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Synchronized state of coupled dynamics on time-varying networks

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We consider synchronization properties of coupled dynamics on time-varying networks and the corresponding time-average network. We find that if the different Laplacians corresponding to the time-varying networks commute with each other then the stability of the synchronized state for both the time-varying and the time-average topologies are approximately the same. On the other hand for noncommuting Laplacians the stability of the synchronized state for the time-varying topology is in general better than the time-average topology. *© 2006 American Institute of Physics.*
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Synchronization is an important property of dynamical systems. Several diverse systems such as chemical reactions, electronic circuits, array of Josephson junctions, neurons, different body functions like heart rate and respiration, flapping of wings by the birds are known to show synchronization in some form. Roughly synchronization corresponds to the coherent evolution of different coupled dynamical systems. It is clearly of interest to study synchronization on different networks. The network represents the underlying geometrical structure of entities (nodes) and links (edges) in a given system. Recent investigations of several systems in different fields ranging from physical, biological and chemical systems to social and economic systems has shown that networks are ubiquitous in nature. Networks are often associated with dynamical variables evolving with time and it is possible to show that they show synchronization. However, natural networks are not static in time and the structure of nodes and links changes with time. In this paper we investigate the synchronization properties of networks with time-varying structure and compare it with the synchronization in static time-average networks. Network structure can be represented by the adjacency matrix whose elements are unity if the corresponding nodes are connected and zero otherwise. We can construct the Laplacian matrix from the adjacency matrix by subtracting the diagonal matrix of the degrees of different nodes. We find that if the Laplacians of the different time-varying networks commute with each other then the synchronization properties of the time-varying and time-average networks are approximately similar. On the other hand, if the Laplacians do not commute then the synchronization property of the time-varying network is more stable then that of the time-average network. We demonstrate the effect using an example of coupled Rössler systems.

I. INTRODUCTION

Several networks in the real world consist of dynamical elements interacting with each other and the interactions can be used to define the links of the network. Several of these networks have a large number of degrees of freedom and it is important to understand their dynamical behavior.[1] One of the important dynamical property of the coupled networks is the synchronization of the dynamical variables associated with individual nodes. The earlier studies of synchronization in networks concentrated on regular networks such as lattices with nearest neighbor or short range couplings or globally coupled networks.[2–7] Recently, it has been recognized that several complex systems have underlying structures that are described by networks or graphs which are not regular but have some random element and may have some universal properties such as small-world length scales or scale free degree distribution.[1,8] This has led to the study of synchronization properties of different networks.[9–15] In particular, it was shown that a state with several synchronized clusters is possible. Two main types of clusters can be identified, namely driven clusters that have mostly intercluster couplings and self-organized clusters that have mostly intracluster couplings.[14]

In spite of several studies of synchronization on networks, most of the studies have concentrated on static networks where the nodes and edges (couplings) are constant in time. However, in several naturally occurring networks the topology of the networks changes with time. Both the number of nodes and the edges connecting the nodes can vary with time. Such a time-varying topology can occur in social networks, computer networks, WWW, biological systems, spread of epidemics, etc. Recently, there have been studies of dynamics of time-varying network topologies.[16,17] It is shown that if the topology switches sufficiently fast between different networks, then the synchronized state can become

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stable even when the individual networks do not support the synchronized state. In this paper we investigate the synchronization properties of time-varying networks. We find that such time-varying networks cannot only stabilize the synchronized state but also can lead to better stability in some situations.

