FLOW OF A REINER-RIVLIN FLUID BETWEEN TWO CONCENTRIC, ROUGH CIRCULAR CYLINDERS ROTATING ABOUT THEIR COMMON AXIS

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INTRODUCTION

CITRON[1] studied the slow motion of an incompressible viscous fluid between two rough circular cylinders rotating about their common axis. The variation of the radii of the bounding cylinders was taken to be axially symmetric and small. The radial and axial velocities arising due to roughness were absent because of the neglect of inertial terms. In applying the Fourier transform technique, he assumed the azimuthal velocity to be zero when the axial co-ordinate $Z \rightarrow \pm \infty$. Such an assumption is not appropriate in the case of sinusoidal roughness extending over the region $-\infty < Z < \infty$.

In the present note, we shall take into account the inertial terms for the slow motion in a limited manner by maintaining the first powers of the Reynolds number $R = \rho a^2 \Omega/\eta$, where ρ is the density of the fluid, η the coefficient of viscosity and a and Ω are the radius and angular velocity of the inner cylinder. The effect of cross-viscosity is studied through the parameter $S = \zeta\Omega/\eta$, where ζ is the coefficient of cross-viscosity taken to be small.

We assume the roughness and the resulting velocities to be in the form of Fourier series in the axial co-ordinate Z.

1. FORMULATION OF THE PROBLEM

In a system of cylindrical co-ordinates, the equations determining the axially symmetric flow of a Reiner-Rivlin fluid are

$$R\left[U\frac{\partial U}{\partial X} + W\frac{\partial U}{\partial Z} - \frac{V^2}{X}\right]$$

$$= -\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial X^2} + \frac{1}{X}\frac{\partial U}{\partial X} - \frac{U}{X^2} + \frac{\partial^2 U}{\partial Z^2} + 2S\left[4\frac{\partial U}{\partial X}\frac{\partial^2 U}{\partial X^2}\right.$$

$$- \frac{U}{X}\left(\frac{\partial^2 W}{\partial X \partial Z} + \frac{\partial^2 U}{\partial Z^2}\right) + \left(\frac{\partial U}{\partial Z} + \frac{\partial W}{\partial X}\right)\left(\frac{\partial^2 U}{\partial X \partial Z} + \frac{\partial^2 W}{\partial X^2}\right)$$

$$- \frac{2U^2}{X^3} + \frac{2}{X}\left(\frac{\partial U}{\partial X}\right)^2 + \left(\frac{\partial V}{\partial X} - \frac{V}{X}\right)\frac{\partial}{\partial X}\left(\frac{\partial V}{\partial X} - \frac{V}{X}\right)$$

$$+ \frac{\partial V}{\partial Z}\frac{\partial}{\partial Z}\left(\frac{\partial V}{\partial X} - \frac{V}{X}\right) - \frac{1}{X}\left(\frac{\partial V}{\partial Z}\right)^2 + \frac{\partial^2 V}{\partial Z^2}\left(\frac{\partial V}{\partial X} - \frac{V}{X}\right)\Bigg], \tag{1.1}$$

$$R\left[U\left(\frac{\partial V}{\partial X} + \frac{V}{X}\right) + W\frac{\partial V}{\partial Z}\right]$$

$$= \frac{\partial^2 V}{\partial X^2} + \frac{1}{X}\frac{\partial V}{\partial X} - \frac{V}{X^2} + \frac{\partial^2 V}{\partial Z^2} + S\Bigg[\frac{\partial^2 V}{\partial X \partial Z}\left(\frac{\partial U}{\partial Z} + \frac{\partial W}{\partial X}\right)$$

$$+ \frac{\partial V}{\partial Z}\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial Z} + \frac{\partial W}{\partial X}\right) - 2\frac{\partial^2 U}{\partial X \partial Z}\frac{\partial V}{\partial Z} + \frac{\partial}{\partial Z}$$

$$\times \left\{\left(\frac{\partial U}{\partial Z} + \frac{\partial W}{\partial X}\right)\left(\frac{\partial V}{\partial X} - \frac{V}{X}\right)\right\} - 2\frac{\partial W}{\partial Z}\frac{\partial}{\partial X}\left(\frac{\partial V}{\partial X} - \frac{V}{X}\right)$$

