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# Stability of periodic orbits of coupled-map lattices

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We consider the stability properties of spatial and temporal periodic orbits of one-dimensional coupled-map lattices. The stability matrices for them are of the block-circulant form. This helps us to reduce the problem of stability of spatially periodic orbits to the smaller orbits corresponding to the building blocks of spatial periodicity, enabling us to obtain the conditions for stability in terms of those for smaller orbits.

Spatially extended nonlinear dynamical systems have recently attracted considerable attention [1-3]. This is because of their wide range of applications such as turbulence, pattern formation in natural systems, solitons, etc. They also exhibit a very rich phenomenology including a wide variety of both spatial as well as temporal periodic structures, intermittency, chaos, domain walls, kink dynamics, etc.

In this Rapid Communication we address the problem of stability of spatial and temporal periodic structures. We specifically consider coupled-map lattices with nearest-neighbor couplings. Detailed numerical studies show that the coupled-map lattices give rise to a variety of rich spatial and temporal structures [1]. Consider following the general model:

$$x_{t+1}(i)=h_0 f_0(x_t(i))+h_1 f_1(x_t(i+1))+h_{-1}f_{-1}(x_t(i-1)), \tag{1}$$

where $x_t(i)$ is the variable associated with the $i$th lattice point at time $t$ taking values in a suitably bounded phase space. The maps $f_0$, $f_1$, $f_{-1}$ are maps, such as a logistic map, that describe the evolution of an otherwise isolated system. The parameters $h_0$, $h_1$, and $h_{-1}$ represent the coupling strengths and are chosen so that $x_{t+1}(i)$ lies in the same phase space (e.g., [0,1] for the logistic map $f(x)=\mu x(1-x)$, $0\le\mu\le 4$). Henceforth, we assume that $h_0$, $h_1$, $h_{-1}$ are positive. However, almost all our results are valid even otherwise.

Let $\mathcal{C}_N$ denote a closed chain of $N$ lattice points in which the right-hand neighbor of the $N$th point is the first lattice point. We note that for $N=1$ the chain $\mathcal{C}_1$ consists of a single point which is to be understood as a neighbor of itself. Let $R_t=(x_t(1),\ldots,x_t(N))$ denote the state of the system for the chain $\mathcal{C}_N$ at time $t$. Let $S_\tau(N,1)$ denote a solution of Eq. (1) with temporal periodicity $\tau$ for the chain $\mathcal{C}_N$, i.e., $S_\tau(N,1)=\{R_1,R_2,\ldots,R_\tau,R_1,R_2,\ldots\}$. Now consider a closed chain of twice the length, i.e., $\mathcal{C}_{2N}\equiv\mathcal{C}_{2\times N}$. Obviously the spatially periodic sequence

$$S_\tau(N,2)=\{\langle R_1R_1\rangle_2,\langle R_2R_2\rangle_2,\ldots,\langle R_\tau R_\tau\rangle_2,\langle R_1R_1\rangle_2,\ldots\}$$

built from the states $\{R_t\}$ as the building blocks is a solution of Eq. (1) for the closed chain $\mathcal{C}_{2\times N}$ with temporal periodicity $\tau$. Here the ordered pair $\langle R_tR_t\rangle_2$ represents the state $(x_t(1),\ldots,x_t(N),\ldots,X_t(2N))$, with $x_t(N+i)=x_t(i)$, $i=1,2,\ldots,N$, which is made up of two replicas of the state $R_t$. In general, the sequence

$$S_\tau(N,k)=\{\langle R_1R_1,\ldots,R_1\rangle_k,\ldots,\langle R_\tau,\ldots,R_\tau\rangle_k,\langle R_1,\ldots,R_1\rangle_k,\ldots\}$$

represents a solution of Eq. (1) for the closed chain $\mathcal{C}_{k\times N}$ with temporal periodicity $\tau$. Here the ordered pair $\langle R_t\cdots R_t\rangle_k$ represents a state made up of $k$ replicas of the state $R_t$. We call $S_\tau(N,k)$ the $k$ replica solution of $S_\tau(N,1)$. The problem we address concerns the stability properties of such spatially and temporally periodic solutions $S_\tau(N,k)$, from the analysis of the stability matrices for $S_\tau(N,1)$ of the building blocks. In other words, we question what the effect is of enlargement of phase space and the couplings on the stability of the replica solutions.

We note that Waller and Kapral (Ref. [4]) have considered a similar problem for some very specific maps and couplings and for simple homogeneous and small period solutions. Here we analyze the problem in a very general way and obtain conditions for the stability of the spatially extended solution.

