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Constraints On Radiative Neutrino Mass Models From Oscillation Data

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Abstract

The three neutrino Zee model and its extension including three active and one sterile species are studied in the light of new neutrino oscillation data. We obtain analytical relations for the mixing angle in solar oscillations in terms of neutrino mass squared differences. For the four neutrino case, we obtain the result $\sin^2 2\theta_\odot \approx 1 - \left[(\Delta m^2_{Atm})^2/(4\ \Delta m^2_{LSND}\Delta m^2_\odot)\right]^2$, which can accommodate both the large and small mixing scenarios. We show that within this framework, while both the SMA-MSW and the LMA-MSW solutions can easily be accommodated, it would be difficult to reconcile the LOW-QVO solutions. We also comment on the active-sterile admixture within phenomenologically viable textures.

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I. INTRODUCTION

Recent results from SNO [1] and super-Kamiokande [2] experiments have confirmed the presence of a non-electron flavor in the measured solar ν_e flux on earth implying the existence of neutrino oscillations. A detailed analysis of the data favors the Large Mixing Angle (LMA) solution [3] within the Mikheev-Smirnov-Wolfenstien (MSW) framework, though other solutions are still not ruled out. A simplified two-flavor oscillation picture prefers [3] a neutrino mass squared difference $\Delta m^2_\odot \sim 4 \times 10^{-5}$ eV^2 with $\sin^2 2\theta_\odot \sim 0.66$ for the mixing angle $\theta_\odot$. On the other hand, results [4] from the super-K atmospheric neutrino experiments indicate a non-muonic component in the atmospheric ν_μ flux. Again, in terms of the difference of squared masses between two oscillating neutrinos, these imply $\Delta m^2_{Atm} \sim 3 \times 10^{-3}$ eV^2 with a near maximal mixing $\sin^2 2\theta_{Atm} \sim 1$. These two independent scales and mixing angles can be accommodated within the standard model of three neutrinos ν_e, ν_μ, ν_τ. However, in addition to these, another scale pertaining to neutrinos is indicated by $\bar{\nu_\mu} \leftrightarrow \bar{\nu_e}$ (and $\nu_\mu \leftrightarrow \nu_e$) flavor oscillations seen by the LSND experiment [5]. The parameters of a two-flavor oscillation that explain these results are $\Delta m^2_{LSND} \sim \mathcal{O}(1 \text{ eV}^2)$ and $\sin^2 2\theta_{LSND} \sim \mathcal{O}(10^{-3})$. A simultaneous explanation of all the three types of oscillations is not possible with only three types of neutrinos.

An additional light neutrino, which is not electroweak active and is hence called the sterile neutrino ν_s, is often proposed to understand all the above anomalies simultaneously. Within this four neutrino framework, only two distinct mass patterns (the **3 + 1** and the **2 + 2**) were originally allowed by the data. The former has three nearly degenerate active neutrinos, with mass differences equal to the solar and atmospheric oscillation scales, separated as a whole in mass from the sterile flavor by the large LSND mass difference. However, it is not favored when fitted [6] to the latest LSND results, together with data from reactor experiments like BUGEY; therefore, we discard it. In the **2 + 2** case, two doublets, each nearly degenerate with mass splits $\mathcal{O}(\sqrt{\Delta m^2_\odot})$ and $\mathcal{O}(\sqrt{\Delta m^2_{Atm}})$ and corresponding mixing angles $\theta_\odot$ and θ_{Atm} respectively, are separated by a large mass difference $\mathcal{O}(\sqrt{\Delta m^2_{LSND}})$. The mixing between the two doublets is controlled by the angle θ_{LSND}.

It is true that, strictly within two flavor oscillations, neither the SNO nor the super-K results support the occurrence of a sterile species in either solar or atmospheric neutrino oscillation. Nevertheless, this simple picture changes when one considers all four neutrinos

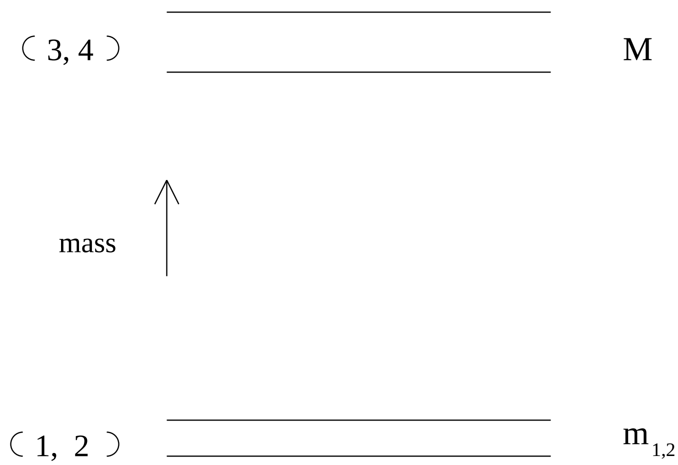

FIG. 1: The $2+2$ neutrino mass pattern. The heavier (lighter) pair is split by the mass scale of the atmospheric (solar) neutrino anomaly while the two pairs are separated by the LSND scale.

$\nu_e, \nu_\mu, \nu_\tau, \nu_s$. A comprehensive analysis within a four neutrino framework of all (including the latest) data on solar, atmospheric and LSND oscillations, also taking into account extant reactor and accelerator constraints, has recently been performed by Gonzalez-Garcia *et.al* [7]. To summarize their results, it is convenient to define the four species neutrino mixing matrix as

$$\mathsf{U} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} & U_{e4} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} & U_{\mu 4} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} & U_{\tau 4} \\ U_{s1} & U_{s2} & U_{s3} & U_{s4} \end{pmatrix}. \tag{1}$$

In eq.(1), the subscripts $(1,2)$ represent the lighter pair participating in solar neutrino oscillations, while $(3,4)$ refer to the heavier pair relevant to atmospheric neutrino oscillations. These pairs are separated in mass by the LSND mass scale. We present a schematic diagram of this scenario in Fig.1. For that situation, the BUGEY experiment [8] provides the maximum constraint on the ν_e content in the heavier $(3,4)$ pair namely

$$|U_{e3}|^2 + |U_{e4}|^2 \lesssim 10^{-2}. \tag{2}$$

In addition, the CCFR [9] and the CDHSW [10] experiments constrain the ν_μ content in the lighter $(1,2)$ pair by

$$|\mathsf{U}_{\mu 1}|^2 + |\mathsf{U}_{\mu 2}|^2 \lesssim 0.2, \tag{3}$$

again for an LSND scale mass separation between the pairs.

The rest of the constraints on U arrive from solar and atmospheric neutrino oscillation data. Eqs. (2) and (3), show that in the $\mathbf{2+2}$ mass pattern, ν_{e} is mostly confined to the $(1,2)$ pair whereas ν_μ largely resides in the $(3,4)$ pair. Three alternative situations are now possible [11]:

(i) Solar neutrino oscillations take place into the purely active neutrino ν_τ; atmospheric neutrinos oscillate into the purely sterile neutrino ν_{s} with maximal mixing.

(ii) Solar neutrino oscillations take place into the purely sterile neutrino ν_{s}; atmospheric neutrinos get converted to the purely active neutrino ν_τ with maximal mixing.

(iii) Both solar and atmospheric neutrinos oscillate into linear combinations of ν_{s} and ν_τ; the combination pertaining to atmospheric neutrino oscillations is maximally mixed with ν_μ.

As mentioned earlier, recent data from both the SNO [1] and the super-K [4] experiments disfavor two flavor neutrino oscillations into only a sterile species as an explanation for either the solar or the atmospheric anomaly. Consequently, configurations (i) and (ii) are severely constrained. In contrast, configuration (iii) can still be realized [7] even with new data from SNO [1]. The favorite $\mathbf{2+2}$ configuration, most favored by the data, is when the linear combination of ν_{s} and ν_τ in the state to which ν_{e} from the Sun oscillates, is not maximal but in the ratio $1:2$, *i.e,* $20:80$ in the probabilities. We shall define this 'active-sterile admixture' as the sterile content in the solar sector:

$$\mathsf{A} \equiv |\mathsf{U}_{\mathsf{s}1}|^2 + |\mathsf{U}_{\mathsf{s}2}|^2. \tag{4}$$

When the solar mixing angle is within the LMA region, corresponding to the best-fit solution, the active-sterile admixture is required to be [7]:

$$\mathsf{A} \approx 0.18 - 0.2. \tag{5}$$

This means that the 'atmospheric' pair $(3,4)$ has a dominant sterile content. To be precise, the sterile species participates in about 20% (80%) of the solar (atmospheric) neutrino oscillations. It should be emphasized that the SNO results have played a key role in these conclusions[1] reached by the authors of Ref.[7]. This configuration leads to what has been called [7] *close to active solar plus close to sterile atmospheric* (**CAS + CSA**) neutrino oscillations.

