Proc. Indian Acad. Sci., Vol. 86 A, No. 6, December 1977, pp. 537-547, © Printed in India.

Temperature distribution in Couette flow past a permeable bed

N RUDRAIAH and R VEERABHADRAIAH
Department of Mathematics, Central College, Bangalore University, Bangalore 560 001

MS received 29 September 1976

Abstract. The temperature distribution in a steady plane Couette flow having one permeable bounding wall is investigated in the presence of buoyancy force N_0 when $N_0 > 0$, it is shown that heat is transported both by convection and diffusion. The effect of convection is to increase the magnitude of the temperature distribution both in the free and Darcy flows. In particular, it is shown that the wall shear has no significant effect on the temperature distribution. The rate of heat transfer between the fluid and the surface is also calculated and it is shown that, it increases with the porous parameter σ. Although the viscous dissipation has very little effect on the temperature distribution yet its effect is significant on heat transfer.

Keywords. Couette flow; permeable, Navier-Stokes equation; Darcy law; Boussinesq approximation; slip parameter; Poiseuille flow.

1. Introduction

Recently Rudraiah and Veerabhadraiah (1976) have investigated the problem of Couette flow past a permeable bed in the presence of buoyancy force with the object of developing a proper theory for the experimental work of Rajasekhara (1974). They found an excellent agreement between their theoretical results and the experimental results of Rajasekhara (1974). The work of Rudraiah and Veerabhadraiah (1976) is silent about the prediction of the effect of slip on the temperature distribution. The object of this paper is to find the effect of boundary layer thickness, at the nominal surface, on the heat transfer coefficient with the following two motives:

Cooling problems assume a continuously growing importance in the development of high speed vehicles (like space vehicles, aircrafts, missiles and so on) as the flight velocity increases. It is well known that part of the power which is necessary to overcome the drag of the vehicle is converted into heat by internal friction within the boundary layer which surrounds the vehicle. This heat flows partially from the air layer into the surface of the vehicle in an amount which increases rapidly with increase of vehicle speed. As a consequence, cooling problems arise in almost every component of the space vehicle. The basis of any engineering-design calculations whose aim is to determine the cooling requirements is always a determination of the convective heat transfer from the heated boundary layer into the skin of the space vehicle. Accordingly, an extensive literature has been devoted to the subject of determining the convective heat transfer in boundary layer flow along surfaces of idealized shapes by calculations or by experiments. Of the many solutions proposed (see Rudraiah 1966) transpiration cooling gives an effective method where the surfaces to be protected against the influence of a hot fluid stream are manufactured from a porous material and cold fluid is ejected through the wall to form a protective layer along the surface. In this case the velocity distribution is obtained using the no-slip boundary condition. If the surfaces to be protected against the influence of a hot fluid stream are lined with porous material of large thickness, the cooling effect may be enhanced due to the slip that exists (see Beavers and Joseph 1967) at the porous surface.

The results of this investigation may also be used in electric rotating induction motors. More specifically, we know that in electric rotating motors, the heat transfer coefficient on the housing is proportional to the reciprocal of the boundary layer thickness at the housing. The smaller the boundary layer thickness, the larger the heat transfer coefficient and hence smaller the temperature difference. The available literature (see Rudraiah and Natarajan 1976) states that the calculation of boundary layer thickness on the housing of an electric totally enclosed fan cooling induction motor is based on the flat plate theory which is at least three times larger than the predicted value. Recently, Rudraiah and Natarajan (1976) have modified this flat plate theory concept and calculated the boundary layer thickness on the housing of an electric motor using the cross flow which shows a close agreement with the experimental value. In this paper, we propose a model to line the housing of an electric induction motor by a porous material called the nominal surface. There exists a slip at this nominal surface due to the transfer of momentum (see Beavers and Joseph 1967, Beavers *et al* 1970, Rajasekhara 1974) and the effect of this slip is to increase the velocity and to decrease the boundary layer thickness just at the nominal surface in the porous material. This velocity distribution, determined by using the slip at the nominal surface, is used in this paper to determine the temperature distribution using the proper boundary condition. It is shown that the effect of slip with a favourable temperature gradient is to increase the heat transfer coefficient and hence to decrease the temperature distribution.