We begin with the simplest case of $N=1$ so that $R_t$ consists of a single lattice point $x_t(1)=x_t$ and consequently we suppress the lattice index. The replica solution $S_\tau(1,k)$ for the chain $\mathcal{C}_k=\mathcal{C}_{k\times 1}$ is a homogeneous solution with $\{x_t\}$ as building blocks. Now the solution $S_\tau(1,1)=\{x_1,x_2,\ldots,x_\tau,x_1,x_2,\ldots\}$ for the building block is a stable solution provided

$$|f'(x_1)f'(x_2),\ldots,f'(x_\tau)|<1\,, \tag{2}$$

where $f(x)=h_0f_0(x)+h_1f_1(x)+h_{-1}f_{-1}(x)$ and $f'(x)=df(x)/dx$. For the homogeneous solution $S_\tau(1,k)$, the stability condition is that all eigenvalues of the $k\times k$ stability matrix $J=J_1J_2\cdots J_\tau$ have magnitude less than one. Here $J_t$ is a $k\times k$ Jacobian matrix given by

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$$J_t=\begin{pmatrix} h_0f_0' & h_1f_1' & 0 & \cdots & 0 & 0 & h_{-1}f_{-1}' \\ h_{-1}f_{-1}' & h_0f_0' & h_1f_1' & \cdots & 0 & 0 & 0 \\ 0 & h_{-1}f_{-1}' & h_0f_0' & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & h_{-1}f_{-1}' & h_0f_0' & h_1f_1' \\ h_1f_1' & 0 & 0 & \cdots & 0 & h_{-1}f_{-1}' & h_0f_0' \end{pmatrix}. \quad (3)$$

The matrix $J_t$ is a circulant matrix whose eigenvalues are given by [5]

$$\lambda_{t,r}=(h_0f_0'+\omega_r h_1f_1'+\omega_r^{k-1}h_{-1}f_{-1}')f'(x_t)\,, \quad r=1,2,\ldots,k\,, \quad (4)$$

where $\omega_r$ is the $k$th root of unity given by $\omega_r=e^{i[2\pi(r-1)/k]}$. Thus the eigenvalues of the stability matrix $J$ are

$$\lambda_r=\prod_{t=1}^{\tau}\lambda_{t,r}=\prod_{t=1}^{\tau}[h_0f_0'(x_t)+\omega_r h_1f_1'(x_t)+\omega_r^{k-1}h_{-1}f_{-1}'(x_t)]\,. \quad (5)$$

Now $|\lambda_r|<1$, for all $r$, ensures the stability of the homogeneous solution $S_\tau(1,k)$.

Consider the special case when all the maps are the same, i.e., $f_0(x)=f_1(x)=f_{-1}(x)$. For a single point, i.e., the chain $\mathcal{C}_1$, this implies that $h_0+h_1+h_{-1}=1$. Assuming that condition (2) is satisfied, the homogeneous solution $S_\tau(1,k)$ for the chain $\mathcal{C}_N$ is stable if

$$|h_0+\omega_r h_1+\omega_r^{k-1}h_{-1}|\le 1\,. \quad (6)$$

Condition (6) is satisfied provided $|h_0+h_1+h_{-1}|\le 1$, which is obviously true since $h_0+h_1+h_{-1}=1$. Thus the stability of the homogeneous solution $S_\tau(1,k)$ is guaranteed by the stability of the single-point solution $S_\tau(1,1)$ for the same parameters of the map exhibiting no effect of enlargement of phase space and the couplings.

As a specific example, for this special case, we take the logistic map. This map has several stable periodic orbits depending on the value of $\mu$ (Ref. [6]). In particular, it shows a period-doubling structure leading to a period-doubling attractor [6]. The above analysis shows that for the coupled logistic map the entire period-doubling structure and the structure of other periodic windows will be lifted to the chain $\mathcal{C}_k$ for the same values of $\mu$ together with the same stability properties for all $k$.

Our second example is that considered by Waller and Kapral [4]. They consider the maps

$$h_0f_0(x)=\mu x(1-x)-2\gamma x\,, \quad (7a)$$

$$h_1f_1(x)=h_{-1}f_{-1}(x)=\gamma x\,. \quad (7b)$$

Using Eq. (5) for the fixed point and the condition $\lambda=\pm 1$, i.e., the condition for marginal stability, we obtain the boundaries of the stability region of the fixed point and the periodic solution in the $\mu$-$\gamma$ plane. Our results coincide with those of Ref. [4]. For example, for the fixed point homogeneous solution for $x=0$, the stability criterion using Eq. (5) is given by

$$\mu=\pm 1+\gamma(2-e^{i\theta}-e^{-i\theta})\,, \quad (8)$$

where $\theta=2\pi i/k$, $i=0,1,2,\ldots,k$. This coincides with Eqs. (2) and (6) of Ref. [4].

