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On the Feigenbaum-Cvitanović equation in the theory of chaotic behaviour

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Abstract. We propose an analytic perturbative approach for the determination of the Feigenbaum-Cvitanović function and the universal parameter α occurring in the Feigenbaum scenario of period doubling for approach to chaotic behaviour. We apply the method to the case $Z = 2$ where Z is the order of the unique local maximum of the nonlinear map. Our third order approximation gives $\alpha = 2{\cdot}5000$ as compared to "exact" numerical value $\alpha = 2{\cdot}5029 \ldots$. We also obtain a reasonably accurate value of the Feigenbaum-Cvitanović function.

Keywords. Feigenbaum-Civitanović equation; chaotic behaviour; analytic perturbative approach.

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1. Introduction

The discovery by M Feigenbaum of universal quantitative behaviour, in the maps of an unit interval onto itself, has generated enormous excitement as it opens up the possibility of quantitatively understanding turbulence and other chaotic natural phenomenon. In Feigenbaum scenario a system follows period doubling route to chaos (Feigenbaum 1978, 1979, 1980).

In this theory asymptotically, at each period doubling the separation between any two corresponding adjacent elements of a periodic attractor is scaled by a constant ratio α. This leads to the existence of a function $g(x)$ which reproduces itself under the mapping except for the relevant scaling. This function satisfies the equation,

$$-\alpha g(g(x/\alpha)) = g(x), \tag{1}$$

and is normalised by

$$g(0) = 1. \tag{2}$$

We shall refer to this equation and $g(x)$ as Feigenbaum-Cvitanović equation and function respectively (Feigenbaum 1978). For a generalisation to period n-tuplings of this equation we refer to Cvitanović-Myrheim (1983). Equation (1) together with normalisation (2) determines both the functional form of $g(x)$ and α. We shall be interested in the solution which has a local maximum at $x = 0$ of the order Z. Such a solution is unique (Feigenbaum 1980, p. 15; Collet *et al* 1980).

The universal function $g(x)$ and α do, of course, depend on the order of local maximum *i.e.* if the transformation T considered is

$$T: x \to x' = \lambda(1 - a|x|^Z),$$

then they depend on Z. For $Z=2$, which is the most interesting case, we have

$$\alpha = 2{\cdot}502907875\ldots$$

The function $g(x)$ has also been determined numerically for this case (Feigenbaum 1979).

Another important universal constant δ, which determines the rate of period-doublings, is given by

$$\delta = 4{\cdot}6692016\ldots$$

for $Z=2$. A consideration of asymptotic period-doubling leads to the equation

$$-\alpha[h(g(x/\alpha))+g'(g(x/\alpha))h(x/\alpha)] = \delta h(x)$$
$$\delta > 1 \tag{3}$$

for a determination of the universal function $h(x)$ and universal constant δ. Again numerical determination of $h(x)$ and δ has been carried out (Feigenbaum 1979).

In view of the importance of these functions it would be desirable to have an analytical approach for their determination as opposed to one involving pure numerical computation. We propose an analytical perturbative scheme towards this purpose in the present paper. It is a systematic procedure and can be improved successively. Among other attempts to determine α, but not the $g(x)$, by analytic approximations, we may mention those of Helleman (1983) and Hu and Mao (1982) among others.

2. Basic equations and method

Replacing x by αx we get

$$g(\alpha x)+\alpha g[g(x)] = 0, \tag{4}$$

$$g(0) = 1. \tag{5}$$

Let
$$g(x) = 1+p(x), \tag{6}$$

$$p(x) = \sum_{n=1}^{\infty} p_n y^n, \tag{7}$$

$$y = |x|^Z \qquad p_1 \neq 0, \tag{8}$$

where Z is the order of local maximum of the transformation.
Further let

$$p(\alpha x) = \sum_{n=1}^{\infty} C_n[p(x)]^n. \tag{9}$$

Using (6) and (9) in (4) we obtain

$$1+\sum_{n=1}^{\infty} C_n[p(x)]^n+\alpha g[1+p(x)] = 0. \tag{10}$$

Equating the coefficients of $[p(x)]^n$ to zero in (10) we obtain

$$1+\alpha g(1) = 0, \tag{11}$$

and

$$n!C_n + \alpha f^{(n)}(1) = 0 \qquad (n = 1, 2, \ldots). \tag{12}$$

On the other hand, if we expand both sides of (9) in powers of y and equate the coefficients we obtain ($n = 1, 2, \ldots$)

$$p_n\beta^n = \sum_{l, m_1, m_2, \ldots, m_l \geqslant 1} C_l p_{m_1} p_{m_2} \ldots p_{m_l} \delta_{m_1+m_2+\ldots+m_l, n} \tag{13}$$

where $\beta = |\alpha|^Z$.

