# First-passage times and hysteresis in multivariable stochastic processes: The two-mode ring laser

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The decay of metastable states of a system obeying an $n$-variable Fokker-Planck equation is considered by evaluating the mean first-passage time $T_p$ using the asymptotic method of Schuss and Matkowsky. The statistics of the first-passage time $\tau$ in the small-noise high-barrier limit is shown to follow $\langle \tau^r \rangle \simeq r! \langle \tau \rangle^r$, where $T_p = \langle \tau \rangle$, independent of the number of degrees of freedom, $n$. The time $T_p$ in the low-barrier, high-noise limit is also calculated. The hysteresis window for the control-parameter sweep rate is generalized to multivariate systems. These general results are applied to a model of a two-mode laser, where $n=4$. A comparison with recent results of Mandel and co-workers is made, and experimental tests of the predictions are suggested.

## I. INTRODUCTION

The decay of metastable states is a problem of interest in many fields of condensed-matter physics and quantum optics. Examples include the rate of droplet formation in a supersaturated vapor,[1] the decay rate of superconductor currents,[2] and spontaneous switching times between two coherent modes in a ring laser.[3]

Generally, the statistical dynamics of a large class of systems that exhibit metastable behavior can be characterized by a multivariate Fokker-Planck equation of the form

$$\frac{\partial P}{\partial t} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[A_i(\vec{x},\mu)P] + \sum_{i,j=1}^{n} \frac{\partial^2 [D_{ij}(\vec{x})P]}{\partial x_i \partial x_j} \equiv L_{\vec{x}} P \tag{1.1}$$

whose steady-state solution $P_0(\vec{x})$ has a bimodal character. $\mu$ is a control parameter for the transition.

One would like to calculate the rate of transition from one minimum of $P_0(\vec{x})$ to another. For a single-variable Fokker-Planck system, various methods for computing such a rate are known—the Kramers method based on probability current flow,[4] the variational approach[5] and the first-passage-time formalism.[6] These methods yield very similar results, and have been applied extensively to one-variable situations. They have been reviewed elsewhere.[7]

The generalization of these methods to multivariate systems is far from trivial since no general methods for solving multivariate Fokker-Planck equations appear to be known. Schuss and Matkowsky, in a series of important papers,[8–10] have shown how the average first-passage time (FPT) can be asymptotically evaluated in a systematic fashion, in the limit of weak noise and a high barrier.

An extension to $n$ variables of the Kramers[11] and variational[12] methods, by similar arguments,[7] is also possible. However, in what follows, we discuss the FPT formalism because of the following:

(i) it is directly relevant to some experimental results;[3]

(ii) it treats metastable lifetimes and dissipative phase transitions[13] in a single unified framework;

(iii) it is relatively versatile, and can be generalized to situations where detailed balance does not hold;[9] and

(iv) it answers questions regarding the statistics of passage times,[6] as well.

The FPT formalism focuses on the time $\tau$ for a fluctuating $\vec{x} = (x_1, x_2, \ldots, x_n) \in \Omega$ to first cross a specified boundary $\partial\Omega$, with $\tau$ being a stochastic variable. The mean first-passage time

$$T_p(\vec{x}_0) = \langle \tau(\vec{x}_0) \rangle \tag{1.2}$$

depends on the initial value $\vec{x} = \vec{x}_0$ and satisfies a differential equation in $\vec{x}_0$ that is exactly soluble in the single-variable $n=1$ case:[6]

$$D \vec{\nabla}^2_{\vec{x}_0} T_p(\vec{x}_0) - \vec{A}(\vec{x}_0) \cdot \vec{\nabla}_{\vec{x}_0} T_p(\vec{x}_0) = -1 \ . \tag{1.3}$$

Here $A(\vec{x})$ is the drift term and $D$ is a diffusion term, constant and diagonal, for simplicity. (A more general case is considered later.)

In the problem of interest, $\vec{x}_0$ is in a well around a metastable minimum $\vec{x}_m^{(1)}$ and jumps occur over a barrier at a saddle point $\vec{x}_s$ to a stable minimum $\vec{x}_m^{(2)}$. The barrier height is $\Delta U = U(\vec{x}_s) - U(\vec{x}_m^{(1)})$ where the potential is defined by the stationary solution $P_0(x) = e^{-U(\vec{x})/D}$ of the Fokker-Planck equation. The small parameter in the approximation methods for $T_p$ is then

$$\epsilon \equiv D/\Delta U \ll 1 \tag{1.3$'$}$$

in a weak-noise, high-barrier regime.