II. COUPLED DYNAMICS ON NETWORKS

A. Static network

We introduce the notation by first considering a model of static network. Consider a network of N nodes (oscillators). Let $\mathbf{x}^i(t) \in R^m$ be the m-dimensional variable of the ith node. Let the uncoupled dynamics of each node be defined by the function $\mathbf{f}(\mathbf{x}^i(t))$ and the coupling by the function $\mathbf{h}: R^m \to R^m$. Let G be the $N \times N$ adjacency matrix of the network such that $G_{ij}=1$ if an edge connects the nodes i and j and $G_{ij}=0$ otherwise. The Laplacian L of the network is given by

$$L = \mathrm{diag}(d) - G, \tag{1}$$

where ith element of $d \in R^N$ is the degree of node i. Thus, the dynamics of the ith node is given by

$$\dot{\mathbf{x}}^i(t) = \mathbf{f}(\mathbf{x}^i(t)) + \sigma \sum_j L_{ij} \mathbf{h}(\mathbf{x}^j(t)), \tag{2}$$

where σ is the coupling strength. Let $\mathbf{x}=(\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^N)$, $\mathbf{f}(\mathbf{x})=(\mathbf{f}(\mathbf{x}^1),\mathbf{f}(\mathbf{x}^2),\ldots,\mathbf{f}(\mathbf{x}^N))$, $\mathbf{h}(\mathbf{x})=(\mathbf{h}(\mathbf{x}^1),\mathbf{h}(\mathbf{x}^2),\ldots,\mathbf{h}(\mathbf{x}^N))$. The complete dynamics can be expresses as

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{f}(\mathbf{x}(t)) + \sigma(L \otimes I_m)\mathbf{h}(\mathbf{x}(\mathbf{t})), \tag{3}$$

where $\otimes$ is the direct product.

Let $\mathbf{x}^0=\mathbf{x}^1=\mathbf{x}^2=\cdots=\mathbf{x}^N$ be the synchronized state. Linearizing each oscillator (2) about the synchronized trajectory $\mathbf{x}^0(t)$ gives

$$\dot{\mathbf{z}}(\mathbf{t}) = [\mathbf{I}_N \otimes F(t) + \sigma L \otimes H]\mathbf{z}(t) = J(t)\mathbf{z}(t), \tag{4}$$

where $\mathbf{z}=(\mathbf{z}^1,\mathbf{z}^2,\ldots,\mathbf{z}^N)$, $\mathbf{z}^i=\mathbf{x}^i-\mathbf{x}^0$, and $\mathbf{F}(t)=D\mathbf{f}$ evaluated at $\mathbf{x}^0$, $\mathbf{H}=D\mathbf{h}$. Henceforth, we take the coupling function $\mathbf{h}$ to be linear and hence $\mathbf{H}$ becomes a constant $m\times m$ matrix. The Jacobian $J=\mathbf{I}_N\otimes F+\sigma L\otimes H$ is an $mN\times mN$ matrix. A formal solution of Eq. (4) can be written as

$$\begin{aligned}\mathbf{z}(t+\tau) &= \Phi_L(t+\tau,t)\mathbf{z}(t)\\ &= \exp\left[\int_t^{t+\tau} I_N \otimes F(t')\mathrm{d}t' + \sigma L \otimes H\tau\right]\mathbf{z}(t)\\ &= \exp\left[\int_t^{t+\tau} J(t')\mathrm{d}t'\right]\mathbf{z}(t).\end{aligned} \tag{5}$$

We can also simplify the linearized equation (4) by noting that the transformation which diagonalizes L does not affect the first term in the equation since it is already block diagonal with identical blocks of size $m\times m$. Thus the linearized equation (4) can be block diagonalized with each block $(m\times m)$ having the form

$$\dot{\mathbf{z}}_{\mathbf{k}}(\mathbf{t}) = [F(t) + \sigma\gamma_k H]\mathbf{z}_k(t), \tag{6}$$

where γ_k is an eigenvalue of L, $k=0,1,\ldots,N-1$. The largest eigenvalue $\gamma_0=0$ and corresponds to the eigenvector $(1,1,\ldots,1)$. This eigenvector defines the synchronization manifold and the rest of the eigenvalues and the corresponding eigenvectors define the transverse manifold. Using Eq. (6) we can define the master stability function and the associated master stability function determines the stability of the synchronized state.[18] For the stability of the synchronized state the system must be stable along the transverse manifold. We can also write a formal solution for Eq. (6) as

$$\begin{aligned}\mathbf{z}_k(t+\tau) &= \Phi_k(t+\tau,t)\mathbf{z}_k(t)\\ &= \exp\left(\int_t^{t+\tau} F(t')\mathrm{d}t' + \sigma\gamma_k H\tau\right)\mathbf{z}_k(t).\end{aligned} \tag{7}$$

