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Linear and non-linear magnetoconvection in a porous medium

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Abstract. Linear and non-linear magnetoconvection in a sparsely packed porous medium with an imposed vertical magnetic field is studied. In the case of linear theory the conditions for direct and oscillatory modes are obtained using the normal modes. Conditions for simple- and Hopf-bifurcations are also given. Using the theory of self-adjoint operator the variation of critical eigenvalue with physical parameters and boundary conditions is studied. In the case of non-linear theory the subcritical instabilities for disturbances of finite amplitude is discussed in detail using a truncated representation of the Fourier expansion. The formal eigenfunction expansion procedure in the Fourier expansion based on the eigenfunctions of the corresponding linear stability problem is justified by proving a completeness theorem for a general class of non-self-adjoint eigenvalue problems. It is found that heat transport increases with an increase in Rayleigh number, ratio of thermal diffusivity to magnetic diffusivity and porous parameter but decreases with an increase in Chandrasekhar number.

1. Introduction

The linear and non-linear magnetoconvection in an electrically conducting fluid has been extensively studied by many authors ([1], [2], [17], [18], [21], [22]) but the corresponding problem in a porous medium has not been given much attention, inspite of its various engineering and geophysical applications. Rudraiah [12], Rudraiah and Prabhamani [13], and Prabhamani and Rudraiah [9–11], have considered this problem using a universal stability analysis (*i.e.*, stability for arbitrary disturbances) which gives, as in infinitesimal disturbances, only the criterion for the onset of convection but is silent about the prediction of heat transfer. The study of heat transfer is essential to understand the mechanism of heat transfer from the deep interior of the earth to shallow depth in geothermal region. In other words, the problem considered in this paper is similar to a situation that exists in the geothermal regions where the surface liquid possesses a general upward convective drift due to the buoyancy induced by Joule heat and interior temperature. Since the rising liquid is cooled as it approaches the surfaces where heat is removed by evaporation, radiation and movement in surface streams an unstable state may be induced and complicated convective motions appear in the layers near the surface. Therefore convection through a porous medium in the presence of a geothermal magnetic field is studied in this paper because the results of such a study is useful in understanding the field behaviour in the extraction of energy in the geothermal regions.

2. Mathematical formulation

The physical configuration considered in this paper is shown in figure 1 which consists of a horizontal thin porous layer of permeability k_p and of infinite extent filled with a

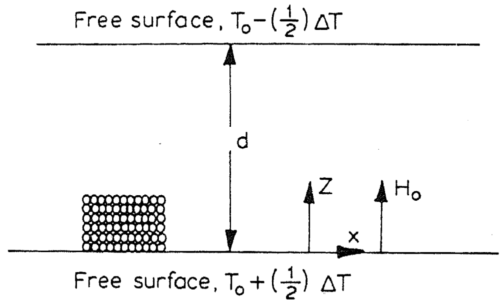

Figure 1. Physical configuration.

conducting Boussinesq fluid heated from below and permeated by an externally applied uniform vertical magnetic field H_0. The layer has thickness d and is bounded by two free surfaces. The upper surface is at a constant temperature $T_0 - \Delta T/2$ and the lower at $T_0 + \Delta T/2$. We write the total temperature as

$$T_{\text{total}} = T_0 - \Delta T(z/d - 1/2) + T(x, y, z, t)$$

where $T(x, y, z, t)$ is the deviation of the temperature from the linear profile and we assume that all the physical quantities are independent of y. In other words, we consider here the two-dimensional horizontal rolls.

For a creeping electrically conducting Boussinesq fluid flowing through a porous medium, the momentum equation is the modified Darcy's law

$$\frac{\partial \mathbf{q}}{\partial t} = -\frac{1}{\rho_0}\nabla P + \frac{\rho}{\rho_0}\mathbf{g} - \frac{\nu}{k_p}\mathbf{q} + \frac{\mu_m}{\rho_0}(\mathbf{H}\cdot\nabla)\mathbf{H}, \tag{1}$$

where $\mathbf{q}$ is the Darcy velocity, ρ is the density, ρ_0 is the density at the reference temperature $T = T_0$, $\mathbf{H}$ is the magnetic field, P is the total pressure, ν is the kinematic viscosity, μ_m is the magnetic permeability and $\mathbf{g}$ is the acceleration due to gravity. This equation neglects non-linear inertia effects. At high Reynolds number this equation should be generalized to include inertia effects. The effect of inertia must appear as a drag proportional to the square of the velocity and is of the form

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{1}{\rho_0}\nabla P + \frac{\rho}{\rho_0}\mathbf{g} - \frac{\nu}{k_p}\mathbf{q} - \frac{c_b}{k_p^{1/2}}|\mathbf{q}|\mathbf{q} + \frac{\mu_m}{\rho_0}(\mathbf{H}\cdot\nabla)\mathbf{H}, \tag{2}$$

where c_b is a non-dimensional constant and has a more or less universal value for a particular family of materials. This neglects shear effects. In a sparsely distributed porous medium made of spherical particles of uniform size, the porosity is very high and we have to include the usual shear effects produced by the distortion of velocity. In that case, considered in this paper, the basic momentum equation is the modified Brinkman equation

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{1}{\rho_0}\nabla P + \frac{\rho}{\rho_0}\mathbf{g} - \frac{\nu}{k_p}\mathbf{q} - \frac{c_b}{k_p^{1/2}}|\mathbf{q}|\mathbf{q}$$
$$+ \frac{\mu_m}{\rho_0}(\mathbf{H}\cdot\nabla)\mathbf{H} + \nu_m\nabla^2\mathbf{q}. \tag{3}$$

In this paper since we deal with pure magnetoconvection problem, we neglect the drag proportional to the square of the velocity and consider the basic equations:

$$\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{1}{\rho_0}\nabla P+\frac{\rho}{\rho_0}\mathbf{g}-\frac{\nu}{k_p}\mathbf{q}+\nu\nabla^2\mathbf{q}+\frac{\mu_m}{\rho_0}(\mathbf{H}\cdot\nabla)\mathbf{H}, \tag{4}$$

$$\nabla\cdot\mathbf{q} = 0, \tag{5}$$

$$\frac{\partial \mathbf{H}}{\partial t}+(\mathbf{q}\cdot\nabla)\mathbf{H} = (\mathbf{H}.\nabla)\mathbf{q}+\nu_m\nabla^2\mathbf{H}, \tag{6}$$

$$\rho = \rho_0[1-\beta(T-T_0)], \tag{7}$$

$$\frac{\partial T}{\partial t}+(\mathbf{q}\cdot\nabla)T+(\Delta T/d)w = K_d\nabla^2 T, \tag{8}$$

where β is the volumetric expansion coefficient, ν_m is the magnetic viscosity and K_d is the thermal diffusivity.

These equations, using

$$u = \frac{\partial\psi}{\partial z},\ w = -\frac{\partial\psi}{\partial x},\ H_x = \frac{\partial\phi}{\partial z},\ H_z = -\frac{\partial\phi}{\partial x}$$

and making the resulting equations dimensionless, take the form

$$\frac{\partial\eta}{\partial t} = J(\psi,\eta)-\frac{\Pr Q}{S}J(\phi,\xi)-\Pr\,\mathrm{Ra}\frac{\partial T}{\partial x}$$

$$+\frac{\Pr Q}{S}\nabla^2(\partial\phi/\partial z)-\frac{\Pr}{P_l}\eta+\Pr\nabla^2\eta, \tag{9}$$

$$\frac{\partial\phi}{\partial t} = J(\psi,\phi)+\frac{\partial\psi}{\partial z}+\frac{1}{S}\nabla^2\phi, \tag{10}$$

$$\frac{\partial T}{\partial t} = J(\psi,T)-\frac{\partial\psi}{\partial x}+\nabla^2 T, \tag{11}$$

where the equations are made dimensionless using d, d^2/K_d, K_d/d, ΔT and H_0 the scales for length, time, velocity, temperature and magnetic field respectively, Ra $= (\beta g\Delta T d^3)/K_d$ is the Rayleigh number, $\Pr = \nu/K_d$ is the Prandtl number, $P_l = k_p/d^2$ is the porous parameter, $Q = \mu_m H_0^2 d^2/\rho_0\nu\nu_m$ is the Chandrasekhar number, $S = K_d/\nu$, $\eta = \nabla^2\psi$, $\xi = \nabla^2\phi$ and $J(.,.)$ is the Jacobian.

