Loss of memory in a chaotic dynamical system

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A chaotic signal loses the memory of the initial conditions with time, and the future behavior becomes unpredictable. Here we propose a method to understand the loss of memory with time from a time series. This is done by introducing time-dependent generalized exponents. The asymptotic behavior of these exponents is interesting and can distinguish between chaotic systems that lose memory of the initial conditions completely, those that partially retain the memory, and those (borderline of chaos) that fully retain the memory. We discuss these features with some illustrative examples.

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I. INTRODUCTION

Sensitivity to initial conditions is probably the most important characteristic that differentiates a chaotic system from an integrable system [1]. This sensitivity to initial conditions is reflected in the fact that errors grow exponentially with time. This exponential growth is characterized by the Lyapunov exponent. Equivalently, two close-by trajectories diverge exponentially in time. This leads to the unpredictability associated with the chaotic systems in the following sense. A small but finite uncertainty in the initial conditions grows very rapidly with time and after some time it becomes almost impossible to predict the phase-space trajectory. We may say that the system progressively loses the memory of the initial conditions. This loss of memory of the initial conditions obviously depends on the exponential divergence of the trajectories and also on the amount of uncertainty of the initial conditions.

The initial exponential divergence of close-by trajectories is well characterized by the Lyapunov exponent [1]. However, the Lyapunov exponent does not remain a very useful parameter for larger times. This is because, as the distance between the trajectories increases and reaches scales of the size of the attractor, the trajectories start folding back on the attractor and a simple description in terms of the exponential divergence is no longer valid. Thus a different quantity is needed to characterize the long-time behavior of a chaotic signal. One would also like to know whether the memory of the initial conditions is completely lost and how much time is required for this to happen. The purpose of this paper [2] is to introduce a method of quantitatively analyzing loss of memory of a chaotic signal which is suitable for all the times. The method makes use of the notion of the fractal dimension and generalized dimensions [3–8] and introduces time-dependent generalized exponents. It reduces to the analysis given by the Lyapunov exponent for shorter times. For larger times the analysis gives the progressive loss of memory of initial conditions. We also find that in some chaotic signals there is never a complete loss of memory. Our method has an advantage of being able to characterize this situation.

In Sec. II we introduce the time-dependent generalized exponents for a one-dimensional case and discuss their properties. Section III gives illustrative examples. The higher-dimensional case and examples are discussed in Sec. IV. We conclude with a discussion in Sec. V.

II. TIME-DEPENDENT GENERALIZED EXPONENTS

A. Definitions and formalism

Consider a time series $\{x_k\}$, $k=1,2,\ldots$ specifying values of a physical observable x at successive times t_k. We assume that the transients, if any, have already died out and our time series gives the points of a trajectory on the attractor. We begin by discussing a one-dimensional situation first. Divide the maximum range of the variable in N parts of equal length l. Here the length scale l specifies the uncertainty or error in the initial conditions. Let p_i be the probability that the variable x lies in the ith interval. We define a joint probability of $p_{i,j}(t)$, $i,j=1,2,\ldots,N$, as the probability that the variable x lies in the ith interval at some time t' and in the jth interval at time $t+t'$, i.e., after a time t. The joint probability of $p_{i,j}(t)$ will be independent of t' since we assume translational invariance in time.

We now introduce the time-dependent generalized exponents $D_{q,l}(t)$ (Refs. [2–8]) for the above cover by the following relation ($-\infty<q<\infty$):

$$D_{q,l}(t)=\frac{1}{(q-1)}\tau_{q,l}(t) \quad (2.1)$$

where

$$\tau_{q,l}(t)=\frac{\ln\left[\sum_{i,j}{}' p_{i,j}^{q}(t)\right]}{\ln l} . \quad (2.2)$$

The prime in the summation of Eq. (2.2) means that the sum is over only those i and j values for which the probability $p_{i,j}(t)$ is nonzero. In particular, for $q=0$ we get the time-dependent fractal exponent $D_{0,l}(t)$ [6], and for $q=2$ we get the time-dependent correlation exponent $D_{2,l}(t)$ [4]. For $q=1$ we get the time-dependent information exponent $D_{1,l}(t)$ [8], and its expression is obtained by taking the limit $q\rightarrow 1$ in (2.1) and is given by

$$D_{1,l}(t)=\frac{\sum_{i,j}' p_{i,j}\ln_{10}(p_{i,j})}{\ln(l)} . \tag{2.3}$$

The exponents defined above use the capacity notion of exponents since we use equal length scales. We also note that $D_{q,l}(t)$ defined by Eq. (2.1) can be treated as the generalized exponents in the two-dimensional space defined by the vectors $(x_k, x_{k+t}), k=1,2,\ldots$.

Let us consider the following two limiting cases.

(a) $t=0$. In this case, the joint probability $p_{i,j}(0)=p_i\delta_{i,j}$. Clearly our time-dependent generalized exponents reduce to the usual time-independent generalized dimensions of the attractor [9].

$$D_{q,l}(0)=D_q . \tag{2.4}$$

(b) $t=\infty$. The asymptotic behavior is more complicated. Consider the following two extreme possibilities.

(i) Suppose there is no loss of memory. Then writing $p_{i,j}(t)=p_{j|i}(t)p_i$ where $p_{j|i}(t)$ is the conditional probability, we note that for no loss of memory $p_{j|i}=1$ for some value of j and 0 for the remaining values. Thus from Eqs. (2.1) and (2.2), we find that the time-dependent generalized exponents do not change with time.

(ii) Consider the situation when the system has completely lost the memory of the initial conditions. Then we get $p_{i,j}(t)=p_ip_j$ and from Eqs. (2.1) and (2.2) we see that $D_{q,l}(t)=2D_q$. Thus the doubling of the generalized exponents indicates complete loss of memory.