$$- 2\frac{\partial^2 V}{\partial Z^2}\frac{\partial U}{\partial X} - 2\frac{\partial^2 W}{\partial X \partial Z}\left(\frac{\partial V}{\partial X} - \frac{V}{X}\right) - \frac{4}{X}\frac{\partial W}{\partial Z}\left(\frac{\partial V}{\partial X} - \frac{V}{X}\right)$$

$$+ \frac{2}{X}\frac{\partial V}{\partial Z}\left(\frac{\partial U}{\partial Z} + \frac{\partial W}{\partial X}\right)\Bigg], \tag{1.2}$$

$$R\left[U\frac{\partial W}{\partial X} + W\frac{\partial W}{\partial Z}\right]$$

$$= -\frac{\partial P}{\partial Z} + \frac{\partial^2 W}{\partial X^2} + \frac{1}{X}\frac{\partial W}{\partial X} + \frac{\partial^2 W}{\partial Z^2} + 2S\Bigg[4\frac{\partial W}{\partial Z}\frac{\partial^2 W}{\partial Z^2} - \frac{U}{X}$$

$$\times \left(\frac{\partial^2 U}{\partial X \partial Z} + \frac{\partial^2 W}{\partial X^2}\right) + \left(\frac{\partial U}{\partial Z} + \frac{\partial W}{\partial X}\right)$$

$$\times \left(\frac{\partial^2 U}{\partial Z^2} + \frac{\partial^2 W}{\partial X \partial Z} - \frac{1}{X}\frac{\partial U}{\partial X}\right)\Bigg] + S\Bigg[\frac{\partial V}{\partial Z}\left(2\frac{\partial^2 V}{\partial Z^2} + \frac{\partial^2 V}{\partial X^2}\right)$$

$$+ \frac{\partial^2 V}{\partial X \partial Z} \times \left(\frac{\partial V}{\partial X} - \frac{V}{X}\right)\Bigg], \tag{1.3}$$

and

$$\frac{\partial U}{\partial X} + \frac{U}{X} + \frac{\partial W}{\partial Z} = 0, \tag{1.4}$$

where R is the Reynolds number, S the cross-viscosity parameter and the various physical quantities have been non-dimensionalized by means of the length 'a', a velocity $a\Omega$ and a stress $\eta\Omega$.

2. The Primary Motion

If we neglect the effect of the axial roughness of the bounding cylinders, we have the motion of a Reiner-Rivlin fluid between two concentric smooth circular cylinders rotating about their common axis. Quantities pertaining to this primary motion are denoted by the suffix zero.

The primary motion is given by

$$-\frac{RV_0^2}{X} = -\frac{dP_0}{dX} + S\frac{d}{dX}\left(\frac{dV_0}{dX} - \frac{V_0}{X}\right)^2, \tag{2.1}$$

$$O = \frac{d^2V_0}{dX^2} + \frac{1}{X}\frac{dV_0}{dX} - \frac{V_0}{X^2}, \tag{2.2}$$

and

$$U_0 = W_0 = 0, \tag{2.3}$$

with the boundary conditions

$$V_0(1) = 1, \tag{2.4}$$

$$V_0(l) = ml, \tag{2.5}$$

where $l (= b/a)$ is the radius ratio of the bounding cylinders.

The velocity can easily be obtained as

$$V = AX + \frac{B}{X}, \tag{2.6}$$

where

$$A = \frac{l^2m - 1}{l^2 - 1}, \tag{2.7}$$

and

$$B = \frac{l^2(1 - m)}{l^2 - 1}, \tag{2.8}$$

The pressure is given by

$$P_0 = \frac{4SB^2}{X^4} + R\left[\frac{A^2X^2}{2} - \frac{B^2}{2X^2} + 2AB\log X\right]. \qquad (2.9)$$