The boundary conditions for the problem, when the boundaries $z = 0$ and $z = 1$ taken as flat, are the magnetic and velocity stress free,

$$\psi = 0,\ \frac{\partial^2\psi}{\partial z^2} = 0,\ T = 0,\ \frac{\partial\phi}{\partial z} = 0 \quad \text{at} \quad z = 0, 1. \tag{12}$$

3. Linear stability analysis

In this section, following the analysis of Rudraiah and coworkers [15, 18], we discuss the linear stability analysis considering both the marginal and overstable states. The solutions of this analysis are of great importance in the finite amplitude study, using the local non-linear stability analysis.

The required stability equation for this case, after neglecting the Jacobian terms in (9) and (10) and eliminating ϕ and T, is

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)\left(\frac{\partial}{\partial t}+\frac{1}{S}\nabla^2\right)\left(\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t}+\left(\frac{1}{P_l}-\nabla^2\right)\right)\nabla^2\psi$$
$$=\left(\frac{\partial}{\partial t}-\nabla^2\right)\frac{Q}{S}\frac{\partial^2\nabla^2\psi}{\partial z^2}+\left(\frac{\partial}{\partial t}-\frac{1}{S}\nabla^2\right)\mathrm{Ra}\frac{\partial^2\psi}{\partial x^2}. \tag{13}$$

We look for the solution of the form

$$\psi=\exp(p_0 t)\sin\pi\alpha x\sin\pi z, \tag{14}$$

where p_0 $(=p_{0r}+ip_{0i})$ is the frequency. Substituting (14) into (13) and after some simplification, we obtain the dispersion relation

$$p_0^3+\left[\mathrm{Pr}\left(\frac{1}{P_l}+\pi^2(\alpha^2+1)\right)+\frac{\pi^2(\alpha^2+1)}{S}+\pi^2(\alpha^2+1)\right]p_0^2$$
$$+\left[\mathrm{Pr}\left(\frac{1}{P_l}+\pi^2(\alpha^2+1)\right)\pi^2(\alpha^2+1)+\frac{\pi^4(\alpha^2+1)^2}{S}\right.$$
$$\left.+\frac{\mathrm{Pr}Q\pi^2}{S}-\frac{\mathrm{Ra}\alpha^2\mathrm{Pr}}{\alpha^2+1}+\frac{\mathrm{Pr}}{S}\left(\frac{1}{P_l}+\pi^2(\alpha^2+1)\right)\pi^2(\alpha^2+1)\right]p_0$$
$$+\left[\frac{\mathrm{Pr}}{S}\left(\frac{1}{P_l}+\pi^2(\alpha^2+1)\right)\pi^4(\alpha^2+1)^2-\frac{\mathrm{Pr}}{S}\pi^2(\alpha^2+1)\right.$$
$$\left.\times\left(\frac{\mathrm{Ra}\alpha^2}{\alpha^2+1}-Q\pi^2\right)\right]=0. \tag{15}$$

3.1 *Marginal state*

Linear theory predicts the condition for the onset of convection at the critical Rayleigh number (see [2]). The unstable thermal stratification tends to produce steady convection at $\mathrm{Ra}=\mathrm{Ra}_c^m$, the critical Rayleigh number, given by

$$\mathrm{Ra}_c^m=R_L^m+R_M^m, \tag{16}$$

where
$$R_L^m=\frac{\pi^2(\alpha^2+1)^2}{\alpha^2 P_l} \tag{17}$$

is the Lapwood [6] Rayleigh number and

$$R_M^m=\frac{\pi^2(\alpha^2+1)}{\alpha^2}[\pi^2(\alpha^2+1)^2+Q] \tag{18}$$

is the Rayleigh number for magnetoconvection given by Chandrasekhar [2] and the superscript m denotes the marginal state. The critical wave number for the direct mode is obtained from the cubic equation

$$x^3+\frac{1}{2\pi^2}\left(3\pi^2+\frac{1}{P_l}\right)x^2-\left\{\frac{\pi^2+Q}{2\pi^2}+\frac{1}{P_l\pi^2}\right\}=0, \tag{19}$$

with $x=\alpha^2$. Equation (16) reveals that the effects of permeability and magnetic field are to suppress steady convection. Similarly (19) shows that the wave number x is influenced by the permeability parameter P_l and the Chandrasekhar number Q. Therefore, both permeability and magnetic field influence the cell pattern.

3.2 *Overstability*

The striking similarity that exists on several occasions between the results pertaining to a fluid in the presence of a magnetic fluid and a porous medium makes one to anticipate overstable motions in the present study. Therefore, in this section, we discuss infinitesimal overstable motions.

The condition for overstable motion is that the product of the coefficient of p_0^2 and p_0 in (15) must be equal to the coefficient of p_0^0 (*i.e.*, constant term) and is given by

$$\mathrm{Ra}_c^0 = \frac{\pi^2(\alpha^2+1)(1+S)[(\mathrm{Pr}S/P_l\pi^2(\alpha^2+1))+1]}{\alpha^2\mathrm{Pr}S^2} \times\left[\pi^2(\alpha^2+1)^2+\frac{\mathrm{Pr}QP_l\pi^2(\alpha^2+1)}{(P_l\pi^2(\alpha^2+1)+\mathrm{Pr})(1+S)}\right]. \tag{20}$$

Also, for overstable motions the coefficient of p_0 in (15) should be real and positive, and hence it leads to the condition

$$Q \geqslant \frac{\pi^2(\alpha^2+1)^2(1+(\mathrm{Pr}/P_l\pi^2(\alpha^2+1)))}{\mathrm{Pr}(S-1)} \tag{21}$$

for overstability which is valid only when $S > 1$ *i.e.*, $\mathrm{Pm} > \mathrm{Pr}$. The minimum values of Ra_c^0 and the corresponding values of α^2 for given values of Q, P_l, S and $\mathrm{Pr} = 1$ are obtained by varying the values of α^2 in (20) and satisfying the condition (21).

We note that this analysis reduces to the Darcy case if $\pi^2(\alpha^2+1) \ll (1/P_l)$ *i.e.*, neglecting $\pi^2(\alpha^2+1)$ compare to $(1/P_l)$ in $((1/P_l)+\pi^2(\alpha^2+1))$ of equation (15). Similarly it reduces to magnetoconvection in the absence of a porous medium if $1/P_l \ll \pi^2(\alpha^2+1)$ *i.e.*, neglecting $1/P_l$ compare to $\pi^2(\alpha^2+1)$ in $(1/P_l)+\pi^2(\alpha^2+1)$ of (15). This is the same as the limiting case of $P_l \to \infty$ in (15).

3.3 *Bifurcations from the static solution*

The dispersion relation (12) may be used to examine the bifurcations from the static solution. Since this dispersion relation is a cubic polynomial in p_0 with either zero or two sign changes. Hence, by Descarte's rule of signs, either there are no bifurcation points or there are two. In the latter case there will be two types of bifurcation; simple—(or direct) and Hopf bifurcation.

We first seek the condition for a simple-bifurcation for which $p_0 = 0$. Traditionally, this is known as an exchange of stabilities. To obtain the condition we rewrite (12) in the form

$$p_0^3+\left\{\mathrm{Pr}\left(1+\frac{1}{P_l'}\right)+1+\frac{1}{S}\right\}\delta^2p_0^2+\left[\left\{\mathrm{Pr}\left(1+\frac{1}{S}\right)\left(1+\frac{1}{P_l'}\right)+\frac{1}{S}\right\}\delta^4 - \mathrm{Ra}\,\mathrm{Pr}\frac{\pi^2\alpha^2}{\delta^2}+\frac{Q\mathrm{Pr}\pi^2}{S}\right]p_0+\frac{\mathrm{Pr}}{S}\left[\left(1+\frac{1}{P_l'}\right)\delta^6+Q\delta^2\pi^2-\pi^2\alpha^2\mathrm{Ra}\right]=0 \tag{22}$$

where $\delta^2 = \pi^2(\alpha^2+1)$, $P_l' = P_l\delta^2$.

