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The Kosterlitz-Thouless transition and vortex dynamics

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Abstract. The physics of the Kosterlitz-Thouless vortex-unbinding transition in two-dimensional superfluids is discussed, and the $N \times N$ Josephson junction array is considered as a prototype system. Dynamical behaviour is considered in two cases: (a) the complex impedance shows structure at a frequency-dependent transition temperature, similar to the dynamic susceptibility of a spin glass; (b) with a perpendicular non-uniform magnetic field, of a particular 'self-similar' hierarchical pattern, a scaling argument gives non-exponential relaxational dynamics of a prepared non-equilibrium vortex distribution.

Keywords. Kosterlitz-Thouless transition: vortex dynamics; Josephson junction array.

1. Introduction

Extended defects are familiar in the context of crystalline solids. In particular, edge and screw dislocations are line defects, that cause strain fields falling off as $1/r$ throughout the crystal. In superfluids, the complex order parameter $\psi(r) = |\psi(r)| \exp[i\theta(r)]$ can also exhibit a screw dislocation or 'vortex', with a jump of $\theta(r)$ by $2n\pi$ on moving around the vortex line, where $n = \pm 1, \pm 2 \ldots$ denoting the vorticity (analogous to a Burgers vector). Since there is an energy cost per unit length, and the lines can only terminate on boundaries, these line defects in 3D are created by external rotation in superfluids and by external magnetic fields for superconductors. In 2D, however, with superfluid/superconductor films of small thickness compared to a coherence length, thermal energies $k_B T$ are sufficient to create pairs of ± 1 vortex 'points' (line segments along the thickness). These have a (logarithmic) effective interaction, can participate in the novel type of Berezinskii-Kosterlitz-Thouless phase transition, and play an important role in the dynamic response of superfluids, superconductors and Josephson junction arrays.

In this paper we consider the vortex defect dynamics of Josephson arrays, that exhibit a Kosterlitz-Thouless transition (Rice 1965; Mermin 1967; Berezinskii 1972; Kosterlitz and Thouless 1973; Kosterlitz 1974).

2. The Kosterlitz-Thouless transition

2.1 *Statics*

For a planar ferromagnet with $d < d_c = 2$, a lower critical dimension, there is no long-range-order (LRO) in the spin-spin correlation function (Rice 1965; Mermin 1967) $<\exp\{i[\theta(r) - \theta(0)]\}> \equiv C(r)$. A phase transition occurs at $T < T_{\mathrm{KT}}$ by the unbinding of pairs of vortex excitations of opposite vorticity. This is the Kosterlitz-Thouless transition (Berezinskii 1972; Kosterlitz and Thouless 1973; Kosterlitz 1974).

The vortex pairs are ± 1 charges that interact via a $\ln r$ interaction, and the 2D planar ferromagnet (XY model) is thus mapped onto a 2D Coulomb gas. The transition shows up as a discontinuity in the inverse dielectric constant (Nelson and Kosterlitz 1977) at $T = T_{\mathrm{KT}}$ where the dielectric to 'plasma' transition takes place.

The Coulomb gas partition function onto which the original 2D XY model

$$\beta H = -K_0 \Sigma_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

is mapped to:

$$Z_v = \sum_{\substack{\{m_i\} \\ \text{Configurations}}} y_0^{\Sigma_i m_i^2} \exp\left[\pi K_0 \sum_{k \neq l} m_k m_l \ln(r_{kl}/a_0)\right].$$

The lattice scale a_0 is the minimum scale and all separations $r_{kl} > a_0$. Here $\Sigma_i m_i$ … and K_0 enters the coupling between the (dual lattice) vortex charges $m_i = \pm 1$, th… interact logarithmically. $y_0^2 = \exp(-\pi^2 K_0)$ is the fugacity for thermally creating a … pair.

The vortices bind in $\pm$ pairs, but are nested within each other, rather than bei… widely separated 'molecules'. Thus a scaling procedure (Berezinskii 1972; Kosterl… and Thouless 1972; Kosterlitz 1973) iteratively integrating out pairs of separation … $a + da$ is needed. One gets scale-dependent couplings and fugacities, $K(a) \equiv K_l$, $y(a) \equiv y_l$; $l \equiv \ln(a/a_0)$ and a set of scaling equations due to Kosterlitz (Kosterlitz and Thouless 1972; Kosterlitz 1973):

$$dK_l/dl = -4\pi^3 K_l^2 y_l^2,$$

$$dy_l/dl = -(\pi K_l - 2)\, y_l.$$

The force, at large distances between test charges is

$$\lim_{r \to \infty} (2\pi K(r)/r)$$

and this goes to $2\pi K_\infty / r$ in the dielectric phase for $\pi K_\infty(T) - 2 > 0$ i.e. $T < T_{\mathrm{KT}}$. For $T > T_{\mathrm{KT}}$ or $\pi K_\infty(T) - 2 < 0$, there are free vortices to provide plasma screening and $2\pi K(r)/r \to 0$ as $r \to \infty$. Here $K_\infty \equiv \lim_{r \to \infty} K(r)$.