2. Mathematical formulation

A physical model, illustrating the problem under consideration, is shown in figure 1. It consists of a parallel plate channel of height h where the lower bounding wall is permeable while the upper is rigid, moving with a uniform velocity U_0. To discuss

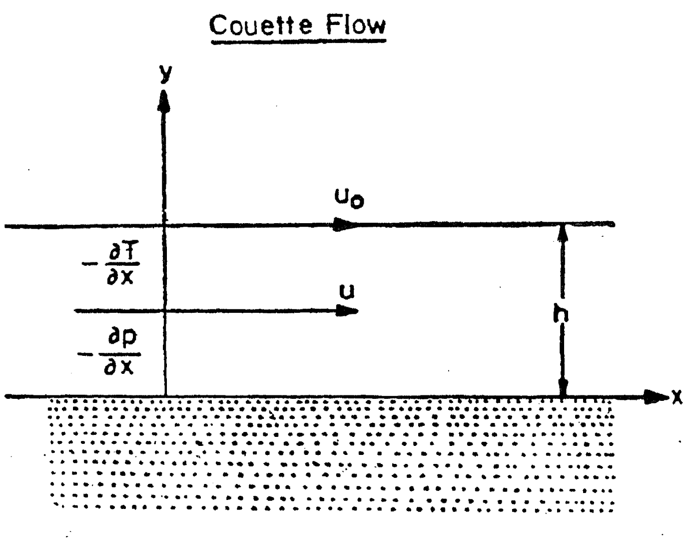

Figure 1. Physical model

the solutions, the flow regime is divided into two zones; in zone one, from the impermeable upper rigid plate to the permeable bed, the flow is laminar and is governed by the Navier-Stokes equations and in zone two, the flow in the permeable bed below the nominal surface is governed by the Darcy law. Hereafter, we call the former as zone 1 and the latter zone 2. The axial and transverse coordinates are respectively x and y, the latter being measured vertically upwards from the nominal surface (Beavers and Joseph 1967).

To derive the basic equations, for this physical model, we make the following approximations:

(i) The flow in zones 1 and 2, in the x-direction, is driven by a common pressure gradient $\partial p/\partial x$, the shear produced by the motion of the upper rigid plate and by the buoyancy force $\partial T/\partial x$.

(ii) The fluid is viscous and satisfies the Boussinesq approximation which is valid only when the speed of flow is very much less than that of sound and the accelerations are slow compared with those associated with sound waves. This means that fluctuations in density occur principally as a result of thermal, rather than pressure variations which is valid in the case of liquid considered in this problem.

(iii) The flow is steady and fully developed so that all the physical quantities except the pressure and temperature are functions of y only.

(iv) The porous medium is assumed to be homogeneous and isotropic so that its permeability k is constant.

Under these approximations, the basic equations in zones 1 and 2 are:

Zone 1

$$\frac{\partial^2 u}{\partial y^2}=\frac{1}{\mu}\frac{\partial p}{\partial x} \tag{1}$$

$$\frac{\partial p}{\partial y}=-\rho g \tag{2}$$

$$\rho=\rho_0[1-\beta(T'-T_0)] \tag{3}$$

$$\rho_0 C_p u\frac{\partial T'}{\partial x}=K\frac{\partial^2 T'}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{4}$$

Zone 2

$$Q=-\frac{k}{\mu}\frac{\partial p}{\partial x} \tag{5}$$

$$\frac{\partial p}{\partial y}=-\rho g \tag{5a}$$

$$\frac{\partial^2 u}{\partial y^2}-\frac{1}{k}u=\frac{1}{\mu}\frac{\partial p}{\partial x} \tag{6}$$

$$\rho_0 C_p u\frac{\partial T'}{\partial x}=K\frac{\partial^2 T'}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2+\frac{\mu}{k}u^2 \tag{7}$$

Here u is the velocity in the axial direction, p the pressure, ρ the density, T' the temperature; μ the viscosity, C_p the specific heat at constant pressure; K the thermal conductivity of the fluid; ρ_0 the density at $T'=T_0$; β is the thermal expansion coefficient; T_0 is the ambient temperature; Q is the Darcy velocity and k is the permeability of the porous material.