In addition to the above 'best-fit' solution, configuration (ii) with the solar neutrino oscillations occurring into purely sterile neutrinos is still allowed by the data. The active-sterile admixture A in this case is :

$$\mathsf{A} = |\mathsf{U}_{s1}|^2 + |\mathsf{U}_{s2}|^2 \approx 0.91 - 0.97. \tag{6}$$

However, here the solar neutrino mixing angle lies in the SMA region. For both of these fits, it is seen that

$$|\mathsf{U}_{\mu 1}|^2 + |\mathsf{U}_{\mu 2}|^2 \approx 0. \tag{7}$$

In this work, we focus on these mass patterns and study the viability of the 4×4 neutrino mass matrix within a radiative Zee-type model [14] extended to include the sterile neutrino [15]. An extension from the standard 3×3 mass matrix to the 4×4 mass matrix can be realized either by a conventional seesaw type of mechanism [16] with heavy right-handed states or in a radiative model [15]. The three neutrino Zee model has been quite popular in analyzing neutrino oscillation data (minus those of the LSND experiment) on account of its predictivity. However, this model has run into a serious problem with the data on the mixing angle pertinent to solar neutrino oscillations. It has been shown [17] that $\sin^2 2\theta_\odot$ in this model is forced to be close to unity [18] within $\mathcal{O}[(\Delta m^2_\odot/\Delta m^2_A)^2]$ in disagreement with the best-fit value [3] $\sin^2 2\theta_\odot \sim 0.66$. More specifically, if $\sin^2 2\theta_{Atm} \sim 1$ is used as an input, the Zee model is found to allow [17, 18] only solutions with bimaximal mixing [19]. This result of the Zee model is a natural consequence of the structure of its mass matrix - specifically, its vanishing diagonal elements, rather than the details of the model. It is thus

[1] The active (sterile) content in the solar neutrino sector can be decreased (increased) [12] to more than what is implied by eq.(5) if the $^8\mathsf{B}$ flux is suitably renormalized. Such a renormalization of up to 30% may be allowed [13] by the theoretical uncertainties in the calculation of the said flux. It has been shown recently [3], however, that the inclusion of the SK data on the day and night spectral energy distribution disfavor such a scenario at 3σ level.

essential to understand the consequences of such a mass matrix when extended to the four neutrino case. To be precise, one needs to compute the allowed values of the mixing angle relevant to the solar sector the four neutrino radiative model.

Within the standard three neutrino set up, various extensions of the Zee model have been presented in literature to evade the compulsion of $\sin^2 2\theta_\odot \approx 1$. Most of these include additional couplings and/or fields [20] and then they can avoid bimaximal mixing and incorporate the LMA solution. Here we consider the extension of the Zee model by an additional sterile species [15]. This would enable us to include solutions corresponding to the LSND anomaly. Using the property of vanishing diagonal elements, we are able to make a prediction on the solar neutrino mixing angle involving all the three squared mass differences, which is compatible with the present data. We show that the model is suitable for explaining all the three neutrino anomalies. The precise value of $\theta_\odot$ depends on the ratio between the square of the atmospheric neutrino mass squared difference and the product of the solar and LSND mass squared differences. Present experimental errors on these measured mass squared differences allow the solar neutrino oscillations to take place with MSW conversion either with small or large mixing. However, we find it difficult to reconcile the LOW-QVO solutions with this scenario, though they might not be completely ruled out.

The rest of the paper is organized as follows. We show in section **II** how the near-maximality constraint on $\theta_\odot$ arises from the vanishing diagonal elements of the 3×3 Zee mass matrix, establishing a method that can be extended to the 4×4 case. This extension is done in section **III** where we derive the result for $\sin^2 2\theta_\odot$ mentioned in the abstract. In section **IV**, we comment on textures of the mass matrix within this model which are phenomenologically viable. We then show that the active-sterile admixture in solar neutrinos and the LSND mass scale within these models have a common origin within the mass matrix. The final section, section V summarizes our conclusions.

II. RESTRICTION ON SOLAR NEUTRINO MIXING FROM THE THREE NEUTRINO ZEE MASS MATRIX

As already mentioned, the three neutrino Zee model has a serious problem on account of the current neutrino oscillation data, specifically with the data from the solar sector. The reason for this can be seen, without going in to the details of the model, from one feature of it - namely, its vanishing diagonal elements. The Zee mass matrix in three generations is given, in the $\nu_{\mathsf{e}}, \nu_\mu, \nu_\tau$ flavor basis, by

$$\mathcal{M}_\nu^{(3)} = \begin{pmatrix} 0 & \mathsf{a} & \mathsf{b} \\ \mathsf{a} & 0 & \mathsf{c} \\ \mathsf{b} & \mathsf{c} & 0 \end{pmatrix}. \tag{8}$$

The elements $\mathsf{a},\ \mathsf{b},\ \mathsf{c}$ are three[2] real parameters determining two mass squared differences and three mixing angles relevant to solar and atmospheric neutrino oscillation phenomenology. Evidently, this leads to a constrained pattern of neutrino masses and mixing which has to face the challenge of the emerging data. Three relations emerge from the vanishing diagonal terms[3] of $\mathcal{M}_\nu^{(3)}$ if we write $\mathcal{M}_\nu^{(3)} = \mathsf{O_{MNS}}\ \mathsf{diag}\{\mathsf{m}_{\nu_1}, \mathsf{m}_{\nu_2}, \mathsf{m}_{\nu_3}\}\ \mathsf{O}^{\mathsf{T}}_{\mathsf{MNS}}$:

$$\begin{aligned} \mathsf{m}_{\nu_1}\mathsf{O}^2_{\mathsf{e}1} + \mathsf{m}_{\nu_2}\mathsf{O}^2_{\mathsf{e}2} + \mathsf{m}_{\nu_3}\mathsf{O}^2_{\mathsf{e}3} &= 0, && (9) \\ \mathsf{m}_{\nu_1}\mathsf{O}^2_{\mu1} + \mathsf{m}_{\nu_2}\mathsf{O}^2_{\mu2} + \mathsf{m}_{\nu_3}\mathsf{O}^2_{\mu3} &= 0, && (10) \\ \mathsf{m}_{\nu_1}\mathsf{O}^2_{\tau1} + \mathsf{m}_{\nu_2}\mathsf{O}^2_{\tau2} + \mathsf{m}_{\nu_3}\mathsf{O}^2_{\tau3} &= 0. && (11) \end{aligned}$$

Because of the orthogonality of $\mathsf{O_{MNS}}$, eqs. (9), (10), (11) lead to [17]:

$$\begin{aligned} \mathsf{m}_{\nu_1} + \mathsf{m}_{\nu_2} + \mathsf{m}_{\nu_3} &= 0, && (12) \\ \left(\mathsf{O}^2_{\mathsf{e}1} - \mathsf{O}^2_{\mathsf{e}3}\right)\ \left(\mathsf{O}^2_{\mu2} - \mathsf{O}^2_{\mu3}\right) &= \left(\mathsf{O}^2_{\mu1} - \mathsf{O}^2_{\mu3}\right)\left(\mathsf{O}^2_{\mathsf{e}2} - \mathsf{O}^2_{\mathsf{e}3}\right), && (13) \\ \frac{\mathsf{O}^2_{\mathsf{e}2} - \mathsf{O}^2_{\mathsf{e}3}}{\mathsf{O}^2_{\mathsf{e}1} - \mathsf{O}^2_{\mathsf{e}3}} &= -\frac{\mathsf{m}_{\nu_1}}{\mathsf{m}_{\nu_2}}. && (14) \end{aligned}$$

Eq.(12) is a trivial consequence of the tracelessness of $\mathcal{M}_\nu^{(3)}$ while eq.(13) implies that one of the mixing angles is not independent. If $\Delta\mathsf{m}^2_{\mathsf{ij}} \equiv\ |\mathsf{m}^2_{\nu_\mathsf{i}} - \mathsf{m}^2_{\nu_\mathsf{j}}|$ and is positive by definition, let

[2] For three neutrinos, the Zee mass matrix has only three parameters. Thus all phases can be rotated away from the mixing matrix, by absorbing them in the three neutrino fields. Hence one can choose the orthogonal matrix $\mathsf{O_{MNS}}$ for mixing instead of the unitary Maki-Nakagawa-Sakata matrix $\mathsf{U_{MNS}}$.

