Stability of periodic orbits of coupled-map lattices

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(Received 22 March 1991)

We consider the stability properties of spatial and temporal periodic orbits of one-dimensional coupled-map lattices. The stability matrices for them are of the block-circulant form. This helps us to reduce the problem of stability of spatially periodic orbits to the smaller orbits corresponding to the building blocks of spatial periodicity, enabling us to obtain the conditions for stability in terms of those for smaller orbits.

Spatially extended nonlinear dynamical systems have recently attracted considerable attention [1−3]. This is because of their wide range of applications such as turbulence, pattern formation in natural systems, solitons, etc. They also exhibit a very rich phenomenology including a wide variety of both spatial as well as temporal periodic structures, intermittency, chaos, domain walls, kink dynamics, etc.

In this Rapid Communication we address the problem of stability of spatial and temporal periodic structures. We specifically consider coupled-map lattices with nearest-neighbor couplings. Detailed numerical studies show that the coupled-map lattices give rise to a variety of rich spatial and temporal structures [1]. Consider following the general model:

\[ x_{i+1}(i) = h_0 f_0(x_i(i)) + h_1 f_1(x_i(i+1)) + h_{-1} f_{-1}(x_i(i-1)), \]

where \( x_i(i) \) is the variable associated with the \( i \)th lattice point at time \( t \) taking values in a suitably bounded phase space. The maps \( f_0, f_1, f_{-1} \) are maps, such as a logistic map, that describe the evolution of an otherwise isolated system. The parameters \( h_0, h_1, \) and \( h_{-1} \) represent the coupling strengths and are chosen so that \( x_{i+1}(i) \) lies in the same phase space (e.g., \([0,1]\) for the logistic map \( f(x) = \mu x(1-x), 0 \leq \mu \leq 4 \)). Henceforth, we assume that \( h_0, h_1, h_{-1} \) are positive. However, almost all our results are valid even otherwise.

Let \( \mathcal{C}_N \) denote a closed chain of \( N \) lattice points in which the right-hand neighbor of the \( N \)th point is the first lattice point. We note that for \( N = 1 \) the chain \( \mathcal{C}_N \) consists of a single point which is to be understood as a neighbor of itself. Let \( R_t = (x_1, \ldots, x_N) \) denote the state of the system for the chain \( \mathcal{C}_N \) at time \( t \). Let \( \mathcal{S}_N(N,1) \) denote a solution of Eq. (1) with temporal periodicity \( \tau \) for the chain \( \mathcal{C}_N \), i.e., \( \mathcal{S}_N(N,1) = \{ R_{1}, R_{2}, \ldots, R_{N}, R_{1}, R_{2}, \ldots \} \).

Now consider a closed chain of twice the length, i.e., \( \mathcal{C}_{2N} \). Obviously the spatially periodic sequence \( \mathcal{S}_2(N,2) = \{ R_{1} R_{2}, R_{3} R_{4}, \ldots, R_{N-1} R_{N}, R_{1} R_{2}, \ldots \} \)

represents a solution of Eq. (1) for the closed chain \( \mathcal{C}_{2N} \) with temporal periodicity \( \tau \). Here the ordered pair \( (R_t \cdots R_t) \) represents a state made up of \( k \) replicas of the state \( R_t \). We call \( \mathcal{S}_N(N,k) \) the \( k \) replica solution of \( \mathcal{S}_N(N,1) \).

The problem we address concerns the stability properties of such spatially and temporally periodic solutions \( \mathcal{S}_N(N,k) \), from the analysis of the stability matrices for \( \mathcal{S}_N(N,1) \) of the building blocks. In other words, we question what the effect is of enlargement of phase space and the couplings on the stability of the replica solutions.

We note that Waller and Kapral (Ref. [4]) have considered a similar problem for some very specific maps and couplings and for simple homogeneous and small period solutions. Here we analyze the problem in a very general way and obtain conditions for the stability of the spatially extended solution.