From (22) $p_0 = 0$ when $\mathrm{Ra} = \mathrm{Ra}^{(e)}$, where

$$\mathrm{Ra}^{(e)} = [\delta^6(1+1/P_l') + \pi^2\delta^2 Q]/\pi^2\alpha^2 \tag{23}$$

When $Q = 0$, $\mathrm{Ra}^{(e)} = \mathrm{Ra}_0$, where

$$\mathrm{Ra}_0 = \frac{\delta^6}{\pi^2\alpha^2}(1+1/P_l'). \tag{24}$$

The first term on the right side is the boundary layer correction arising due to the use of modified Brinkman equation. It will prove convenient, especially when we come to consider non-linear solutions to introduce a more compact notation by defining a normalised Rayleigh number

$$r = \mathrm{Ra}\frac{\pi^2\alpha^2}{\delta^6};$$

$$\omega = p_0/\delta^2; \quad \eta' = 1 + 1/P_l'$$

$$q = Q\pi^2/\delta^4;$$

So that (22) simplifies to

$$\omega^3 + \left\{\mathrm{Pr}\eta' + 1 + \frac{1}{S}\right\}\omega^2 + \left[\mathrm{Pr}\left(\eta' - r + \frac{q}{S}\right) + \frac{1}{S}(\mathrm{Pr}\eta' + 1)\right]\omega$$

$$+\frac{\mathrm{Pr}}{S}(\eta' + q - r) = 0. \tag{25}$$

The condition for simple bifurcation is now $r = r^{(e)}$, where

$$r^{(e)} = \eta' + q. \tag{26}$$

Of the three roots of (25), one is always real and negative. The other pair may be real, or complex conjugates. In the latter case there is a Hopf bifurcation from the static solution (when $\mathrm{Re}(\omega) = 0$, $\omega = \pm i\omega_0$) at $r = r^{(0)}$, where

$$r^{(0)} = \eta' + \frac{(\mathrm{Pr}+1)}{\mathrm{Pr}S(\mathrm{Pr}\eta'+1)}\left(1 + \mathrm{Pr}\eta' + \frac{1}{S}\right) + \frac{(\mathrm{Pr}\eta' + 1/S)q}{S(\mathrm{Pr}\eta'+1)} \tag{27}$$

provided that

$$\omega_0^2 = \frac{\mathrm{Pr}}{S(1+\mathrm{Pr}\eta'+1/S)}(r^{(e)} - r^{(0)}) = \frac{(\mathrm{Pr}+1)}{(\mathrm{Pr}\eta'+1)S^2} + \frac{(S-1)\mathrm{Pr}q}{S^2(\mathrm{Pr}\eta'+1)} \tag{28}$$

is positive.

Hence, if a Hopf bifurcation is possible it always occurs at a lowest value of r than the simple bifurcation. In traditional terms, overstability sets in at $r = r^{(0)} < r^{(e)}$. It is clear from (28) that a necessary condition for the existence of a Hopf bifurcation is

$$S > 1, q > q_0 \equiv \frac{(\mathrm{Pr}+1)}{\mathrm{Pr}(S-1)}. \tag{29}$$

However, for $S < 1$ or $q < q_0$, the only bifurcation from the static solution occurs at $r = r^{(e)}$ and the static solution is unstable for all $r > r^{(e)}$. If, on the other hand, (29) is satisfied, so that $\omega_0^2 > 0$, the static solution is unstable for all $r > r^{(0)}$. The behaviour of

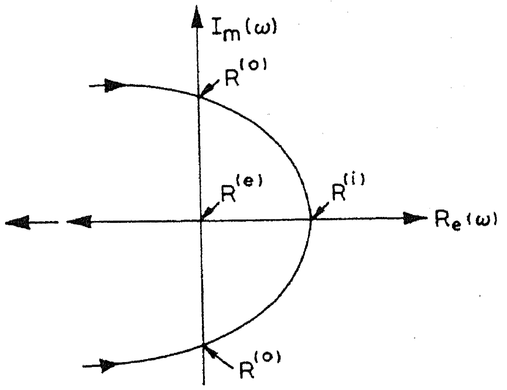

Figure 2. Behaviour of the eigen values when convection sets in *via* a Hopf bifurcation at $R = R^{(0)}$. The arrows indicate the evolution of the eigenvalues ω with increasing R in the complex plane.

the eigenvalues in the complex plane is illustrated in figure 2; as r passes through $r^{(0)}$, the real part of the complex conjugate pair changes sign. As r further increases, $\mathrm{Re}(\omega)$ increases while $|\mathrm{Im}(\omega)|$ decreases. At $r = r^{(i)}$, say $\mathrm{Im}(\omega) = 0$, if the two roots coalesce. Thereafter both eigenvalues remain on the real axis; one increases with increasing r, but the other decreases and passes through zero at $r = r^{(e)}$. In linear theory, therefore, the transition from oscillatory to direct instability occurs at $r = r^{(i)}$. The condition for (25) to have two equal roots is

$$\left(\frac{A_1}{3}-\frac{A_2^2}{9}\right)^3+\left\{\frac{1}{6}(A_1A_2-3A_3)-\frac{1}{27}A_2^3\right\}^2=0$$

where
$$A_1 = \mathrm{Pr}\,\eta' + 1 + \frac{1}{S}$$

$$A_2 = \mathrm{Pr}\left(\eta' - r + \frac{q}{S}\right) + \frac{1}{S}(\mathrm{Pr}\,\eta' + 1)$$

$$A_3 = \frac{\mathrm{Pr}}{S}(\eta' + q - r).$$

Values of $r^{(i)}$, if needed, must in general be found numerically.

3.4 *Parametric perturbation methods*

The analysis of the previous sections pertain only to the case when the boundaries are isothermal (*i.e.* boundary condition on temperature is of first kind). In many practical problems discussed in §1, the realistic boundary condition on temperature is of the radiation type (*i.e.* boundary condition on temperature is of the third kind). Therefore the effect of radiation boundary condition on the eigenvalues is studied in this section without going into the detailed numerical computation. Using the concept of self adjoint operator we try to find the effect of Biot number, Chandrasekhar number and porous parameter on the monotonicity of Rayleigh number. Our main object here is to show that under what conditions the critical Rayleigh number for marginal convection is an upper bound for finite Biot number and to find an analytical relation for the variation of Rayleigh number with respect to Chandrasekhar number. As an extra result we also obtain expressions for obtaining first order effects on the eigen-functions. We employ a method which has been used by Narayanan *et al* [7] in a more general context. The first order effects can be calculated using a modified Greens matrix as the kernel of a matrix differential operator.

For this purpose we define an operator L as follows:

$$\mathrm{L} \equiv \begin{bmatrix} (D^2-\pi^2\alpha^2)\{(D^2-\pi^2\alpha^2)-P_l^{-1}\} & \mathrm{Rm}\,(D^2-\pi^2\alpha^2)D & -\mathrm{Ra}^0\pi\alpha \\ -\mathrm{Rm}\,(D^2-\pi^2\alpha^2)D & -\frac{\mathrm{Rm}}{S}(D^2-\pi^2\alpha^2)^2 & 0 \\ -\mathrm{Ra}^0\,\pi\alpha & 0 & \mathrm{Ra}^0(D^2-\pi^2\alpha^2) \end{bmatrix}$$

where $D = \frac{\partial}{\partial z}$ and $\mathrm{Rm} = Q/S.$ (30)

We define a vector $\mathbf{v}$ such that

$$\mathbf{v}^t = [\psi, \phi, T],$$

where the three vectors represent the dependent variables. Then from (4) to (8) after neglecting the Jacobians, we have

$$L\mathbf{v} = 0. \tag{31}$$

To visualize the variation of Ra^0 with respect to Q, we differentiate (31) with respect to Q and obtain

$$L\hat{\mathbf{v}} = \hat{\mathbf{h}}, \tag{32}$$

where

$$\hat{\mathbf{h}}^t = \left[-\frac{1}{S}(D^2-\pi^2\alpha^2)D\phi + \hat{\mathrm{R}}\mathrm{a}^0\pi\alpha T, 0, 0\right] \tag{33}$$

and '∧' represents the derivative with respect to Q.