We display first few of these equations.

$$p_1\beta = C_1 p_1, \tag{14}$$

$$p_2\beta^2 = C_1 p_2 + C_2 p_1^2, \tag{15}$$

$$p_3\beta^3 = C_1 p_3 + C_2(2p_1p_2) + C_3 p_1^3, \tag{16}$$

$$p_4\beta^4 = C_1 p_4 + C_2(2p_1p_3 + p_2^2) + C_3(3p_1^2 p_2) + C_4 p_1^4, \tag{17}$$

$$\ldots .$$

We note that since $p_1 \neq 0$,

$$C_1 = |\alpha|^Z. \tag{18}$$

It is also useful to note that

$$f^{(l)}(1) = \sum_{r=1}^{\infty} \frac{(rZ)!}{l!(rZ-l)!} p_r \tag{19}$$

for $l = 1, 2, 3, \ldots$, and

$$f(1) = 1 + \sum_{r=1}^{\infty} p_r. \tag{20}$$

If we combine (11), (13), (19) and (20), and further define

$$p_n\alpha^n = S_n|\alpha|^Z \tag{21}$$

$$n = 1, 2, \ldots$$

We obtain the basic equations to be solved

$$\frac{1}{\alpha} + 1 + |\alpha|^Z \sum_{r=1}^{\infty} \frac{S_r}{\alpha^r} = 0 \tag{22}$$

and

$$S_n + \sum_U \frac{(rZ)!}{l!(rZ-l)!} \cdot \frac{S_r}{\alpha^{r-1}} \cdot \frac{S_{m_1} S_{m_2} \ldots S_{m_l}}{|\alpha|^{Z(n-l)}} = 0$$

$$\text{for} \quad n \geqslant 1. \tag{23}$$

The summation in (23) is over the set U given by

$$r \geqslant 1, \, l \geqslant 1,$$

$$m_1 \geqslant 1, \, m_2 \geqslant 1, \ldots, m_l \geqslant 1$$

and

$$m_1 + m_2 + \ldots + m_l = n.$$

We now note that (23) for $n = 1$ leads to

$$S_1 + \sum_{r>1}^{\infty} \binom{rZ}{1} \frac{S_r S_1}{\alpha^{r-1}} = 0$$

and since $S_1 \neq 0$ we get

$$\frac{1}{Z} + \sum_{r>1} \frac{r S_r}{\alpha^{r-1}} = 0. \tag{24}$$

In fact using this equation we can simplify the remaining equations (23) for $n \geqslant 2$ to read

$$S_n\left(1 - \frac{1}{|\alpha|^{Z(n-1)}}\right) + \sum_{l\geqslant 2}^{l=n} \sum_{r\geqslant 1}^{\infty} \binom{rZ}{l} \frac{S_r}{\alpha^{r-1}}$$

$$\times \frac{\sum_{m_1\geqslant 1,\ldots,m_l\geqslant 1} S_{m_1} S_{m_2} \ldots S_{m_l} \delta_{m_1+m_2+\ldots+m_l,n}}{|\alpha|^{Z(n-l)}} = 0$$

$$\text{for } n = 2, 3, \ldots \tag{25}$$

Equations (22) and (24) and (25) together constitute a system of coupled nonlinear equations to determine α and $S_1, S_2, \ldots$ We regard (22) as the eigenvalue equation for α while (24) and (25) are to be used to determine $S_1, S_2, \ldots$ in terms of α.

These equations are, however, highly nonlinear and coupled. One needs a systematic procedure for their solution. We now notice that (24)–(25) allow us to conclude that, as $\alpha \to \infty$, we have a solution for which S_n tends to constants $S_{n,0}$ *i.e.*

$$S_n(\alpha) \underset{\alpha\to\infty}{\Longrightarrow} S_{n,0}.$$

In particular S_n's admit, as $\alpha \to \infty$, an expansion in the inverse powers of α, of the form, for Z a nonrational number,

$$S_n(\alpha) = \sum_{p,q=0,1,2,\ldots} \frac{S_{n;p;q}}{|\alpha|^{zq+p}},$$

where $S_{n,p,q}$ are numerical constants. When Z is integral or a rational number simpler expansions can be written down.

We propose to use such $\alpha \to \infty$ expansions and use them to $S_n(\alpha)$ in terms of α by using (24) and (25). These expansions for $S_n(\alpha)$ when substituted in (22) would give us the eigenvalue equation for the determination of α.

It is known that as $Z \to 1$, α does tend to infinity. We thus expect this procedure to clearly work when $Z \to 1$ (Collet *et al* 1980; Derrida *et al* 1978, 1979). We will see in the next section, where we apply it to the physically most interesting case *i.e.* $Z = 2$ that the method is still quite effective.