We will see that in some cases this doubling does not occur though the time-dependent exponent increases, i.e., the memory is never completely lost even though we have initial exponential divergence of trajectories as indicated by a positive Lyapunov exponent.

We note that our condition of complete loss of memory corresponds to the mixing property which states that $\lim_{t\to\infty}\Pr(\phi^{-t}B\cap A)=\Pr(B)\Pr(A)$ for all sets A and B and ϕ^t is a dynamical map [10].

Let us compare the time-dependent generalized exponents of Eqs. (2.1) and (2.2) with the autocorrelation function, $\langle x_kx_{k+t}\rangle-\langle x_k\rangle\langle x_{k+t}\rangle$. It is easy to see that these quantities, though similar in nature, behave differently. For example, consider a sine function for which the autocorrelation function oscillates with time giving no indication of this perfectly predictable system. On the other hand, our time-dependent generalized exponents remain constant with time indicating a perfectly predictable signal. Thus, for studying the loss of memory, correlation function is not a good quantity.

B. Relations between invariants of the attractor and the time-dependent generalized exponents

1. Time-dependent fractal exponent and Lyapunov exponent

It is possible to approximately relate the time-dependent fractal exponent $D_{0,l}(t)$, the Lyapunov exponent λ, and the length scale l. In t time steps an interval of width l will be mapped into length $R\simeq le^{\lambda t}$. If for times larger than some time $\bar{t}$, R becomes of the order of the size of the attractor, we can roughly say that the memory of the initial conditions is completely lost, i.e., given an uncertainty of l in the starting value of the variable x, the value of x after time $\bar{t}$ may lie anywhere in the attractor and is thus completely unpredictable. The size of the attractor when measured with the scale l is $l^{-D_0}l$. Thus we get the relation [11]

$$\bar{t}\simeq -D_0\frac{\ln l}{\lambda} . \tag{2.5}$$

For time $t<\bar{t}$, R is less than the size of the attractor. The number of lengths that R covers is R/l. If we start from M initial lengths they are mapped into MR/l lengths. Starting with the entire attractor, i.e., $M\simeq l^{-D_0}$, we get

$$D_{0,l}(t)\simeq D_0-\frac{\lambda t}{\ln(l)} . \tag{2.6}$$

2. Time-dependent information exponent and metric entropy

Equation (2.3) for the information exponent can be rewritten as

$$D_{1,l}(t)=\frac{\sum_{i,j} p_{i,j}\ln[p_{j|i}(t)]}{\ln(l)}+D_1 \tag{2.7}$$

where

$$D_1=\frac{\sum_i p_i\ln(p_i)}{\ln(l)}$$

and $p_{j|i}(t)$ is the conditional probability that the variable x is in the jth interval at time $t+t'$ given that it was in the ith interval at time t'. Now if we assume that we are able to approximately write

$$p_{j|i}\simeq p_{j|i,i',i'',\ldots}$$

where $p_{j|i,i',i'',\ldots}$ is the probability that the variable x is in the jth interval at time $t'+t$ given that it was in the ith interval at time t', (i')th interval at time $t'-t$, (i'')th interval at time $t'-2t$, etc., then Eq. (2.3) can be written as

$$\begin{aligned} D_{1,l}(t)&\simeq D_1+\frac{\sum_{i,j,i',\ldots} p_{i,j,i',i'',\ldots}\ln p_{j|i,i',i'',\ldots}}{\ln(l)} ,\\ D_{1,l}(t)&\simeq D_1-\frac{th}{\ln(l)} \end{aligned} \tag{2.8}$$

where h is the metric entropy [12]. The relation between the metric entropy and the probabilities used to obtain Eq. (2.8) holds provided the original partition is a generating partition, otherwise Eq. (2.8) can be treated as an approximate relation.

3. Other time-dependent exponents

Noting the similarity of the form of Eqs. (2.6) and (2.8) above we may conjecture that for any q we may have

$$D_{q,l}(t)\cong D_q-\frac{\lambda_q t}{\ln(l)} \tag{2.9}$$

where λ_q is some generalized exponent and $\lambda_0=\lambda$ and $\lambda_1=h$.

C. f-α structure

We can introduce the time-dependent singularity index $\alpha(t)$ through the scaling relation

$$p_{i,j}=l^{\alpha_l(t)} . \tag{2.10}$$

It is possible to show that [3–5,13] $\tau_{q,l}(t)$ defined in Eq. (2.1) is related to $\alpha_l(t)$ through a Legendre transform

$$\tau_{q,l}(t)=q\alpha_l(t)-f_l(t) \tag{2.11}$$

where $f_l(t)$ is the fractal dimension of the support of the singularity index $\alpha_l(t)$. For $t=0$, we get the usual time-independent $f(\alpha)$ curve for the attractor. For large times if there is a complete loss of memory, $f_l(t)$ and $\alpha_l(t)$ will go to twice their values at $t=0$. In the other extreme case of no loss of memory for large times the values of $f_l(t)$ will remain unchanged in time.

III. EXAMPLES

We will now illustrate our formalism with various examples.

A. Tent map

Consider the tent map defined by

$$f(x)=\begin{cases} px, & 0\le x\le \frac{1}{2} \\ p(1-x), & \frac{1}{2}<x\le 1 \end{cases} \tag{3.1}$$

where $0\le p\le 2$. The tent map gives chaotic solutions for $1<p\le 2$. We study the following two cases.