B. Time-varying networks

We now consider the time-varying topology where the network periodically switches between graph Laplacians $L_1, L_2, \ldots, L_g$ with periods $\tau_1, \tau_2, \ldots, \tau_g$, respectively, and the total period $T=\Sigma_i \tau_i$. Thus,

$$L(t) = \sum_{i=1}^{g} L_i \chi_{[t_{i-1},t_i]}, \tag{8}$$

where $\chi_{[t_{i-1},t_i)}$ is the indicator function with support $[t_{i-1},t_i)$ and $t_i=t_{i-1}+\tau_i$. The time averaged $L(t)$ is

$$\bar{L} = \frac{1}{T}\int_0^T L(t)\mathrm{d}t = \frac{1}{T}\sum_{i=1}^{g} L_i\tau_i. \tag{9}$$

In Ref. 17 it is shown that if the network synchronizes for the static time average of the topology, i.e., with $\bar{L}$, then the network will synchronize with the time-varying topology if the time variation is done sufficiently fast.

We will refer to the dynamics obtained using time-varying topology as t-varying case and the dynamics using static time-average topology as t-average case. Using Eq. (5) we can write a formal solution for the t-varying case as

$$\begin{aligned}\mathbf{z}(t+T) &= \Phi_L(t_0+T,t_0)\mathbf{z}(t_0)\\ &= \mathcal{T}\left[\prod_{i=1}^{g} \exp\left(\int_{t_{i-1}}^{t_i} J_i(t')\mathrm{d}t'\right)\right]\mathbf{z}(t_0),\end{aligned} \tag{10}$$

where $\mathcal{T}(\cdots)$ represents a suitable time ordering and $\mathbf{z}(t_0)$ is the initial condition. A formal solution for the t-average case with the same initial condition can be written as

$$\bar{\mathbf{z}}(t+T) = \Phi_{\bar{L}}(t_0+T,t_0)\mathbf{z}(t_0) = \exp\left(\int_{t_{i-1}}^{t_i} \bar{J}(t')\mathrm{d}t'\right)\mathbf{z}(t_0). \tag{11}$$

C. Commuting Laplacians

Let us now make the approximation that the total period T is sufficiently small so that for the evolution from t_{i-1} to t_i the Jacobian can be approximately treated as independent of time. Thus we can write Eqs. (10) and (11) as

$$\mathbf{z}(t+T) \approx \mathcal{T}\left(\prod_{i=1}^{g} \exp[J_i\tau_i]\right)\mathbf{z}(t_0), \tag{12a}$$

$$\bar{\mathbf{z}}(t+T) \approx \exp[\bar{J}T]\mathbf{z}(t_0). \tag{12b}$$

From the above equations we see that the t-varying and t-average cases will have similar evolution for the linearized equations provided the different Jacobians or the corresponding Laplacians commute with each other, i.e.,

$$[L_i, L_j] = 0 \quad \text{for } i,j = 1, \ldots, g. \tag{13}$$

Thus, when condition (13) is satisfied, the linearized variables $\mathbf{z}$ will have approximately similar dynamics for the t-varying and t-average cases. Hence the stability conditions and the range of stability of the synchronized state for these two cases will be approximately the same.