3. Application of the method for $Z = 2$

We shall now apply our method for the $Z = 2$ case. In this case we can use the simpler expansion given by

$$S_n(\alpha) = \sum_{m=0,\,1,\,2,\,\ldots} \frac{S_{n,\,m}}{\alpha^m}. \tag{26}$$

Using (24) we obtain, as $\alpha \to \infty$

$$S_1(\alpha) \to -1/2. \tag{27}$$

Similarly from (5) and using (27) we obtain the following leading behaviours, as $\alpha \to \infty$

$$S_2 \to \frac{1}{8},\quad S_3 \to \frac{1}{16\alpha},\quad S_4 \to -\frac{1}{128\alpha}\ \text{etc.} \tag{28}$$

Proceeding further and equating powers of α on both sides of (24) and (25) we obtain

$$\begin{aligned} O &= S_{1,\,1} + 2S_{2,\,0} \\ O &= S_{2,\,1} + 3S_{1,\,0}^2 S_{1,\,1} + 6(S_{1,\,0})^2 S_{2,\,0}, \\ O &= S_{1,\,2} + 2S_{2,\,1} + 3S_{3,\,0}, \\ O &= S_{2,\,2} - S_{2,\,0} + (S_{1,\,0})^2[S_{1,\,2} + 6S_{2,\,1} + 15S_{3,\,0}] \\ &\quad + 2S_{1,\,0}S_{1,\,1}[S_{1,\,1} + 6S_{2,\,0}] + [2S_{1,\,0}S_{1,\,2} + S_{1,\,1}^2]S_{1,\,0}, \\ O &= S_{1,\,3} + 2S_{2,\,2} + 3S_{3,\,1} + 4S_{4,\,0},\ \text{etc.} \end{aligned} \tag{29}$$

Combining the information contained in (28) with that contained in (29) we obtain

$$\begin{aligned} &S_{1,\,0} = -1/2; \\ &S_{1,\,1} = -1/4,\ S_{2,\,0} = 1/8; \\ &S_{1,\,2} = 0,\ S_{2,\,1} = 0,\ S_{3,\,0} = 0; \\ &S_{1,\,3} = -1/4,\ S_{2,\,2} = 1/32,\ S_{3,\,1} = 1/16,\ S_{4,\,0} = 0; \quad \text{etc.} \end{aligned} \tag{30}$$

Using these coefficients we obtain the approximation

$$\begin{aligned} g(x) = 1 - \left(\frac{\alpha}{2} + \frac{1}{4} + \frac{1}{4\alpha^2} + \ldots\right)x^2 + \left(\frac{1}{8} + \frac{1}{32\alpha^2} + \ldots\right)x^4 \\ + \left(\frac{1}{16\alpha^2} + \ldots\right)x^6 + \ldots \end{aligned} \tag{31}$$

where all the coefficients of the powers of x^2 have been kept to the order $(1/\alpha^2)$.

The eigenvalue equation is

$$J(\alpha) = 0, \tag{32}$$

where $J(\alpha) \equiv \frac{1}{\alpha} + 1 + \alpha \sum_{n=1}^{\infty} \frac{S_n}{\alpha^{n-1}}$ and we have

$$J(\alpha) \underset{\alpha\to\infty}{=} -\frac{\alpha}{2} + \frac{7}{8} + \frac{1}{\alpha} - \frac{5}{32\alpha^2} + 0(1/\alpha^3). \tag{33}$$

To this order in approximation we obtain from (32) using (33)

$$\alpha = 5/2 = 2{\cdot}5000 \tag{34}$$

to be compared with the result of the "exact" numerical computation $\alpha = 2{\cdot}5029\ldots$.

Using the approximate value $\alpha = 5/2$ we get from the expression (31),

$$g(x) \approx 1 - 1{\cdot}5400x^2 + 0{\cdot}1300x^4 + 0{\cdot}0100x^6 + \ldots \quad (35)$$

We also quote, for comparison, the result of the "exact" numerical computation (Feigenbaum 1979)

$$g(x) \approx 1 - 1{\cdot}5276x^2 + 0{\cdot}1048x^4 + 0{\cdot}0267x^6 - 0{\cdot}0035x^8 + \ldots \quad (36)$$

where we have kept only the four figures after the decimal sign.

4. An exact solution

An exact solution of (4) is given by

$$g(x) = (1 + |x|^Z)^{1/Z}, \quad (37)$$

$$\alpha = -\frac{1}{2^{1/Z}}. \quad (38)$$

This solution is, however, of no interest for the Feigenbaum scenario for the onset of the chaotic behaviour since $g(x)$ has a local *minimum* rather than local *maximum* at $x = 0$. It however implies that the eigenvalue equation for α should have a root given by (38). For $Z = 2$ we should thus have a root at $\alpha = -1/\sqrt{2} \approx -0{\cdot}707\ldots$ for the exact eigenvalue equation. Our approximate eigenvalue equation given by (32) and (33) *i.e.*

$$O = -\frac{\alpha}{2} + \frac{7}{8} + \frac{1}{\alpha} - \frac{5}{32\alpha^2} \quad (39)$$

has a negative root at $\alpha = -(\sqrt{17} + 3)/8 \approx -0{\cdot}8904$ which presumably a reflection of the root $\alpha \approx -0{\cdot}707\ldots$ for the exact equation.

5. Concluding remarks

We thus see that the present method produces reasonably accurate results. Since the method is a systematic one the accuracy can be improved by taking higher order approximations.

The real utility of the method would be to further allow us to calculate the Z-dependance of the universal parameter α. Moreover whenever one needs $g(x)$ as an input, such as for example in the equations for $h(x)$, it is useful to have such analytic approximations to $g(x)$ as we have obtained here. These and similar investigations will be reported later.

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