3. SECONDARY MOTIONS DEVELOPED BY ROUGHNESS

We now obtain the modification of the primary motion due to small axial roughness of the cylinders. The axial roughness is given by the following expressions for the radii $X_1(Z)$ and $X_2(Z)$ of the inner and outer cylinders respectively:

$$X_1 = 1 + \epsilon\left[\sum_1^\infty a_n \cos nhZ + \sum_1^\infty b_n \sin nhZ\right], \qquad (3.1)$$

and

$$X_2 = l + \epsilon l\left[\sum_1^\infty a_n' \cos nhZ + \sum_1^\infty b_n' \sin nhZ\right], \qquad (3.2)$$

where

$$\epsilon \ll 1. \qquad (3.3)$$

Accordingly, the modifications in the velocities U, V, W and the pressure P may be taken in the following form:

$$U = \epsilon U'(X, Z), \qquad (3.4)$$

$$W = \epsilon W'(X, Z), \qquad (3.5)$$

$$V = V_0 + \epsilon V'(X, Z) \qquad (3.6)$$

and

$$P = P_0 + \epsilon P'(X, Z). \qquad (3.7)$$

Substituting for U, V, W and P in the equations (1.1)–(1.4) and neglecting squares of ϵ, we have the following equations for the determination of U', V', W' and P':

$$-\frac{2RV_0V'}{X}$$

$$= -\frac{\partial P'}{\partial X} + \frac{\partial^2 U'}{\partial X^2} + \frac{1}{X}\frac{\partial U'}{\partial X} - \frac{U'}{X^2} + \frac{\partial^2 U'}{\partial Z^2}$$

$$+ 2S\left[\left(\frac{dV_0}{dX} - \frac{V_0}{X}\right)\frac{\partial}{\partial X}\left(\frac{\partial V'}{\partial X} - \frac{V'}{X}\right) + \left(\frac{\partial V'}{\partial X} - \frac{V'}{X}\right)\right.$$

$$\left. \times \frac{d}{dX}\left(\frac{dV_0}{dX} - \frac{V_0}{X}\right) + \frac{1}{2}\frac{\partial^2 V'}{\partial Z^2}\left(\frac{dV_0}{dX} - \frac{V_0}{X}\right)\right], \qquad (3.8)$$

$$RU'\left(\frac{dV_0}{dX} + \frac{V_0}{X}\right)$$

$$= \frac{\partial^2 V'}{\partial X^2} + \frac{1}{X}\frac{\partial V'}{\partial X} - \frac{V'}{X^2} + \frac{\partial^2 V'}{\partial Z^2} + S\left[\left(\frac{dV_0}{dX} - \frac{V_0}{X}\right)\right.$$

$$\times \frac{\partial}{\partial Z}\left(\frac{\partial U'}{\partial Z} + \frac{\partial W'}{\partial X}\right) - 2\frac{\partial W'}{\partial Z}\frac{d}{dX}\left(\frac{dV_0}{dX} - \frac{V_0}{X}\right)$$

$$\left. - 2\frac{\partial^2 W'}{\partial X \partial Z}\left(\frac{dV_0}{dX} - \frac{V_0}{X}\right) - \frac{4}{X}\frac{\partial W'}{\partial Z}\left(\frac{dV_0}{dX} - \frac{V_0}{X}\right)\right], \qquad (3.9)$$

$$O = -\frac{\partial P'}{\partial Z} + \frac{\partial^2 W'}{\partial X^2} + \frac{1}{X}\frac{\partial W'}{\partial X} + \frac{\partial^2 W'}{\partial Z^2} + S\left[\frac{d^2 V_0}{dX^2}\frac{\partial V'}{\partial Z}\right.$$

$$\left. + \left(\frac{dV_0}{dX} - \frac{V_0}{X}\right)\frac{\partial^2 V'}{\partial X \partial Z}\right], \qquad (3.10)$$

and

$$\frac{\partial U'}{\partial X} + \frac{U'}{X} + \frac{\partial W'}{\partial Z} = 0. \qquad (3.11)$$

The solution of the above equations may be written in the form:

$$U' = \sum_1^\infty \cos nhZ\, U_n'(X) + \sum_1^\infty \sin nhZ\, \bar{U}_n'(X), \qquad (3.12)$$

$$V' = \sum_1^\infty \cos nhZ\, V_n'(X) + \sum_1^\infty \sin nhZ\, \bar{V}_n'(X), \qquad (3.13)$$

$$W' = \sum_1^\infty \sin nhZ\, W_n'(X) + \sum_1^\infty \cos nhZ\, \bar{W}_n'(X) \qquad (3.14)$$

and

$$P' = \sum_1^\infty \cos nhZ\, P_n'(X) + \sum_1^\infty \sin nhZ\, \bar{P}_n'(X). \qquad (3.15)$$

We shall obtain typical solutions for $\bar{U}_n'$, $\bar{V}_n'$, $\bar{W}_n'$ and $\bar{P}_n'$, the method of obtaining the other solutions being the same.