The above analysis can be made clearer by block diagonalizing the Jacobians J_i using the transformations which diagonalize L_i. For commuting Laplacians the same transformation will diagonalize all of them and also the average $\bar{L}$. Thus all the L_i and $\bar{L}$ will have the same eigenvectors, though different eigenvalues. Hence, using Eq. (7) and the approximate solutions Eqs. (12a) and (12b) we can write

$$\mathbf{z}_k(t+T) \approx \mathcal{T}\left(\prod_{i=1}^{g} \exp[(F_i + \sigma\gamma_k^i H)\tau_i]\right)\mathbf{z}_k(t_0), \tag{14a}$$

$$\bar{\mathbf{z}}_k(t+T) \approx \exp[(F + \sigma\bar{\gamma}_k)T]\mathbf{z}_k(t_0). \tag{14b}$$

We now make a further approximation of replacing F by its time average value. If $\bar{\lambda}_{kj}$ and λ_{kj}^i, $j=1,\ldots,m$ are the Lyapunov exponents then from Eqs. (14a) and (14b) we can write to a first approximation

$$\bar{\lambda}_{kj} \approx \frac{1}{T}\sum_{i=1}^{g} \lambda_{kj}^i \tau_i. \tag{15}$$

We thus see that the stability range for the t-varying and commuting Laplacians should be approximately the same as that for the average Laplacian.

D. Noncommuting Laplacians

We now consider the case when the different t-varying Laplacians do not satisfy the condition (13). In this case the eigenvectors corresponding to the different Laplacians are in general not the same. Note that the largest eigenvalue ($\gamma_0 = 0$) and the corresponding eigenvector $(1,\ldots,1)$ which define the synchronization manifold are the same for all the Laplacians. Let the eigenvectors of different L_i be denoted by $\mathbf{z}_k^{(i)}$, $i=1,\ldots,g$, $k=1,\ldots,N$ and the corresponding eigenvalues by γ_k^i. Let the different sets of eigenvectors be related to each other by the transformations

$$\mathbf{z}^{(i)} = R^{ij}\mathbf{z}^{(j)}. \tag{16}$$

In general, the transformation R^{ij} relating eigenvectors of L_i and L_j will consist of several sets of rotations. Now we can express the formal time evolution of different eigenvectors by the following equation:

$$\mathbf{z}_k^{(1)}(t_0+T) = \mathcal{T}\left[\prod_{i=1}^{g}\sum_{k_i=1}^{g-1} R_{k_{i+1}k_i}^{i+1\,i} \times \exp\left(\int_{t_{i-1}}^{t_i} F(t')\mathrm{d}t' + \sigma\gamma_{k_i}^j\tau_i\right)\right]\mathbf{z}_{k_1}^{(1)}(t_0), \tag{17}$$

where $k_{g+1}=k$. The action of each graph Laplacian L_i is to cause a rotation and an evolution of the rotated eigenvectors. We note that this rotation takes place only in the transverse manifold. The evolution of the synchronization manifold is unaffected by these rotations and can be expressed as

$$\mathbf{z}_0^{(1)}(t_0+T) = \mathcal{T}\left[\prod_{i=1}^{g} \exp\left(\int_{t_{i-1}}^{t_i} F(t')\mathrm{d}t' + \sigma\gamma_0^j\tau_i\right)\right]\mathbf{z}_0^{(1)}(t_0). \tag{18}$$

To understand the effect of the evolution of Eq. (17) let us consider a simple case of $m=1$, $g=2$, $N=3$, and $\tau_1=\tau_2=\tau$. We further assume that the period T is sufficiently small and the exponential in Eq. (17) can be replaced by the time average value $\exp[\lambda_{k_i}^i\tau_i]$. We also assume that the two Laplacians have eigenvectors pointing in different directions but the transverse Lyapunov exponents are the same and they are denoted by λ_1 and λ_2 ($\lambda_1 > \lambda_2$). Let, the transformation $R = R^{21}$ cause a rotation by an angle θ in the two-dimensional transverse manifold. Thus the evolution in the transverse manifold can be expressed as

$$\mathbf{z}^{(1)}(t_0+T) = R^{-1}ERE\mathbf{z}^{(1)}(t_0) = M\mathbf{z}^{(1)}(t_0), \tag{19}$$

where E is a diagonal matrix $\mathrm{diag}(\exp[\lambda_1\tau], \exp[\lambda_2\tau])$. The eigenvalues of the matrix M are given by

