Wavelength Doubling Bifurcations in Coupled Map Lattices

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We report an interesting phenomenon of wavelength doubling bifurcations in the model of coupled (logistic) map lattices. The temporal and spatial periods of the observed patterns undergo successive period doubling bifurcations with decreasing coupling strength. The universality constants α and δ appear to be the same as in the case of period doubling route to chaos in the uncoupled logistic map. The analysis of the stability matrix shows that period doubling bifurcation occurs when an eigenvalue whose eigenvector has a structure with doubled spatial period exceeds unity.

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The study of nonlinear dynamical systems has led to a considerable understanding of low dimensional chaotic systems. However, the understanding in the case of spatially extended systems is far from satisfactory. One of the heuristic ways in which the understanding of low dimensional chaos can be utilized in understanding spatially extended systems is to couple such systems on a lattice and study the coupled system, e.g., the oscillator chains, coupled map lattices, and cellular automata. The model of coupled map lattices shows many interesting phenomena such as kink dynamics, solitons, frozen random patterns, periodic patterns, traveling wave solutions, intermittency, chaos, etc. [1]. The phenomenon of spatiotemporal intermittency in Rayleigh-Bénard convection has been modeled by coupled map lattices [2]. There have been several studies of temporal period doublings in coupled map lattices since its introduction by Kaneko [3]. A renormalization group approach for these period doublings has been developed by Kuznetsov [4].

Here we report a novel and interesting phenomenon in the model of coupled map lattices. It may give insight about the routes to spatial inhomogeneity in spatiotemporal systems such as turbulence. The phenomenon is a spatial analog of the well known route to chaos via temporal period doubling. We consider a one dimensional coupled map lattice with logistic maps coupled symmetrically. The model has several spatially and temporally periodic stable solutions [5,6]. Starting with a stable solution with a spatial period two we find that the temporal and spatial periods of the observed patterns undergo successive period doubling bifurcations as the coupling strength is decreased. Using the standard procedure, the universality constants α and δ are obtained and they appear to be the same as in the case of period doubling route to chaos in an uncoupled logistic map [7]. We also analyze the stability matrix and determine the condition for spatial period doubling bifurcations to occur.

Let us consider the following model of one dimensional coupled map lattices:

$$x_{t+1}(i)=(1-\epsilon)f(x_t(i))+\tfrac{1}{2}\epsilon f(x_t(i+1))+\tfrac{1}{2}\epsilon f(x_t(i-1)), \qquad (1)$$

where $x_t(i)$ is the variable associated with the ith lattice point at time t taking values in a suitably bounded phase space, $i=1,\ldots,m$. For the map f we take the logistic map, $f(x)=\mu x(1-x)$, where $0\le x\le 1$ and $0\le\mu\le 4$. The parameter ϵ represents the coupling strength and $0\le\epsilon\le 1$. For $\epsilon=0$ the dynamics of the lattice is one of the uncoupled logistic maps.

Let $S_\tau(N)$ denote a solution of Eq. (1) with time period τ and space period N. Consider the solution $S_2(2)=\{x_1(1),x_1(2)\}$ with $x_1(1)\ne x_1(2)$. It is possible to show that there is a range of parameter values where $S_2(2)$ is a stable solution and is given by

$$x_1(1)=\frac{(\mu+1-2\mu\epsilon)+\sqrt{(\mu+1-2\mu\epsilon)(\mu-3-2\mu\epsilon+4\epsilon)}}{2\mu(1-2\epsilon)},$$
$$x_1(2)=\frac{(\mu+1-2\mu\epsilon)}{\mu(1-2\epsilon)}-x_1(1). \qquad (2)$$

This solution has time period two and $x_2(1)=x_1(2)$, $x_2(2)=x_1(1)$. It may also be treated as a traveling wave solution with velocity one. The stability of this solution can be determined by the eigenvalues of the stability or the Jacobian matrix. The stability criterion is discussed afterwards.