The basic point is to recognize that, although $\epsilon$ is small, it is a singular perturbation as far as a mean FPT is concerned, as the noise kicks the particle over the barrier. The escape time must go to infinity as the scaled noise strength goes to zero, with $\epsilon$ playing the role of a temperature in an activated process,

$$T_p(\vec{x}_0) = v(\vec{x}_0; \epsilon) e^{K/\epsilon} \ . \tag{1.4}$$

With the essential singularity extracted, $v(\vec{x}_0;\epsilon)$ is expandable in powers of $\epsilon$. It is natural to consider a "preferred" reference frame centered on the saddle point $\vec{x}_s$ and rotated so one axis $z$ is along the line of steepest decents of $U(x)$. Then, scaling the new variables in $\epsilon^{1/2}$, the differential equation for $v(x_0;\epsilon)$ becomes effectively one dimensional, in $z$, to leading order in $\epsilon$. The asymptotic mean FPT is thus found.[8–10]

In this paper we outline the general approximation scheme of Schuss and Matkowsky for $n$ variables, and for $\vec{x}$-dependent, nondiagonal diffusion, but with detailed balance. We show that FPT statistics can also be analyzed by the same methods. The general formalism is applied to calculate $T_p$ for an explicit model, namely a two-mode ring laser,[14,15] where $n=4$. The FPT can be directly measured[3] and the equations are particularly simple, so the model is of both pedagogical[16] and experimental interest. The degeneracy of the potential with respect to the phase reduces this to a $n=2$ problem with an intensity-dependent diffusion term. We also generalize our ideas for the limits of observable hysteresis,[17] to $n$-dimensions, and calculate the "hysteresis window" for the ring laser from the decay and relaxation times. The mean FPT and hysteresis window are also calculated for the $\epsilon \gg 1$ or low-barrier, high-noise limit, from a scaled expansion of the FPT equation in $\vec{x}_m^{(1)}-\vec{x}_s$, near the limit of metastability. The mean FPT goes to zero quadratically in the difference of the control parameter from its critical value.

The outline of the paper is as follows. In Sec. II we present the Schuss-Matkowsky argument and show that it can also be applied to higher moments of the FPT $T_p^{(r)}=\langle \tau^r\rangle$. In Sec. III we generalize the hysteresis window to many dimensions. Explicit applications to the two-mode laser are made in Secs. IV, V, and VI that deal with, respectively, the preferred reference frame, the mean FPT and its moments (for $\epsilon \ll 1$), and the hysteresis window. Finally, in Sec. VII we summarize our results and discuss possible experimental tests of these theoretical ideas.

## II. FIRST-PASSAGE-TIME DISTRIBUTIONS IN THE HIGH-BARRIER, WEAK-NOISE LIMIT

As mentioned in the Introduction, the equation obeyed by the mean FPT $T_p(\vec{x}_0)$ in $n$ variables is

$$L^{\dagger}_{\vec{x}_0} T_p(\vec{x}_0) \equiv \left[ \sum_{i,j=1}^{n} D_{ij}(\vec{x}_0)\frac{\partial^2}{\partial x_{0i}\partial x_{0j}} - \sum_{i=1}^{n} A_i(\vec{x}_0)\frac{\partial}{\partial x_{0i}} \right] T_p(\vec{x}_0) = -1 , \tag{2.1}$$

where $A_i(\vec{x})$ and $D_{ij}(\vec{x})$ are the drift and diffusion terms, assumed to satisfy the potential conditions,[18] and $L^{\dagger}$ is the Hermitian conjugate of the Fokker-Planck operator. Here $\vec{x}_0 \in \Omega$, the $n$-dimensional volume containing a single metastable minimum $\vec{x}_m$. The boundary $\partial\Omega$ contains a saddle point $\vec{x}_s$. The boundary condition is that the mean FPT vanishes on the boundary[6]

$$T_p(\vec{x}_0 \in \partial\Omega) = 0 . \tag{2.2}$$

Furthermore, $T_p(\vec{x}_0)$ tends to a constant for $\vec{x}_0$ deep inside $\Omega$.