Define a suitable inner product between two vectors $\mathbf{a}$ and $\mathbf{b}$ as $\langle \mathbf{a}, \mathbf{b}\rangle$ such that

$$\langle \mathbf{a}, \mathbf{b}\rangle = \int_v \mathbf{a}^{*t}\cdot\mathbf{b}\,\mathrm{d}V, \tag{34}$$

where V is the domain of the integral operator in which $\mathbf{a}$ and $\mathbf{b}$ are defined and the asterisks represent complex conjugates. We can easily see that L is selfadjoint and the boundary conditions are also selfadjoint. Thus applying a Fredholm alternative condition to (32) we get

$$\pi\alpha Q\left\{\hat{\mathrm{R}}\mathrm{a}^0 - \frac{\mathrm{Ra}^0}{Q}\right\}\int_v \psi^* T\,\mathrm{d}v = -\int_v |D^2\psi|^2\,\mathrm{d}v$$

$$-(2\pi^2\alpha^2+P_l^{-1})\int_v |D\psi|^2\,\mathrm{d}V - \pi^2\alpha^2(\pi^2\alpha^2+P_l^{-1})\int_v |\psi|^2\,\mathrm{d}v. \tag{35}$$

We use the energy and continuity equation to arrive at the conclusion that $\hat{\mathrm{R}}\mathrm{a}^0 > 0$ if Ra^0 is positive. From this it is clear that an increase in strength of the magnetic field causes an increase in Ra^0 if we have the most unstable case.

We can consider the dependence of Ra^0 on the porous parameter by taking the derivatives of (31) with respect to P_l and obtain

$$L\bar{\mathbf{v}} = \bar{\mathbf{h}}, \tag{36}$$

where

$$\bar{\mathbf{h}}^t = [(D^2 - \pi^2\alpha^2)\psi + \overline{\mathrm{Ra}^0}\,\pi\alpha T, 0, 0] \tag{37}$$

and overbars ($-$) represent differentiation with respect to P_l. On applying the solvability condition we get

$$\pi\alpha\overline{\mathrm{Ra}^0}\int_v \psi^* T\,\mathrm{d}v = \int_v |D\psi|^2\,\mathrm{d}v + \pi^2\alpha^2\int_v |\psi|^2\,\mathrm{d}v. \tag{38}$$

This yields the condition that Ra^0 increases if P_l decreases.

Finally, we consider the effect of boundary parameter, the Biot number Bi, on Ra^0 we shall see that the obtained Rayleigh numbers are upper bounds for finite Biot number. We note that the effect of surface tension gradient is not considered here.

We may differentiate (31) and the corresponding boundary conditions with respect to Bi and obtain

$$L\tilde{\mathbf{V}} = [\tilde{\mathrm{R}}\mathrm{a}^0\,\pi\alpha T, 0, 0]^t \tag{39}$$

where tilde ($\sim$) overbars represent the differentiation with respect to Bi and at the boundary we have

$$D\tilde{T} + Bi\,\tilde{T} + T = 0. \tag{40}$$

Thus on applying the solvability condition to (39) in the light of (40), we get after much algebraic manipulation

$$\pi\alpha\frac{\tilde{\mathrm{R}}\mathrm{a}^0}{\mathrm{Ra}^0}\int_v \psi^* T\,\mathrm{d}v = \int_v (-|T|^2)\,\mathrm{d}v. \tag{41}$$

On using the energy equation, we get from (41)

$$\frac{\partial \ln \mathrm{Ra}^0}{\partial Bi} > 0.$$

Thus if $\mathrm{Ra}^0 < 0$ (such as the case of heated from above) then Ra^0 will decrease with an increase in Bi. In the present problem we are concerned with the case of heated from below and so the case of Bi tending to infinity provides an upper bound on Ra^0. Equation (39) is a vehicle for calculating the first-order effects and a modified Green matrix is needed for this calculation. Such approximations are of a practical value.

3.5 *A completeness theorem*

The non-linear magnetoconvection in a porous medium studied in the next section is based on expanding the Fourier components of a spatially periodic disturbance in the eigen-functions of the corresponding linear stability problem. This is, usually, known as Stuart's shape assumption. For a general discussion of this procedure and an application to the problem of the stability of plane parallel flow and of viscous flow between rotating concentric cylinders see Diprima and Habetler [4]; for an application to the stratified viscous shear flows problem see Herron [5]. Their completeness theorem is the generalization of Naimark's theorem [8]. In this section following Diprima and Habetler [4] we will justify the formal eigenfunction expansion procedure used in the next section by proving a completeness theorem for a general class of non-selfadjoint eigenvalue problems which includes the matrix differential operators.

It is not difficult to show that the system (4) to (8), neglecting the Jacobian terms and using the solution of the form

$$f(z)\exp\{i\alpha(x-ct)\},$$

may be written as

$$L\Phi = \lambda M\Phi, \tag{42}$$

where

$$L = \begin{bmatrix} L_0 - \dfrac{Q}{S}D^2(D^2-\alpha^2) & -\dfrac{i\alpha\mathrm{Ra}}{S}(D^2-\alpha^2) \\ i\alpha & -(D^2-\alpha^2) \end{bmatrix},$$

$$M = \begin{bmatrix} L_1 & \alpha\mathrm{Ra} \\ 0 & 1 \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \psi \\ T \end{bmatrix},$$

$$L_0 = \left[-\frac{\alpha^2C^2}{\mathrm{Pr}} - \frac{1}{S}(D^2-\alpha^2)\left\{\frac{1}{P_l} - (D^2-\alpha^2)\right\}\right](D^2-\alpha^2)$$

$$L_1 = \left[\frac{1}{P_l} - (D^2-\pi^2\alpha^2) - \frac{1}{\mathrm{Pr}S}(D^2-\pi^2\alpha^2)\right](D^2-\pi^2\alpha^2),$$

$$D = \mathrm{d}/\mathrm{d}z \quad \text{and} \quad \lambda = i\alpha C.$$

This is the type of non-standard non-selfadjoint eigenvalue problem studied by Diprima and Habetler [4] and Herron [5]. The theorem that we try to establish concerns eigenvalue problems of the form (42), in a Hilbert space H where L and M are matrix differential operators with $dmn\ L \subset dmn\ M$, and M is positive definite. Another Hilbert space H_M is embedded in H where $[f, g] = \langle f, Mg\rangle$ (for $f, g \in dmn\ M$) is the corresponding inner product and $[f,f]^{1/2} = \|f\|_M$. We have the following theorem

Theorem: Let $L = L_s + L_B$ (43)

where

$$L_s = \begin{bmatrix} \dfrac{1}{S}(D^2-\alpha^2)^3 & 0 \\ 0 & -(D^2-\alpha^2) \end{bmatrix} \tag{44}$$

$$L_B = \begin{bmatrix} -\dfrac{\alpha^2C^2}{\mathrm{Pr}} - \dfrac{1}{S\,\mathrm{Pr}}(D^2-\alpha^2)^2 - \dfrac{Q}{S}(D^2-\alpha^2)D^2 & -\dfrac{i\alpha\mathrm{Ra}}{S}(D^2-\alpha^2) \\ i\alpha & 0 \end{bmatrix} \tag{45}$$

are linear operators with $dmn\ L_s = dmn\ L_B \subset dmn\ M$ dense in a Hilbert space H such that

(i) M and L_s are positive bounded below with $rng\ M = rng\ L_s$.

(ii) $M^{-1}L_B$ is bounded in H_M and *dmn* L_s is dense in H_M.
(iii) G^{-1} is compact and *rng* $G^{-1} \subset$ *dmn* M where G^{-1} is the inverse of the selfadjoint extension of $M^{-1}L_s$ in H_M.
(iv) For some γ, *rng* $(L_s+L_B+\gamma M) =$ *rng* M and $(G+M^{-1}L_B+\gamma I)$ has an inverse.
(v) There exists a sequence of concentric circles $\{C_k\}$ with radii $\{r_k\}$ such that

(a) $\lim_{k\to\infty} r_k = \infty$.