Let us consider the case when $\mu=\mu_\infty=3.569\ldots$ which is the accumulation point of the period doubling cascade in an uncoupled logistic map and the coupling parameter ϵ is allowed to vary. The period-two solution $S_2(2)$ is stable in the range $\epsilon_0=0.13418\ldots$ to $\epsilon_1=0.038890\ldots$ [see Eqs. (9) and (10)]. These values are listed in Table I. For $\epsilon<\epsilon_1$ the solution $S_2(2)$ becomes unstable and undergoes a period doubling bifurcation. A new solution $S_4(4)$ with space period four and time period four be-

TABLE I. The values of ϵ_n at successive bifurcation points for different n. The table also lists values of d_n, α_n, and δ_n.

n	ϵ_n	d_n	α_n	δ_n
1	0.0388908	0.34932	−3.0396	3.855
2	0.009765	−0.11492	−2.5682	4.434
3	0.002182	0.04475	−2.4355	4.621
4	0.000472	−0.01837	−2.5216	4.642
5	0.000102	0.00728	−2.4884	
6	0.0000223	−0.00292		

comes stable. The solution $S_4(4)$ is stable in the range ϵ_1 to $\epsilon_2=0.0097649\ldots$. At ϵ_2 we have one more period doubling bifurcation leading to the period-eight solution $S_8(8)$ for $\epsilon<\epsilon_2$. Further numerical investigations show that the period doubling cascade continues and probably leads to the accumulation point at $\epsilon_\infty=0.0$. At each bifurcation point both the space and the time periods double. Since we have a spatial period doubling cascade starting with space period two, it was necessary to choose the lattice size in powers of two in numerical simulations. The maximum lattice size used was 1024. The stability of the solutions was checked by giving small perturbations and also by eigenvalues of the matrices $M(\theta)$ [Eq. (5)].

In Table I the ϵ_n values at the successive bifurcation points are listed. Let δ_n be given by

$$\delta_n=\frac{\epsilon_n-\epsilon_{n+1}}{\epsilon_{n+1}-\epsilon_{n+2}}\,. \tag{3}$$

The values of δ_n are listed in Table I. Though these values are still inadequate to conclude about the asymptote $\delta=\delta_\infty$, they are clearly consistent with the value $\delta=4.6692\ldots$ obtained from the period doubling sequence of an uncoupled logistic map as a function of μ [7].

In Fig. 1 we plot the values of $x_t(1)$ for different values of t, as a function of ϵ. The bifurcation diagram has a striking similarity to the one in the case of an uncoupled logistic map as a function of μ. To determine the scaling parameter α_n, we determine the value of ϵ for each period for which one value of $x_t(1)$ is 0.5. This defines the analog of the superstable orbit for an uncoupled logistic map. Let d_n be the separation of the point $x_t(1)=0.5$ from the nearest x value (see Fig. 1) for the period 2^n. Define the scaling parameter α_n by

$$\alpha_n=d_n/d_{n+1}\,. \tag{4}$$

The values of d_n and α_n are listed in Table I. We again note that the values of α_n are consistent with the asymptotic value $\alpha=\alpha_\infty=2.5029\ldots$ for the uncoupled logistic map as a function of μ [7].

The period doubling solutions that we observe can also be treated as traveling wave solutions. The velocities of the solutions with periods 2, 4, and 8 that we have obtained are 1, 3, and 5, respectively. For higher order periods the observed velocity is 11. We note that the sequence of traveling wave speeds (1,3,5,11) corresponds to the succession $v_{n+1}=v_n+2v_{n-1}$. This is characteristic of the main frequency in the frequency doubling cascade in nonorientable manifolds [8].