Note that eq. (7) is the modified Darcy equation derived by Tam (1969) and Lundgren (1972) which is needed to derive the expression for the boundary layer thickness at the permeable surface. Eliminating the pressure p in eq. (1) using eq. (2), we get

$$\frac{\partial^3 u}{\partial y^3}=\frac{g\beta}{\gamma}\frac{\partial T'}{\partial x} \text{ i.e. } \frac{\partial^2 \zeta}{\partial y^2}=-\frac{\beta g}{\gamma}\frac{\partial T'}{\partial x} \tag{8}$$

where $\zeta=-(\partial u/\partial y)$ is the vorticity component, γ is the kinematic viscosity. Physically, eq. (8) represents the balance between potential energy released by the horizontal temperature gradient and the viscous dissipation of flow. Equation (8) describes the fully developed flow only when the left hand side is independent of x. To satisfy this condition, we assume that the temperature varies linearly in the direction of flow. This means that the heat flux is constant in the direction of flow. Mathematically, this can be expressed as

$$T'(x,y)=Ax+T(y) \tag{9}$$

where $T(y)$ is the entrance temperature and A is the axial temperature gradient.

Recently, Rudraiah and Veerabhadraiah (1976) have obtained the velocity distribution in zones (1) and (2) using the Beavers and Joseph (1974) slip-boundary condition

$$\frac{du}{dy}=\frac{\alpha(u_B-Q)}{\sqrt{k}} \tag{10}$$

where α is the slip-parameter and u_B is the slip-velocity. The expressions for velocity are:

$$u=\frac{Q\sigma^2}{12}\left[\left\{2N_0\left(\frac{1}{1+\alpha\sigma}+\eta+\eta^2\right)+6(\eta+f_0)+\frac{\alpha\sigma}{1+\alpha\sigma}P_0\right\}(1-\eta)-P_0\right] \tag{11}$$

in zone 1, and

$$u=u_B+Q(1-\cosh\sigma\eta-\sinh\sigma\eta+N_0\eta) \tag{12}$$

in zone 2, where $G=A\beta g/\gamma$

$$N_0=-\frac{Gh^3}{Q\sigma^2}=\frac{\mu Gh}{\partial p/\partial x},\ \sigma=\frac{h}{\sqrt{k}},\quad \eta=\frac{y}{h}$$

$$f_0=\frac{\sigma+2\alpha}{\sigma(1+\alpha\sigma)},\ P_0=-\frac{2\mu u_0}{h^2\,(\partial p/\partial x)}>0$$

$$u_B=\frac{Q\sigma^2}{6}\left[\frac{N_0}{1+\alpha\sigma}+3f_0-\frac{1}{1+\alpha\sigma}\cdot\frac{P_0}{2}\right]$$

$$=\frac{Q\sigma^2}{12}\left[\frac{\sigma(2N_0-P_0+6)+12\alpha}{\sigma(1+\alpha\sigma)}\right]$$

The object of this paper is to determine, using the above expressions for velocity, the temperature distribution in zones 1 and 2. The crux of the problem is to specify the proper boundary conditions on temperature. These may be one of several types: Dirichlet, Neumann, Fourier or mixed. In practice, the impermeable material is usually made with quasi-isothermal surface and hence we impose the Dirichlet type boundary condition at the rigid moving upper plate. Since the problem considered in this paper involves the entry temperature, care must be taken to specify the proper Dirichlet condition. The condition that we impose is

$$T'=T_1 \text{ at } y=h \text{ and } x=0 \tag{13}$$

and at all other values of x

$$T'=Ax+T_1 \text{ at } y=h. \tag{14}$$

It is important to note that we cannot specify constant temperature T_1 all along the plate for all x, otherwise to be inconsistent with eq. (9), namely the temperature field varies linearly with x, we have to allow A to vary with y or set $A=0$ neither of which is suitable to our problem. Therefore, we have to impose the boundaty conditions of the type (13) together with (14). The other boundary condition on temperature can be obtained from the physical consideration of heat balance for an element at the nominal surface. The heat conducted away from the channel through the nominal surface must be equal to the heat absorbed from the porous medium and hence

