ON SUPERPOSABLE FLOWS

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Let (q_1, p_1, Ω_1) and (q_2, p_2, Ω_2) be the two flows of an incompressible fluid of uniform density ρ and kinematic viscosity ν, where as usual $\vec{q}$ is the velocity vector, p the pressure and Ω the force potential from which the external forces are assumed to be derived. Following Ram Ballabh[1] we shall call these flows as superposable if it is possible to determine a pressure $p_1 + p_2 + \pi$ such that $(\vec{q}_1 + \vec{q}_2, \; p_1 + p_2 + \pi, \; \Omega_1 + \Omega_2)$ is also a solution of Stokes-Navier equations with the necessary modification in the initial and boundary conditions. In the present note we shall consider the two-dimensional flows due to a (i) vortex, (ii) spiral-vortex, (iii) vortex-doublet and (iv) a radial flow from the point of view of superposability. We may note that in a recent paper Prem Prakash[2] has studied radial flow but we are reconsidering it here in order to bring out some important points about it.

In the first section we have obtained the general condition for superposability of two flows and other necessary expressions working through vectors in order to make the treatment independent of the choice of co-ordinate system. In particular, we deduce the condition of superposability of axi-symmetrical flows and discuss them from the point of view of superposability. As far as the plane flows are concerned we have not obtained any new result in this section but the discussion of axi-symmetrical flows appears to be new. Besides, we have included this deduction here as it is short and elegant and we use the condition of superposability in the text in the form deduced here.

1. Denoting vorticity by $\vec{\omega}$, the equations of continuity and motion for the two flows are:

$$\operatorname{div} \vec{q}_1 = 0 \tag{1.1}$$

$$\frac{\partial q_1}{\partial t} + \operatorname{grad}\left(\tfrac{1}{2}\vec{q}_1^{\,2} + \frac{p_1}{\rho} + \Omega_1\right) + \vec{\omega}_1 \times \vec{q}_1 = -\nu \operatorname{curl} \vec{\omega}_1 \tag{1.2}$$

$$\operatorname{div} \vec{q}_2 = 0 \tag{1.3}$$

$$\frac{\partial \vec{q}_2}{\partial t} + \text{grad}\left(\tfrac{1}{2}\vec{q}_2{}^2 + \frac{p_2}{\rho} + \Omega_2\right) + \vec{\omega}_2 \times \vec{q}_2 = -\nu \,\text{curl}\, \vec{\omega}_2 \tag{1.4}$$

We first note that condition of integrability of the equation of motion is

$$\frac{d\vec{\omega}}{dt} = (\vec{\omega} \cdot \nabla)\vec{q} + \nu \triangle \vec{\omega} \tag{1.5}$$

as will be seen by taking the curl of both sides of the equation of motion.

Adding (1.1) and (1.3) we find that the equation of continuity

$$\text{div}\,(\vec{q}_1 + \vec{q}_2) = 0 \tag{1.6}$$

for the superposed motion is satisfied.

Adding (1.2) and (1.4), we have

$$\begin{aligned}\frac{\partial}{\partial t}(\vec{q}_1 + \vec{q}_2) &+ \text{grad}\left[\tfrac{1}{2}(\vec{q}_1 + \vec{q}_2)^2 + \frac{p_1 + p_2 + \pi}{\rho} + (\Omega_1 + \Omega_2)\right] \\ &+ (\vec{\omega}_1 + \vec{\omega}_2) \times (\vec{q}_1 + \vec{q}_2) - \left[\text{grad}\left(\frac{\pi}{\rho} + \vec{q}_1 \cdot \vec{q}_2\right)\right. \\ &\left. + \vec{\omega}_2 \times \vec{q}_1 + \vec{\omega}_1 \times \vec{q}_2\right] \\ = &- \nu \,\text{curl}\,(\vec{\omega}_1 + \vec{\omega}_2)\end{aligned} \tag{1.7}$$

Hence the two flows are superposable if

$$\text{grad}\left[\frac{\pi}{\rho} + \vec{q}_1 \cdot \vec{q}_2\right] + \vec{\omega}_2 \times \vec{q}_1 + \vec{\omega}_1 \times \vec{q}_2 = 0] \tag{1.8}$$

Or

$$\text{grad}\,\pi = -\rho\,[\text{grad}\,(\vec{q}_1 \cdot \vec{q}_2) + \vec{\omega}_2 \times \vec{q}_1 + \vec{\omega}_1 \times \vec{q}_2] \tag{1.9}$$

The equation (1.9) determines π. Performing curl operation on (1.8) we have

$$\text{curl}\,(\vec{\omega}_2 \times \vec{q}_1) + \text{curl}\,(\vec{\omega}_1 \times \vec{q}_2) = 0 \tag{1.10}$$

(1.10) gives the necessary condition for superposability of the two flows.

