

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\nu_c\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2 + 0(\delta)\\ &= -\frac{1}{\rho}\frac{\partial}{\partial x}\mathrm{P} + \nu\frac{\partial^2 u}{\partial y^2} + 0(\delta),\end{aligned} \tag{6}$$

where

$$\mathrm{P} = p - \frac{1}{2}\mu_c\left(\frac{\partial u}{\partial y}\right)^2. \tag{7}$$

A similar consideration of order in equation (3) gives us

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{2}\nu_c\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 = 0(\delta) \tag{8}$$

or

$$\frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial y} = 0(\delta). \tag{9}$$

Thus we find that to our approximation, P (and not p as usual in Newtonian fluids) does not vary with y across the boundary layer.

Further, since at the edge of the boundary layer $\partial u/\partial y = 0$,

$$\mathrm{P} = p - \frac{1}{2}\rho\nu_c\left(\frac{\partial u}{\partial y}\right)^2 = p_0, \tag{10}$$

where p_0 is the corresponding pressure in the potential flow. Hence we may write the equation (6) in boundary layer approximation as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p_0}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{11}$$

and then the pressure in the boundary layer is given by

$$p = p_0 + \frac{1}{2}\rho\, \nu_c \left(\frac{\partial u}{\partial y}\right)^2 . \tag{12}$$

From (12) we conclude that the pressure in the non-Newtonian boundary layer has to be more than in the corresponding boundary layer of the Newtonian fluid with the same coefficient of viscosity by the amount $\frac{1}{2}\mu_c\,(\partial u/\partial y)^2$ which decreases from $\frac{1}{2}\mu_c/\mu^2\,\tau_w{}^2$ on the wall to 0 at the edge of the boundary layer. The value of p_0 as usual is determined by the equation

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p_0}{\partial x} . \tag{13}$$

In the case of the steady flow, the boundary layer equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{14}$$

where the pressure in the boundary layer is given by

$$p = p_{0\infty} + \frac{1}{2}\rho\,(U_\infty{}^2 - U^2) + \frac{1}{2}\mu_c\left(\frac{\partial u}{\partial y}\right)^2 . \tag{15}$$

3. From the above discussion, we conclude that the velocity distribution in the case of the present class of non-Newtonian fluids is the same as in the Newtonian fluids with the same coefficient of viscosity μ. However, the cross-viscosity exhibits itself through the increase in pressure at each point of the boundary layer.

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