We begin with the simplest case of \( N = 1 \) so that \( R_t \) consists of a single lattice point \( x_1(t) = x_t \) and consequently we suppress the lattice index. The replica solution \( \mathcal{S}_1(1,k) \) for the chain \( \mathcal{C}_k = \mathcal{C}_{1 \times k} \) is a homogeneous solution with \( \{ x_t \} \) as building blocks. Now the solution \( \mathcal{S}_2(1,1) = \{ x_{1,1}, x_{1,2}, \ldots, x_{N,1}, x_{N,2}, \ldots \} \) for the building block is a stable solution provided

\[ |f'(x_1) f'(x_2) \cdots f'(x_N)| < 1, \]

where \( f(x) = h_0 f_0(x) + h_1 f_1(x) + h_{-1} f_{-1}(x) \) and \( f'(x) = df(x)/dx \). For the homogeneous solution \( \mathcal{S}_1(1,k) \), the stability condition is that all eigenvalues of the \( k \times k \) stability matrix \( J = J_1 J_2 \cdots J_k \) have magnitude less than one. Here \( J_i \) is a \( k \times k \) Jacobian matrix given by

\[ \begin{bmatrix} J_1 & \ldots & J_k \end{bmatrix} \]
The matrix $J_i$ is a circulant matrix whose eigenvalues are given by \[ \lambda_n = \left( h_0 f_0 + \omega_r h_1 f_1 + \omega_r^{k-1} h_{-1} f_{-1} \right) f'(x_i), \]
where $\omega_r = e^{i2\pi r/(k+1)}$. Thus the eigenvalues of the stability matrix $J$ are
\[ \lambda_r = \prod_{i=1}^{r} \lambda_i = \prod_{i=1}^{r} \left( h_0 f_0(x_i) + \omega_r h_1 f_1(x_i) + \omega_r^{k-1} h_{-1} f_{-1}(x_i) \right). \]
Now $|\lambda_r| < 1$, for all $r$, ensures the stability of the homogeneous solution $S_i(1,k)$.

Consider the special case when all the maps are the same, i.e., $f_0(x) = f_1(x) = f_{-1}(x)$. For a single point, i.e., the chain $\mathcal{E}_1$, this implies $h_0 + h_1 + h_{-1} = 1$. Assuming that condition (2) is satisfied, the homogeneous solution $S_i(1,k)$ for the chain $\mathcal{E}_N$ is stable if
\[ |h_0 + h_1 + h_{-1}| \leq 1. \]
Condition (6) is satisfied provided $|h_0 + h_1 + h_{-1}| \leq 1$, which is obviously true since $h_0 + h_1 + h_{-1} = 1$. Thus the stability of the homogeneous solution $S_i(1,k)$ is guaranteed by the stability of the single-point solution $S_i(1,1)$ for the same parameters of the map exhibiting no effect of enlargement of phase space and the couplings.

As a specific example, for this special case, we take the logmap. This map has several stable periodic orbits depending on the value of $\mu$ (Ref. [6]). In particular, it shows a period-doubling structure leading to a period-doubling attractor [6]. The above analysis shows that for the coupled logmap the entire period-doubling structure and the structure of other periodic windows will be lifted to the chain $\mathcal{E}_k$ for the same values of $\mu$ together with the same stability properties for all $k$.

Our second example is that considered by Waller and Kapral [4]. They consider the maps
\[ h_0 f_0(x) = \mu x(1-x) - 2\gamma x, \]
\[ h_1 f_1(x) = h_{-1} f_{-1}(x) = \gamma x. \]
Using Eq. (5) for the fixed point and the condition $\lambda = \pm 1$, i.e., the condition for marginal stability, we obtain the boundaries of the stability region of the fixed point and the periodic solution in the $\mu - \gamma$ plane. Our results coincide with those of Ref. [4]. For example, for the fixed point homogeneous solution for $x = 0$, the stability criterion using Eq. (5) is given by
\[ \mu = \pm 1 + \gamma (2 - e^{i\theta} - e^{-i\theta}), \]
where $\theta = 2\pi i / k$, $i = 0, 1, 2, \ldots, k$. This coincides with Eqs. (2) and (6) of Ref. [4].