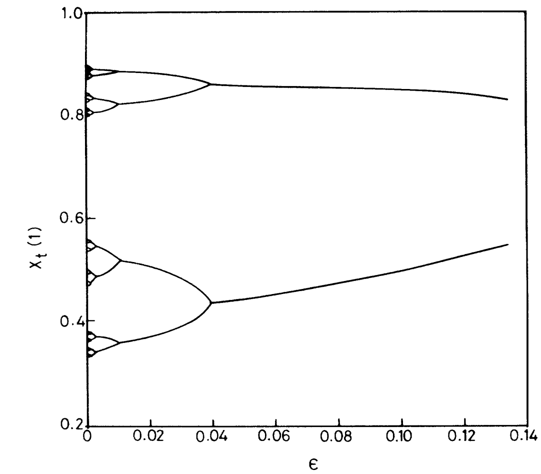

FIG. 1. The variables $x_t(1)$ at one site at different times as a function of ϵ at $\mu=\mu_\infty$.

We now consider the stability of the periodic solution $S_\tau(N)$. This problem can be simplified by using the results of Refs. [5] and [6]. We first consider a one dimensional lattice chain $\mathcal{C}_M$ of length M with cyclic boundary conditions; i.e., the first and the Mth lattice points are neighbors of each other. Let $R_t=(x_t(1),\ldots,x_t(N))$ denote the state of the system of the chain $\mathcal{C}_N$ at time t. Let $S_\tau(N,1)$ denote a solution of Eq. (1) with temporal periodicity τ for the chain $\mathcal{C}_N$, i.e.,

$$S_\tau(N,1)=\{R_1,R_2,\ldots,R_\tau,R_1,R_2,\ldots\}\,.$$

Now consider a closed chain $\mathcal{C}_{kN}$ of length kN, $k=1,2,\ldots$. Obviously the spatially periodic sequence

$$S_\tau(N,k)=\{\langle R_1,\ldots,R_1\rangle_k,\ldots,\langle R_\tau,\ldots,R_\tau\rangle_k,\ \langle R_1,\ldots,R_1\rangle_k,\ldots\}$$

of wavelength N built from the states $\{R_t\}$ as the building blocks is a solution of Eq. (1) for the closed chain $\mathcal{C}_{kN}$ with temporal periodicity τ. Here the ordered pair $\langle R_t,\ldots,R_t\rangle_k$ represents a state made up of k replicas of the state R_t. We call $S_\tau(N,k)$ the k replica solution of $S_\tau(N,1)$. The stability criterion for the k replica solution was discussed in Ref. [5]. It was shown that the problem of the eigenvalues of the $kN\times kN$ stability matrix of the k replica solution can be simplified to the analysis of k matrices of size $N\times N$ which are constructed using the stability matrix for the solution $S_\tau(N,1)$, the building block of spatial periodicity. The problem can be further simplified for a traveling wave solution [6]. If v is the velocity of the traveling wave, then the problem of stability analysis of the k replica solution reduces to the analysis of the eigenvalues of the $N\times N$ matrices [6]

$$M(\theta)=(\Pi_\theta)^v J_\theta\,, \tag{5}$$

where $\theta=0,2\pi/k,\ldots,(k-1)2\pi/k$. Here Π_θ and J_θ are $N\times N$ matrices given by

$$\Pi_\theta=\begin{bmatrix} 0 & 0 & \cdots & 0 & e^{i\theta} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{6}$$

and

$$J_\theta=\begin{bmatrix} (1-\epsilon)f'(x_1(1)) & \frac{1}{2}\epsilon f'(x_1(2)) & \cdots & \frac{1}{2}\epsilon f'(x_1(N))e^{i\theta} \\ \frac{1}{2}\epsilon f'(x_1(1)) & (1-\epsilon)f'(x_1(2)) & \cdots & 0 \\ \vdots & & & \vdots \\ \frac{1}{2}\epsilon f'(x_1(1))e^{-i\theta} & 0 & \cdots & (1-\epsilon)f'(x_1(N)) \end{bmatrix}. \tag{7}$$

Now the k replica solution is stable if all the eigenvalues of the matrices $M(\theta)$ have magnitude less than one. As $k\to\infty$ or as the size of the lattice increases, θ takes continuous values between 0 and 2π. It is easy to show that it is sufficient to check the eigenvalues of $M(\theta)$ in the range $0\le\theta\le\pi$ to determine the stability of the solution as $k\to\infty$. Let $S_\tau(N)=\lim_{k\to\infty}S_\tau(N,k)$.