The dielectric constant, that has a discontinuity at T_{KT}, is $K_\infty / K_0 = 1/\varepsilon$.

2.2 *Dynamics*

Helium films, with hamiltonians $\beta H = K \int d^2 r (\nabla \theta(r))^2$ also exhibit a KT vortex induced phase transition. This has been probed by Bishop and Reppy (1980) in a replay of the Andronikashvili torsional oscillator experiment, using a jelly roll of plastic film immersed in a superfluid helium bath. The frequency shift ω showed a roll-off, and the damping showed a peak, at a frequency-dependent temperature $T_\omega > T_{\mathrm{KT}}$, rather than at $T = T_{\mathrm{KT}}$ where the transition occurred.

The physics of such behaviour was understood in terms of the phenomenology of Ambegaokar *et al* (1980). For an oscillating probe there is a probe length $r_\omega = [(D_v/a_0^2)/\omega]^{\frac{1}{2}}$ above which vortex pairs respond too slowly to contribute, they are thus 'frozen out'. Thus only scales $< r_\omega$ are within an observation window. T_{KT} corresponds to the unbinding of *infinite* sized pairs, outside the observation window $< r_\omega$, so no singular behaviour is seen. Note that D_v is a phenomenological vortex diffusion constant and is a fitted parameter. In the microscopic calculation described later, one obtains similar results, with microscopic quantities only.

For $T > T_{KT}$, $\xi_+(T)$ is the length scale separating bound pairs (of separation $<\xi_+$) from free vortices ($>\xi_+$). Here

$$\xi_+(T) = a_0 \exp[\text{const}(T - T_{KT})^{-\frac{1}{2}}]$$

and T_ω is defined by, when the free vortices first enter the observation window, $r_\omega = \xi_+(T_\omega)$; so $T_\omega > T_{KT}$, but $T_\omega \to T_{KT}$ as $\omega \to 0$.

The Josephson junction arrays (JJA) in 2D (Lobb 1984) are grains of superconducting material below their grain transition temperature, placed in an $N \times N$ array connected by oxide (or normal metal) junctions (figure 1). The hamiltonian is of the XY form

$$\beta H = -K_0 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

and therefore, this system has a KT transition. Here $K_0 \equiv \hbar I_J / 2eK_B T$ where I_J is the maximum Josephson current of a single junction.

The JJA system (currently prepared in 1000×1000 arrays by photolithographic techniques) is more than just a physical realization of the 2D XY model. Magnetic fields perpendicular to the array enter in minimally coupled form through a vector potential A_{ij}:

$$\beta H = -K_0 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j - A_{ij});$$

i, j on a 2D lattice (figure 1).

The analogue of the torsional oscillator experiment is the electromagnetic response of the array to an oscillating transverse vector potential. The response function is the dynamic conductivity $\sigma(\omega, T)$.

3. The dynamic response of a 2D Josephson junction array

The dynamic conductivity $\sigma(\omega, T) \equiv \sigma_1 + i\sigma_2$ for an array has been calculated (Shenoy 1985) and an experimental measurement has been performed (Leeman *et al* 1986). As predicted, the real part of the conductivity σ_1, the peaks, and the imaginary

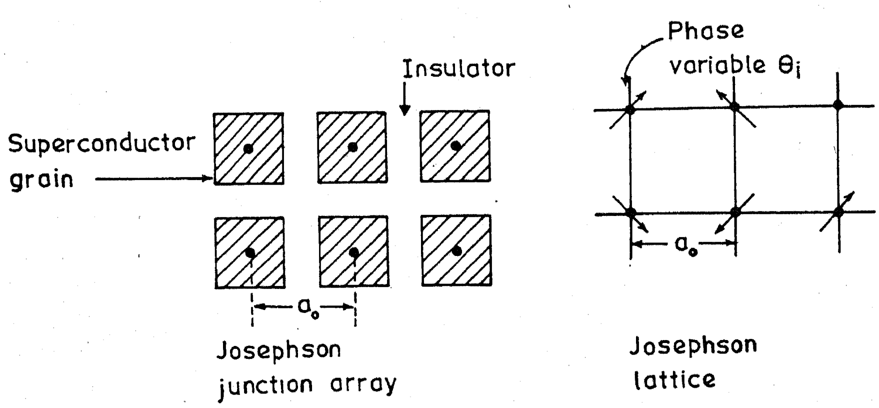

Figure 1. Josephson junction array and lattice model.