Case (a), $p=2$. For this case we obtain a fully developed chaos and have a natural partition of the interval [0,1] into $N=2^m$ cells of size $l=2^{-m}$. The probability $p_i=2^{-m}$. It is easy to see that after t iterations each cell expands into 2^t cells with equal weights. Hence for each such cell $p_{i,j}(t)=2^{-m-t}$. For time $t\ge m$, the number of cells to which a single cell expands becomes a constant equal to 2^m. Thus we obtain

$$D_{q,l}(t)=D_q-\frac{t\ln(2)}{\ln(l)}$$
$$=1+\frac{t}{m} \quad \text{for } t\le m \tag{3.2a}$$

and

$$D_{q,l}(t)=2 \quad \text{for } t\ge m . \tag{3.2b}$$

Equations (3.2) are q independent. The Lyapunov exponent and metric entropy for the present case ($p=2$) are both ln(2). Thus our expression for $D_{q,l}(t)$ matches exactly with Eqs. (2.6) and (2.8) and complete loss of memory is obtained in $\bar{t}=m$ steps. [See Eq. (2.5).]

Case (b), $p=\sqrt{2}$. In this case the tent map has a chaotic attractor which is split into two bands A and B given by the open intervals $(\sqrt{2}-1,2-\sqrt{2})$ and $(2-\sqrt{2},1/\sqrt{2})$, respectively. The variable x keeps on jumping from one band to the other at each time step. Let us cover the attractor in the following way. Divide each band into 2^{m-1} equal parts [14]. The probability p_i of each box is equal to 2^{-m}. It is easy to see from the map that one length of band A maps into two lengths of band B while one length of band B maps into only one length of band A. This allows us to calculate our time-dependent generalized dimensions and we obtain

$$D_{q,l}(t)=\begin{cases} -\dfrac{\ln(2^{m+t/2})}{\ln(l)} & \text{if } t \text{ is even} \\ -\dfrac{\ln(2^{m+(t-1)/2})}{\ln(l)}+\dfrac{\ln(2^{-q}+2^{-1})}{(q-1)\ln(l)} & \text{if } t \text{ is odd} . \end{cases} \tag{3.3}$$

If we approximately take $l\simeq 2^{-m}$ Eq. (3.3) gives

$$D_{q,l}(t)\simeq\begin{cases} 1+\dfrac{t}{2m}, & t \text{ even} \\ 1+\dfrac{(t-1)}{2m}-\dfrac{\ln(2^{-q}+2^{-1})}{(q-1)m\ln(2)}, & t \text{ odd} . \end{cases} \tag{3.4}$$

The asymptotic behavior is obtained for $\bar{t}=2m-2$ steps and is given by

$$D_{q,l}(t)\simeq 2-\frac{1}{m}, \quad t\ge\bar{t} . \tag{3.5}$$

Thus we see that asymptotically $D_{q,l}(t)$ does not give twice the original value and the correction is dependent on the logarithm of the length scale or the uncertainty in the initial conditions. This is clear due to the band periodicity of the attractor mentioned above. A trajectory completely loses the memory of its location inside the band but retains the memory of the band permanently.

B. Logistic map

Consider the logistic map [15] defined by

$$x_{n+1}=\mu x_n(1-x_n) . \tag{3.6}$$

Here we discuss four different values of μ.

Case (a), $\mu=4.0$. This is a case of fully developed chaos. Here we observe doubling of the time-dependent exponents $D_{q,l}(t)$ for larger times. Figure 1 shows the plots of the fractal exponent $D_{0,l}(t)$ as a function of t for different values of l. The curves start from $D_{0,l}(0)=1$. There is a monotonic increase and a subsequent flattening as we approach $D_{0,l}(t)\simeq 2$, i.e., twice the original value.

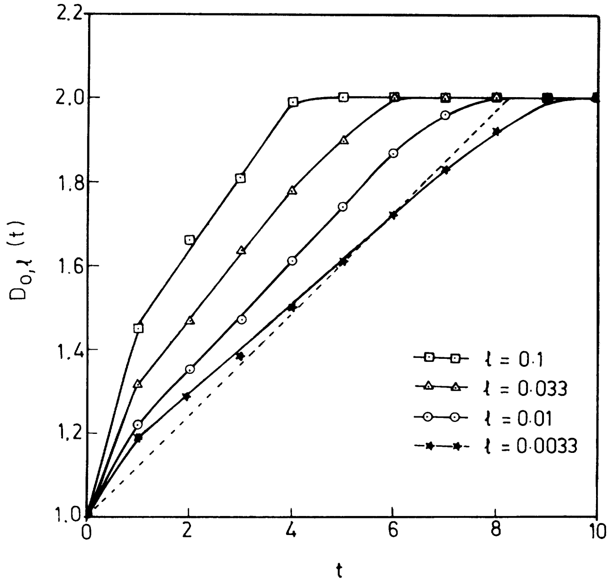

FIG. 1. The fractal exponent $D_{0,l}(t)$ is plotted as a function of t for four different values of l for the logistic map with $\mu=4.0$. The dashed line shows the behavior of $D_{0,l}(t)$ for $l=0.003$ and $\lambda=\ln 2$ according to Eq. (2.6). The number of points of the time series chosen in this and all the other calculations is such that each box has at least 40 points on the average.