[3] Two loop corrections can induce very small nonzero diagonal entries [21] in the Zee mass matrix of eq. (8). Strictly speaking, they would change eqs. (12), (13 and (14). However estimates [21] of these tiny changes suggest that they would not significantly affect the statement of eq.(21) on the solar neutrino mixing angle $\theta_\odot$ [18].

us choose $\Delta\mathsf{m}^2_{32}$ ($=\ \Delta\mathsf{m}^2_{\mathsf{Atm}}$) and $\Delta\mathsf{m}^2_{21}$ ($=\ \Delta\mathsf{m}^2_{\odot}$) as the two independent mass squared differences. We know from condition (12) that one the eigenvalues of $\mathcal{M}_\nu^{(3)}$ is not independent. The latter is chosen to be m_{ν_3} and can be eliminated. Thus one can now write

$$|\mathsf{m}^2_{\nu_1} + 2\mathsf{m}_{\nu_1}\mathsf{m}_{\nu_2}| \ = \ \Delta\mathsf{m}^2_{\mathsf{Atm}} \ , \tag{15}$$

$$|\mathsf{m}^2_{\nu_2} - \mathsf{m}^2_{\nu_1}| \ = \ \Delta\mathsf{m}^2_{\odot} \ . \tag{16}$$

We emphasize here that the quantities appearing in the RHS of eqs. (15) and (16), namely $\Delta\mathsf{m}^2_{\odot},\ \Delta\mathsf{m}^2_{\mathsf{Atm}}$ are positive. These equations yield m_{ν_1} and m_{ν_2} in terms of $\Delta\mathsf{m}^2_{\mathsf{Atm}}$ and $\Delta\mathsf{m}^2_{\odot}$. Let us define

$$\alpha \equiv \frac{\Delta\mathsf{m}^2_{\odot}}{\Delta\mathsf{m}^2_{\mathsf{Atm}}} \ . \tag{17}$$

In order to determine the solar mixing angle, we can take $\theta_{23} = \theta_{\mathsf{Atm}}$, *i.e* the atmospheric mixing angle, to be exactly maximal. Furthermore, on account of the CHOOZ [22] experimental result, we can put $\mathsf{O}_{\mathsf{e3}} = \epsilon \lesssim 0.1$, keeping only $\mathcal{O}(\epsilon)$ terms in $\mathsf{O}_{\mathsf{MNS}}$. Choosing the mixing angle $\theta_{12} = \theta_\odot$ for solar neutrino oscillations, $\mathsf{O}_{\mathsf{MNS}}$ now gets reduced to

$$\mathsf{O}_{\mathsf{MNS}} = \begin{pmatrix} \cos\theta_\odot & \sin\theta_\odot & \epsilon \\ (-\epsilon\cos\theta_\odot - \sin\theta_\odot)/\sqrt{2} & (-\epsilon\sin\theta_\odot + \cos\theta_\odot)/\sqrt{2} & 1/\sqrt{2} \\ (-\epsilon\cos\theta_\odot + \sin\theta_\odot)/\sqrt{2} & (-\epsilon\sin\theta_\odot - \cos\theta_\odot)/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} + \mathcal{O}(\epsilon^2) \ . \tag{18}$$

The substitution of eq.(18) into eq.(14) leads after some algebra to

$$\sin^2 2\theta_\odot = -\frac{4\ \mathsf{m}_{\nu_1}/\mathsf{m}_{\nu_2}}{(1-\mathsf{m}_{\nu_1}/\mathsf{m}_{\nu_2})^2} + \mathcal{O}\left(\epsilon^2(1+\mathsf{m}_{\nu_1}/\mathsf{m}_{\nu_2})\right) \ . \tag{19}$$

The physical solutions[4] of $\mathsf{m}_{\nu_1},\ \mathsf{m}_{\nu_2}$ of eqs. (15) and (16) yield

$$-\frac{\mathsf{m}_{\nu_1}}{\mathsf{m}_{\nu_2}} = \frac{1}{\alpha + \sqrt{1-\alpha+\alpha^2}} \ . \tag{20}$$

[4] We choose only those solutions which yield positive values for $\mathsf{m}^2_{\nu_1},\ \mathsf{m}^2_{\nu_2}$ and $\sin^2 2\theta_\odot$. There are four sets with $\mathsf{m}_{\nu_1} = \pm(\Delta\mathsf{m}^2_{\mathsf{Atm}}/3)^{\frac{1}{2}}\ (1+2\alpha+2\sqrt{1+\alpha+\alpha^2})$ and $\mathsf{m}_{\nu_2} = -(-\alpha+\sqrt{1+\alpha+\alpha^2})\ \mathsf{m}_{\nu_1}$, $\mathsf{m}_{\nu_1} = \pm(\Delta\mathsf{m}^2_{\mathsf{Atm}}/3)^{\frac{1}{2}}\ (1-2\alpha+2\sqrt{1-\alpha+\alpha^2})$ and $\mathsf{m}_{\nu_2} = -(\alpha+\sqrt{1-\alpha+\alpha^2})\ \mathsf{m}_{\nu_1}$, expressed as functions of α while m_{ν_3} is always $-\mathsf{m}_{\nu_1} - \mathsf{m}_{\nu_2}$. The solutions are physical only for $|\mathsf{m}_{\nu_1}| \sim |\mathsf{m}_{\nu_2}| \gg |\mathsf{m}_{\nu_3}|$ so that $\sqrt{\Delta\mathsf{m}^2_{\mathsf{Atm}}} \sim |\mathsf{m}_{\nu_{1,2}} - \mathsf{m}_{\nu_3}|$ and $\sqrt{\Delta\mathsf{m}^2_{\odot}} \sim |\mathsf{m}_{\nu_1} - \mathsf{m}_{\nu_2}|$. It is not difficult to see that this implies $\mathsf{a} \sim \mathsf{b} \approx \mathcal{O}(\sqrt{\Delta\mathsf{m}^2_{\mathsf{Atm}}})$ and $\mathsf{c} \approx \mathcal{O}(\Delta\mathsf{m}^2_{\odot}/\sqrt{\Delta\mathsf{m}^2_{\mathsf{Atm}}})$.

Eqs. (19) and (20) lead to the following constraint on the solar mixing angle[5]

$$\sin^2 2\theta_\odot = 1 - \frac{1}{16}\alpha^2 + \ \mathcal{O}\left(\epsilon^2\alpha\right), \tag{21}$$

matching with the result of Ref. [18]. Since [3, 4] $\alpha \equiv (\Delta m^2_\odot/\Delta m^2_{Atm}) \lesssim 6\times 10^{-1}$ and $\epsilon \lesssim 10^{-1}$, we see that $\sin^2 2\theta_\odot$ is forced to be very close to unity ($i.e$ $|m_{\nu_1}| \approx |m_{\nu_2}|$) - a situation disfavored [3] by the data[6], though not completely ruled out. A point to note is that eq.(21) follows from eq.(18) by use of only the elements in the first row of O_{MNS}. This derivation does not use the fact that $\theta_{23} \equiv \theta_{Atm}$ is $\approx \pi/4$. It is, of course, true that the extra input of the latter immediately forces near bimaximal mixing in the Zee model, as mentioned in the Introduction. Another point to keep in mind is that the texture $|c| \ll |a| \sim |b|$, forced by the neutrino mass solutions given in footnote 4, implies an approximate global $L_e - L_\mu - L_\tau$ symmetry [17].