part σ_2 has a roll-off, at a frequency-dependent temperature,

$$T_\omega = T_{KT} + \mathrm{const}/(\ln(\Gamma_0/\omega))^2$$

where the phase relaxation rate $\Gamma_0 = 2eI_J R_s/\hbar$ is proportional to the single-junction normal shunt resistance R_s.

Since details have been published elsewhere (Shenoy 1985), only an outline will be given here. The starting point is the total current through a bond i.e. a junction, as a sum of super—normal—and fluctuation terms ($\theta_{ij} \equiv \theta_i - \theta_j$)

$$j_{ij} = K_0 \sin(\theta_{ij} - A_{ij}) + \frac{K_0}{\Gamma_0}(\dot{\theta}_{ij} - \dot{A}_{ij}) + \hat{f}_{ij}(t). \tag{3}$$

Here the (Johnson) noise current terms are delta-correlated, $\langle \hat{f}_{ij}(\tau)\hat{f}_{kl}(0)\rangle = 2(K_0/\Gamma_0)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\,\delta(t)$. The procedure is as follows:

(a) A Langevin equation for $\dot{\theta}_i$ is derived from (3) by imposing current conservation,

$$\sum_{\hat{x}=1}^{4} j_{i,i+\hat{x}} = 0.$$

An equivalent Fokker-Planck (FP) equation $\dot{P} = -\mathscr{L}_0 P$ for the probability $P(\{\theta_{ij} - A_{ij}\}, t)$ is linearized in $A_{ij}(t) = A_{ij}(\omega)\exp(-i\omega t)$ and solved formally, in terms of an inverse FP operator. This enables the dynamic conductivity, essentially a super current-current correlation $\sigma \sim \langle \sin\theta_{ij} \sin\theta_{kl}\rangle$ to be formally written out, in terms of the inverse FP operator $(\mathscr{L}_0 - i\omega)^{-1}$.

(b) Standard dual transformation (José *et al* 1977; Savit 1980) techniques are invoked to extract the important vortex contribution to the current-current correlation. The remaining part depends on spin waves only and a gaussian spin wave truncation makes the FP operator $\mathscr{L}_0$ tractable; so the spin waves can now be integrated out. One obtains the $\sigma_{ij,kl}$ dynamic current correlation in terms of a ratio

$$Z_v(\{m_R + i\chi_{R(\omega)}\})/Z_v(\{m_R\}).$$

The numerator is just the vortex partition function, with frequency-dependent test charges at four points, $R = I, J, K, L$. The denominator is the pure static vortex partition function.

(c) Standard Kosterlitz scaling methods (Berezmskii 1972; Kosterlitz and Thouless 1973; Kosterlitz 1974) for the static partition function of the vortex Coulomb gas can now be applied, supplemented by a physically motivated frequency rescaling.

The final result $\sigma(\omega, T) \equiv \sigma_1 + i\sigma_2$ is

$$\sigma_1/\sigma_0 = \begin{cases} \Pi K_\infty \Gamma_0 \delta(\omega) - \dfrac{\Pi}{4}\dfrac{\Gamma_0}{|\omega|}\dfrac{dK_{l\omega}}{dl_\omega}, & T < T_{KT} \\[2ex] K_{l_+}\dfrac{\Gamma_0}{|\omega|}\dfrac{\rho_+}{1+\rho_+^2} - \dfrac{\Gamma_0}{2|\omega|}\dfrac{dK_{l\omega}}{dl_\omega}\tan^{-1}\rho_+, & T > T_{KT} \end{cases}$$

$$\sigma_2/\sigma_0 = \begin{cases} K_{l\omega}\Gamma_0/|\omega|, & T < T_{KT}, \\[1ex] K_{l\omega}\dfrac{\Gamma_0}{|\omega|}\dfrac{\rho_+^2}{1+\rho_+^2}, & T > T_{KT}. \end{cases}$$

Here the scaling $\sigma_0 = \langle \cos\theta_{ij} \rangle / \mathrm{d}K_0 R_s$, and the variables are $\rho_+ \equiv (\xi_+ (T)/r_\omega)^2$, $l_\omega \equiv \ln(r_\omega/a_0)$, $l_+ \equiv \ln(\xi_+/a_0)$, K_l is the static scaled coupling at a separation $l \equiv \ln(r/a_0)$, obtained from (2).