Now we turn to the case of higher values of $N$. Consider the solution $S_\tau(N,1)$ for the closed chain $\mathcal{C}_N$. Stability of the solution is determined by the eigenvalue with largest magnitude of $N\times N$ matrix

$$j=j_1j_2\cdots j_\tau\,, \quad (9)$$

where $j_t$ is the Jacobian matrix given by

$$j_t=A_t+B_t+C_t\,. \quad (10)$$

Here $A_t$ is a tridigonal matrix given by

$$A_t=\begin{pmatrix} h_0f_0'(x_t(1)) & h_1f_1'(x_t(2)) & 0 & \cdots \\ h_{-1}f_{-1}'(x_t(1)) & h_0f_0'(x_t(2)) & h_1f_1'(x_t(3)) & \cdots \\ \vdots & \vdots & & \end{pmatrix} \quad (11a)$$

and the matrices $B_t$ and $C_t$ have only a single nonzero element and are given by

$$(B_t)_{ij}=h_1f_1'(x_t(1))\delta_{iN}\delta_{j1}\,, \quad (11b)$$

$$(C_t)_{ij}=h_{-1}f_{-1}'(x_t(N))\delta_{i1}\delta_{jN}\,. \quad (11c)$$

Let us now consider the solution $S_\tau(N,k)$ of the closed chain $\mathcal{C}_{k\times N}$ which is obtained by $k$ replicas of the solution $S_\tau(N,1)$ for $\mathcal{C}_N$. The stability of $S_\tau(N,k)$ is determined by the eigenvalues of $kN\times kN$ stability matrix $J=J_1J_2\cdots J_\tau$ where $J_t$ is a $kN\times kN$ Jacobian matrix given by

$$J_t=\begin{pmatrix} A_t & B_t & 0 & \cdots & 0 & C_t \\ C_t & A_t & B_t & \cdots & 0 & 0 \\ 0 & C_t & A_t & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & A_t & B_t \\ B_t & 0 & 0 & \cdots & C_t & A_t \end{pmatrix} \quad (12)$$

for $k>2$. For $k=2$,

$$J_t=\begin{pmatrix} A_t & B_t+C_t \\ B_t+C_t & A_t \end{pmatrix} \quad (13)$$

and for $k=1$, $J_t=A_t+B_t+C_t=j_t$. We note that Jacobian matrices $J_t$ [Eqs. (12) and (13)] are block circulant where each block is an $N\times N$ matrix. This observation is crucial for our analysis of stability properties. A block-circulant matrix can be put into a block diagonal form by a unitary transformation [5]. The block diagonal form is

$$D_t=\begin{pmatrix} M_t^1 & 0 & \cdots & 0 \\ 0 & M_t^2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & M_t^k \end{pmatrix}, \quad (14)$$

where the matrices, $M_t^{(r)}$, $r=1,\ldots,k$, are $N\times N$ matrices given by

$$M_t^{(r)}=A_t+\omega_r B_t+\omega_r^{k-1}C_t\,. \tag{15}$$

Note that this form is a generalization of Eq. (4). The matrix $M_t^{(1)}$ is the same as the matrix $j_t$ of Eq. (9) since $\omega_1=1$. The unitary matrix which affects the above block diagonalization is a direct product of Fourier matrices of sizes $k\times k$ and $N\times N$. The elements of Fourier matrices are only roots of unity and thus are independent of the matrix being diagonalized. Consequently, the same unitary matrix block diagonalizes the product of $J_t$'s. Thus the block diagonal form of the product matrix $J=J_1,J_2\cdots J_\tau$ is given by

$$D_t=\begin{bmatrix}\prod_{t=1}^{\tau}M_t^{(1)} & & & \\ & \prod_{t=1}^{\tau}M_t^{(2)} & & \\ & & \ddots & \\ & & & \prod_{t=1}^{\tau}M_t^{(k)}\end{bmatrix}. \tag{16}$$

The first block $\prod_{t=1}^{\tau}M_t^{(1)}$ is the same as the matrix $j=j_1j_2\cdots j_\tau$ of Eq. (9). The stability properties of the solution $S_\tau(N,k)$ are determined by the eigenvalues of matrix (16) of which $j$ is only one constituent block. In addition to the eigenvalues of $j$, we now must look at the eigenvalues of the remaining $k-1$ blocks of Eq. (16). Thus the effects on the stability due to the enlargement of the phase space and couplings manifest themselves through the eigenvalues of the additional blocks. A general block $M$ (size $N\times N$) has the following structure:

$$M=\prod_{t=1}^{\tau}(A_t+e^{i\theta}B_t+e^{-i\theta}C_t)\,, \tag{17}$$

where $\omega_r=e^{i\theta}$ and $\omega_r^{k-1}=e^{-i\theta}$.