$$K\frac{\partial T'}{\partial y}=h_e\,(T_B-T_0)$$

$$\frac{\partial T'}{\partial y}=H\frac{(T_B-T_0)}{\sqrt{k}} \tag{15}$$

where $H=h_e\sqrt{k}/K$ is the Biot number, h_e is the heat transfer coefficient from the porous medium into the channel, T_0 is the ambient temperature and T_B is the temperature at the bed. Physically H represents the rate of heat loss through the channel relative to the conductors in the porous media. If H is large, the interface must be a nominal surface in order to supply the heat lost from the porous media. If H is small, the heat losses are small and the interface is really the free boundary. In other words, H is the controlling parameter because of its relation to the overall thermal balance. Since the boundary condition (10) postulated by Beavers and Joseph (1967) is based on the nominal surface, H has to be large in our problem. Since eqs (10) and (15) are analogous, H can also be defined as thermal slip-parameter.

The boundary conditions on temperature in zone 2 are:

$$T' = T_B \text{ at } y = 0 \tag{16}$$

$$T' = T_0 \text{ at } y = -\delta \tag{17}$$

where δ is the boundary layer thickness just below the bed and we assume that this δ is the same for both velocity and temperature distributions.

3. Temperature distribution

The temperature distribution in the presence of viscous dissipation for zone 1, by solving (4) using the boundary conditions (13) to (15) is

$$\theta'(\xi, \eta) = a\,\xi + \theta(\eta) \tag{18}$$

Where $\theta = (T'-T_0)/(T_1-T_0)$, $a = AL/(T_1-T_0)$, L is the length of the channel, $\xi = x/L$

$$\theta(\eta) = \theta_B + a_1\,\eta + a_2\,\eta^2 + a_3\,\eta^3 + a_4\,\eta^4 + a_5\,\eta^5 + a_6\,\eta^6 \tag{19}$$

where

$$\theta_B = \frac{1}{1+H\sigma}\left[1 - \frac{P_e A_0}{720}(4\,N_0 - 5\,P_0 + 24) + \frac{P_r\,E}{720}\left\{(N_0^2 + N_0 + 60) - \frac{5\alpha\,(N_0+2)\,\{\sigma^2\,(2\,N_0 - P_0 + 6) - 12\}}{\sigma\,(1+\alpha\,\sigma)} + \frac{5\alpha^2\,\{\sigma^2\,(2\,N_0 - P_0 + 6) - 12\}^2}{2\sigma^2\,(1+\alpha\,\sigma)^2}\right\}\right]$$

$$a_1 = H\,\sigma\,\theta_B$$

$$a_2 = \frac{P_e\,A_0\,\{\sigma\,(2\,N_0 - P_0 + 6) + 12\alpha\}}{24\sigma\,(1+\alpha\,\sigma)} - \frac{P_r\,E\alpha^2\,\{\sigma^2\,(2\,N_0 - P_0 + 6) - 12\}^2}{288\,\sigma^2\,(1+\alpha\,\sigma)^2}$$

$$a_3 = \frac{\alpha\,(P_e\,A_0 + 2\,P_r\,E)\,\{\sigma^2\,(2\,N_0 - P_0 + 6) - 12\}}{72\sigma\,(1+\alpha\,\sigma)}$$

$$a_4 = -\frac{1}{12}\left[\frac{P_e\,A_0}{2} + \frac{P_r\,E}{12}\,\frac{12\sigma\,(1+\alpha\,\sigma) - \alpha\,N_0\,\{\sigma^2\,(2\,N_0 - P_0 + 6) - 12\}}{\sigma\,(1+\alpha\,\sigma)}\right]$$

$$a_5 = -\frac{N_0}{120}(P_e\, A_0 + 6\, P_r\, E)$$

$$a_6 = -\frac{N_0^{\,2}}{120} P_r\, E$$

$$P_e = \frac{\rho_0\, C_p\, Q\, \sigma^2\, h}{K} = \text{Peclet number}$$

$$P_r = \frac{\mu\, C_p}{K} = \text{Prandtl number}$$

$$E = \frac{(Q\,\sigma^2)^2}{C_p\,(T_1 - T_0)} = \text{Eckert number}$$

$$A_0 = \frac{A}{(T_1 - T_0)/h} = \text{Dimensionless temperature gradient.}$$

We observe that, when we neglect viscous dissipation term in (4), the expression for temperature distribution will be simplified.
In this case the expression for temperature is