4. A glass prototype: Josephson array in a hierarchical field

Nonexponential decays ('power law' $\sim t^{-\alpha}$ or 'stretched exponential' $\sim \exp(-\tau^\beta)$ are standard in glassy systems (Mezei 1983). Hierarchical (Palmer *et al* 1984; Huberman and Kerszberg 1985; Kumar and Shenoy 1986) nested-cluster models of relaxation have been developed to explain the slow decays: the cluster generations relax sequentially with longer time scales for larger sizes, and so the cumulative decay envelope decays slowly e.g. $\sim 1/t^T$ in time ('cooler = slower', as temperature T decreases).

The 2D Josephson array in a particular spatially varying perpendicular magnetic field is a useful prototype model for glasses. It is modelled by the hamiltonian $\beta H = -K_0 \Sigma \cos(\theta_{ij} - A_{ij})$. Dual transforms (José *et al* 1977; Savit 1980) as in the zero-field case give a Coulomb gas partition function but with a flux background $\{\Phi_I\}$ fixed in space, determined by the external magnetic field. The partition function is as in (1) but with $m_I \to m_I + \Phi_I$, $\{I\}$ on a dual lattice.

If the transverse array size is small enough compared to an effective array magnetic screening length (Berezinskii 1972; Kosterlitz and Thouless 1973; Kosterlitz 1974) $\lambda_\perp \propto \lambda_J^2 \varepsilon/d$, A_{ij} can be taken as the external magnetic field only. Here the flux variable Φ_I is per unit cell, $2\pi\Phi_I = \oint_I A_{kl}$ with the $\oint_I$ sum round a loop joining grain centres, and centred at a point I at the grain corners. (λ_J is the single-junction Josephson length, d is the grain thickness).

The model then reduces to one of mobile ± 1 charges floating in a fixed background of $\{\Phi_I\}$ frustration points, with a logarithmic interaction between all elements, for all separations, for $T < T_{\mathrm{KT}}$. (For $T > T_{\mathrm{KT}}$ there is screening beyond separations $> \xi_+(T)$). There is overall neutrality, $\Sigma_I (m_I + \Phi_I) = 0$. We choose (Shenoy 1987) a non-uniform, neutral, flux $\Sigma_I \Phi_I = 0$, with $\Phi_I = \pm\Phi$ $(\Phi < \frac{1}{2})$ at various sites I in a particular pattern. Then $\Sigma_I m_I = 0$ separately and since vortex pair creation is activated, the equilibrium vortex population tends to zero. We consider the time decay of prepared excess vortex populations, placed on the $\{\Phi_I\}$ background.

The ± 1 charges, when placed $\mp\Phi$ frustration sites separated by $r_0 > a_0$, the minimum scale, will annihilate by jumping over the $\ln(r_0/a_0)$ barrier under the action of a thermal random force, with a typical time $\tau \propto \exp(2\pi K_0 \ln(r_0/a_0))$. The annihilation time scale depends on the frustration separation. Since we have overall $\Sigma_I \Phi_I = 0$, the most obvious distribution is to put down $+\Phi$, $-\Phi$, $+\Phi$ . . . alternating sign background, all of separation r_0. However vortices -1, $+1$, -1 . . . placed on these, would all face one single $\sim \ln r_0$ barrier and hence one single time scale—the annihilation would be conventional.

The next best thing is triplets, rather than $+\Phi$, $-\Phi$ pairs, each of which is non-neutral, but clusters of which are 'as neutral as possible' or *quasineutral*, in a way now described, in 1D. The pattern generated is *non-unique*, and is just the simplest of possibly some larger class of such hierarchically self-similar $\pm\Phi$ patterns.

(i) start with a seed clusters $+\Phi$, $+\Phi$, $-\Phi$. Mirror reflects to the right and generates another cluster $-\Phi$, $+\Phi$, $+\Phi$.

(ii) Translate this generated cluster to the right and charge-conjugate it, $+\Phi$, $-\Phi$, $-\Phi$.

(iii) Repeat, using the 9-member cluster as a seed. Note that the excess charge for any 3^n cluster $n=1, 2, 3$ is unity i.e. the excess charge density $\sim 3^{-n}$ scales to zero.