Now we turn to the case of higher values of $N$. Consider the solution $S_i(N,1)$ for the closed chain $\mathcal{E}_N$. Stability of the solution is determined by the eigenvalue with largest magnitude of $N \times N$ matrix
\[ J = \begin{pmatrix} f'(x_i) & 0 & \cdots & 0 \\ -1 & f'(x_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f'(x_i) \end{pmatrix}, \]
where $J_i$ is the Jacobian matrix given by
\[ J_i = A_i + B_i + C_i, \]
Here $A_i$ is a tridiagonal matrix given by
\[ A_i = \begin{pmatrix} h_0 f_0(x_i(1)) & h_1 f_1(x_i(2)) & 0 & \cdots \\ -1 & h_0 f_0(x_i(2)) & h_1 f_1(x_i(3)) & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_0 f_0(x_i(N)) \end{pmatrix}, \]
and the matrices $B_i$ and $C_i$ have only a single nonzero element and are given by
\[ (B_i)_{ij} = h_1 f_1(x_i(1)) \delta_{iN} \delta_{j1}, \]
\[ (C_i)_{ij} = h_{-1} f_{-1}(x_i(N)) \delta_{i1} \delta_{jN}. \]
Let us now consider the solution $S_i(N,k)$ of the closed chain $\mathcal{E}_{k \times N}$ which is obtained by $k$ replicas of the solution $S_i(N,1)$ for $\mathcal{E}_N$. The stability of $S_i(N,k)$ is determined by the eigenvalues of $k \times k \times N$ stability matrix $J = J_1 J_2 \cdots J_k$ where $J_k$ is a $k \times k \times N$ Jacobian matrix given by
\[ J_k = \begin{pmatrix} A_i & B_i & 0 & \cdots & 0 & C_i \\ C_i & A_i & B_i & \cdots & 0 & 0 \\ 0 & C_i & A_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_i & B_i \\ B_i & 0 & 0 & \cdots & C_i & A_i \end{pmatrix}, \]
for $k > 2$. For $k = 2$,
\[ J_2 = \begin{pmatrix} A_i & B_i + C_i \\ B_i + C_i & A_i \end{pmatrix}, \]
and for $k = 1$, $J_1 = A_i + B_i + C_i$. We note that Jacobian matrices $J_i$ [Eqs. (12) and (13)] are block circulant where each block is an $N \times N$ matrix. This observation is crucial for our analysis of stability properties. A block-circulant matrix can be put into a block diagonal form by a unitary transformation [5]. The block diagonal form is
\[ D_i = \begin{pmatrix} M_i^1 & 0 & \cdots & 0 \\ 0 & M_i^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & M_i^k \end{pmatrix}, \]
where the matrices, $M_i^{(r)}$, $r=1, \ldots, k$, are $N \times N$ matrices given by

$$M_i^{(r)} = A_i + \omega_i B_i + \omega_i^{k-1} C_i.$$  \hspace{1cm} (15)

Note that this form is a generalization of Eq. (4). The matrix $M_i^{(1)}$ is the same as the matrix $j_i$ of Eq. (9) since $\omega_i = 1$. The unitary matrix which affects the above block diagonalization is a direct product of Fourier matrices of sizes $k \times k$ and $N \times N$. The elements of Fourier matrices are only roots of unity and thus are independent of the matrix being diagonalized. Consequently, the same unitary matrix block diagonalizes the product of $j_i$'s. Thus the block diagonal form of the product matrix $J = \prod_{i=1}^r J_i$ is given by

$$D_r = \prod_{i=1}^r M_i^{(1)} \prod_{i=1}^r M_i^{(2)} \cdots \prod_{i=1}^r M_i^{(k)}.$$  \hspace{1cm} (16)