The equations giving $\bar{U}_n'$, $\bar{V}_n'$, $\bar{W}_n'$ and $\bar{P}_n'$ are

$$-\frac{2R\bar{V}_n'}{l^2-1}\left[l^2m-1+\frac{l^2(1-m)}{X^2}\right]$$

$$=-\frac{d\bar{P}_n'}{dX}+\left[\frac{d^2}{dX^2}+\frac{1}{X}\frac{d}{dX}-\frac{1}{X^2}-n^2h^2\right]\bar{U}_n'$$

$$-\frac{4Sl^2(1-m)}{X^2(l^2-1)}\frac{d}{dX}\left(\frac{d}{dX}-\frac{1}{X}\right)\bar{V}_n'+\frac{8Sl^2(1-m)}{X^3(l^2-1)}$$

$$\times\left(\frac{d}{dX}-\frac{1}{X}\right)\bar{V}_n'+\frac{2Sl^2(1-m)}{(l^2-1)X^2}n^2h^2\bar{V}_n', \tag{3.16}$$

$$\frac{2R(l^2m-1)}{l^2-1}\bar{U}'_n$$

$$=\left(\frac{d^2}{dX^2}+\frac{1}{X}\frac{d}{dX}-\frac{1}{X^2}-n^2h^2\right)\bar{V}_n'+\frac{2Snhl^2(1-m)}{X^2(l^2-1)}$$

$$\times\left(nh\bar{U}'_n-\frac{d\bar{W}_n'}{dX}\right), \tag{3.17}$$

$$0=-nh\bar{P}_n'+\left(\frac{d^2}{dX^2}+\frac{1}{X}\frac{d}{dX}-n^2h^2\right)\bar{W}_n'+\frac{2l^2(1-m)nh}{X^2(l^2-1)}$$

$$\times\left(\frac{1}{X}-\frac{d}{dX}\right)\bar{V}_n', \tag{3.18}$$

and

$$\left(\frac{d}{dX}+\frac{1}{X}\right)\bar{U}_n'-nh\bar{W}_n'=0. \tag{3.19}$$

For the slow motion of a Reiner-Rivlin fluid with a small value of the parameter S, we can obtain solutions in the form:

$$\bar{U}_n'=R\bar{u}_{1,n}+S\tilde{u}_{1,n}+R^3\bar{u}_{2,n}+\ldots, \tag{3.20}$$

$$\bar{V}_n' = \bar{v}_{1,n} + R^2\bar{v}_{2,n} + RS\bar{v}_{3,n} + \ldots, \tag{3.21}$$

$$\bar{W}_n' = R\bar{w}_{1,n} + S\bar{w}_{1,n} + R^3\bar{w}_{2,n} + \ldots, \tag{3.22}$$

and

$$\bar{P}_n' = R\bar{p}_{1,n} + S\bar{p}_{1,n} + R^3\bar{p}_{2,n} + \ldots \tag{3.23}$$

Retaining only the terms involving the first powers of R and S, we obtain the following solutions of equations (3.16)-(3.19):

$$\begin{aligned} \bar{U}_n' = &- \frac{Sn^2h^2l^2(1-m)}{X(l^2-1)}\left[B_1K_0(x) - A_1I_0(x)\right] \\ &+ \frac{R}{l^2}\left[\frac{(l^2m-1)x^2}{4n^2h^2}\{A_1I_1(x) + B_1K_1(x)\}\right] \\ &+ A_3I_1(x) + B_3K_1(x) \\ &- \frac{R}{2}(1-m)\Big[\log x\{A_1I_1(x) + B_1K_1(x)\} \\ &- \frac{1}{x}\{A_1I_0(x) - B_1K_0(x)\}\Big] \\ &+ \frac{x}{2}\Big[A_2I_0(x) - B_2K_0(x)\Big], \end{aligned} \tag{3.24}$$