$$\theta(\eta) = \frac{1+\sigma\, H\eta}{1+\sigma\, H} + \frac{P_e}{24}(k_0 + k_1\eta + k_2\eta^2 + k_3\eta^3 - \eta^4)$$
$$+ \frac{P_0\, P_e}{6}(l_0 + l_1\eta + l_2\eta^2 + l_3\eta^3)$$
$$+ \frac{N_0\, P_e}{360}(n_0 + n_1\eta + n_2\eta^2 + n_3\eta^3 - 3\eta^5) \tag{20}$$

where

$$k_0 = -\frac{1+4\, f_0}{1+\sigma\, H}, \qquad l_0 = -\frac{f_1}{1+H\sigma}, \qquad n_0 = -\frac{27+7\alpha\,\sigma}{(1+\alpha\,\sigma)\,(1+\sigma\, H)},$$

$$k_1 = \sigma\, H\, k_0, \qquad l_1 = \sigma\, H\, l_0, \qquad n_1 = \sigma\, H\, n_0,$$

$$k_2 = 6\, f_0, \qquad l_2 = \frac{3}{1+\alpha\,\sigma}, \qquad n_2 = \frac{30}{1+\alpha\,\sigma},$$

$$k_3 = 2\,(1-f_0), \qquad l_3 = \tfrac{1}{3}\,\alpha\,\sigma\, l_2, \qquad n_3 = \tfrac{1}{3}\,\alpha\,\sigma\, n_2.$$

$$f_1 = \frac{3+\alpha\,\sigma}{2\,(1+\alpha\,\sigma)}.$$

The temperature distribution for Poiseuille flow can easily be obtained from eq. (18) by setting $P_0=0$. The first term in eq. (20) represents the heat transport due to diffusion and the remaining terms represents the transport of heat due to convection. When $P_e=0$, i.e. in the absence of convection, eq. (20) becomes

$$\theta(\eta)=\frac{1+\sigma H\eta}{1+\sigma H}.$$

This shows that as $H\to\infty$ (i.e. perfectly conducting permeable interface)

$$\theta(\eta)=\eta$$

However, when $H=0$ (i.e. insulating permeable interface)

$$\theta(\eta)=1$$

The temperature distribution in the presence of viscous dissipation for zone 2, solving eq. (7) using the boundary conditions (16) and (17) is

$$\theta(\eta)=b_0+b_1\eta+b_2\eta^2+b_3\eta^3+b_4\eta^4+(b_5+b_6\eta)(\cosh\sigma\eta+\sinh\sigma\eta)$$
$$+b_7(\cosh 2\sigma\eta+\sinh 2\sigma\eta)+(1+\alpha\sigma\eta)\,\theta_B \qquad (21)$$

where

$$b_0=\frac{P_e A_0}{\sigma^4}+\frac{P_r E}{2\sigma^5}(4N_0-3\sigma-4\sigma^3 v_B),$$

$$b_1=\frac{P_e A_0}{6\alpha^2\sigma^4}\{3\alpha\sigma(1+2\alpha^2+\sigma^2 v_B)-N_0\}+\frac{P_r E}{12\alpha^3\sigma^5}[24\alpha^4\sigma N_0-(1+6\alpha^2)N_0^2$$
$$+2\alpha(1+\sigma^2 v_B)\{2N_0-3\alpha\sigma(1+\sigma^2 v_B)-12\alpha^3\sigma\}],$$

$$b_2=\frac{P_e A_0}{2\sigma^2}(1+\sigma^2 v_B)-\frac{P_r E}{2\sigma^2}\left\{\frac{N_0^2}{\sigma^2}+(1+\sigma^2 v_B)^2\right\},$$

$$b_3=\frac{N_0}{6\sigma^2}[P_e A_0-2P_r E(1+\sigma^2 v_B)],$$

$$b_4=-\frac{P_r E N_0^2}{12\sigma^2},$$

$$b_5=\frac{1}{\sigma^5}[2P_r E\{\sigma(1+\sigma^2 v_B)-N_0\}-\sigma P_e A_0],$$

$$b_6=\frac{2P_r E N_0}{\sigma^4},\qquad b_7=-\frac{P_r E}{2\sigma^4},$$

$$v_B=\frac{1}{6}\left[\frac{N_0}{1+\alpha\sigma}-\frac{P_0}{2(1+\alpha\sigma)}+\frac{3(\sigma+2\alpha)}{\sigma(1+\alpha\sigma)}\right]$$