The first block $\prod_{i=1}^r M_i^{(1)}$ is the same as the matrix $j = j_1 j_2 \cdots j_r$ of Eq. (9). The stability properties of the solution $S_i(N,k)$ are determined by the eigenvalues of matrix (16) of which $j$ is only one constituent block. In addition to the eigenvalues of $j$, we now must look at the eigenvalues of the remaining $k - 1$ blocks of Eq. (16). Thus the effects on the stability due to the enlargement of the phase space and couplings manifest themselves through the eigenvalues of the additional blocks. A general block $M$ (size $N \times N$) has the following structure:

$$M = \prod_{i=1}^r (A_i + e^{i\theta} B_i + e^{-i\theta} C_i),$$  \hspace{1cm} (17)

where $\omega_i = e^{i\theta}$ and $\omega_i^{k-1} = e^{-i\theta}$.

We note that the elements of $M$ are just combinations of entries of $j$ (Eqs. (9)–(11)). Thus the problem of stability analysis of larger orbits is reduced to that of the entries of $j$. This corresponds to the reduction of the analysis of $kN \times kN$ matrices to that of $N \times N$ matrices.

Coming back to Eq. (17), it is clear that if we check the eigenvalues of $M$ for $\theta$ between 0 and $2\pi$, it ensures the stability for all values of $k$. Actually, it is sufficient to check for $0 \leq \theta \leq \pi$. Of course, for a given value of $k$ it is sufficient to check for a maximum of $[(k/2) + 1] \theta$ values [7].

A further simplification occurs for $r = 1$, i.e., for a fixed-point solution. In this case it can be shown that the eigenvalues of Eq. (17) need to be checked only for $\theta = 0$ and $\theta = \pi$ for the following two cases. (a) $N$ is 2 or 3. (b) The largest (absolute) eigenvalue is real as a function of $\theta$.

We now illustrate our procedure with coupled logistic maps with $\mu = 4$, i.e., $f(x) = \mu x (1 - x)$ and $x_{r+1}(i) = (1 - \epsilon) f(x_r(i)) + \frac{1}{2} \epsilon f(x_r(i+1))$

$$+ \frac{1}{2} \epsilon f(x_r(i-1)),$$  \hspace{1cm} (18)

where $0 \leq \epsilon \leq 1$. We discuss the stability of two different solutions. First, consider a fixed-point solution $S_i(2,1) = (x_1(1), x_1(2))$ of Eq. (18) for the chain $e_i$ with $x_1(1) \neq x_1(2)$. This solution is stable for the values of $\epsilon$ in the range $(4 + \sqrt{6})/8 \approx 0.806 \ldots$ to $(4 + \sqrt{7})/32 \approx 0.860 \ldots$. As noted above, to check the stability of the solution $S_i(2,1)$ for the chain $e_{i+2}$ obtained by $k$ replicas of the solution $S_i(2,1)$, it is sufficient to consider only two values of $\theta$, namely, $\theta = 0$ and $\pi$ in Eq. (14). The condition for stability for $\theta = 0$ is the same as that for the solution $S_i(2,1)$ and one needs to check only for $\theta = \pi$ additionally. Explicit calculation shows that the solution $S_i(2,1)$ remains stable for the same range of $\epsilon$ values for all $k$.

Second, we consider a period-two solution of Eq. (18) for the closed chain $e_2$, namely, $S_i(2,1) = [R_i, R_j]$ where $R_1 = (x_1(1), x_1(2))$ and $R_2 = (x_1(1), x_1(2), x_1(2))$ with $x_1(1) \neq x_1(2)$. This solution is stable for $\epsilon$.