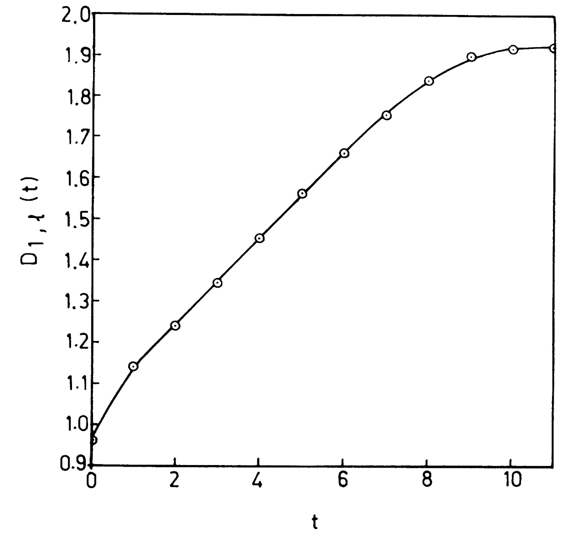

FIG. 2. The information exponent $D_{1,l}(t)$ is plotted as a function of t for $l=0.003$ for the logistic map with $\mu=4.0$.

The rate of rise and the time required for doubling clearly depend on the length scale. The sharp rise in $D_{0,l}(t)$ from $t=0$ to 1 is an artifact of choosing equal length scales and not the natural length scales of the system. Using the region of steady rise of $D_{0,l}(t)$ to obtain an estimate of the Lyapunov exponent we get $\lambda \simeq 0.60$ while the exact analytical value is $\lambda=\ln 2 \simeq 0.69$. In Fig. 1 we have also plotted the behavior of $D_{0,l}(t)$ for $l=0.003$ expected from Eq. (2.6) with $\lambda=\ln 2$ (dashed line). We see that the numerical values show a systematic deviation. They are larger than those given by Eq. (2.6) for small t and smaller for large t. Hence the estimate of the Lyapunov exponent is smaller than the actual value and serves as a lower bound in this case. Though the cause of this systematic deviation is not clear, it is probably due to the jump at the first time step. We also find that the estimate of λ improves as l decreases.

In Fig. 2 we plot the time-dependent information exponent $D_{1,l}(t)$ as a function of time. Using the region of almost linear rise of $D_{1,l}(t)$ in time and Eq. (2.8), we get the metric entropy $h \simeq 0.60$.

The behavior of the time-dependent exponents for other values of q is similar to Figs. 1 and 2. However, the plots are not very smooth for larger values of q. Figure 3 plots $\tau_{q,l}(t)$ as a function of q for various times for $l=0.01$. At around $t \simeq 9$, $\tau_{q,l}(t)$ values reach the values which are twice those at $t=0$. For $q>0$, the $\tau_{q,l}(t)$ curves increase steadily with time. However, for $q<0$, the $\tau_{q,l}(t)$ values first decrease rapidly and then increase. The behavior for negative q values corresponds to low probabilities. In this region numerical errors are larger and it is difficult to interpret the result. Similar behavior to that of Fig. 3 is obtained for $f_l(t)-\alpha_l(t)$ curves also. Using Eq. (2.9) we have found λ_q for $0 \le q \le 5$ using linear portions of the graphs of $D_{q,l}(t)$ versus t for $l=0.003$. (Numerical errors for the negative q values are higher as explained earlier.) The behavior of λ_q as a function of q is shown in Fig. 4.

Case (b), $\mu=3.5699\ldots$. For this value of μ we have a period-doubling attractor. The time-dependent fractal exponent $D_{0,l}(t)$ of this map is plotted as a function of time in Fig. 5. The Lyapunov exponent is zero in this

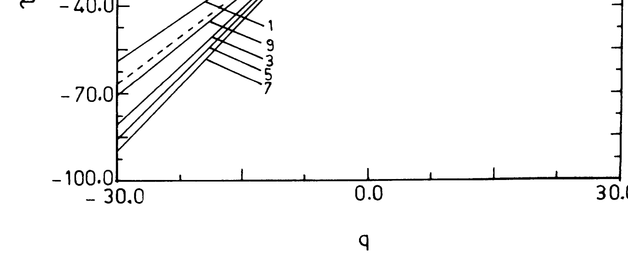

FIG. 3. The $\tau_{q,l}(t)$ vs q plots are shown for different times for the logistic map with $\mu=4.0$ and $l=0.01$. The two dashed curves correspond to the $\tau_{q,l}(t)$ values for $t=0$ and values obtained by doubling its values [16]. The solid curves correspond to $\tau_{q,l}(t)$ values for different times.

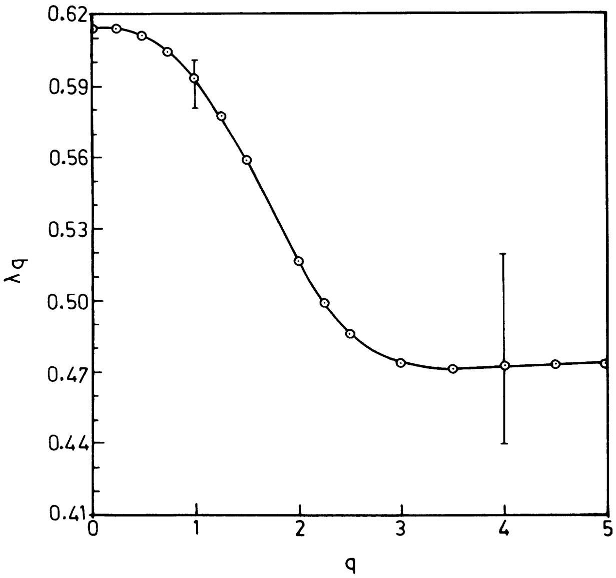

FIG. 4. The λ_q values are plotted as a function of q for the logistic map with $l=0.0033$ and $\mu=4$. The values of λ_q are calculated using the linear portion of the $D_{q,l}(t)-t$ curve and Eq. (2.9).