From (1.10) we conclude that there exists a scalar function χ such that

$$\vec{\omega}_2 \times \vec{q}_1 + \vec{\omega}_1 \times \vec{q}_2 = \text{grad}\,\chi \tag{1.11}$$

so that from (1.8) we have

$$\frac{\pi}{\rho} = \text{constant} - \vec{q}_1 \cdot \vec{q}_2 - \chi \tag{1.12}$$

In view of (1.11) and (1.12) it is easy to see that (1.10) is also a sufficient condition.

For the self-superposability of the flow $(\vec{q}, p, \Omega)$ the necessary and sufficient condition is

$$\text{curl}\,(\vec{\omega} \times \vec{q}) = 0 \tag{1.13}$$

so that

$$\vec{\omega} \times \vec{q} = \text{grad}\,\bar{\chi} \tag{1.14}$$

and

$$\frac{\pi}{\rho} = \text{constant} - \vec{q}^{\,2} - \bar{\chi} \tag{1.15}$$

We may note the condition of superposability and the expression for π do not contain t explicitly and hence the difference in the consideration of steady and non-steady cases arises only through the condition of integrability (1.5).

If one flow, say with suffix 1, be irrotational then the condition of superposability of a rotational flow, with suffix 2, on it is

$$(\vec{q}_1 \cdot \nabla)\,\vec{\omega}_2 = (\vec{\omega}_2 \cdot \nabla)\,\vec{q}_1 \tag{1.16}$$

Considerable simplification results if we restrict our consideration to only the two-dimensional flows. In this particular case the above conditions reduce to

$$(\vec{q}_2 \cdot \nabla)\,\vec{\omega}_1 + (\vec{q}_1 \cdot \nabla)\,\vec{\omega}_2 = 0, \tag{1.17}$$

for the superposition of rotational flows,

$$(\vec{q}_1 \cdot \nabla)\,\vec{\omega}_2 = 0 \tag{1.18}$$

for the superposition of a rotational flow on an irrotational flow, and

$$(\vec{q} \cdot \nabla)\,\vec{\omega} = 0 \tag{1.19}$$

for the self-superposition of a rotational flow.

We may notice that the two irrotational flows always satisfy the condition of superposability.

We shall now consider the axi-symmetrical viscous flows. We shall take the x-axis along the axis of symmetry and as usual denote the perpendicular distance from the axis by ϖ. The discussion simplifies considerably if we work through Stokes' stream function ψ such that

$$q_x = -\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}, \quad q_{\varpi} = \frac{1}{\varpi}\frac{\partial\psi}{\partial x} \tag{1.20}$$

and

$$\omega = \frac{1}{\varpi} E^2 \psi, \quad E^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial\varpi}. \tag{1.21}$$

In terms of the stream function the condition of integrability (1.5) for steady motion reduces to

$$\varpi\, J\begin{pmatrix}\psi, & \varpi^{-2} E^2\psi \\ x, & \varpi\end{pmatrix} = \nu E^4\psi, \tag{1.22}$$

while the condition of superposability of axi-symmetrical flows with stream functions ψ_1 and ψ_2 takes the form

$$J\begin{pmatrix}\psi_1, & \varpi^{-2} E^2\psi_2 \\ x\,, & \varpi\end{pmatrix} + J\begin{pmatrix}\psi_2, & \varpi^{-2} E^2\psi_1 \\ x\,, & \varpi\end{pmatrix} = 0, \tag{1.23}$$

where J denotes the Jacobian. We may note that in deducing (1.23) we have used the equations of continuity for both the flows.

From (1.23) we find that the condition of self-superposability is

$$J\begin{pmatrix}\psi, & \varpi^{-2} E^2\psi \\ x, & \varpi\end{pmatrix} = 0 \tag{1.24}$$

while the condition of superposability of a rotational flow, with suffix 2, on an irrotational flow, with suffix 1, is

$$J\begin{pmatrix}\psi_1, & \varpi^{-2} E^2\psi_2 \\ x\,, & \varpi\end{pmatrix} = 0 \tag{1.25}$$

From (1.23) we conclude that the two flows will be superposable if

(i) they are irrotational,

(ii) vorticity of each flow is proportional to ϖ,

(iii) ω_1/ϖ is constant along the stream lines of flow 2, while ω_2/ϖ is constant along the stream lines of flow 1.