FIG. 1. This figure shows kink-type solutions for various $N$ and their replica solutions for $k = 10$. (a) A kink solution $S_2(5,1)$ on the left-hand side and its replica solution for $k = 10$ on the right-hand side for $\epsilon = 0.08$. (b),(c) Similar figures for $N = 7$ and 8, respectively.
in the range from \((13 - \sqrt{73})/32 = 0.1392\ldots\) to
\((4 - \sqrt{6})/8 = 0.1938\ldots\). The stability of the \(k\) replica
solution \(S_2(2,k)\) has been verified numerically using Eq.
(17). For even \(k\), the lower bound shifts to 0.14037\ldots For
odd \(k\), the lower bound approaches this value according
to the sequence 0.14009 \ldots for \(k = 3, 0.14026 \ldots\), for
\(k = 5, 0.14031 \ldots\) for \(k = 7\), etc.

Our next example is the kink-type solutions [3] to Eq.
(18). We have considered several kink-type solutions.
Here we discuss the following solutions for \(\mu = 3.41\): (i)
\(N = 5\). Consider the basic unit \(S_2(5,1)\) shown in Fig.
1(a). Figure 1(a) also shows the replica solution with
\(k = 10\). The basic unit \(S_2(5,1)\) is stable in the \(\epsilon\) range
from 0 to 0.0967\ldots We use Eq. (17) to determine the
stability of the replica solutions. The higher-order solutions
are stable in the same range within computational
accuracy. This has been confirmed by actual numerical
simulations for replica solutions with many \(k\) values. (ii)
\(N = 7\). Consider the basic unit \(S_2(7,1)\) shown in Fig.
1(b). Figure 1(b) also shows the replica solution with
\(k = 10\). The basic unit \(S_2(7,1)\) is stable in the range from
0 to 0.33772\ldots We analyze the stability of replica solutions
using Eq. (16). We find that for even \(k\), the stability
range reduces from \(\epsilon = 0\) to \(\epsilon = 0.33762\ldots\) For odd \(k\),
the lower limit remains the same, i.e., \(\epsilon = 0\) and the upper
limit approaches 0.33762\ldots by the sequence 0.33766\ldots
for \(k = 5, 0.33764\ldots\) for \(k = 5\), etc. Again this result has
been confirmed by actual numerical simulations. (iii)
\(N = 6\) and 8. We consider the kink solutions, each with an
equal number of consecutive points in the upper and lower
branches, i.e., 3 and 4 points for \(N = 6\) and 8, respectively.
The basic unit \(S_2(8,1)\) for \(N = 8\) and its replica solution
with \(k = 10\) are shown in Fig. 1(c). In this case, using Eq.
(17) and actual numerical simulations we find that the
stability of the replica solutions remains unchanged by en-
largement of the phase space.

We have discussed above the conditions that ensure the
stability of spatially and temporally periodic orbits. In
addition, our analysis also leads to the following important
conclusion about unstable periodic orbits. As noted in a
comment after Eq. (16) the matrix \(J\) appears as a block of
the matrix \(D\) of Eq. (16). Hence, a solution built out of
replicas of unstable periodic orbits will also be unstable.
Enlargement of phase space and the effect of couplings
cannot stabilize an unstable replica solution.

To conclude, we have shown that the stability of spati-
ally and temporally periodic orbits can be analyzed in
terms of smaller orbits made up of building blocks of spa-
tial periodicity. We find that for the homogeneous solu-
tion no further conditions are imposed if \(f_0 = f_1 = f_-1\) and
the stable solution for a single point remains stable on the
enlargement of phase space and the introduction of cou-
lings. However, solutions with larger spatial periodicities
require additional conditions for stability. These condi-
tions depend on the stability matrices for the building
block of spatial periodicity and the roots of unity. We also
find that replica solution of unstable periodic orbits
remain unstable. It is clear that these replica solutions
can be used to construct a hierarchy of unstable periodic
orbits based on the orbits for building blocks. This may
help in organization of spatiotemporal chaos on the lines
of arguments in Ref. [8].

One of us (V.M.N.) thanks the University of Poona for
the hospitality at Poona. R.E.A. and A.D.G. thank the
Department of Science and Technology (India) and
A.D.G. and P.M.G. thank the University Grants Com-
mission (India) for financial assistance.

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