III. SOLAR NEUTRINO MIXING IN THE RADIATIVE MASS MODEL WITH FOUR NEUTRINOS

The Majorana mass matrix with three active and one sterile species can be generally represented in the $\{\nu_e,\ \nu_\mu,\ \nu_\tau,\ \nu_s\}$ flavor basis as

$$\mathcal{M}_\nu^{(4)} = \begin{pmatrix} 0 & a & b & d \\ a & 0 & c & e \\ b & c & 0 & f \\ d & e & f & 0 \end{pmatrix}. \tag{22}$$

The elements $a,\ b,\ c,\ d,\ e,\ f$ appearing in eq.(22) get generated radiatively at one-loop and are given by

$$a \ = \ f_{e\mu}(m_\mu^2 - m_e^2)\left(\frac{\mu v_2}{v_1}\right) F(m^2_{\chi 1}, m^2_{\phi_1}), \tag{23}$$

[5] We should also mention that eq.(21) was initially derived in Ref.[23] within a seesaw model which has a mass texture different from that of the Zee model. We thank K. R. Balaji for pointing this out.

[6] Some detailed numerical analyses of the Zee model, confronting the present data, have argued that the possibility $\sin^2 2\theta_\odot \approx 1$ is not completely ruled out [24, 25], if one insists on 99% C.L. limits.

$$
\begin{aligned}
\mathsf{b} &= \mathsf{f}_{\mathsf{e}\tau}(\mathsf{m}_\tau^2 - \mathsf{m}_\mathsf{e}^2)\left(\frac{\mu \mathsf{v}_2}{\mathsf{v}_1}\right)\mathsf{F}(\mathsf{m}_{\chi_1}^2, \mathsf{m}_{\phi_1}^2), && (24)\\
\mathsf{c} &= \mathsf{f}_{\mu\tau}(\mathsf{m}_\tau^2 - \mathsf{m}_\mu^2)\left(\frac{\mu \mathsf{v}_2}{\mathsf{v}_1}\right)\mathsf{F}(\mathsf{m}_{\chi_1}^2, \mathsf{m}_{\phi_1}^2), && (25)\\
\mathsf{d} &= \left(\mathsf{f}_{\mathsf{e}\tau}\mathsf{f}'_\tau \mathsf{m}_\tau + \mathsf{f}_{\mathsf{e}\mu}\mathsf{f}'_\mu \mathsf{m}_\mu\right)\mu' \mathsf{u}\ \mathsf{F}(\mathsf{m}_{\chi_1}^2, \mathsf{m}_{\chi_2}^2), && (26)\\
\mathsf{e} &= \left(\mathsf{f}_{\mu\tau}\mathsf{f}'_\tau \mathsf{m}_\tau + \mathsf{f}_{\mu\mathsf{e}}\mathsf{f}'_\mathsf{e} \mathsf{m}_\mathsf{e}\right)\mu' \mathsf{u}\ \mathsf{F}(\mathsf{m}_{\chi_1}^2, \mathsf{m}_{\chi_2}^2), && (27)\\
\mathsf{f} &= \left(\mathsf{f}_{\tau\mu}\mathsf{f}'_\mu \mathsf{m}_\mu + \mathsf{f}_{\tau\mathsf{e}}\mathsf{f}'_\mathsf{e} \mathsf{m}_\mathsf{e}\right)\mu' \mathsf{u}\ \mathsf{F}(\mathsf{m}_{\chi_1}^2, \mathsf{m}_{\chi_2}^2), && (28)
\end{aligned}
$$

where $\mathsf{f},\ \mathsf{f}'$ ($\mu,\ \mu'$) are dimensionless (dimensional) couplings in the model, $\mathsf{v}_{1,2}$ are two Higgs VEVs and $\mathsf{m}_{\chi_{1,2}}, \mathsf{m}_{\phi_1}$ are the three Higgs scalar masses, as explained in Ref.[15]. Moreover, the function $\mathsf{F}(\mathsf{m}_1, \mathsf{m}_2)$ is defined as

$$
\mathsf{F}(\mathsf{m}_1, \mathsf{m}_2) = \frac{1}{16\pi^2}\frac{1}{\mathsf{m}_1^2 - \mathsf{m}_2^2}\ln\left(\frac{\mathsf{m}_1^2}{\mathsf{m}_2^2}\right). \tag{29}
$$

For simplicity, we assume the matrix elements of eq. (22) to be real and ignore the presence of any CP-violation in the neutrino sector.

In the same manner, as shown in the previous section, one can arrive[7] at the following relations by equating $\mathcal{M}_\nu^{(4)}$ with $\mathsf{U}\ \mathsf{diag}\{\mathsf{m}_{\nu_1},\ \mathsf{m}_{\nu_2},\ \mathsf{m}_{\nu_3},\ \mathsf{m}_{\nu_4}\}\ \mathsf{U}^\mathsf{T}$:

$$
\begin{aligned}
\mathsf{m}_{\nu_1}\mathsf{U}_{\mathsf{e}1}^2 + \mathsf{m}_{\nu_2}\mathsf{U}_{\mathsf{e}2}^2 + \mathsf{m}_{\nu_3}\mathsf{U}_{\mathsf{e}3}^2 + \mathsf{m}_{\nu_4}\mathsf{U}_{\mathsf{e}4}^2 &= 0, && (30)\\
\mathsf{m}_{\nu_1}\mathsf{U}_{\mu 1}^2 + \mathsf{m}_{\nu_2}\mathsf{U}_{\mu 2}^2 + \mathsf{m}_{\nu_3}\mathsf{U}_{\mu 3}^2 + \mathsf{m}_{\nu_4}\mathsf{U}_{\mu 4}^2 &= 0, && (31)\\
\mathsf{m}_{\nu_1}\mathsf{U}_{\tau 1}^2 + \mathsf{m}_{\nu_2}\mathsf{U}_{\tau 2}^2 + \mathsf{m}_{\nu_3}\mathsf{U}_{\tau 3}^2 + \mathsf{m}_{\nu 4}\mathsf{U}_{\tau 4}^2 &= 0, && (32)\\
\mathsf{m}_{\nu_1}\mathsf{U}_{\mathsf{s}1}^2 + \mathsf{m}_{\nu_2}\mathsf{U}_{\mathsf{s}2}^2 + \mathsf{m}_{\nu_3}\mathsf{U}_{\mathsf{s}3}^2 + \mathsf{m}_{\nu 4}\mathsf{U}_{\mathsf{s}4}^2 &= 0, && (33)
\end{aligned}
$$

where U_{ij} are the elements of the matrix U defined in eq.(1). Since we ignore the phases present in the elements of U, we can utilize the unitarity of U to obtain the following relations from eq. (30) - (33).

$$
\mathsf{m}_{\nu_1} + \mathsf{m}_{\nu_2} + \mathsf{m}_{\nu_3} + \mathsf{m}_{\nu_4} = 0, \tag{34}
$$

[7] As in the three neutrino radiative Zee model, we expect that the effect of the two loop corrections to the four neutrino radiative model mass matrix would be small and thus of not much consequence to the results presented hereafter.

$$m_{\nu_2} = -\frac{(U_{e1}^2 - U_{e4}^2)\,(U_{\mu 3}^2 - U_{\mu 4}^2) - (U_{e3}^2 - U_{e4}^2)\,(U_{\mu 1}^2 - U_{\mu 4}^2)}{(U_{e2}^2 - U_{e4}^2)\,(U_{\mu 3}^2 - U_{\mu 4}^2) - (U_{e3}^2 - U_{e4}^2)\,(U_{\mu 2}^2 - U_{\mu 4}^2)}\, m_{\nu_1}, \tag{35}$$

$$m_{\nu_3} = -m_{\nu_1}\,\frac{U_{e1}^2 - U_{e4}^2}{U_{e3}^2 - U_{e4}^2} - m_{\nu_2}\,\frac{U_{e2}^2 - U_{e4}^2}{U_{e3}^2 - U_{e4}^2}, \tag{36}$$

$$\begin{aligned}&\frac{(U_{e2}^2 - U_{e4}^2)(U_{\mu 3}^2 - U_{\mu 4}^2) - (U_{e3}^2 - U_{e4}^2)\,(U_{\mu 2}^2 - U_{\mu 4}^2)}{(U_{e1}^2 - U_{e4}^2)\,(U_{\mu 3}^2 - U_{\mu 4}^2) - (U_{e3}^2 - U_{e4}^2)(U_{\mu 1}^2 - U_{\mu 4}^2)}\\ = &\frac{(U_{\tau 2}^2 - U_{\tau 4}^2)(U_{e3}^2 - U_{e4}^2) - (U_{e2}^2 - U_{e4}^2)(U_{\tau 3}^2 - U_{\tau 4}^2)}{(U_{\tau 1}^2 - U_{\tau 4}^2)(U_{e3}^2 - U_{e4}^2) - (U_{e1}^2 - U_{e4}^2)(U_{\tau 3}^2 - U_{\tau 4}^2)}.\end{aligned} \tag{37}$$

These are the four neutrino versions of eqs. (12), (13) and (14).