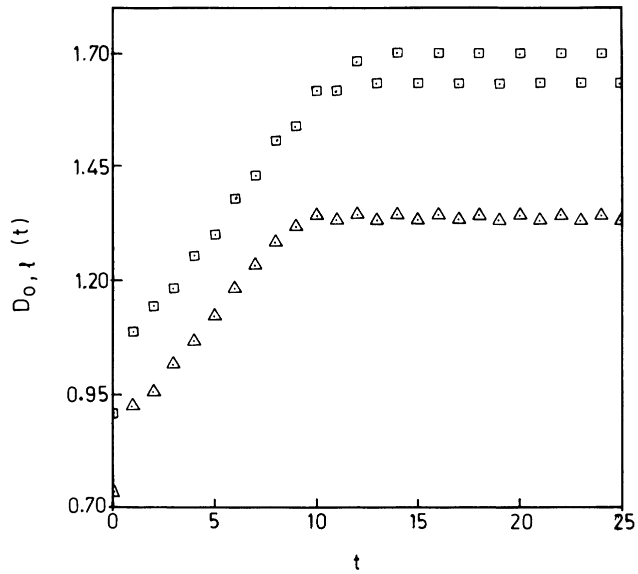

FIG. 6. The fractal exponent $D_{0,l}(t)$ vs t is plotted for the band-merging case of logistic map (squares) and band-merging case of tent map (triangles) for $l=0.01$.

case. Thus we expect time-dependent fractal exponent $D_{0,l}(t)$ to remain invariant in time. We find that this is indeed the case. [There are some small oscillations which may be due to not using natural length scales and the $D_{0,l}(t)$ values are larger than fractal dimension for the attractor due to finite value of l.] This indicates that there is no loss of memory and the future is completely predictable [17].

Case (c), $\mu \simeq 3.678\,573\,5$. This value of μ shows the band-merging case of the logistic map. This value separates between two-band and one-band attractor regimes of the logistic map. As in the band-joining tent map, in the band-merging case of the logistic map also the third iterate of the maximum falls on the unstable fixed point and the points on the right and left sides of the unstable fixed point are visited alternately. The plot of $D_{0,l}(t)$ versus time is shown in Fig. 6 for $l=0.01$ by points represented by squares. The behavior for small t is like the $\mu=4$ case. However, asymptotic behavior is quite different. First, the asymptotic value of the time-dependent fractal exponent is clearly less than twice its value at $t=0$. This shows that the system never loses memory of the initial conditions completely. This is because the variable x_n keeps on alternating between the two bands on the right and left sides of the unstable fixed point and we always know the band in which it lies at any time once the initial band is known. Secondly, asymptotically the $D_{0,l}(t)$ alternates between two values. This is because the two bands have different widths and we have used capacity notion of dimensions. For small t a straight line fit to the linear portion of the data in Fig. 6 yields $\lambda \simeq 0.27$ against the actual value 0.34.

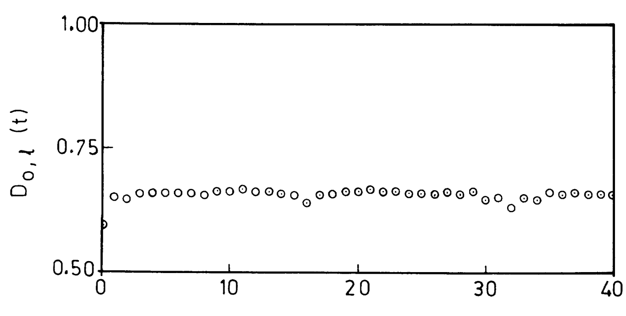

FIG. 5. The fractal exponent $D_{0,l}(t)$ vs t is plotted for the period-doubling attractor of logistic map for $l=0.003$.

Let us compare this result for the time-dependent fractal exponent with the one obtained for case (b) for the tent map $p=\sqrt{2}$ discussed above. The two are obviously similar and the band periodicity is reflected in the fact that the fractal exponent does not double asymptotically. In Fig. 6 we also plot $D_{0,l}(t)$ as a function of time for the band-merging tent map for $l=0.01$ (shown by triangles in the figure). Both have almost the same Lyapunov exponent. From Fig. 6 we can see that points in the case of the band-merging tent map and band-merging logistic map evolve parallel to each other for short times since both maps have the same Lyapunov exponent and asymptotically $D_0(l,t)$ alternates between two values less than twice the value at $t=0$ in both cases.

Figure 7 shows the plot of $D_{1,l}(t)$ as a function of t for the present case. We notice that the asymptotic value does not double but is almost a constant nonfluctuating value in contrast with $D_{0,l}(t)$. In fact, investigation of

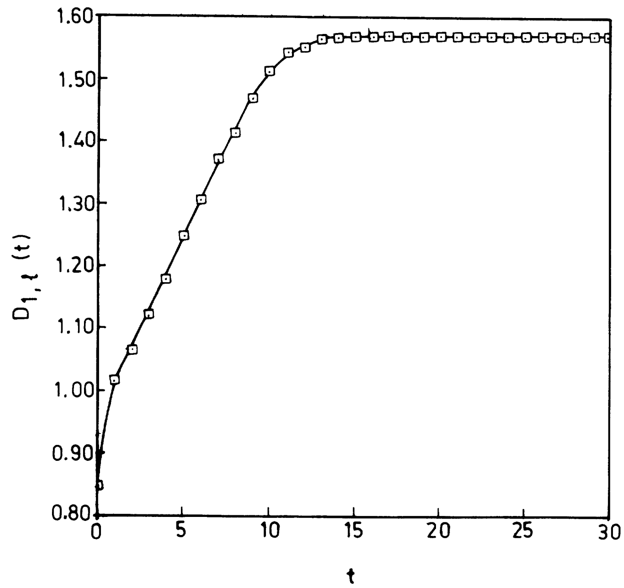

FIG. 7. The information exponent $D_{1,l}(t)$ vs t is plotted for the band-merging case of logistic map for $l=0.01$.