We note that the elements of $M$ are just combinations of entries of $j$ [Eqs. (9)–(11)]. Thus the problem of stability analysis of larger orbits is reduced to that of the entries of $j$. This corresponds to the reduction of the analysis of $kN\times kN$ matrices to that of $N\times N$ matrices.

Coming back to Eq. (17), it is clear that if we check the eigenvalues of $M$ for $\theta$ between 0 and $2\pi$, it ensures the stability for all values of $k$. Actually, it is sufficient to check for $0\le\theta\le\pi$. Of course, for a given value of $k$ it is sufficient to check for a maximum of $[(k/2)+1]\theta$ values [7].

A further simplification occurs for $\tau=1$, i.e., for a fixed-point solution. In this case it can be shown that the eigenvalues of Eq. (17) need to be checked only for $\theta=0$ and $\theta=\pi$ for the following two cases. (a) $N$ is 2 or 3. (b) The largest (absolute) eigenvalue is real as a function of $\theta$.

We now illustrate our procedure with coupled logistic maps with $\mu=4$, i.e., $f(x)=\mu x(1-x)$ and

$$x_{t+1}(i)=(1-\epsilon)f(x_t(i))+\tfrac{1}{2}\epsilon f(x_t(i+1))+\tfrac{1}{2}\epsilon f(x_t(i-1))\,, \tag{18}$$

where $0\le\epsilon\le 1$. We discuss the stability of two different solutions. First, consider a fixed-point solution $S_1(2,1)=(x_1(1),x_1(2))$ of Eq. (18) for the chain $\mathcal{C}_2$ with $x_1(1)\neq x_1(2)$. This solution is stable for the values of $\epsilon$ in the range $(4+\sqrt{6})/8=0.806\ldots$ to $(4+\sqrt{73})/32=0.860\ldots$. As noted above, to check the stability of the solution $S_1(2,k)$ for the chain $\mathcal{C}_{k\times 2}$ obtained by $k$ replicas of the solution $S_1(2,1)$, it is sufficient to consider only two values of $\theta$, namely, $\theta=0$ and $\pi$ in Eq. (14). The condition for stability for $\theta=0$ is the same as that for the solution $S_1(2,1)$ and one needs to check only for $\theta=\pi$ additionally. Explicit calculation shows that the solution $S_1(2,k)$ remains stable for the same range of $\epsilon$ values for all $k$.

Second, we consider a period-two solution of Eq. (18) for the closed chain $\mathcal{C}_2$, namely, $S_2(2,1)=\{R_1,R_2\}$ where $R_1=(x_1(1),x_1(2))$ and $R_2=(x_2(1)=x_1(2),x_2(2)=x_1(1))$ with $x_1(1)\neq x_1(2)$. This solution is stable for $\epsilon$