$$\bar{V}_n' = A_1I_1(x) + B_1K_1(x), \tag{3.25}$$

$$\begin{aligned} \bar{W}_n' = &\frac{Sl^2(1-m)}{(l^2-1)}\frac{n^2h^2}{x}\left[A_1I_1(x) + B_1K_1(x)\right] \\ &+ \frac{R}{l^2-1}\Big[\frac{l^2m-1}{4n^2h^2}\Big\{xA_1\left[xI_0(x) + 2I_1(x)\right] \\ &+ xB_1\left[2K_1(x) - xK_0(x)\right] \\ &- \frac{l^2}{2}(1-m)\log x\left[A_1I_0(x) - B_1K_0(x)\right]\Big\}\Big] \\ &+ \tfrac{1}{2}\left[A_2\{2I_0(x) + xI_1(x)\} + B_2\{xK_1(x) - 2K_0(x)\}\right] \\ &+ A_3I_0(x) - B_3K_0(x), \end{aligned} \tag{3.26}$$

and

$$\bar{P}_n' = \frac{Sl^2}{l^2-1}(1-m)\,nh\Big[\frac{A_1}{x^2}\Big\{\frac{3I_1(x)}{x} - I_0(x)\Big\}$$

$$+\frac{B_1}{x^2}\Big\{\frac{3K_1(x)}{x} + K_0(x)\Big\}$$

$$+\frac{R(l^2m-1)}{4n^3h^3}\Big[A_1\Big\{6I_0(x) + 2\Big(x+\frac{1}{x}\Big)I_1(x)\Big\}$$

$$+B_1\Big\{2\Big(x+\frac{1}{x}\Big)K_1(x) - 6K_0(x)\Big\}$$

$$-\frac{l^2}{x}(1-m)\{A_1I_1(x) + B_1K_1(x)\}\Big]$$

$$+\frac{1}{nh}\{A_2I_0(x) + B_2K_0(x)\}$$

$$+\frac{2l^2(1-m)}{n^2h^2(l^2-1)}\Big[A_1\Big\{\frac{2I_1(x)}{x} - I_0(x)\Big\}$$

$$+B_1\Big\{\frac{2K_1(x)}{x} + K_0(x)\Big\}\Big], \qquad (3.27)$$

where A_1, A_2, A_3, B_1, B_2 and B_3 are arbitrary constants to be determined from the boundary conditions and $x = nhX$.

4. The Boundary Conditions

The boundary conditions on U, V, W are

$$U = W = 0, \text{ when } X = X_1, X_2; \qquad (4.1)$$

$$V(X_1) = X_1, \qquad (4.2)$$

$$V(X_2) = mX_2. \qquad (4.3)$$

Accordingly, we obtain the following boundary conditions on a typical set of velocities $\bar{U}_n'$, $\bar{V}_n'$, $\bar{W}_n'$:

$$\bar{U}_n' = \bar{W}_n' = 0,$$

when

$$X = 1, l; \qquad (4.4)$$

$$[\bar{V}_n']_{X=1} = [1 + B - A]\, b_n, \tag{4.5}$$

$$[\bar{V}_n']_{X=l} = \left[m + \frac{B}{l^2} - A\right] l b_n'. \tag{4.6}$$

5. A Particular Case of Sinusoidal Roughness

We have studied in detail a case of sinusoidal roughness with

$$b_1 = b_1' = 1,$$

$$b_i = b_i' = 0\ (i \neq 1).$$

The velocity components and the pressure in the cases of a Newtonian fluid with

$$R = 0{\cdot}1,\ S = 0$$

and a Reiner-Rivlin fluid with

$$R = S = 0{\cdot}1,$$

have been determined.

We have chosen, for a sample numerical work, the values of the ratios l and m as $l = m = 2$.