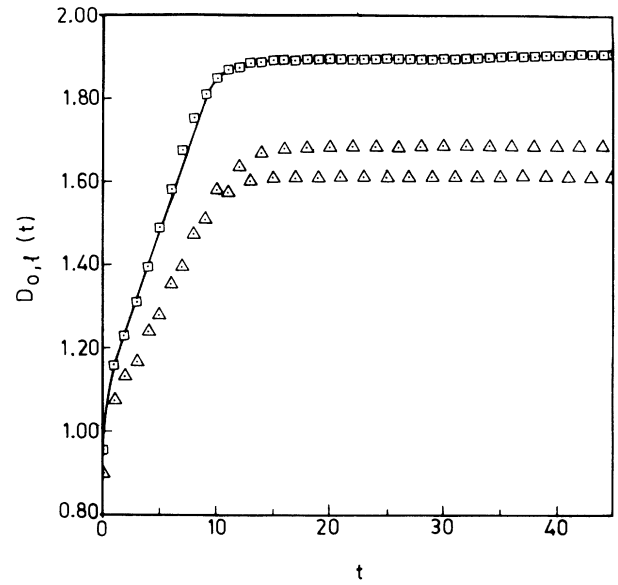

FIG. 8. The fractal exponent $D_{0,l}(t)$ vs t is plotted for the intermittency case ($\mu=3.827\,79$) and two-band case ($\mu=3.674$) of the logistic map for $l=0.01$. The points in the intermittency case are represented by squares while those in the two-band case are represented by triangles.

other q values reveals that $q=1$ is a special case and for all other values of q the asymptotic value fluctuates.

Case (d), $\mu=3.827\,79$. We know that for $\mu=1+\sqrt{8}$, we get a tangent bifurcation. We have period 3 window for larger values in μ and intermittency for smaller values. In the case of intermittency we find the doubling of fractal exponent. The time-dependent fractal exponent against time is plotted in Fig. 8 for $l=0.01$ (shown by squares in the figure). The Lyapunov exponent calculated from the linear portion of the curve in Fig. 8 gives $\lambda \simeq 0.37$ while the actual value is 0.32. We find that the fractal exponent doubles around $t \simeq 43$. This shows that there is a complete loss of memory of the initial conditions. The numerical calculations reveal that this doubling takes place for the entire range of q values for large enough time. The time required for the doubling is far higher than predicted by Eq. (2.5). We also find that this time increases indefinitely as μ approaches the tangent bifurcation value. This is expected since $\lambda \to 0$.

The intermittency serves as an example to illustrate the fact that our formalism gives certain information not given by the Lyapunov exponent alone. We choose another value of μ having almost the same value of Lyapunov exponent, namely, $\mu=3.674$ but in the two-band attractor regime. In Fig. 8 we also plot for the time-dependent fractal exponent as a function of time in this case (see triangles in figure). There is a clear distinction in asymptotic behavior. In the case of intermittency, we see the doubling of the fractal exponent asymptotically whereas in the case of the two-band attractor we do not see the doubling asymptotically.

C. Limiting behavior

We now discuss the effect of the two limits $l \to 0$ and $t \to \infty$. We find that these two limits are noncommuting. The behavior in the limit $t \to \infty$ is clear from the examples discussed above. The asymptotic behavior gives us information about the extent of loss of memory. In the case of the limit $l \to 0$ we can define time-dependent generalized dimensions $D_q(t)$ as [Eqs. (2.1) and (2.2)]

$$D_q(t)=\frac{1}{(q-1)}\lim_{l\to 0}\tau_{q,l}(t) \ . \tag{3.7}$$

From Eq. (2.6) we see that the slope of $D_{0,l}(t)$ versus t tends to 0 as $l \to 0$ and hence for any finite t, $D_{0,l}(t)$ tends to D_0 and not $2D_0$ as $l \to 0$. The slopes of actual curves of $D_{0,l}(t)$ in Fig. 1 are also decreasing as $l \to 0$ and seem to support this conclusion. Similar behavior is seen for other values of q. [See also Eqs. (2.8), (3.2a), and (3.4).] Based on this let us conjecture that $\lim_{l\to 0} D_{q,l}(t)=D_q$. This conjecture implies that there is no loss of memory if the initial conditions are specified with infinite precision. This is natural since we have deterministic chaos. Thus the above conjecture allows us to conclude that the loss of memory of the initial conditions is a property of coarse graining [18]. Thus the two limits $l \to 0$ and $t \to \infty$ are not interchangeable. Taking the $l \to 0$ limit first and $t \to \infty$ limit afterwards gives us the generalized dimensions D_q. On the other hand, taking the $t \to \infty$ limit first we get different types of asymptotic behavior as discussed above.