Let us apply the above stability analysis to the period-two solution $S_2(2,k)$ [Eq. (2)] which has velocity one. For $N=2$ and $v=1$, matrices $M(\theta)$ [Eq. (5)] are given by

$$M(\theta)=\begin{bmatrix} 0 & e^{i\theta} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} (1-\epsilon)f'(x_1(1)) & \frac{1}{2}\epsilon(1+e^{i\theta})f'(x_1(2)) \\ \frac{1}{2}\epsilon(1+e^{-i\theta})f'(x_1(1)) & (1-\epsilon)f'(x_1(2)) \end{bmatrix}. \tag{8}$$

We first consider the stability of the solution for $k=1$, i.e., the solution $S_2(2,1)$. The stability criterion is that the eigenvalues of the matrix $M(0)$ have magnitude less than one. From the eigenvalues we find that the solution $S_2(2,1)$ is stable in the range $\epsilon_0<\epsilon<\epsilon'$, where

$$\epsilon_0=\frac{1}{2}\left[1-\left(\frac{3}{\mu(\mu-2)}\right)^{1/2}\right],\qquad \epsilon'=\frac{2\mu^2-4\mu-3-(8\mu^2-16\mu+9)^{1/2}}{4\mu(\mu-2)}. \tag{9}$$

At the lower limit ϵ, one of the eigenvalues of $M(0)$ becomes -1, while at the upper limit ϵ_0, both the eigenvalues are complex and have unit magnitude. For $k=2$ we must consider both $\theta=0$ and π, i.e., the matrices $M(0)$ and $M(\pi)$. The analysis of the eigenvalues of $M(\pi)$ shows that the stability range of ϵ values shrinks, with the upper limit ϵ_0 remaining unchanged and the lower limit shifting to ϵ_1, which is one of the solutions of the equation

$$4(1-\epsilon)^3-(1-2\epsilon)^2-\mu(\mu-2)(1-\epsilon)^2(1-2\epsilon)^2=0\,. \tag{10}$$

At ϵ_1 the eigenvalues of $M(\pi)$ are ±1. For $k>2$ the eigenvalues of $M(\theta)$ with θ in the range 0 to π must be considered. By obtaining eigenvalues for θ values for $k=3,4,\ldots$, we find that there is no further reduction in the stability range (ϵ_1,ϵ_0) of ϵ values for the solution $S_2(2,k)$ as $k\to\infty$ [5]. We have also numerically checked the stability of the solution in this range by random perturbations of the solution.

Let us consider the point ϵ_1 where we have a period doubling bifurcation and for $\epsilon<\epsilon_1$ we have a stable solution $S_4(4)$ of period four. At $\epsilon=\epsilon_1$ the eigenvalues of the matrix $M(\pi)$ are ±1. To understand how an eigenvalue of $M(\pi)$ shows a tendency towards period doubling, consider the 4×4 matrix W whose eigenvalues determine the stability of the solution $S_2(2,2)$ [6],

$$W=\begin{bmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{bmatrix}\begin{bmatrix} (1-\epsilon)f'(x_1(1)) & \frac{1}{2}\epsilon f'(x_1(2)) & 0 & \frac{1}{2}\epsilon f'(x_1(2)) \\ \frac{1}{2}\epsilon f'(x_1(1)) & (1-\epsilon)f'(x_1(2)) & \frac{1}{2}\epsilon f'(x_1(1)) & 0 \\ 0 & \frac{1}{2}\epsilon f'(x_1(2)) & (1-\epsilon)f'(x_1(1)) & \frac{1}{2}\epsilon f'(x_1(2)) \\ \frac{1}{2}\epsilon f'(x_1(1)) & 0 & \frac{1}{2}\epsilon f'(x_1(1)) & (1-\epsilon)f'(x_1(2)) \end{bmatrix}. \tag{11}$$