$$\Lambda_{1,2} = \frac{1}{2}\left(e_1^2 + e_2^2 \pm (e_1^2 - e_2^2) \times \sqrt{1 - 2s^2\frac{e_1^2+e_2^2}{(e_1+e_2)^2} + s^4\frac{(e_1-e_2)^4}{(e_1+e_2)^2}}\right), \tag{20}$$

where $s=\sin(\theta)$ and $e_i=\exp(\lambda_i\tau)$. By writing $\Lambda_i=\exp(\lambda_i^r\tau)$ we get for small θ and τ,

$$\lambda_{1,2}^r = 2\lambda_{1,2} \mp s^2\frac{\lambda_1-\lambda_2}{2}. \tag{21}$$

Thus, the larger exponent (λ_1) decreases while the smaller one (λ_2) increases. Hence, the effect of periodic switching between Laplacians appears to reduce the spread of the transverse Lyapunov exponents and in particular the larger exponents will decrease. This should in general enhance the stability of the synchronized state. In the next section we demonstrate this by using an example of coupled Rössler systems.

III. ILLUSTRATION

As an illustration we consider a system of coupled Rössler oscillators,

$$\dot{x}_i(t) = - y_i(t) - z_i(t) - \sigma \sum_{j=1}^{N} (L(t))_{ij} x_j(t),$$

$$\dot{y}_i(t) = x_i(t) + a y_i(t), \tag{22}$$

$$\dot{z}_i(t) = b + z_i(t)(x_i(t) - c),$$

where $L(t)$ is given by Eq. (8), $i=1,\ldots,N$, $a=0.2$, $b=0.2$, $c=7.0$. We consider several graphs with $N=10$. Indicating each edge by a pair of nodes we define the following graphs. (1) $G_1=\{(i+1,i)|i=1,\ldots,N,\&N+1=1\}$, i.e., graph with nearest neighbor couplings on a ring. (2) $G_2=\{\}$, i.e., graph with zero edges. (3) $G3=\{(1,2),(3,4),(5,6),(7,8),(9,10)\}$. (4) $G4=\{(2,3),(4,5),(6,7),(8,9),(10,1)\}$. (5) $G_5=\{(1,2),(2,3),(3,4),(4,5),(5,6)\}$. (6) $G_6=\{(6,7),(7,8),(8,9),(9,10),(10,1)\}$. For simplicity we report here the results for the combination of two graphs each ($g=2$).

(a) The combination (G_1,G_2) represents commuting Laplacians. Both the t-varying and t-average cases show a stable synchronized state in the range $\sigma \in (0.75,2.30)$.

(b) The combination (G_3,G_4) represents noncommuting Laplacians. The t-varying case is stable in the range $\sigma \in (0.70,2.30)$ while the t-average case is stable in the range $\sigma \in (0.75,2.30)$. Thus the lower limit which corresponds to the long-wavelength instability[18] gets extended for the t-varying case. We expect the difference between the t-varying and t-average cases to come from the commutator of the t-varying Laplacians. Hence, to understand the stability of the synchronized state we apply a perturbation of the type

$$\delta C_{ij} x_j(t),$$

where $C=[L_3,L_4]$ is the commutator of the two Laplacians and add it to the first equation of (22). Note that C is an antisymmetric matrix with the property $\Sigma_j C_{ij}=0$ and does not affect the division of the entire space into a combination of the synchronization manifold and the transverse manifold. The range of stability of the synchronized state in the σ-δ plane is shown in Fig. 1(a). The switching time between different networks used for Fig. 1 is $\tau=0.3$.

(c) The combination (G_5,G_6) again represents noncommuting Laplacians. The t-varying case is stable in the range $\sigma \in (0.75,2.45)$ while the t-average case is stable in the range $\sigma \in (0.75,2.30)$. Thus the upper limit which corresponds to the short-wavelength instability gets extended for the t-varying case. We again consider a perturbation of the type $\delta C_{ij} x_j(t)$ where now we choose $C=[L_5,L_6]$. The range of stability of the synchronized state in the σ-δ plane is shown in Fig. 1(b).