The stationary solution of the Fokker-Planck equation is $P_0(x)=e^{-\Phi(\vec{x})}$, defining a "potential," expressible as a line integral of the drift and diffusion terms

$$\Phi(\vec{x}) = \int^{\vec{x}} \sum_{i,j=1}^{n} dx_i (\underline{D}^{-1})_{ij} \left[ A_j + \sum_{k=1}^{n} \frac{\partial D_{jk}}{\partial x_k} \right] . \tag{2.3}$$

As the two-mode laser involves $\vec{x}$-dependent diffusion, we generalize the treatment[8] slightly.[9] Scaling in $\Delta U$ defined by

$$\Delta U \equiv [\Phi(\vec{x}_s) - \Phi(\vec{x}_m)] \frac{1}{n} \sum_{i=1}^{n} D_{ii}(\vec{x}_s) \tag{2.4}$$

leads to

$$L^{\dagger}_{\vec{x}} \tau_p(\vec{x}) = \left[ \epsilon \sum_{i,j=1}^{n} d_{ij}(\vec{x}) \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{n} g_i(\vec{x}) \frac{\partial}{\partial x_i} \right] \tau_p(\vec{x}) = -1 , \tag{2.5}$$

where

$$d_{ij}(\vec{x}) \equiv \left[ \frac{1}{n} \sum_{i=1}^{n} D_{ii}(\vec{x}_s) \right]^{-1} D_{ij}(\vec{x}) , \tag{2.6a}$$

$$g_i(\vec{x}) \equiv A_i(\vec{x})/\Delta U , \tag{2.6b}$$

$$\tau_p(\vec{x}) \equiv \Delta U\, T_p(\vec{x}) . \tag{2.6c}$$

The small parameter $\epsilon$ is

$$\epsilon \equiv \left[ \frac{1}{n} \sum_{i=1}^{n} D_{ii}(\vec{x}_s) \right] \Big/ \Delta U . \tag{2.7}$$

Note that $\epsilon$ depends on $D_{ij}$, as $\Phi$ depends on the diffusion. For a constant diagonal diffusion term $D_{ij}(\vec{x})=D\delta_{ij}$, one has $\Phi(\vec{x})=\int^{\vec{x}} \vec{A}\cdot d\vec{l}/D = U(\vec{x})/D$, $\Delta U$ is independent of $D$, $d_{ij}(\vec{x})=\delta_{ij}$, and $\epsilon = D/\Delta U$.

As noted in (1.2), the Schuss-Matkowsky starting point is to write

$$\tau_p(\vec{x}) = \lambda e^{K/\epsilon} v(\vec{x};\epsilon) , \tag{2.8}$$

where $K$ is a constant that can be found later. From (2.2),

$$v(\vec{x} \in \partial\Omega;\epsilon) = 0 . \tag{2.9}$$

Away from the boundary $\partial\Omega$, i.e., deep inside $\Omega$, the functions $v$ tends to unity. We will use the symbol $\infty$ for deep inside the volume even though $\Omega$ is finite. Note that if a function has a width $\Delta x$, then deep inside would mean $x \gg \Delta x$. Now we write

$$v(\vec{x};\epsilon) \to 1 ; \quad |\vec{x}-\vec{x}_b| \to \infty , \quad \vec{x} \in \Omega , \quad \vec{x}_b \in \partial\Omega \tag{2.10}$$

so that the asymptotic mean FPT is, in this limit,

$$\tau_p(\vec{x}) \to \tau_p(\infty) \equiv \lambda e^{K/\epsilon} .$$

The constant $\lambda$ can be related to $v(\vec{x};\epsilon)$ through the identity

$$\int_{\Omega} d^n x\, P_0(\vec{x}) [L^{\dagger}_{\vec{x}} T_p(\vec{x}) + 1] = 0 . \tag{2.11}$$

Using (2.7) and (2.9) and Green's theorem on the second derivative terms of (2.1), the asymptotic mean FPT is expressible as a ratio between a volume and a surface integral:

$$T_p(\infty) = -\frac{\int_\Omega d^n x\, P_0(\vec{x})}{\epsilon \int_{\partial\Omega} d^{n-1}x\, P_0(\vec{x}) \sum_{i,j=1}^{n} \hat{\nu}_i(\vec{x}) d_{ij}(\vec{x}) \frac{\partial}{\partial x_j} v(\vec{x};\epsilon)}, \tag{2.12}$$

where $\hat{\nu}(\vec{x})$ is an outward normal to the boundary $\partial\Omega$, and in the denominator $\vec{x}\in\partial\Omega$. The asymptotic FPT $T_p(\infty)$ depends on $v(\vec{x};\epsilon)$ only through its derivative, evaluated on the boundary $\partial\Omega$.