Eq. (34) shows us that the $2+2$ mass pattern depicted in Fig.1 can be easily made compatible with this model if $|m_{\nu_1}| \sim |m_{\nu_2}| \ll |m_{\nu_3}| \sim |m_{\nu_4}|$ or if $|m_{\nu_1}| \ll |m_{\nu_2}| \ll |m_{\nu_3}| \sim |m_{\nu_4}|$. In either case, there are three independent mass squared differences which we choose to be $\Delta m_{21}^2 \equiv \Delta m_{\odot}^2$, $\Delta m_{43}^2 \equiv \Delta m_{\mathrm{Atm}}^2$ and $\Delta m_{32}^2 \equiv \Delta m_{\mathrm{LSND}}^2$. We can now perform an extended version of the same eigenvalue analysis as done in Section **II** for the three neutrino case. The three independent eigenvalues are chosen here to be $m_{\nu_1}, m_{\nu_2}, m_{\nu_3}$. Eqs. (15) and (16) now extend to

$$|(m_{\nu_1} + m_{\nu_2})^2 + 2m_{\nu_3}(m_{\nu_1} + m_{\nu_2})| = \Delta m_{\mathrm{Atm}}^2, \tag{38}$$

$$|m_{\nu_3}^2 - m_{\nu_1}^2| = \Delta m_{\mathrm{LSND}}^2, \tag{39}$$

$$|m_{\nu_2}^2 - m_{\nu_1}^2| = \Delta m_{\odot}^2, \tag{40}$$

where the RHS quantities of the above equations, namely $\Delta m_{\mathrm{LSND}}^2$, $\Delta m_{\odot}^2$, $\Delta m_{\mathrm{Atm}}^2$, are positive as in eqs. (15) and (16). The physical mass eigenvalue solutions to the above three equations form four sets as detailed in the Appendix. From these we see that

$$\begin{aligned}-\frac{m_{\nu 1}}{m_{\nu_2}} &= -\frac{N_1^{\pm}}{N_2^{\pm}}\\ &\approx \frac{4\,\Delta m_{\odot}^2\,\Delta m_{\mathrm{LSND}}^2 \mp (\Delta m_{\mathrm{Atm}}^2)^2}{4\,\Delta m_{\mathrm{LSND}}^2\,\Delta m_{\odot}^2 \pm (\Delta m_{\mathrm{Atm}}^2)^2} + \mathcal{O}(\alpha),\end{aligned} \tag{41}$$

where $N_{1,2}^{\pm}$ are defined in the Appendix and the upper (lower) sign corresponds to sets 1, 2 (3, 4).

In order to arrive at the prediction on the solar neutrino mixing angle in the four neutrino scenario, we can choose[8] a parameterization of the 4×4 generalization of U_{MNS} matrix as [26]

$$U = \begin{pmatrix} \cos\theta_\odot & \sin\theta_\odot & \epsilon & \epsilon' \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} & U_{\mu 4} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} & U_{\tau 4} \\ U_{s1} & U_{s2} & U_{s3} & U_{s4} \end{pmatrix} + \mathcal{O}(\epsilon^2, \epsilon'^{\,2}, \epsilon\epsilon'). \tag{42}$$

In eq.(42), the solar neutrino oscillations have been chosen to be between the physical states $|\nu_1>$ and $|\nu_2>$ where the BUGEY constraint eq.(2) has been taken care of by choosing $U_{e3} = \epsilon$ and $U_{e4} = \epsilon'$, both[9] being $\lesssim \mathcal{O}(10^{-2})$. We keep only $\mathcal{O}(\epsilon\ ,\epsilon')$ terms in the above mixing matrix. The rest of the parameters, while otherwise arbitrary, will be required to satisfy unitarity and other experimental constraints later.

One can now determine the mixing angle in the solar sector in much the same manner as in the three neutrino case. Eq.(19) is still valid to $\mathcal{O}(\epsilon^2, \epsilon'^{\,2})$ and can be rewritten as[10]

$$\sin^2 2\theta_\odot = \frac{-4 m_{\nu_1}/m_{\nu_2}}{(1 - m_{\nu_1}/m_{\nu_2})^2} + \mathcal{O}(\epsilon^2,\ \epsilon'^{\,2}) \tag{43}$$

Eqs. (41) and (43) together yield:

$$\sin^2 2\theta_\odot \approx 1 - \rho^2 + \mathcal{O}\Big(\alpha,\ \epsilon^2,\ \epsilon'^{\,2}\Big), \tag{44}$$

where ρ is defined as

$$\rho \equiv \frac{(\Delta m^2_{Atm})^2}{4\ \Delta m^2_\odot\ \Delta m^2_{LSND}}. \tag{45}$$

Eq.(44) leads us to the following conclusions.

- Because of eq.(44), we have the condition

$$\rho \equiv \frac{(\Delta m^2_{Atm})^2}{4\ \Delta m^2_\odot\ \Delta m^2_{LSND}} \lesssim 1, \tag{46}$$

---

[8] The matrix U has six independent angles and two independent phases in general. But one has the freedom to choose a parameterization in which the phases do not appear in U_{e1} and U_{e2}. In any event, we are ignoring the phases.

[9] It should be noted that in the analysis presented in [7] both U_{e3} and U_{e4} are taken to be zero.

[10] Since $m_{\nu_1}/m_{\nu_2} = \tan^2\theta_\odot + \mathcal{O}(\epsilon^2,\ \epsilon'^{\,2})$, it follows that the RHS of eq.(41) must be real and positive modulo $\epsilon^2,\ \epsilon'^{\,2}$ terms.

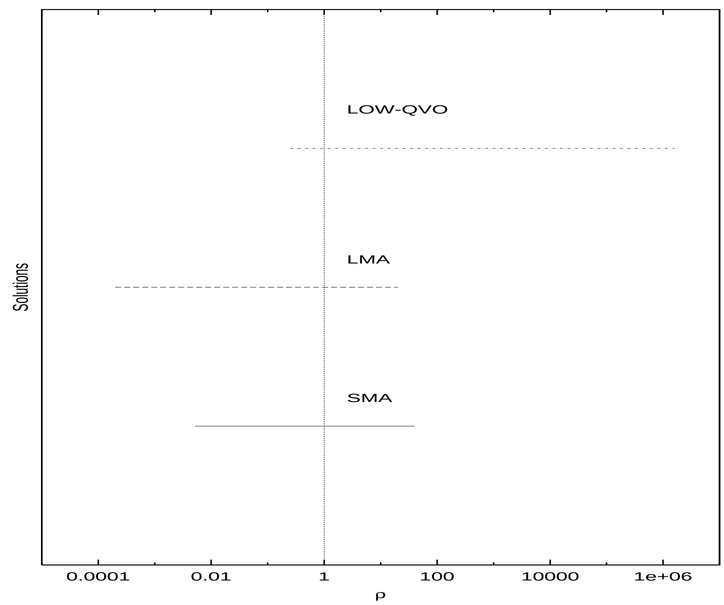

FIG. 2: The range of $\rho \equiv \frac{(\Delta m^2_{Atm})^2}{4\ \Delta m^2_{\odot}\ \Delta m^2_{LSND}}$, shown on a logarithmic scale, for different solar neutrino solutions.

upto $\alpha, \epsilon^2, \epsilon^{'\ 2}$ terms. In Fig.2 we plot the ranges of ρ for various proposed solutions of solar neutrino oscillations. We have chosen $\Delta m^2_{LSND} \approx (0.2-2)\text{eV}^2$ [5], $\Delta m^2_{Atm} \approx (1-8)\times 10^{-3}\text{eV}^2$ [4] and, for the various solar neutrino solutions, $\Delta m^2_{\odot}(\text{SMA}) \approx (2\times 10^{-6} - 2\times 10^{-5})$ eV^2, $\Delta m^2_{\odot}(\text{LMA}) \approx (4\times 10^{-6} - 6\times 10^{-4})$ eV^2 $\Delta m^2_{\odot}$ $(\text{LOW}-\text{QVO}) \approx (5\times 10^{-7} - 5\times 10^{-11})$ eV^2 [3]. From the figure, we see that eq.(46) is clearly valid for large domains of SMA and LMA solutions. However, in the case of LOW-QVO solutions, only a small domain obeys eq.(46) making it difficult to be accommodated within the 4×4 radiative [15] model of neutrino masses. A detailed numerical analysis of all oscillation data within the four neutrino model would be able to show the confidence level at which the LOW-QVO solution is valid given the mass matrix of eq. (22).