(b) $d[c_k, \sigma\theta(L_s, M)] > 0$ for all k.

(c) $\lim_{k\to\infty} d[c_k, \sigma\theta(L_s, M)] = \infty$

Then (i), the eigenvalues of (42) lie within the circles of radii $\|M^{-1}L_B\|$ about those of $L_s = \lambda M\Phi$, and (ii) the generalised eigenfunctions of (42) span H_M such that for $\Phi \in H_M$

$$\phi = \sum_{i=1}^{\infty}\sum_{j=1}^{n_i} a_{ij}\phi_{ij},$$

where $\phi_{k_1}, \phi_{k_2}, \ldots, \phi_{k_{n_k}}$ are eigenfunctions corresponding to eigenvalues in $r_{k-1} < |\gamma| < r_k$ for $k > 1$ and $|\gamma| < r_1$ for $k = 1$.

Following Diprima and Habetler [4] with L_s given by (44) it is straight forward to show that the hypothesis (i), (iii) and (v) are satisfied. If $\|M^{-1}L_B\|$ is bounded hypothesis (ii) holds and (iv) follows. In (45) L_B is different from the one given by Diprima and Habetler [4] and also of Herron [5]. However following [5] it is easy to show that $M^{-1}L_B$ is bounded. Hence all the hypotheses of Diprima and Habetler [4] is satisfied and its conclusions justify the use of spatial expansion in the non-linear problem based on the eigenfunctions of the linear stability problem.

4. Finite amplitude analysis with a limited representation

In this section we discuss, following Rudraiah and Vortmeyer [14] the finite amplitude analysis by considering a truncated representation of Fourier series for velocity, magnetic and temperature fields and try to understand the physics of the problem with minimum mathematics. We note that the results obtained from such a simple analysis can be used as a starting value in solving a general non-linear convection problem.

The first effect of non-linearity is to distort the temperature field through the interaction of ψ and T and the zonal current field through the interaction of ψ and ϕ. The distortion of temperature field will correspond to a change in the horizontal mean, *i.e.*, a component of the form $\sin 2\pi z$ will be generated. Similarly, zonal current field will be distorted by a component of the form $\sin 2\pi\alpha x$. Thus, a minimal system which describes finite amplitude convection is given by

$$\psi = A(t)\sin\pi\alpha x \sin\pi z, \tag{46}$$

$$T = B(t)\cos\pi\alpha x\sin\pi z + C(t)\sin 2\pi z, \tag{47}$$

$$\phi = D(t)\sin\pi\alpha x\cos\pi z + E(t)\sin 2\pi\alpha x, \tag{48}$$

where the amplitudes A, B, C, D and E are generally functions of time and are to be determined by the dynamics of the system. Substituting (46) to (47) into (4) to (8) and equating like terms, we get

$$\dot{A} = \frac{\pi\alpha Q(3\alpha^2-1)}{\alpha^2+1} DE - \Pr\left(\frac{1}{P_l} + \pi^2(\alpha^2+1)\right)A - \pi QD - \frac{\alpha \Pr \mathrm{Ra}\, B}{\pi(\alpha^2+1)}, \tag{49}$$

$$\dot{B} = -\pi^2(\alpha^2+1)B - \pi\alpha A - \pi^2\alpha AC, \tag{50}$$

$$\dot{C} = \frac{\pi^2}{2}\alpha AB - 4\pi^2 C, \tag{51}$$

$$\dot{D} = \pi^2\alpha AE + \pi A - \frac{\pi^2}{S}(\alpha^2+1)D, \tag{52}$$

$$\dot{E} = -\frac{\pi^2}{2}\alpha AD - \frac{4\pi^2\alpha^2}{S}E, \tag{53}$$

where the dot corresponds to a time derivative.

This set of non-linear ordinary differential equations is not amenable to analytical treatment for the general time-dependent variables and we have to solve it using a numerical method. However, in the case of steady motions, these equations can be solved analytically. Such solutions are very useful because they show that a finite amplitude steady solution to the system is possible for subcritical values of the Rayleigh number and that the minimum values of Ra for which steady solution is possible lies below the critical values for instability to either a steady infinitesimal disturbance or an overstable infinitesimal disturbance.

Thus, if the system is steady, (49) to (53) take the form

$$\pi^2(\alpha^2+1)((1/P_l) + \pi^2(\alpha^2+1))A + \pi\alpha \mathrm{Ra} B + (\pi^3/\Pr)Q(\alpha^2+1)D - \frac{\pi^4\alpha Q(3\alpha^2-1)}{\Pr}DE = 0, \tag{54}$$

$$\pi^2\alpha AC + \pi\alpha A + \pi^2(\alpha^2+1)B = 0, \tag{55}$$

$$\frac{\pi^2}{2}\alpha AB - 4\pi^2 C = 0, \tag{56}$$

$$\pi^2\alpha AE + \pi A - \frac{\pi^2}{S}(\alpha^2+1)D = 0, \tag{57}$$

$$\frac{\pi^2}{2}\alpha AD + 4\pi^2\frac{\alpha^2}{S}E = 0. \tag{58}$$

Equations (55) to (58) can be rewritten in the form

$$B = -\frac{\pi\alpha A}{\pi^2(\alpha^2+1+(1/8)\alpha^2A^2)}, \tag{59}$$

$$C = (1/8)\alpha AB, \tag{60}$$

$$D = \frac{\pi SA}{\pi^2(\alpha^2+1+(1/8)S^2A^2)}, \tag{61}$$

$$E = -(SAD/8\alpha). \tag{62}$$

Substituting (59) to (62) into (54) and after some simplification, we get

$$\begin{aligned}&\pi^2 S^2(\alpha^2+1)((1/P_l)+\pi^2(\alpha^2+1))(A^2/8)^3\\ &+[(\pi^2 S^2(\alpha^2+1)^2(1+(2\alpha^2/S^2)))((1/P_l)+\pi^2(\alpha^2+1))\\ &-\alpha^2 S^2\mathrm{Ra}+4\pi^2\alpha^4 Q](A^2/8)^2+[(\pi^2(\alpha^2+1)^3(2+(\alpha^2/S^2)))\times\\ &\times((1/P_l)+\pi^2(\alpha^2+1))-2\alpha^2\mathrm{Ra}(\alpha^2+1)+Q\pi^2\alpha^2(\alpha^2+1)\times\\ &\times(4+((\alpha^2+1)/S^2))](A^2/8)+(\alpha^2(\alpha^2+1)^2/S^2)(\mathrm{Ra}_c^m-\mathrm{Ra}).\end{aligned} \tag{63}$$

We have to look for real and positive roots of this cubic equation in $A^2/8$, otherwise, the amplitude of the stream function becomes imaginary. So we solve this equation numerically to find the real and positive roots.

4.1 *Finite amplitude analysis with small induced magnetic field and small induced current*

Rudraiah and Vortmeyer [14] have studied the finite amplitude analysis using a Darcy model by setting a constant term in (63) to zero. We note that setting the constant term in (63) equal to zero, is equivalent to restricting the value of Ra near to Ra_c^m. However, this can be avoided by assuming that the terms involving induced magnetic field and induced current are small compared to the applied magnetic field; which is usually the case in many practical problems. In this case (12) takes the form

$$\frac{\partial\eta}{\partial t}+(\mathbf{q}\cdot\nabla)\eta=\frac{\mu_m H_0}{\rho_0}\frac{\partial\xi_0}{\partial z}-\alpha g\frac{\partial T}{\partial x}-\frac{\nu}{k_p}\eta+\nu\nabla^2\eta, \tag{64}$$

where $\eta = \nabla^2\psi$ and $\xi = \nabla^2\phi$,

and all other equations remain the same. Now, substituting (46) to (48) into these equations and assuming, as before, the steady case we get

$$\pi^2(\alpha^2+1)((1/P_l)+\pi^2(\alpha^2+1))A+\pi\alpha\mathrm{Ra}B+(\pi^3 Q/S)(\alpha^2+1)D=0 \tag{65}$$

and the other equations are the same as (55) to (58). As before, substituting (59) to (61) into (65) we get

$$\begin{aligned}&A[(\pi^2 S^2\alpha^2(\alpha^2+1)((1/P_l)+\pi^2(\alpha^2+1)))(A^2/8)^2\\ &+(\pi^2 S^2(\alpha^2+1)^2((1/P_l)+\pi^2(\alpha^2+1))+\alpha^4\mathrm{Ra}_c^m-\alpha^2 S^2\mathrm{Ra})(A^2/8)\\ &+\alpha^2(\alpha^2+1)(\mathrm{Ra}_c^m-\mathrm{Ra})]=0.\end{aligned} \tag{66}$$