IV. HIGHER-DIMENSIONAL SYSTEMS

The formalism introduced in Sec. II is easily extended to higher-dimensional systems. For a d-dimensional system, Eqs. (2.1) and (2.2) can be generalized by letting the indices i and j represent d-dimensional boxes. If we have a time series in only one variable, we can use the method of time delays to construct the state vectors $\mathbf{x}_k=(x_k,x_{k+1},\ldots,x_{k+d-1})$, $k=1,2,\ldots$ where d is the embedding dimension [19–20]. The index i for the box is replaced by the string $(i_1,i_2,\ldots,i_d)$ where i_m is the index for the length intervals in the mth direction of the embedding space. The time-dependent generalized exponents are now given by Eqs. (2.1) and (2.2) with the summation indices i and j now representing d-dimensional boxes,

$$D_{q,l}(t)=\frac{1}{q-1}\frac{\ln\left[\sum'_{(i_1,\ldots,i_d),(j_1,\ldots,j_d)}[p^q_{(i_1,\ldots,i_d),(j_1,\ldots,j_d)}(t)]\right]}{\ln l} \tag{4.1}$$

where $p_{(i_1,\ldots,i_d),(j_1,\ldots,j_d)}$ is the probability that the state vector $\mathbf{x}$ lies in the box $(i_1,\ldots,i_d)$, at time t' and in the box $(j_1,\ldots,j_d)$ at time $t'+t$. Again, for $t=0$

$$p_{(i_1,\ldots,i_d),(j_1,\ldots,j_d)}(0)=p_{(i_1,\ldots,i_d)}\delta_{i_1,j_1}\cdots\delta_{i_d,j_d}$$

and the time-dependent generalized exponents are the usual generalized dimensions of the attractor. For large times there are two extreme cases.

(i) When the memory is completely lost the arguments similar to the ones used in the one-dimensional situation show that the time-dependent generalized exponents will double asymptotically.

(ii) The other extreme case is obtained when there is no loss of memory. In this case writing

$$p_{i,j}(t)=p_{(j_1,\ldots,j_d)|(i_1,\ldots,i_d)}(t)p_{i_1,\ldots,i_d}\ ,$$

where $p_{(j_1,\ldots,j_d)|(i_1,\ldots,i_d)}(t)$ is the conditional probability, we note that for no loss of memory $p_{(j_1,\ldots,j_d)|(i_1,\ldots,i_d)}(t)=1$ for some box $(j_1,\ldots,j_d)$ and 0 for the remaining boxes. Thus from Eq. (4.1) we find that the time-dependent generalized exponents do not change with time.

The definition (4.1) above though correct becomes cumbersome to implement in higher dimensions since one has to work in the embedding space which has double the dimension of the space in which the original dynamical system is embedded. For dimension $d>1$ a different version of the time-dependent generalized exponents which gives essentially the same information can be introduced by defining the modified time-dependent generalized exponents $\bar{D}_{q,l}(t)$ as

$$\bar{D}_{q,l}(t)=\frac{1}{q-1}\frac{\ln\left[\sum'_{(i_1,\ldots,i_d),j}[p_{(i_1,\ldots,i_d),j}(t)]^q\right]}{\ln l} \tag{4.2}$$

where $p_{(i_1,\ldots,i_d),j}(t)$ is the probability that the state vector $\mathbf{x}$ lies in the box $(i_1,\ldots,i_d)$ at some time t' and the variable x lies in the length interval j after a time $t+d-1$, i.e., at time $t'+t+d-1$. The definition (4.2) of $\bar{D}_{q,l}(t)$ requires calculations in a space of $d+1$ dimensions only. For $t=0$, we have $p_{(i_1,\ldots,i_d),j}(0)=p_{(i_1,\ldots,i_d)}\delta_{i_d,j}$. Hence we get the usual generalized dimensions $\bar{D}_{q,l}(0)=D_q$. In the other limit of large time if we assume that the variable x completely loses memory of the initial state vector, then $p_{(i_1,\ldots,i_d),j}(t)=p_{(i_1,\ldots,i_d)}p_j$. For a chaotic attractor with $\bar{D}_0>1$, the projection along any direction is expected to be continuous. Thus in this case $\bar{D}_{0,l}(t)=D_{0,l}+1$ if we normalize the total range of the variable to unity. Also if for large times there is no loss of memory the exponents will not change with time.

We now discuss some illustrative examples.

(a) Hénon map: The Hénon map [21] is given by

$$(x_{n+1},y_{n+1})=(y_n+1-ax_n^2,bx_n)\ . \tag{4.3}$$

Let $a=1.4$ and $b=0.3$. We consider the time series in only one variable, say x. We next construct the state vectors $\mathbf{x}_k=(x_k,x_{k+1})$. Using this state vector the time-dependent generalized exponents can be calculated. We have carried out this calculation using both Eqs. (4.1) and (4.2). For Eq. (4.1) the numerical results show that the fractal exponent $D_{0,l}(t)$ doubles asymptotically. Thus there is a complete loss of memory of the initial conditions. The value of the Lyapunov exponent obtained from the slope of the $D_{0,l}(t)$ versus t curve is 0.32 [see

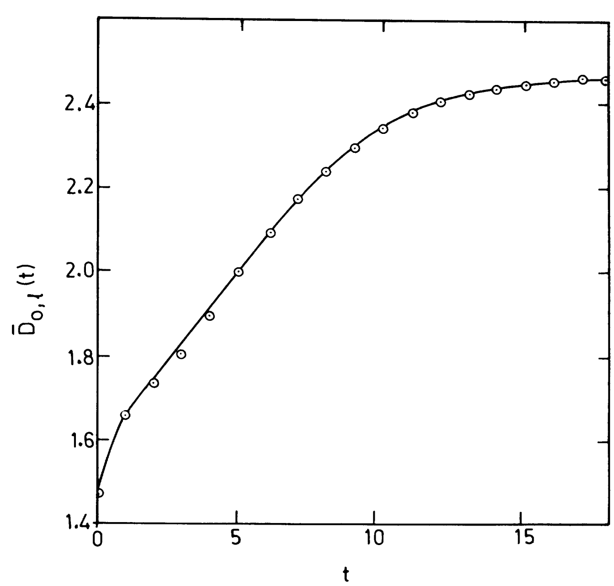

FIG. 9. The figure shows $\bar{D}_{0,l}(t)$ as a function of t for $l=0.05$ for the Hénon map [Eq. (4.3)].