- As in section **III**, we have expressed the solar neutrino mixing angle, $\theta_{\odot}$ in terms of ratios of mass squared differences, though the form is different. However, *unlike in the three generation case, the solar neutrino mixing is not forced to be very nearly maximal.* Instead, its deviation from maximality is controlled[11] by the product of

[11] The special case of eq.(46) for $\theta_{\odot} \approx 0$, namely $2\sqrt{\Delta m^2_{LSND}\Delta m^2_{\odot}} \approx \Delta m^2_{Atm}$ was discovered in Ref. [15].

the two factors in the RHS of (44) and specifically by the LSND mass scale. If we take $\Delta m^2_{LSND} \to \infty$, we are back to the three neutrino result of eq.(21). Moreover, the range of the ratio $(\Delta m^2_{Atm})^2/4\Delta m^2_{\odot}\Delta m^2_{LSND}$ with present experimental errors is such (**Fig 2**) that both the small (SMA) and large mixing (LMA) MSW solutions are equally allowed. This is quite unlike in the three generation case.

- The result, stated in eq.(44), has been derived assuming the absence of any phases in $\mathcal{M}_\nu^{(4)}$ and hence in U. The presence of sizeable phases (*i.e* of a significant amount of CP-violation in the neutrino sector) can change the result.

We infer that, in so far as the prediction for the solar neutrino mixing angle is concerned, the radiative model [15] with four neutrinos is on a significantly different footing as compared to the corresponding three generation version. Indeed, the former is better able to tackle the present data, especially from SNO. We discuss in the next section some phenomenologically allowed textures of $\mathcal{M}_\nu^{(4)}$ from the current data.

IV. TEXTURES OF THE 4×4 RADIATIVE MODEL MASS MATRIX

Our derivation of the phenomenological relation eq. (46) for the mixing angle, $\theta_\odot$ in Sec. **III** made use of two inputs : (1) the $2 + 2$ mass pattern implemented on the neutrino mass matrix of eq. (22) with vanishing diagonal coefficients and (2) the form of eq. (42) for the 4×4 unitary mixing matrix U. In this section, we estimate the orders of magnitude of the nondiagonal entries of $\mathcal{M}_\nu^{(4)}$. We do so by relating them to the mass scales $(\Delta m^2_\odot)^{1/2}$, $(\Delta m^2_{Atm})^{1/2}$ and $(\Delta m^2_{LSND})^{1/2}$ via the elements of U and then constraining the latter from phenomenological inputs. We also favor or disfavor certain mass textures of $\mathcal{M}_\nu^{(4)}$.

Let us parameterize the four real neutrino mass eigenvalues, in a way consistent with eq. (34), as

$$m_{\nu_1} = m_1\ ,\ \ m_{\nu_2} = m_2\ ,\ \ m_{\nu_3} = M - m_1\ ,\ \ m_{\nu_4} = -M - m_2. \tag{47}$$

In the **2 + 2** pattern, M represents the mass scale of the heavier $(3,4)$ pair, being $\mathcal{O}(\sqrt{\Delta m^2_{LSND}})$, while $|m_{1,2}|$ stand for the mass magnitudes of the lighter $(1,2)$ pair. Because of the constraint of eq. (46), we can treat $(\Delta m^2_\odot)^{1/2} \sim |m_1^2 - m_2^2|$ and $(\Delta m^2_{LSND})^{\frac{1}{2}} \sim |M|$ (see Fig. 1) as the two independent controlling mass scales in $\mathcal{M}_\nu^{(4)}$. We have thus found in eq.(47) a parameterization that is convenient for the description of all neutrino oscillation data, obeys the tracelessness condition eq.(34) and follows the mass pattern depicted in Fig. 1.

In order to estimate the orders of magnitude of those among the nondiagonal elements of $\mathcal{M}_\nu^{(4)}$ which are nonvanishing, we resort to the equation $\mathcal{M}_\nu^{(4)} = \mathsf{U}\ \mathsf{diag}.\{\mathsf{m}_1, \mathsf{m}_2, \mathsf{M}-\mathsf{m}_1, -\mathsf{M}-\mathsf{m}_2\}\ \mathsf{U}^\mathsf{T}$. We have already put $\mathsf{U}_{e3} \approx \epsilon,\ \mathsf{U}_{e4} \approx \epsilon'$ in eq.(42) with the expectation that $|\epsilon|,\ |\epsilon'| \lesssim \mathcal{O}(10^{-2})$ in view of eq.(2). We can also put $\mathsf{U}_{\mu 1} \approx \delta,\ \mathsf{U}_{\mu 2} \approx \delta'$ and expect in the light of eq.(3) that $|\delta|,\ |\delta'| \lesssim \mathcal{O}(10^{-1})$; indeed, the best fit requires them to be close to zero, *cf.* eq.(7). Furthermore, we can put $\mathsf{U}_{\mu 3}^2 \approx \mathsf{U}_{\mu 4}^2 \approx 1/2$ on account of the observed maximal mixing in the atmospheric neutrino sector. The various entries of $\mathcal{M}_\nu^{(4)}$ can now be related to the mass eigenvalues and hence to $\mathsf{m}_{1,2}$ and M as follows:

$$\mathsf{a} \approx \mathsf{m}_1\ (\delta\ \mathsf{U}_{e1} - \epsilon\ \mathsf{U}_{\mu 3}) + \mathsf{m}_2\ (\delta'\ \mathsf{U}_{e2} - \epsilon'\ \mathsf{U}_{\mu 4}) + \mathsf{M}\ (\epsilon\ \mathsf{U}_{\mu 3} - \epsilon'\ \mathsf{U}_{\mu 4}), \tag{48}$$

$$\mathsf{b} \approx \mathsf{m}_1\ (\mathsf{U}_{e1}\ \mathsf{U}_{\tau 1} - \epsilon\ \mathsf{U}_{\tau 3}) + \mathsf{m}_2\ (\mathsf{U}_{e2}\ \mathsf{U}_{\tau 2} - \epsilon'\ \mathsf{U}_{\tau 4}) + \mathsf{M}\ (\epsilon\ \mathsf{U}_{\tau 3} - \epsilon'\ \mathsf{U}_{\tau 4}), \tag{49}$$

$$\mathsf{c} \approx \mathsf{m}_1\ (\delta\ \mathsf{U}_{\tau 1} - \mathsf{U}_{\mu 3}\ \mathsf{U}_{\tau 3}) + \mathsf{m}_2\ (\delta'\ \mathsf{U}_{\tau 2} - \mathsf{U}_{\mu 4}\ \mathsf{U}_{\tau 4}) + \mathsf{M}\ (\mathsf{U}_{\mu 3}\ \mathsf{U}_{\tau 3} - \mathsf{U}_{\mu 4}\ \mathsf{U}_{\tau 4}), \tag{50}$$

$$\mathsf{d} \approx \mathsf{m}_1\ (\mathsf{U}_{e1}\ \mathsf{U}_{s1} - \epsilon\ \mathsf{U}_{s3}) + \mathsf{m}_2\ (\mathsf{U}_{e2}\ \mathsf{U}_{s2} - \epsilon'\ \mathsf{U}_{s4}) + \mathsf{M}\ (\epsilon\ \mathsf{U}_{s3} - \epsilon'\ \mathsf{U}_{s4}), \tag{51}$$

$$\mathsf{e} \approx \mathsf{m}_1\ (\delta\ \mathsf{U}_{s1} - \mathsf{U}_{\mu 3}\ \mathsf{U}_{s3}) + \mathsf{m}_2\ (\delta'\ \mathsf{U}_{s2} - \mathsf{U}_{\mu 4}\ \mathsf{U}_{s4}) + \mathsf{M}\ (\mathsf{U}_{\mu 3}\ \mathsf{U}_{s3} - \mathsf{U}_{\mu 4}\ \mathsf{U}_{s4}), \tag{52}$$

$$\mathsf{f} = \mathsf{m}_1\ (\mathsf{U}_{s1}\ \mathsf{U}_{\tau 1} - \mathsf{U}_{s3}\ \mathsf{U}_{\tau 3}) + \mathsf{m}_2\ (\mathsf{U}_{s2}\ \mathsf{U}_{\tau 2} - \mathsf{U}_{s4}\ \mathsf{U}_{\tau 4}) + \mathsf{M}\ (\mathsf{U}_{s3}\ \mathsf{U}_{\tau 3} - \mathsf{U}_{s4}\ \mathsf{U}_{\tau 4}). \tag{53}$$