However, the expression for temperature distribution in the absence of viscous dissipation is

$$\theta(\eta) = (1+\alpha\,\sigma\,\eta)\,\theta_B + \frac{P_e}{\sigma^2}\left\{\frac{1}{\sigma^2}\,(1-\cosh\eta\,\sigma-\sinh\eta\,\sigma) + P_1\eta + p_2\eta^2\right\}$$

$$+\frac{N_0\,P_e}{\sigma^2}(q_1\eta + q_2\eta^2 + \tfrac{1}{6}\eta^3) + \frac{P_0\,P_e}{\sigma^2}(r_1\eta + r_2\eta^2) \qquad (22)$$

where

$$\theta_B = \frac{1}{1+\sigma H} + \frac{P_e}{360}\,(15k_0+60\,l_0P_0+n_0N_0)$$

$$p_1 = \frac{\alpha}{\sigma} + \frac{2\alpha+\sigma}{4\alpha\,(1+\alpha\sigma)} + \frac{1}{2\alpha\sigma},\quad q_1 = \alpha\sigma^3-2\,(1+\alpha\sigma),\quad r_1 = \frac{\sigma}{4\alpha(1+\alpha\sigma)}$$

$$p_2 = \frac{2+4\alpha\sigma+\sigma^2}{4(1+\alpha\sigma)},\qquad q_2 = \frac{\sigma^2}{12(1+\alpha\sigma)},\qquad r_2 = \frac{\sigma^2}{4(1+\alpha\sigma)}$$

In the absence of convection (i.e. $P_e=0$), eq. (22) becomes

$$\theta(\eta) = \frac{1+\alpha\sigma\eta}{1+\sigma H}.$$

Comparing the derivative of this, with the boundary condition (15), we find that $\alpha=H$. This means that when heat is transported only by diffusion, the values of α will depend on the values of H. Since H depends only on the structure of the porous media, we conclude that α depends only on the structure of the porous material and not on the geometrical configuration.

Equation (18), with (19) and (21), represents the temperature distribution with viscous dissipation terms while eq. (18), with (20) and (22), represents the temperature distribution in the absence of viscous dissipation. These are numerically evaluated for different values of a, ξ, N_0 and P_0 for fixed α and σ and are shown in figure 2. For favourable temperature gradient (i.e. $\partial T/\partial x=A<0$), $a>0$ corresponds to the heating of the plate (i.e. $T_1-T_0<0$) for, heat flows from the bed towards the plate and $a<0$, corresponds to the cooling of the plate, because heat flows from the plate towards the bed. But $a=0$ (i.e. $\partial T/\partial x=0$) corresponds to the absence of buoyancy force in which no heat is transported by convection and heat is transported only by diffusion. From figure 2 it is clear that when $a=0$, there exists a thin thermal boundary layer just beneath the nominal surface, with higher temperature in the free flow compared to that in the Darcy flow. However, when $a \neq 0$ with $N_0>0$, heat is transported both by convection and diffusion, and figure 2 shows that the effect of convection is to increase the magnitude of the temperature both in the free and Darcy flows. A similar behaviour is also observed in the case of Poiseuille flow with an overall increase of temperature of about 5% compared to that of Couette