The solution $A = 0$ corresponds to the pure conduction and the other solutions are given by

$$\begin{aligned}A^2/8 = {}&[1/((1/P_l)+\pi^2(\alpha^2+1))2\pi^2\alpha^2 S^2(\alpha^2+1)]\\ &\times[\alpha^2 S^2(\mathrm{Ra}-\mathrm{Ra}_c^m)-\alpha^4\mathrm{Ra}_c^m+\pi^2 S^2(\alpha^2+1)Q\\ &\pm((\alpha^2 S^2(\mathrm{Ra}-\mathrm{Ra}_c^m-\mathrm{Ra}_c^m)-\alpha^4\mathrm{Ra}_c^m+\pi^2 S^2(\alpha^2+1)Q)^2\\ &+4S^2\alpha^4(\alpha^2\mathrm{Ra}_c^m-\pi^2(\alpha^2+1)Q))^{1/2}].\end{aligned} \tag{67}$$

To ensure the amplitude of the stream function to be real, we have to take the positive sign in front of the radical in (67).

Consider the case where finite solutions exist for $\mathrm{Ra} < \mathrm{Ra}_c^m$. We know that (Veronis [19]) the minimum value of Ra for which solutions exist is that value of Ra which makes the discriminant zero provided that

$$\mathrm{Ra}_f = \frac{\pi^2(\alpha^2+1)}{S^2}\left[\left\{\frac{(S^2-\alpha^2)(\alpha^2+1)}{\alpha^2}\left(\frac{1}{P_l}+\pi^2(\alpha^2+1)\right)\right\}^{1/2}+Q^{1/2}\right]^2, \quad (68)$$

where Ra_f denotes the Rayleigh number for finite amplitude motion. With this value of Ra_f the amplitudes are real provided that the first term on the right side of (68) be non-negative, or equivalently

$$Q > \frac{\alpha^2(\alpha^2+1)}{(S^2-\alpha^2)}\left(\frac{1}{P_l}+\pi^2(\alpha^2+1)\right). \quad (69)$$

Conditions (68) and (69) are meaningful only when

$$S > \alpha. \quad (70)$$

We note that the conditions (69) and (70) are readily accessible in a laboratory experiment with fibre material for porous media and with mercury flowing through it.

We note that the analysis of Brinkman model given above reduces to the Darcy case (Rudraiah and Vortmeyer [14]) if $\pi^2(\alpha^2+1) \ll 1/P_l$ *i.e.*, neglecting $\pi^2(\alpha^2+1)$ compare to $1/P_l$ in $(1/P_l)+\pi^2(\alpha^2+1)$. Further it reduces to the magnetoconvection in the absence of a porous medium if $1/P_l \ll \pi^2(\alpha^2+1)$ *i.e.*, neglecting $1/P_l$ compare to $\pi^2(\alpha^2+1)$ in $((1/P_l)+\pi^2(\alpha^2+1))$. This is the same as the limiting case $P_l \to \infty$.

This shows that the Brinkman model generalises the problem in the sense that with a suitable limit on P_l, we can obtain the results for Darcy flow and pure viscous flow.

5. Heat transport

In the study of convection problems the determination of heat transport across the layer plays a very important role. This is becuase the onset of convection as the Rayleigh number is increased is more readily detected by its effect on the heat transfer. In the quiescent state, the heat transfer is usually due to conduction (radiative heat transfer is usually neglected). Hence if H_t is the rate of heat transfer per unit area,

$$H_t = -K_d\left\langle\frac{\partial}{\partial z}T_{\mathrm{total}}\right\rangle_{z=0} \quad (71)$$

where the angular brackets $\langle --- \rangle$ correspond to a horizontal average with the definition of T_{total} defined earlier. Equation (71) can be written in the form

$$H_t = \frac{\Delta T}{d} - \frac{\Delta T}{d}\sum_{n=1}^{N} n\pi b_{\mathrm{on}}, \quad (72)$$

with the restriction $N = 2$. The second term on the right side of (72) represents that the heat which enters at the bottom by conduction is carried on to the top by both conduction and convection and hence the heat transfer increases above that given by conduction alone. This process can be explained physically by the relationship between the driving temperature difference ΔT and the heat transport. In dimensionless variables, this is the Rayleigh-Nusselt-number curve. Thus from (72), the Nusselt number is

$$Nu = \frac{H_t d}{K_d \Delta T} = 1 - \sum_{n=1}^{N} n\pi b_{on} = 1 - 2\pi C \tag{73}$$

with $N = 2$, where C is given by (60). For Rayleigh numbers below the critical value, the heat transport is purely by conduction for $A = 0$ and B, C, D and E are all zero, where B, C, D and E are given by (59) to (62). In that case (73) shows Nu to be unity.

6. Numerical experiments

The minimum values of Ra_c^m, Ra_c^0, Ra_f and their corresponding values of α^2 for given values of Q, P_l, S are calculated by small variation of α^2. Ra_c^0 is calculated with the condition (24) on Q. The results of these calculations are tabulated in table 1 for $P_l = 10^{-3}$ and $S = 4$ and 8. The typical behaviour is observed for other values of P_l.

To evaluate heat transport, we have to solve the cubic equation (63) for $(A^2/8)$. Here we solve the cubic equation (63) by using a system subroutine POLRT of DEC 1090 system and we calculate Nusselt number from (73). The results of heat transport are tabulated in table 2 and the results are discussed in §6.

7. Discussion

Magnetoconvection through a sparsely packed porous medium has been investigated in detail using linear stability analysis and finite amplitude analysis with limited representation. The following conclusions are drawn from the present investigation:

Table 1. Values of Ra_c^m, Ra_c^0, Ra_f and their corresponding values of α^2 for $P_l = 10^{-3}$

S	Q	Ra_c^m ($\times 10^5$)	α^2	Ra_c^0 ($\times 10^5$)	α^2	Ra_f ($\times 10^5$)	α^2
4	10^{-3}	0·402540200	0·980	—	—	0·37615010	1·140
	10^{-2}	0·402542000	0·980	—	—	0·37645490	1·140
	10^{-1}	0·402559990	0·980	—	—	0·37741950	1·140
	10^{0}	0·402739400	0·980	—	—	0·38047490	1·130
	10^{1}	0·404530400	0·985	—	—	0·39019010	1·115
	10^{2}	0·422233500	1·030	—	—	0·42145590	1·065
	10^{3}	0·584842300	1·380	0·55369300	1·070	0·52573740	0·935
	10^{4}	$0·186197800 \times 10^6$	3·175	0·70869840	1·415	0·67628390	0·705
8	10^{-3}	0·402540200	0·980	—	—	0·39634190	1·015
	10^{-2}	0·402542000	0·980	—	—	0·39649360	1·010
	10^{-1}	0·402560000	0·980	—	—	0·39697310	1·010
	10^{0}	0·402739400	0·980	—	—	0·39849160	1·010
	10^{1}	0·404530400	0·985	—	—	0·40330850	1·005
	10^{2}	0·422233500	1·030	—	—	0·41868180	0·985
	10^{3}	0·584842300	1·380	0·47814680	1·030	0·46870360	0·930
	10^{4}	$0·186197800 \times 10^6$	3·175	0·67628385	1·415	0·6406941	0·800

Table 2. The values of Nu and Ra for different values of Q and S for $P_l = 10^{-3}$.