We see from the above that in the expressions for the a, b and d, the explicit occurrence of ϵ and ϵ' in the coefficient of M pulls it down significantly; thus they are much smaller in magnitude compared to M. In contrast, the coefficients of M in the expressions for the entries c, e and f do not manifestly involve such small parameters. This is a consequence of the fact that the coefficients of M in all the entries depend only on the last two columns of the mixing matrix U eq.(42). We see from eqs.(30) - (33), as also from eq.(47) that, in the limit when $\mathsf{m}_{1,2} \rightarrow 0,\ \mathsf{U}_{e3}^2 = \mathsf{U}_{e4}^2,\ \mathsf{U}_{\mu 3}^2 = \mathsf{U}_{\mu 4}^2,\ \mathsf{U}_{\tau 3}^2 = \mathsf{U}_{\tau 4}^2$ and $\mathsf{U}_{s3}^2 = \mathsf{U}_{s4}^2$. In such a limit, when M remains as the only controlling mass scale,

$$|\epsilon| \approx |\epsilon'|, \tag{54}$$

$$|\mathsf{U}_{\mu 3}| \approx |\mathsf{U}_{\mu 4}|, \tag{55}$$

$$|\mathsf{U}_{\tau 3}| \;\approx\; |\mathsf{U}_{\tau 4}|, \tag{56}$$

$$|\mathsf{U}_{s3}| \;\approx\; |\mathsf{U}_{s4}|. \tag{57}$$

We restate that we have assumed no phases to be present in our mixing matrix. In the limit of neglecting terms $\mathcal{O}(\delta^2, \delta'^{\,2})$, we then have[12] from the unitarity condition $1 = \mathsf{U}_{\mu 1}^2 + \mathsf{U}_{\mu 2}^2 + \mathsf{U}_{\mu 3}^2 + \mathsf{U}_{\mu 4}^2$ and eq.(55) that

$$|\mathsf{U}_{\mu 3}| = |\mathsf{U}_{\mu 4}| = 1/\sqrt{2}. \tag{58}$$

In addition to satisfying the above conditions in the limit $\mathsf{m}_{1,2} \to 0$, the parameters U_{s3}, U_{s4}, $\mathsf{U}_{\tau 3}$, $\mathsf{U}_{\tau 4}$ also have to satisfy the off-diagonal unitarity condition:

$$\mathsf{U}_{\mu 3}\ \mathsf{U}_{\mu 4} + \mathsf{U}_{\tau 3}\ \mathsf{U}_{\tau 4} + \mathsf{U}_{s3}\ \mathsf{U}_{s4} = \ 0 \ + \ \mathcal{O}(\epsilon\epsilon'). \tag{59}$$

From eq.(58) we see that this would mean

$$\mathsf{U}_{s3}\ \mathsf{U}_{s4} + \mathsf{U}_{\tau 3}\ \mathsf{U}_{\tau 4} \approx \pm\frac{1}{\sqrt{2}}. \tag{60}$$

Eqs. (55), (56) and (57) admit four possibilities corresponding to the signs each of the mixing matrix elements can take. First, we rewrite the condition (55) in terms of two possibilities as $\mathsf{U}_{\mu 3} = \mathsf{U}_{\mu 4}$ or $\mathsf{U}_{\mu 3} = -\mathsf{U}_{\mu 4}$. The remaining possibilities form sub-cases of these two cases. Each of these (sub)cases is further constrained by the off-diagonal unitarity condition of eq.(59). After some algebra, we realize that three possible cases are consistent with the unitarity conditions. Each of them leads to a different texture which we detail below. In all the following cases, we have a, b, $\mathsf{c} \sim \mathcal{O}(\mathsf{m}_{1,2})$.

- **Case A** Here, we estimate the following orders of magnitude for the other entries of $\mathcal{M}_\nu^{(4)}$.

$$\mathsf{c} \ \approx \ 2\ \mathsf{M}\ \mathsf{U}_{\mu 3}\ \mathsf{U}_{\tau 3} \equiv \mathsf{M}'; \ \ \mathsf{e} \ \approx \ \mathcal{O}(\mathsf{m}_{1,2}); \quad \mathsf{f} \ \approx \ 2\ \mathsf{M}\ \mathsf{U}_{s3}\ \mathsf{U}_{\tau 3} \equiv \mathsf{M}''. \tag{61}$$

  This case comes either of the choices: $\mathsf{U}_{\mu 3} = \mathsf{U}_{\mu 4}$, $\mathsf{U}_{\tau 3} = -\mathsf{U}_{\tau 4}$, $\mathsf{U}_{s3} = \mathsf{U}_{s4}$ or $\mathsf{U}_{\mu 3} = -\mathsf{U}_{\mu 4}$, $\mathsf{U}_{\tau 3} = \mathsf{U}_{\tau 4}$, $\mathsf{U}_{s3} = -\mathsf{U}_{s4}$. The mass texture would be of the form :

[12] It follows from eq.(47) that, when $\mathsf{m}_{1,2} \to 0$, ν_3 and ν_4 form a Dirac neutrino. So it is not surprising that we get maximal mixing in the 3, 4 sector in this limit.

$$\mathcal{M}_\nu^{(4)} \sim \begin{pmatrix} 0 & a & b & d \\ a & 0 & M' & e \\ b & M' & 0 & M'' \\ d & e & M'' & 0 \end{pmatrix}, \tag{62}$$

The active sterile admixture A is now given by

$$A \approx 1 - \frac{f^2}{c^2}. \tag{63}$$

We see that[13] $2\,U_{s3}^2 \leq 1$. Thus we can parameterize f/c as $\cos\beta$ in which case $A = \sin^2\beta$.

- **Case B** Here the various magnitudes that $c,\ e,\ f$ can take are given by

$$c \approx \mathcal{O}(m_{1,2}); \quad e \approx 2\,M\,U_{\mu 3}\,U_{s3} \equiv M'''; \quad f \approx 2\,M\,U_{s3}\,U_{\tau 3} \equiv M''. \tag{64}$$

This situation can arise with either of the choices : $U_{\mu 3} = U_{\mu 4}$, $U_{\tau 3} = U_{\tau 4}$, $U_{s3} = -U_{s4}$ or $U_{\mu 3} = -U_{\mu 4}$, $U_{\tau 3} = -U_{\tau 4}$, $U_{s3} = U_{s4}$. The mass matrix would now take the form:

$$\mathcal{M}_\nu^{(4)} \sim \begin{pmatrix} 0 & a & b & d \\ a & 0 & c & M''' \\ b & c & 0 & M'' \\ d & M''' & M'' & 0 \end{pmatrix}. \tag{65}$$

The active sterile admixture in this case is given by

$$A \approx 1 - \frac{f^2}{e^2}. \tag{66}$$

Once again, $f^2/e^2 = 2\,U_{s3}^2$ and can be parameterized as $\cos^2\beta$ so that $A = \sin^2\beta$.