It may be noted that the introduction of perturbation of the type $\delta C_{ij} x_j(t)$ is equivalent to introducing both the Laplacians at the same time. Hence, as δ increases the t-varying and t-average cases cannot be distinguished on the basis of the eigenvectors of the Laplacians and we cannot decide the stability of the synchronized state using the present analysis. This is seen in Fig. 1(b) where for $1.1<\delta<1.4$ and $0.75<\sigma<1.7$ the t-average synchronized state has a slightly better stability than the t-varying one.

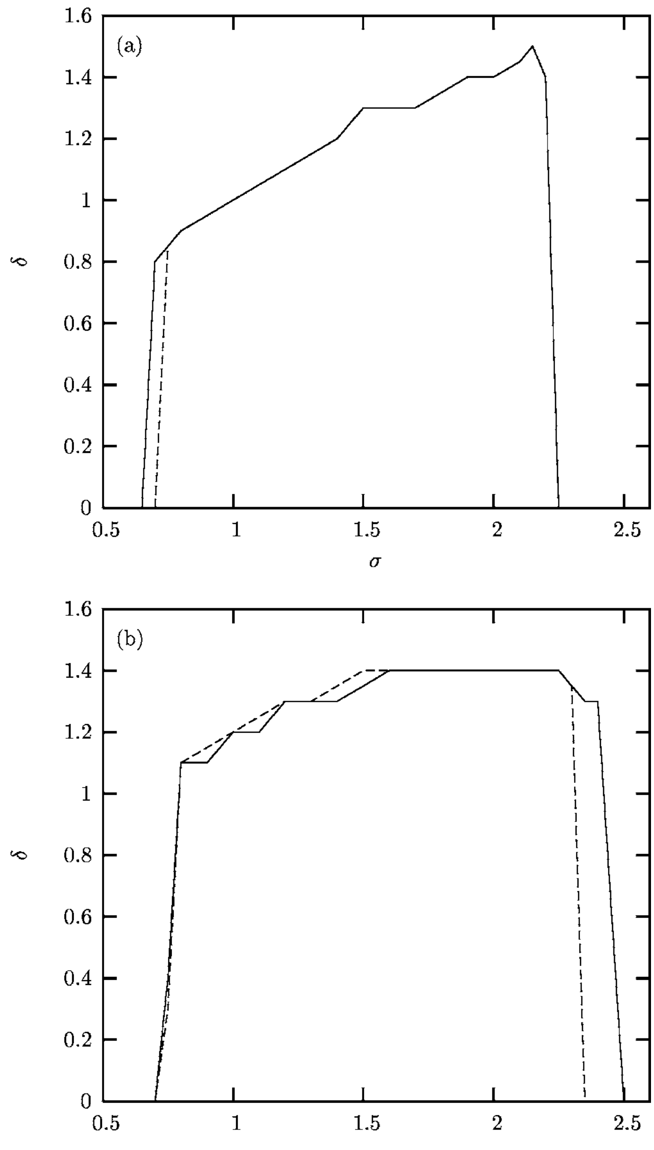

FIG. 1. The figure shows the stability region of the synchronized state in the σ-δ plane for a pair of noncommuting Laplacians. The solid line is for the t-varying case while the dashed line is for the t-average case. (a) is for the graphs G_3 and G_4 and (b) is for the graphs G_5 and G_6.

IV. CONCLUSION

In this paper we have considered coupled dynamics on networks with time varying topology. We analyze the linearized equations and consider its approximate solution for sufficiently fast switching between different Laplacians. We find that for commuting Laplacians the stability of the synchronized state for the t-varying case is mostly unaffected and is almost same the same as that for the average case. On the other hand, the noncommuting Laplacians, in general, lead to better stability of the synchronized state for the t-varying case than the t-average case.

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