From (1.24) we see that an axi-symmetrical flow will be self-superposable if its vorticity ω is of the form

$$\omega = \widehat{\omega} f(\psi) \tag{1.26}$$

and then the condition of integrability (1.22) reduces to

$$E^4 \psi = 0 \tag{1.27}$$

From (1.25) we note that a rotational flow ψ_2 is superposable on an irrotational flow ψ_1, if in general

$$\omega_2 = \widehat{\omega} f(\psi_1) \tag{1.28}$$

and then the condition of integrability becomes

$$\widehat{\omega} \, J \begin{pmatrix} \psi_2, & f(\psi_1) \\ x, & \widehat{\omega} \end{pmatrix} = \nu \, E^2 \, [\widehat{\omega}^2 f(\psi_1)]. \tag{1.29}$$

We may regard (1.29) as an equation to determine ψ_2 when ψ_1 is prescribed.

2. *Two-dimensional vortex.*—Let a two-dimensional vortex of strength K be placed at the origin. Then its complex potential is

$$W = iK \log Z \tag{2.1}$$

so that its velocity potential is

$$\phi = -K\theta \tag{2.2}$$

We investigate the general motion which can be superposed on it. The condition of superposability (1.18) here becomes

$$\frac{\partial \omega_2}{\partial \theta} = 0 \tag{2.3}$$

so that

$$\omega_2 = F(r), \tag{2.4}$$

where F is an arbitrary function of r.

Substituting this in the condition of integrability (1.5) we get

$$\frac{\partial \psi_2}{\partial \theta} = -\nu \left[r \frac{F''}{F'} + 1 \right], \tag{2.5}$$

where ψ_2 is the stream function of the motion with suffix 2, so that

$$\psi_2 = -\nu\theta \, H(r) + G(r), \tag{2.6}$$

where

$$H(r) = r \frac{F''}{F'} + 1 \tag{2.7}$$

and $G(r)$ is an arbitrary function of r.

Now since

$$\omega_2 = \Delta\psi_2 \tag{2.8}$$

We have

$$\omega_2 = -\nu\theta\left[H'' + \frac{1}{r}H'\right] + \left[G'' + \frac{1}{r}G'\right] \tag{2.9}$$

From (2.4) ω_2 is function of r only, this is possible, if

$$H'' + \frac{1}{r}H' = 0 \tag{2.10}$$

and

$$F(r) = G'' + \frac{1}{r}G' \tag{2.11}$$

i.e., if

$$H = A\log r + B. \tag{2.12}$$

Hence from (2.7)

$$\omega_2 = F(r) = D + C\int^{\log r} \exp\left(\tfrac{1}{2}AZ^2 + BZ\right) dZ \tag{2.13}$$

and from (2.6)

$$\psi_2 = \theta\,[A\log r + B] + G(r) \tag{2.14}$$

where $G(r)$ is determined by (2.11) and (2.13).

From (2.13) we conclude that the vorticity in the motion which can be superposed on the flow due to a vortex is constant on concentric circles having the centre at the origin.

3. *Two-dimensional spiral vortex.*—Let a two-dimensional spiral vortex, *i.e.*, a source and a vortex, be situated at the origin, then the complex potential is given by Milne-Thompson[3]

$$W = (-m + iK)\log Z$$

so that its stream function ψ_1 is

$$\psi_1 = -m\theta + K\log r \tag{3.1}$$

We now investigate the general motion which is superposable on this irrotational flow. Here the condition of superposability is

$$mr\frac{\partial\omega_2}{\partial r} + K\frac{\partial\omega_2}{\partial\theta} = 0 \tag{3.2}$$

so that

$$\omega_2 = F(\tau), \tag{3.3}$$

where

$$\tau = re^{-\frac{m}{K}\theta}. \tag{3.4}$$

Also the condition of integrability in terms of the stream function ψ_2 becomes

$$\frac{\partial \psi_2}{\partial \theta} + \frac{m}{K} r \frac{\partial \psi_2}{\partial r} = -\nu\left(1 + \frac{m^2}{K^2}\right)\left(\tau \frac{F''}{F'} + 1\right) \tag{3.5}$$

The complete solution of (3.5) is

$$\psi_2 = -\nu\left(1 + \frac{m^2}{K^2}\right) H(\tau)\,\theta + G(\tau), \tag{3.6}$$

Where $G(\tau)$ is an arbitrary function of τ.