The values of the arbitrary constants in case of the Newtonian fluid are

$$A_1 = -\,0{\cdot}489051,$$

$$A_2 = 0{\cdot}161928,$$

$$A_3 = -\,0{\cdot}049645,$$

$$B_1 = -\,3{\cdot}971162,$$

$$B_2 = -\,0{\cdot}539231,$$

and

$$B_3 = 0{\cdot}004461.$$

In the case of a Reiner-Rivlin fluid, they are

$$A_1 = -\,0{\cdot}489051,$$

$$A_2 = 0{\cdot}224262,$$

$$A_3 = -0\cdot175075,$$
$$B_1 = -3\cdot971162,$$
$$B_2 = 0\cdot020838,$$

and

$$B_3 = 0\cdot544078.$$

6. DISCUSSION OF THE RESULTS

A roughness wave of small amplitude introduces radial and axial velocities and also modifies the azimuthal velocity. In the case of Newtonian fluids and also Reiner-Rivlin fluids with small cross-viscosity, the direction of the axial velocity is unaltered in any cross-section normal to the cylinders. For comparatively larger values of the parameter S, we find that there is a change in the sense of the axial velocity as we pass from one cylinder to the other. Figure 1 shows the axial velocities in the cases of Newtonian and Reiner-Rivlin fluids mentioned in § 5. This aspect may be taken to distinguish the non-Newtonian fluid considered here from the usual Newtonian fluid.

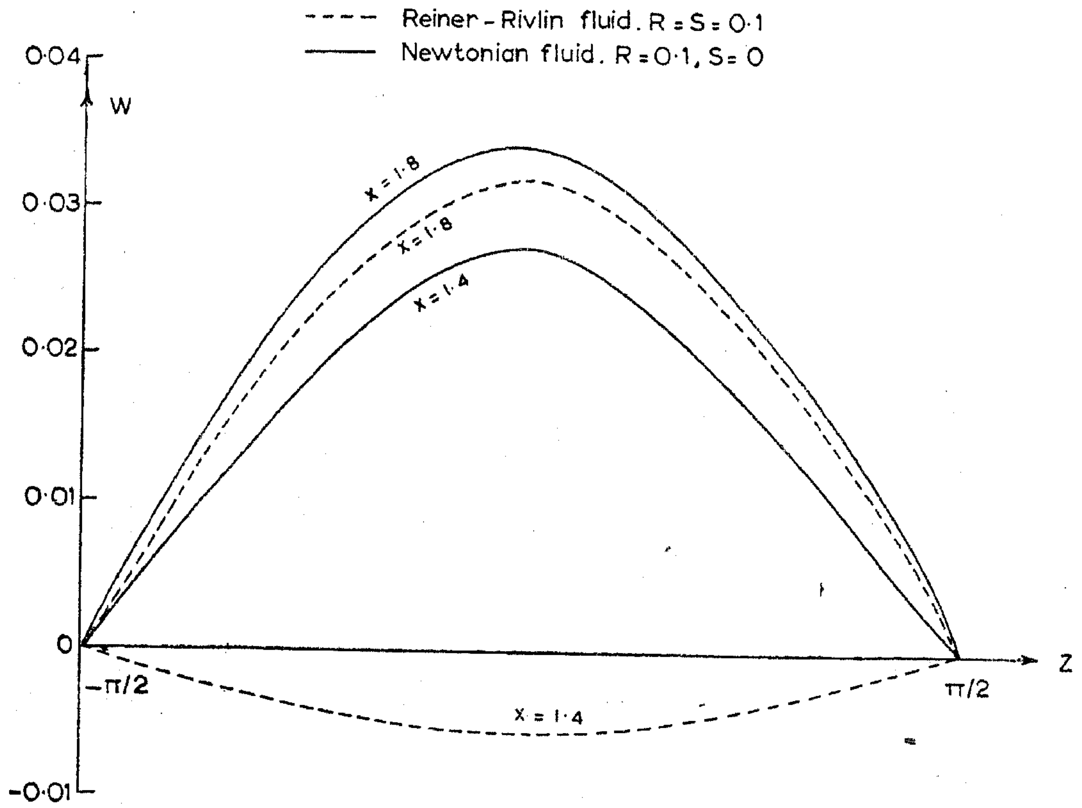

FIG. 1. Axial velocity profiles for Newtonian and Reiner-Rivlin fluids.