S	$Q = 10^{-3}$		$Q = 10$		$Q = 10^3$		$Q = 10^4$	
	Nu	Ra	Nu	Ra	Nu	Ra	Nu	Ra
4	1·012145	$4{\cdot}050 \times 10^4$	1·002325	$4{\cdot}05 \times 10^4$	1·000261	$5{\cdot}850 \times 10^4$	1·001866	$1{\cdot}870 \times 10^5$
	1·385823	$5{\cdot}000 \times 10^4$	1·383947	$5{\cdot}00 \times 10^4$	1·047116	$6{\cdot}000 \times 10^4$	1·041712	$2{\cdot}010 \times 10^5$
	1·651178	$6{\cdot}000 \times 10^4$	1·754479	$6{\cdot}50 \times 10^4$	1·295633	$6{\cdot}025 \times 10^4$	2·193092	$2{\cdot}020 \times 10^5$
	1·841486	$7{\cdot}000 \times 10^4$	2·187899	$1{\cdot}00 \times 10^5$	1·626226	$6{\cdot}500 \times 10^4$	2·474843	$2{\cdot}150 \times 10^5$
	1·984704	$8{\cdot}000 \times 10^4$	2·673922	$2{\cdot}50 \times 10^5$	1·794893	$7{\cdot}000 \times 10^4$	2·588647	$2{\cdot}300 \times 10^5$
	2·672601	$2{\cdot}500 \times 10^5$	2·863926	$6{\cdot}00 \times 10^5$	2·158849	$9{\cdot}000 \times 10^4$	2·738430	$2{\cdot}750 \times 10^5$
	2·882796	$7{\cdot}000 \times 10^5$	2·897917	$8{\cdot}00 \times 10^5$	2.546314	$1{\cdot}500 \times 10^5$	2·958995	$1{\cdot}000 \times 10^6$
	2·917925	$1{\cdot}000 \times 10^6$	2·918320	$1{\cdot}00 \times 10^6$	2·939187	$1{\cdot}005 \times 10^6$	2·993943	$6{\cdot}000 \times 10^6$
8	1·126121	$4{\cdot}300 \times 10^4$	1·002379	$4{\cdot}05 \times 10^4$	1·308728	$5{\cdot}200 \times 10^4$	1·980130	$1{\cdot}138 \times 10^5$
	1·474303	$5{\cdot}300 \times 10^4$	1·384743	$5{\cdot}00 \times 10^4$	1·499350	$5{\cdot}500 \times 10^4$	2·101557	$1{\cdot}151 \times 10^5$
	2·020217	$8{\cdot}300 \times 10^4$	2·187705	$1{\cdot}00 \times 10^5$	2·022799	$7{\cdot}500 \times 10^4$	2·690284	$1{\cdot}800 \times 10^5$
	2·208917	$1{\cdot}030 \times 10^5$	2·836736	$5{\cdot}00 \times 10^5$	2·629199	$1{\cdot}750 \times 10^5$	2·809022	$2{\cdot}500 \times 10^5$
	2·767834	$3{\cdot}530 \times 10^5$	2·883341	$7{\cdot}00 \times 10^5$	2·864103	$4{\cdot}550 \times 10^5$	2·933862	$6{\cdot}000 \times 10^5$
	2·909110	$9{\cdot}030 \times 10^5$	2·909247	$9{\cdot}00 \times 10^5$	2·924147	$8{\cdot}050 \times 10^5$	2·962097	$1{\cdot}000 \times 10^6$
	2·919762	$1{\cdot}023 \times 10^6$	2·918317	$1{\cdot}00 \times 10^6$	2·939438	$1{\cdot}005 \times 10^6$	2·994878	$7{\cdot}000 \times 10^6$

From the linear analysis we find that oscillatory (overstable) motions occur for restricted range of S (>1 which is the ratio of thermal diffusivity to magnetic diffusivity) and Q for given value of the porous parameter P_l, because they can reduce the stabilizing effect of Lorentz force. For given P_l the effect of strong magnetic field is to suppress convection—direct (steady) or oscillatory (overstable) depending on the values of S and Q. From table 1, it is clear that convection sets in as a direct (or marginal) mode at the critical Rayleigh number Ra_c^m when $S < 1$ and it may first appear as an oscillatory mode at $\mathrm{Ra} = \mathrm{Ra}_c^0$ when $S > 1$ in the absence of porous media (see [2, 3] and [20]). In the presence of a porous medium, the same phenomena is also true. Also, the porous medium stabilizes the system to the maximum extent when the usual viscous dissipation is also present in addition to Darcy's resistance. The physical reasons for this is that, the energy released by buoyancy force acting on the fluid must balance the energy released by Darcy's resistance, Joule heating, and the usual viscous dissipation. This can be achieved only at higher temperature gradients.

In general we conclude that for smaller values of Q the motion should be steady, infinitesimal or finite, rather than overstable motion. This should be comparable to that which occurs for hydrodynamic convection through a porous medium (Lapwood [6]). We find that the generation of zonal current by the magnetic field is responsible for oscillatory motions which exist at Rayleigh number smaller than that of infinitesimal steady motions. This zonal current is also responsible for the existence of finite amplitude motions.

Therefore, finite amplitude steady convection is investigated using Veronis's [19] truncated representation. We find that for large values of Q finite amplitude steady convections are not possible because they occur only in a subcritical range of Ra as long as they can reduce the constraint of the magnetic field. This is because, greater amplitudes require more release of potential energy which in turn requires a larger value of Ra. It is also found that, no finite amplitude motions are possible when the

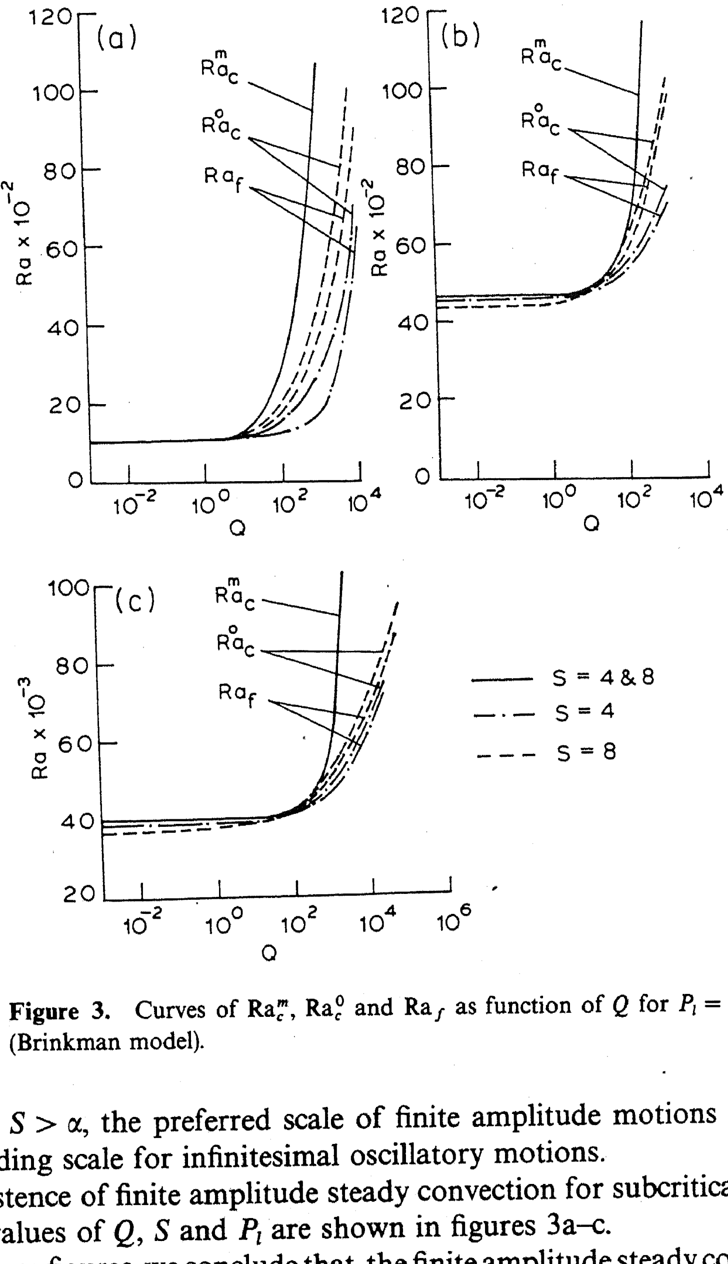

Figure 3. Curves of Ra_c^m, Ra_c^0 and Ra_f as function of Q for $P_l = 10^{-1}$, 10^{-2} and 10^{-3} (Brinkman model).

parameter $S > \alpha$, the preferred scale of finite amplitude motions is larger than the corresponding scale for infinitesimal oscillatory motions.