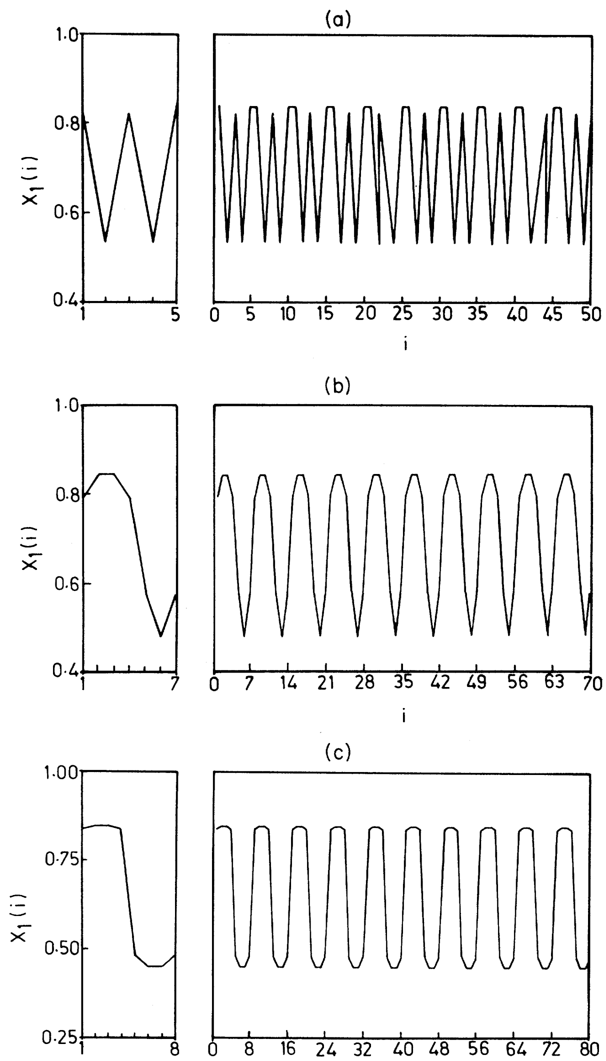


FIG. 1. This figure shows kink-type solutions for various $N$ and their replica solutions for $k=10$. (a) A kink solution $S_2(5,1)$ on the left-hand side and its replica solution for $k=10$ on the right-hand side for $\epsilon=0.08$. (b),(c) Similar figures for $N=7$ and 8, respectively.

in the range from $(13-\sqrt{73})/32=0.1392\ldots$ to $(4-\sqrt{6})/8=0.1938\ldots$. The stability of the $k$ replica solution $S_2(2,k)$ has been verified numerically using Eq. (17). For even $k$, the lower bound shifts to $0.14037\ldots$. For odd $k$, the lower bound approaches this value according to the sequence $0.14009\ldots$ for $k=3$, $0.14026\ldots$, for $k=5$, $0.14031\ldots$ for $k=7$, etc.

Our next example is the kink-type solutions [3] to Eq. (18). We have considered several kink-type solutions. Here we discuss the following solutions for $\mu=3.41$: (i) $N=5$. Consider the basic unit $S_2(5,1)$ shown in Fig. 1(a). Figure 1(a) also shows the replica solution with $k=10$. The basic unit $S_2(5,1)$ is stable in the $\epsilon$ range from 0 to $0.0967\ldots$. We use Eq. (17) to determine the stability of the replica solutions. The higher-order solutions are stable in the same range within computational accuracy. This has been confirmed by actual numerical simulations for replica solutions with many $k$ values. (ii) $N=7$. Consider the basic unit $S_2(7,1)$ shown in Fig. 1(b). Figure 1(b) also shows the replica solution with $k=10$. The basic unit $S_2(7,1)$ is stable in the range from 0 to $0.33772\ldots$. We analyze the stability of replica solutions using Eq. (16). We find that for even $k$, the stability range reduces from $\epsilon=0$ to $\epsilon=0.33762\ldots$. For odd $k$, the lower limit remains the same, i.e., $\epsilon=0$ and the upper limit approaches $0.33762\ldots$ by the sequence $0.33766\ldots$ for $k=3$, $0.33764\ldots$ for $k=5$, etc. Again this result has been confirmed by actual numerical simulations. (iii) $N=6$ and 8. We consider the kink solutions, each with an equal number of consecutive points in the upper and lower branches, i.e., 3 and 4 points for $N=6$ and 8, respectively. The basic unit $S_2(8,1)$ for $N=8$ and its replica solution with $k=10$ are shown in Fig. 1(c). In this case, using Eq. (17) and actual numerical simulations we find that the stability of the replica solutions remains unchanged by enlargement of the phase space.

We have discussed above the conditions that ensure the stability of spatially and temporally periodic orbits. In addition, our analysis also leads to the following important conclusion about unstable periodic orbits. As noted in a comment after Eq. (16) the matrix $j$ appears as a block of the matrix $D$ of Eq. (16). Hence, a solution built out of replicas of unstable periodic orbits will also be unstable. Enlargement of phase space and the effect of couplings cannot stabilize an unstable replica solution.

To conclude, we have shown that the stability of spatially and temporally periodic orbits can be analyzed in terms of smaller orbits made up of building blocks of spatial periodicity. We find that for the homogeneous solution no further conditions are imposed if $f_0=f_1=f_{-1}$ and the stable solution for a single point remains stable on the enlargement of phase space and the introduction of couplings. However, solutions with larger spatial periodicities require additional conditions for stability. These conditions depend on the stability matrices for the building block of spatial periodicity and the roots of unity. We also find that replica solution of unstable periodic orbits remain unstable. It is clear that these replica solutions can be used to construct a hierarchy of unstable periodic orbits based on the orbits for building blocks. This may help in organization of spatiotemporal chaos on the lines of arguments in Ref. [8].

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[7] It is possible to obtain bounds on the eigenvalues of $M$ [Eq. (15)] which are independent of $\theta$, in terms of matrix norms $A_l$ and the absolute values of the nonzero elements of $B_l$ and $C_l$. However, these bounds are, in general, weak.

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