- **Case C** Here we have

$$c \approx 2\,M\,U_{\mu 3}\,U_{\tau 3} \equiv M'; \quad e \approx 2\,M\,U_{\mu 3}\,U_{s3} \equiv M'''; \quad f \approx \mathcal{O}(m_{1,2}), \tag{67}$$

---

[13] The unitarity condition $1 = U_{s1}^2 + U_{s2}^2 + U_{s3}^2 + U_{s4}^2$ and eq. (57) imply that $f^2/c^2 \approx |U_{s3}| \lesssim 1/2$.

arising from either of the choices : $U_{\mu 3} = U_{\mu 4}$, $U_{\tau 3} = U_{\tau 4}$, $U_{s3} = U_{s4}$ or $U_{\mu 3} = -U_{\mu 4}$, $U_{tau3} = -U_{\tau 4}$, $U_{s3} = -U_{s4}$. In this case the mass matrix has the following texture

$$\mathcal{M}_\nu^{(4)} = \begin{pmatrix} 0 & a & b & d \\ a & 0 & M' & M''' \\ b & M' & 0 & f \\ d & M''' & f & 0 \end{pmatrix}. \tag{68}$$

The active sterile admixture is now given by

$$A \approx \frac{c^2}{c^2 + e^2}. \tag{69}$$

The form $A = \sin^2 \beta$ can now be achieved by parameterizing $e/c = U_{\tau 3}/U_{s3}$ as $\tan \beta$. We note that this texture has been recently been studied in Ref. [27] on the basis of two pseudo-Dirac neutrinos and an approximate global $L_e + L_\mu - L_\tau - L_s$ symmetry. A slight variation of eq. (68) with $f = 0$ and a nonzero fourth diagonal element has been proposed in Ref.[28] on the basis of a paired pseudo-Dirac structure and the same global symmetry, but not within a Zee-type radiative model. The textures of our cases **A** and **B** strongly violate the global $L_e + L_\mu - L_\tau - L_s$ symmetry.

All the above textures give rise to the **2 + 2** pattern along with an admixture which in general is between zero or one. The choice $U_{s3} = 0$ or $U_{s3}^2 = 1/2$ will yield the extreme results $A = 0$ or 1 respectively, but such a choice is not favored by the experimental data [7]. In all the above three cases the LSND scale appears in *two* pairs of elements in $\mathcal{M}_\nu^{(4)}$ if β is nonzero. This is not surprising since, for a nonvanishing admixture A, the sterile species ν_s and the tau neutrino ν_τ appear in both the lighter $(1, 2)$ and the heavier (3.4) pair of physical states.

V. CONCLUSIONS

In this paper we have studied the constraints on the neutrino mass matrix $\mathcal{M}_\nu$ for a radiative model [15] of three active and one sterile neutrino species from all the current data on neutrino oscillations. We have established a method to show how the solar neutrino mixing angle is related to the observed squared mass differences. This leads, in the three neutrino case to a near-maximal $\theta_\odot$, not in favor with the latest data. In the four neutrino

Set 1	Set 2	Set 3	Set 4
$m_{\nu_1} = N_1^+/D^+$	$m_{\nu_1} = -N_1^+/D^+$	$m_{\nu_1} = N_1^-/D^-$	$m_{\nu_1} = -N_1^-/D^-$
$m_{\nu_2} = N_2^+/D^+$	$m_{\nu_2} = -N_2^+/D^+$	$m_{\nu_2} = N_2^-/D^-$	$m_{\nu_2} = -N_2^-/D^-$
$m_{\nu_3} = N_3^+/D^+$	$m_{\nu_3} = -N_3^+/D^+$	$m_{\nu_3} = N_3^-/D^-$	$m_{\nu_3} = -N_3^-/D^-$

TABLE I: Neutrino mass solutions

case, however, we derive the result $\sin^2 2\theta_\odot \approx 1 - [(\Delta m^2_{Atm})^2/(4\ \Delta m^2_{LSND}\Delta m^2_\odot)]^2$ which allows a phenomenologically acceptable value for $\theta_\odot$ within the allowed ranges of the three squared mass differences. The above result is compatible with the present solar neutrino data when MSW solutions with either large or small mixing are considered. However, it is difficult to reconcile LOW-QVO solutions within these models.

We have shown then that the radiative model with four neutrinos offers several phenomenologically viable textures of $\mathcal{M}_\nu^{(4)}$. This is quite unlike in the three neutrino case where maximal mixing in the atmospheric sector forces the experimentally disfavored bi-maximal texture.

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APPENDIX

Here we present the neutrino mass solutions of eqs.(38) - (40). There are four sets of solutions. These can be conveniently expressed in terms of numerators $N^\pm_{1,2,3}$ and common denominators $D^\pm$ defined as follows.

$$N_1^+ \equiv -4\ \Delta m^2_{LSND}\ \Delta m^2_\odot + (\Delta m^2_{Atm})^2 - 2\ \Delta m^2_\odot\ \Delta m^2_{Atm} - 3\ (\Delta m^2_\odot)^2,$$

$$N_2^+ \equiv 4\ \Delta m^2_{LSND}\ \Delta m^2_\odot + (\Delta m^2_{Atm})^2 + 2\ \Delta m^2_\odot\ \Delta m^2_{Atm} + (\Delta m^2_\odot)^2,$$

$$N_3^+ \equiv 4\ \Delta m^2_{LSND}\ \Delta m^2_{Atm} + (\Delta m^2_{Atm})^2 + 2\ \Delta m^2_\odot\ \Delta m^2_{Atm} + (\Delta m^2_\odot)^2,$$

$$\mathsf{D}^+ \equiv 2\sqrt{2}\ \left[\left\{(\Delta \mathsf{m}^2_{\mathsf{Atm}})^2 - (\Delta \mathsf{m}^2_{\odot})^2\right\}\left(\Delta \mathsf{m}^2_{\mathsf{Atm}} + 2\ \Delta \mathsf{m}^2_{\mathsf{LSND}} + \Delta \mathsf{m}^2_{\odot}\right)\right]^{\frac{1}{2}},$$

and

$$\mathsf{N}_1^- \equiv\ 4\ \Delta \mathsf{m}^2_{\mathsf{LSND}}\ \Delta \mathsf{m}^2_{\odot} + (\Delta \mathsf{m}^2_{\mathsf{Atm}})^2 + 2\ \Delta \mathsf{m}^2_{\odot}\ \Delta \mathsf{m}^2_{\mathsf{Atm}} -\ 3\ (\Delta \mathsf{m}^2_{\odot})^2,$$

$$\mathsf{N}_2^- \equiv -\ 4\ \Delta \mathsf{m}^2_{\mathsf{LSND}}\ \Delta \mathsf{m}^2_{\odot} + (\Delta \mathsf{m}^2_{\mathsf{Atm}})^2 - 2\ \Delta \mathsf{m}^2_{\odot}\ \Delta \mathsf{m}^2_{\mathsf{Atm}} +\ (\Delta \mathsf{m}^2_{\odot})^2,$$

$$\mathsf{N}_3^- \equiv 4\ \Delta \mathsf{m}^2_{\mathsf{LSND}}\ \Delta \mathsf{m}^2_{\mathsf{Atm}} + (\Delta \mathsf{m}^2_{\mathsf{Atm}})^2 - 2\ \Delta \mathsf{m}^2_{\odot}\ \Delta \mathsf{m}^2_{\mathsf{Atm}} +\ (\Delta \mathsf{m}^2_{\odot})^2,$$

$$\mathsf{D}^- \equiv 2\sqrt{2}\ \left[\left\{(\Delta \mathsf{m}^2_{\mathsf{Atm}})^2 - (\Delta \mathsf{m}^2_{\odot})^2\right\}\left(\Delta \mathsf{m}^2_{\mathsf{Atm}} + 2\ \Delta \mathsf{m}^2_{\mathsf{LSND}} - \Delta \mathsf{m}^2_{\odot}\right)\right]^{\frac{1}{2}}.$$

Note that N_i^- $(\mathsf{i} = 1, 2, 3)$ and D^- can always be obtained from the corresponding N_i^+ and D^+ by putting $\Delta \mathsf{m}^2_{\odot} \to -\Delta \mathsf{m}^2_{\odot}$. The four sets of neutrino mass solutions with m_{ν_4} always being $-\mathsf{m}_{\nu_1} - \mathsf{m}_{\nu_2} - \mathsf{m}_{\nu_3}$, are shown in table I.

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