This procedure generates the Φ background as shown in figure 2. An extra overall neutralizing Φ can be placed somewhere in the system (the inset shows a 2D version). The sequential annihilation of mobile ± 1 charges on this background is illustrated: the same pattern repeats itself on a larger scale. Since barriers depend on separation, this means a (sequentially) larger set of time scales.

One can write down (Shenoy 1987) a kinetic equation for the probability P_α of finding a charge in a cell α using a chemical reaction-rate analogy $\dot{P}_\alpha$ will depend on the rate of annihilation times the product of the annihilating probabilities.

$$\dot{P}_\alpha = -\sum_{\substack{\beta \\ (\neq \alpha)}} \frac{Q_{\alpha\beta}}{\tau(r_{\alpha\beta})} P_\alpha P_\beta. \tag{4}$$

Here the annihilation time depends on the charge separation, $\tau = \tau(r_{\alpha\beta})$, and projection factors $Q_{\alpha\beta} = \frac{1}{2} q_\alpha q_\beta (q_\alpha q_\beta - 1)$ are non-zero only for opposite-sign annihilating charges. Defining a course-grained probability by $P = \Sigma q_\alpha P_\alpha / \Sigma q_\alpha$, where the sum runs over charges *within* a cluster, one can write a scaled set of annihilation equations just as in the previous case, provided the smallest times are now for the next generation (larger) separations, $r_0^{(n)} = 3^n r_0$.

The net result is that the survival probability or charge density envelope varies as

$$\rho(t) \sim \frac{1}{t^{T/T_h}}; \quad \frac{T_h}{T} = 2\pi |\Phi| K_0 / \mathrm{d}, \tag{5}$$

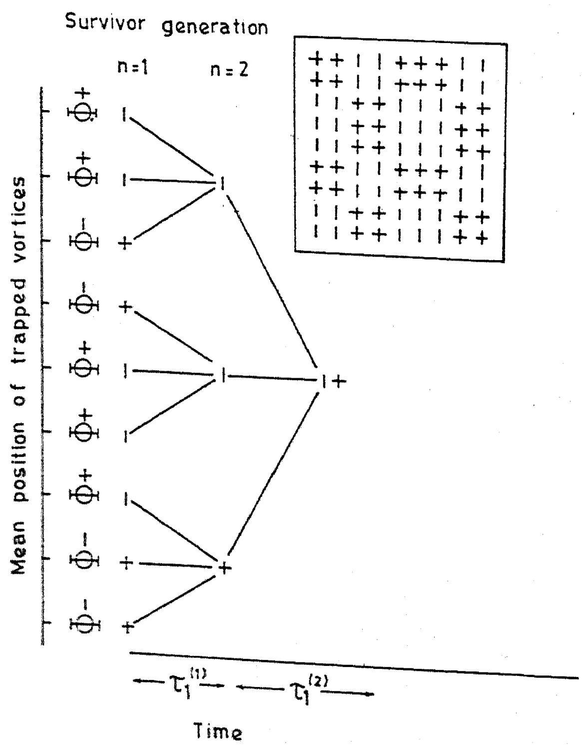

Figure 2. Hierarchical annihilation of mobile ± 1 vortices with time, on a $\pm \Phi$ flux fixed 1D flux pattern. (Inset: 2D vortex initial pattern, flux not shown).

where d is the dimensionality. Thus if a non-equilibrium (opposite-sign) vortex population is placed on a hierarchical, self-similar frustration background, it will annihilate in a slow, non-exponential manner.

Simple kinetic equations, incorporating an effective decay rate $k(t) \equiv -\dot{\rho}(t)/\rho(t)$ can be used to model the accumulation of vortices under a temperature-time cooling ramp. The non-equilibrium trapped fraction accumulates appreciably, below a 'glass transition temperature' $T_G(\dot{T}) \sim T_{KT} + \mathrm{const}/(\ln(|\dot{T}|))^2$, $|\dot{T}| =$ cooling rate.

Finally a separate point; this basic approach of focussing on topological 'disorder' variables, is also useful for the 3D XY model (where one has vortex loops) (Gupte and Shenoy 1986), and related 'lattice superconductor' models (Dasgupta and Halperin 1981; Bartholomew 1983; Shenoy S R and Gupte N, submitted) where fluctuations of the gauge field are included.

In summary, the frequency response and relaxational behaviour of 2D Josephson junction arrays can be usefully described in terms of the motion of disorder variables or vortices, i.e. in terms of the dynamics of extended defects.

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