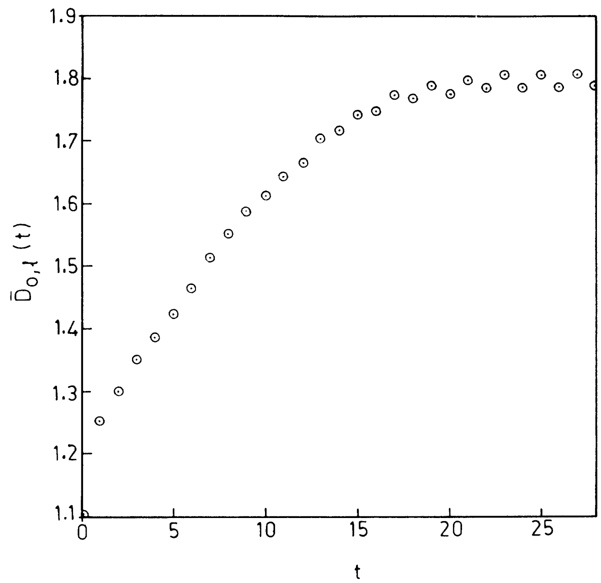

FIG. 10. The figure shows time-dependent fractal exponent $\bar{D}_{0,l}(t)$ as a function of t for $l=0.04$ for the variant of Hénon map [Eq. (4.4)] in the two-band case.

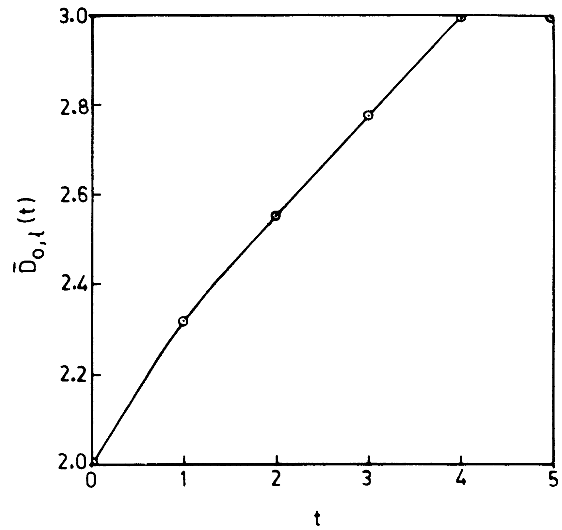

FIG. 11. The figure shows time-dependent fractal exponent $\bar{D}_{0,l}(t)$ as a function of t for $l=0.013$ for the cat map [Eq. (4.5)].

Eq. (2.6)] for $l=0.05$. The known largest value of λ for the Hénon map is 0.418 (Ref. [22]).

Next we use Eq. (4.2) and obtain the modified exponents $\bar{D}_{q,l}(t)$. Figure 9 shows $\bar{D}_{0,l}(t)$ as a function of t for $l=0.05$.... The increase of 1 in fractal exponent confirms the fact that there is complete loss of memory.

(b) Our next example is of a two-band attractor. We consider a modified version of the Hénon map

$$(x_{n+1},y_{n+1})=(p-x_n^2-Jy_n,x_n) \tag{4.4}$$

with $J=0.3$ and $p=2$. Figure 10 shows the time-dependent fractal exponent $\bar{D}_{0,l}(t)$ as a function of t for $l=0.04$. . . . We see that $\bar{D}_{0,l}(t)$ does not increase by one in this case. This is enough to conclude that the memory is not lost completely. The information exponent $D_{1,l}(t)$ also shows a similar trend.

(c) Cat map: We have also analyzed a case of an area-preserving map, the cat map [23]. The cat map is defined by

$$\begin{aligned} x_{n+1}&=(x_n+y_n), \quad \text{mod}1 \ , \\ y_{n+1}&=(x_n+2y_n), \quad \text{mod}1 \ . \end{aligned} \tag{4.5}$$

In this case we find that the memory is lost completely. The time-dependent fractal exponent $D_{0,l}(t)$ [Eq. (4.1)] doubles in a few time steps. In Fig. 11 we have plotted $\bar{D}_{0,l}(t)$ [Eq. (4.2)] as a function of t for $l=0.013$. . . . As shown in Fig. 11 the time-dependent fractal exponent $\bar{D}_{0,l}(t)$ also increases by 1. The largest value for λ in this case is $\ln[(3+\sqrt{5})/2]$, i.e., 0.962. . . whereas the value calculated from the slope of the linear portion in Fig. 11 is 0.98 which is near the actual value.

V. CONCLUSIONS

In this paper we have presented a method of analyzing the time evolution of a chaotic signal. We have dealt with the problem of the loss of memory of initial conditions as time evolves which is one of the central characteristics of a chaotic system. In particular we know how the loss of memory of the initial conditions takes place at each time step. The change in the time-dependent fractal exponent is a measure of this loss of memory. Complete loss of memory is represented by doubling of the time-dependent generalized exponents. Our method is able to distinguish between chaotic signals which lose memory completely and those which retain partial memory of the initial conditions. The partial loss of memory represents some kind of band periodicity still persistent on the attractor. Thus we expect our method to be useful in knowing the kind of chaotic attractor that one has. We do not know of any other simple method which can give the above kind of information from a time series. In addition we also get a rough estimate of the Lyapunov exponent and metric entropy. We also note that the loss of memory is a property of coarse graining.

Our analysis has relevance for predicting a chaotic time series. Farmer and Sidorowich [24,25] find that the normalized error of prediction approaches one for large prediction times in some systems while it remains less than one in the other cases. These two situations will correspond to an asymptotic value of $D_{q,l}(t)$ which is twice the original value and an asymptotic value which is less than twice the original value.

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