and

$$H(\tau) = \tau \frac{F''}{F'} + 1 \tag{3.7}$$

using (3.6) we get

$$\omega_2 = \Delta\psi_2 = -\nu\left(1 + \frac{m^2}{K^2}\right)^2 \left(H'' + \frac{1}{\tau} H'\right)\left(\theta e^{-2\frac{m}{K}\theta}\right)$$

$$+\left(1 + \frac{m^2}{K^2}\right) e^{-\frac{2m}{K}\theta} \left[2 \frac{m\nu}{K} \frac{H'}{\tau} + \left(G'' + \frac{1}{\tau} G'\right)\right] \tag{3.8}$$

From (3.3), ω_2 is function of τ only and in view of (3.8) it is possible only if

$$H'' + \frac{1}{\tau} H' = 0 \tag{3.9}$$

and

$$2\frac{m}{K} \nu \frac{H'}{\tau} + \left(G'' + \frac{1}{\tau} G'\right) = 0 \tag{3.10}$$

but then

$$\omega_2 = 0 \tag{3.11}$$

Therefore we conclude that on the flow due to spiral vortex only an irrotational flow can be superposed.

We may note that when $m = 0$, (3.8) reduces to (2.9) because then $\tau = r$.

4. *Two-dimensional vortex doublet.*—We shall now investigate the general flow which can be superposed on the flow due to a vortex doublet situated at the origin along the axis of y. The velocity potential here is

$$\phi = \mu \frac{\cos\theta}{r} \tag{4.1}$$

so that the condition of superposability of flow (ψ_2, ω_2) on it is

$$r \frac{\partial \omega_2}{\partial r} + \tan\theta \frac{\partial \omega_2}{\partial \theta} = 0 \tag{4.2}$$

The complete solution of (4.2) is

$$\omega_2 = \mathrm{F}(\tau) \tag{4.3}$$

where F is an arbitrary function and

$$\tau = \frac{\sin\theta}{r} \tag{4.4}$$

Also the condition of integrability for the flow with suffix 2 is

$$r \sin\theta \frac{\partial \psi_2}{\partial \theta} + r^2 \cos\theta \frac{\partial \psi_2}{\partial r} = \nu \frac{\mathrm{F}''}{\mathrm{F}'} \tag{4.5}$$

The complete solution of (4.5) is

$$\psi_2 = -\nu \mathrm{H}(\tau) \frac{\cos\theta}{\tau} + \mathrm{G}(\tau), \tag{4.6}$$

where $\mathrm{G}(\tau)$ is an arbitrary function of τ

and

$$\mathrm{H}(\tau) = \mathrm{F}''/\mathrm{F}'. \tag{4.7}$$

From (4.6)

$$\omega_2 = \Delta\psi_2 = -\nu \frac{(1 - r^2\tau^2)^{\frac{1}{2}}}{r^5} \mathrm{H}'' + \frac{1}{r^4} \mathrm{G}'' \tag{4.8}$$

From (4.3) we find that ω_2 is function of τ only. This is possible if in (4.8)

$$\mathrm{H}'' = 0 \tag{4.9}$$

and

$$\mathrm{G}'' = 0 \tag{4.10}$$

but then

$$\omega_2 = 0 \tag{4.11}$$

so that only irrotational flow can be superposed on the flow due to a vortex-doublet.

5. *Radial flow.*—with velocity components[4]

$$(q_r)_2 = \frac{1}{r} f(\theta) \tag{5.1}$$

$$(q_\theta)_2 = 0 \tag{5.2}$$

and vorticity

$$\omega_2 = -\frac{1}{r^2} f'(\theta). \tag{5.3}$$

The condition of integrability of the Stokes Navier equation reduces to

$$2ff' + \nu(f^{\mathrm{IV}} + 4f'') = 0 \tag{5.4}$$

We shall now determine the most general irrotational flow that can be superposed on it. Let the stream function of the irrotational flow be ψ_1, then the condition of superposability is

$$\frac{f'}{f''}\frac{\partial \psi_1}{\partial \theta} + \tfrac{1}{2} r \frac{\partial \psi_1}{\partial r} = 0 \tag{5.5}$$

The complete solution of (5.5) is

$$\psi_1 = \mathrm{F}(\omega_2) \tag{5.6}$$

Now, since ψ_1 is the stream function of irrotational motion,

$$\Delta \psi_1 = 0$$

i.e.,

$$\left[\left(\frac{\partial \omega_2}{\partial r}\right)^2 + \left(\frac{\partial \omega_2}{r\partial \theta}\right)^2\right] \mathrm{F}'' + (\Delta \omega_2)\,\mathrm{F}' = 0 \tag{5.7}$$

or

$$\omega_2 \frac{\mathrm{F}''}{\mathrm{F}'} = -\frac{(4f' + f''')f'}{4(f')^2 + (f'')^2}. \tag{5.8}$$