The radial velocity (Fig. 2) in any cross-section normal to the common axis of the cylinders does not reverse either for a Newtonian or a Reiner-

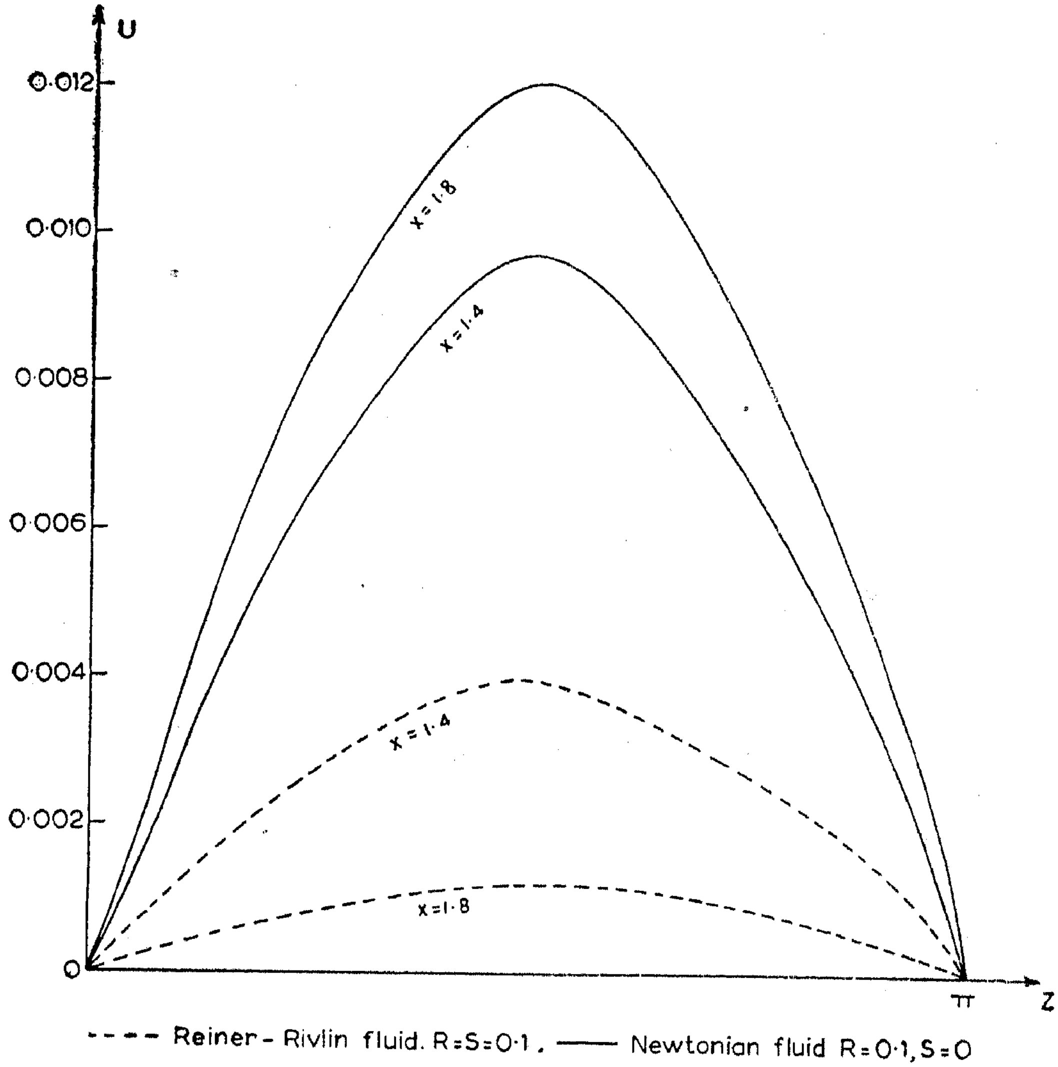

FIG. 2. Radial velocity profiles for Newtonian and Reiner-Rivlin fluids.

Rivlin fluid. The azimuthal velocity is unaffected by any small values of the parameters R and S and depends only on the amplitude of the roughness wave.

REFERENCE

1. Citron, J. S. .. *ASME Jour. App. Mech.*, 1962, **29**, 188.