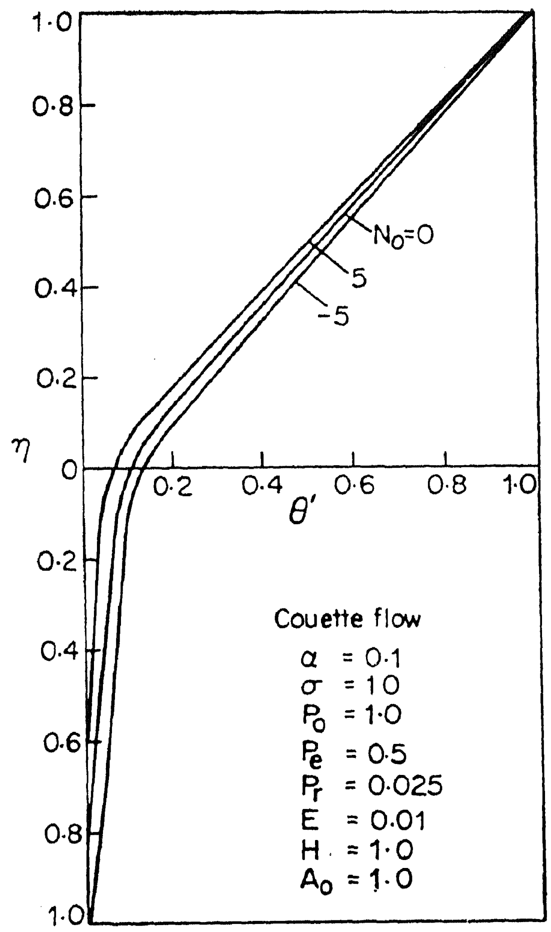

Figure 2. Temperature distribution

flow. Similar conclusions are true for $N_0 < 0$. Therefore, we conclude that the effect of wall shear has no significant influence on the temperature distribution.

From the technological point of view, it is of interest to know the rate of heat transfer q between the fluid and the nominal surface and we get

$$q=\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=0}=\sigma H \theta_B \tag{23}$$

in the presence of viscous dissipation and

$$q=\frac{\sigma H}{1+\sigma H}\left[1-\frac{P_e}{24}(1+4f_0)-\frac{N_0 P_e}{360}\,\frac{27+7\alpha\sigma}{1+\alpha\sigma}-\frac{P_0 P_e}{6}f_1\right] \tag{24}$$

in the absence of viscous dissipation. From eqs (23) and (24) it is clear that heat transfer coefficient is proportional to surface temperature and increases with increasing σ. The q's are plotted against σ and is shown in figure 3. We observe that q increases with σ. We also observe that viscous dissipation has a significant effect on heat transfer between the fluid and the nominal surface.

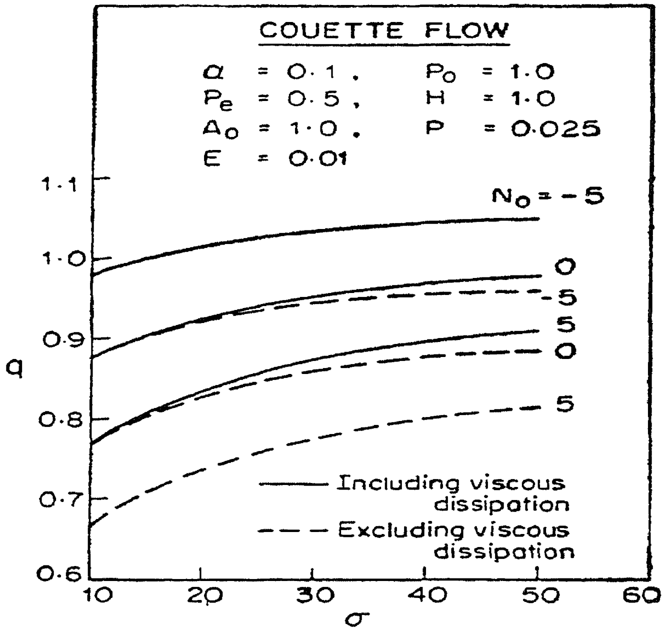

Figure 3. Heat transfer coefficient q against σ

Acknowledgement

This work is supported by UGC, India under research project F. 23-237/75 SR-II.

References

Beavers G S and Joseph D D 1967 *J. Fluid Mech.* 3 197
Beavers G S, Sparrow E M and Magnuson R A 1976 *J. Basic. Engg. Trans. ASME* **92** 843
Christopher K W Tan 1969 *J. Fluid Mech.* **38** 537
Lundgren 1972 *J. Fluid Mech.* **51** 273
Rajasekhara B M 1974 Ph.D. Thesis, Bangalore University
Rudraiah N 1966 NAL Tech. Note TN-PH-4-66
Rudraiah N and Natarajan A 1976 NGEF Int. Rep.
Rudraiah N and Veerabhadraiah R 1976 *Phys. Fluids* (submitted)