Prem Prakash, working through cartesian co-ordinates, obtains an equation corresponding to (5.8) and his satisfying that equation in a particular manner amounts to taking

$$\mathrm{F}'' = 0 \tag{5.9}$$

and

$$\Delta \omega_2 = 0 \tag{5.10}$$

in (5.7). Hence he has investigated only a particular case of the radial motion for which the vorticity is harmonic. Further, in this case from (1.5) the condition of integrability will be independent of viscosity and (5.4) reduces to

$$ff' = 0 \tag{5.11}$$

which means

$$f' = 0, \text{ i.e., } f = \text{constant} \tag{5.12}$$

so that

$$(q_r)_2 = \frac{\text{constant}}{r}. \tag{5.13}$$

Hence the radial flow which he considers is due to either a source or a sink under his assumptions.

We may regard (5.8) as an equation in two independent variables θ and $\omega_2 = -1/r^2 f'(\theta)$ and hence it will be satisfied if each side is taken equal to a constant k, say, *i.e.*, if

$$\omega_2 \frac{F''}{F'} = k \tag{5.14}$$

and

$$\frac{(4f' + f''')f'}{4(f')^2 + (f'')^2} = -k \tag{5.15}$$

The complete solution of (5.14) is

$$\psi_1 = F(\omega_2) = \frac{K}{k+1}\,\omega_2^{k+1} + L \tag{5.16}$$

where K and L are arbitrary constants.

On integrating (5.15) we have

$$f'' = \left[\frac{M^2}{(f')^{2k}} - 4(f')^2\right]^{\frac{1}{2}} \tag{5.17}$$

$$f' = \left(\frac{M}{2}\right)^{\lambda} \sin^{\lambda}\left(\frac{2\theta}{\lambda} + N\right) \tag{5.18}$$

and

$$f = \left(\frac{M}{2}\right)^{\lambda} \int \sin^{\lambda}\left(\frac{2\theta}{\lambda} + N\right) d\theta + P, \tag{5.19}$$

where M, N, P are arbitrary constants, and

$$\lambda = \frac{1}{k+1} \tag{5.20}$$

Now the superposition of an irrotational motion on the radial flow is possible only if f obtained in (5.19) satisfies the condition of integrability (5.4). Substituting the values of f', f'', f^{IV} in it, we get

$$f + \frac{4\nu(\lambda-1)(\lambda-2)}{\lambda^2}\cos\left(\frac{2\theta}{\lambda}+\mathrm{N}\right)\operatorname{cosec}^3\left(\frac{2\theta}{\lambda}+\mathrm{N}\right)=0.$$

Differentiating the above relation and substituting for f we get

$$t^{\lambda+4}+2at^2-3a=0$$

where

$$t=\sin\left(\frac{2\theta}{\lambda}+\mathrm{N}\right)$$

and

$$a=\left(\frac{2}{\mathrm{M}}\right)^{\lambda}\frac{8\nu(\lambda-1)(\lambda-2)}{\lambda^3}.$$

This cannot hold for all t. Hence we conclude that it is not possible to superpose an irrotational motion on the radial flow in general.

6. In conclusion, we may note that in the cases of the flows due to spiral-vortex, vortex-doublet and the radial flow we prove the non-existence of the rotational flows in the case of the first two and of irrotational flow in the case of the third which can be superposed on them. Such a conclusion, however, is not unexpected and it is in the very nature of the inverse approach[5] which we have adopted here. In the case of the flow due to a vortex we could find a family of rotational flows which are superposable on it. In these flows the isocurls are concentric circles.

SUMMARY

In the first section of the present paper we obtain the condition of superposability working through vectors and in particular give explicitly the condition of superposability of axi-symmetrical flows. This enables us to make some general remarks on the possibility of superposition of two axi-symmetrical flows. The rest of the paper is devoted to the consideration of the possibility of superposition of general rotational flow on the flows due to a vortex, spiral-vortex and vortex-doublet and of irrotational flow on a radial flow in two-dimensions.

In case of the flow due to a vortex, we find that a family of rotational flows for which the isocurls are concentric circles is superposable on it. In

the case of the remaining three flows we find that the contemplated types of flows do not exist.

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