The existence of finite amplitude steady convection for subcritical values of Ra for different values of Q, S and P_l are shown in figures 3a–c.

From these figures, we conclude that, the finite amplitude steady convection exists for subcritical values of Ra for all $Q > 7$ for $P_l \leqslant 10^{-1}$. However for $P_l = 10^{-3}$ the finite amplitude steady convection exists for subcritical values of Ra for all Q and these values of Ra for finite amplitude steady convection are small in the case of $S = 4$ compared to that of $S = 8$.

The heat transport given by (73) is numerically evaluated for different values of Ra, Q, S and α^2 and the results are tabulated in table 2 and depicted in figures 4 to 6 and we found that:

(a) For a fixed value of Q, Nu increases with S (for $S = 4$ and 8) which means that a fluid having less magnetic diffusivity transports more heat for a fixed value of Q.

(b) As the value of P_l is decreased, instability sets in at a higher value of Rayleigh number as in the rotating single and two component systems (Rudraiah and Srimani [15]; Srimani [16]). We find that chances of subcritical instabilities occurring are less for smaller values of P_l ($< 10^{-3}$). This is because no fluid exhibits subcritical instability

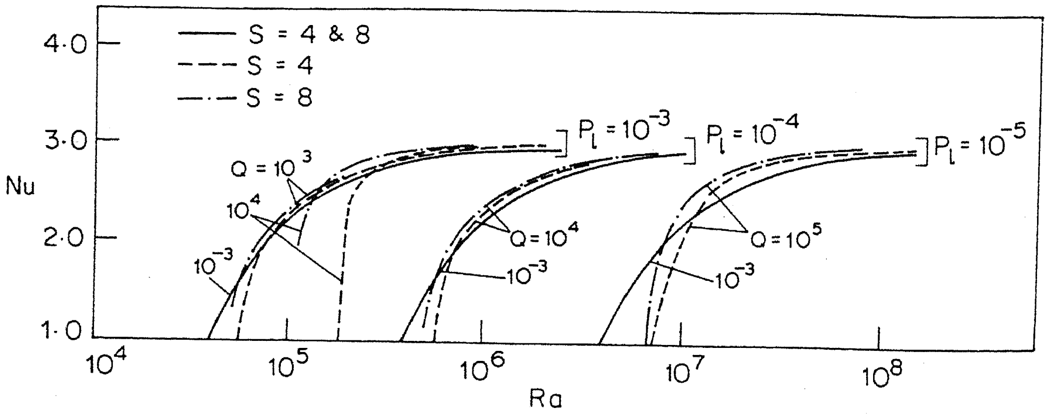

Figure 4. Curves of heat transport Nusselt number (Nu) against Rayleigh number (Ra) for (i) $P_l = 10^{-3}$ (ii) $P_l = 10^{-4}$ and (iii) $P_l = 10^{-5}$.

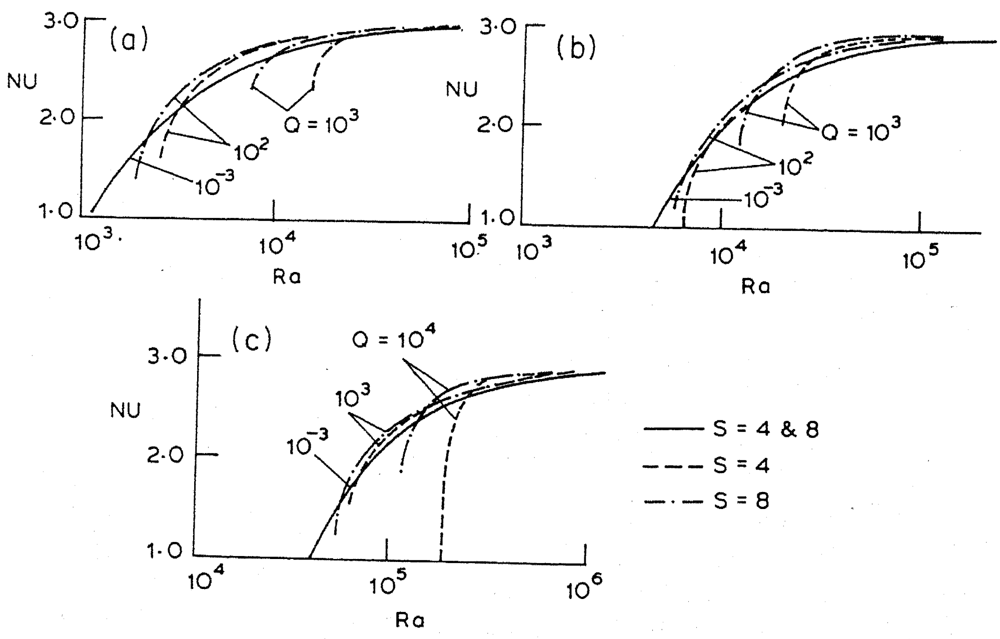

Figure 5. Curves of heat transport Nusselt number (Nu) against Rayleigh number (Ra) for Brinkman model (**a**) $P_l = 10^{-1}$ (**b**) $P_l = 10^{-2}$ and (**c**) $P_l = 10^{-3}$.

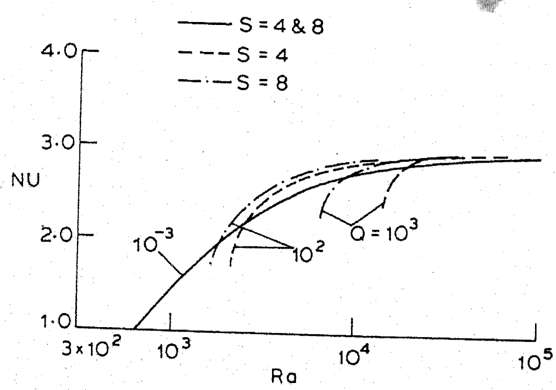

Figure 6. Curves of heat transport Nusselt number (Nu) against Rayleigh number (Ra) for viscous case $P_l = 10^3$.

for $P_l = 10^{-5}$ (see figure 4) in the range $10^{-3} \leqslant Q \leqslant 10^{-5}$. Whereas for $P_l = 10^{-4}$ and 10^{-3}, fluids having $S = 8$ exhibits subcritical instability but not for $S = 4$ (figures 4–6). This means that as P_l increases the system becomes more and more stable to finite amplitude perturbations as the magnetic diffusivity is decreased.

In this context we can note that some fluids ($2 \leqslant S \leqslant 8$) do exhibit subcritical instabilities for $P_l = 10^{-1}$ and 10^{-2} (figures 5a, b). Whereas, for $P_l = 10^{-3}$ instabilities exist only for $S = 8$ (figure 5c) which is similar to the values of $P_l > 10^{-3}$.

In the viscous case ($P_l \to \infty$) all the fluids considered exhibit subcritical instabilities as expected (Weiss [21, 22]). Further, for $Q \leqslant 10^{-3}$ we note that subcritical instabilities do not exist and there is no difference in the values of Nu for $S = 4$ and 8 (figures 4 to 6). We also note that Nu increases with P_l and the values of the heat transport in the case of a porous medium are less compared to the pure viscous case ($P_l \to \infty$) (see figure 6).

For values of $P_l = 10^{-1}$ and 10^{-2}, heat transport decreases with increasing Q for the fluid $S = 4$, whereas it increases with increasing Q for $S = 8$. But for the case $P_l = 10^{-3}$, heat transport decreases with increasing Q for $S = 4$, while it increases with increasing Q for $S = 8$ (figure 5).

Acknowledgement

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