Let (a_i,b_i), $i=1,2$, be the eigenvectors of $M(0)$ [Eq. (8)] with eigenvalues Λ_i and let (c_i,d_i) be the eigenvectors of $M(\pi)$ with eigenvalues ν_i. It is easy to verify that (a_i,b_i,a_i,b_i) are the eigenvectors of W [Eq. (11)] with the eigenvalues Λ_i and $(c_i,d_i,-c_i,-d_i)$ are the eigenvectors of W with eigenvalues ν_i [9]. Thus, we see that the magnitude of eigenvalues of $M(\pi)$ exceeding unity is a clear signal for the instability of $S_2(2,2)$ towards a period doubling bifurcation. Similar arguments can be used for the eigenvectors with larger values of k, e.g., $(c_i,d_i,-c_i,-d_i,c_i,d_i,-c_i,-d_i)$ are eigenvectors of the 8×8 stability matrix for $S_2(2,4)$ $(k=4)$ with eigenvalues ν_i.

Subsequent period doubling bifurcations take place in a similar fashion. At each bifurcation point one of the eigenvalues of the matrix $M(\pi)$ becomes one and the eigenvalues of matrices $M(\theta)$ with other values of θ are still less than one.

Let Λ denote the eigenvalue with largest magnitude of the matrices $M(\theta)$. We define the Lyapunov exponent λ as

$$\lambda=\ln|\Lambda|. \qquad (12)$$

In Fig. 2 we plot the value of Lyapunov exponent as a function of ϵ. We observe a graph similar to the one in the case of period doubling transition to chaos in an uncoupled logistic map as a function of μ with the difference that here λ remains finite since the largest magnitude eigenvalue is never zero. Starting from zero at a bifurcation point the Lyapunov exponent decreases as ϵ decreases, reaches a minimum, and then again arises to zero at the next bifurcation point.

We have also analyzed the behavior at other values of μ. For $\mu<\mu_\infty$ we see a finite number of bifurcations as ϵ decreases until one reaches the correct limit at $\epsilon=0$ (i.e., the lattice has the same wavelength and temporal period as the temporal period of the uncoupled case). For the values of $\mu>\mu_\infty$ we see a similar phenomena of wavelength doubling bifurcations. However, there are difficulties in determining the bifurcation point as μ increases [10]. Kaneko [11] has reported several wave patterns and Brownian motion of defects in the region $\mu>\mu_\infty$. Solutions with almost temporal period two and temporal period four are observed in this region. Further detailed numerical studies are necessary to see how our results extend to this region.

We have also investigated the region in the ϵ-μ plane near the period three window in an uncoupled logistic map. Period doubling bifurcations similar to those reported above are seen as ϵ is decreased.

Thus, we have found a wavelength doubling phenomenon in coupled map lattices. This is a spatial analog of the normal temporal period doubling route to chaos. This wavelength doubling route to spatiotemporal chaos can be very important in our understanding of different phenomena seen in spatiotemporal systems. Experiments on large aspect ratio cells in Raleigh-Bénard convection with annular geometry may be one of the systems where the phenomenon described above may be observed [12].

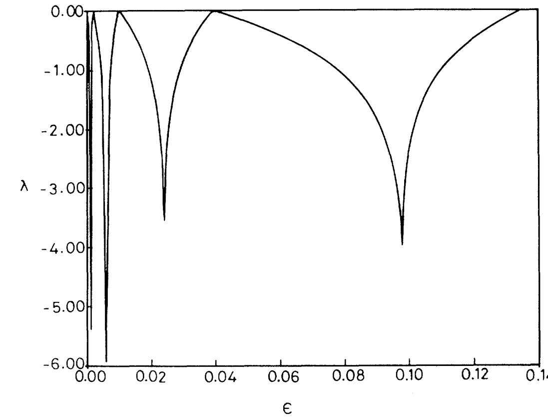

FIG. 2. The Lyapunov exponent λ as a function of ϵ at $\mu=\mu_\infty$.

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