For small $\epsilon$, the function $v(\vec{x};\epsilon)$ can be expanded as

$$v(\vec{x};\epsilon) = v^{(0)}(\vec{x}) + \epsilon v^{(1)}(\vec{x}) + \cdots \tag{2.13}$$

so that

$$\left[\epsilon \sum_{i,j=1}^{n} d_{ij}(\vec{x}) \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{n} g_i(\vec{x}) \frac{\partial}{\partial x_i}\right] v^{(0)}(\vec{x}) \simeq 0 \tag{2.14}$$

neglecting $e^{-K/\epsilon} \ll 1$ since $e^{-x}$ goes to zero faster than $1/x$ in the limit $x\to\infty$. The first term on the left-hand side is retained, as it must contain an $O(1)$ contribution due to rapid variation on a scale $\epsilon^{1/2}$. If it is dropped as being $O(\epsilon)$ then one is led to unacceptable solutions: If $v^{(0)}(\vec{x})$ is a constant where $g(\vec{x})\neq 0$, then from (2.9) this constant must be zero.

To study escape over a barrier near a saddle point $\vec{x}_s$, it is appropriate to transform to a new and preferred set of axes centered at $\vec{x}_s$ and rotated such that one axis, $z$, is along the line of steepest descent of the force $A(\vec{x})$. The other $\vec{\rho}=(\rho_1,\ldots,\rho_{n-1})$ axes are perpendicular to $z$. The saddle point is $(\vec{\rho},z)=(\vec{0},0)$ in the new frame. The $z$ direction increases inward from the boundary so a reflection may be necessary for acute rotation angles (see Fig. 1). Then the old $\vec{x}=(x_1,x_2,\ldots,x_n)$ and new

$$\vec{\alpha} \equiv (\vec{\rho},z) \tag{2.15}$$

coordinates are related by

$$x_i = x_{si} + \sum_{i=1}^{n} \Lambda_{ij}\alpha_j , \tag{2.16}$$

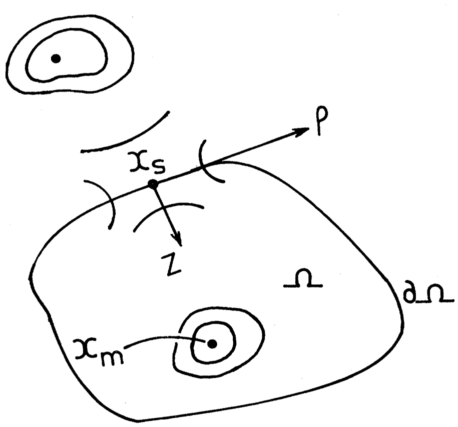


FIG. 1. Schematic plot of equipotentials and preferred axes $(\rho,z)$ for the problems of escape over barrier. $\vec{x}_s$ is the saddle point, $\vec{x}_m$ the metastable minimum, and $\partial\Omega$ the boundary enclosing $\Omega$.

where $\underline{\Lambda}$ is the appropriate rotation-reflection matrix. Then

$$\left[\epsilon \sum_{i,j=1}^{n} \tilde{d}_{ij}(\vec{\alpha}) \frac{\partial^2}{\partial\alpha_i \partial\alpha_j} - \sum_{i=1}^{n} \tilde{g}_i(\vec{\alpha}) \frac{\partial}{\partial\alpha_i}\right] v^{(0)}(\vec{\alpha}) = 0 , \tag{2.17}$$

where

$$\tilde{d}_{ij}(\vec{\alpha}) = \sum_{k,l=1}^{n} \Lambda_{ik}\Lambda_{jl} d_{kl}(\vec{x}_s + \underline{\Lambda}\vec{\alpha}) , \tag{2.18}$$

$$\tilde{A}_i(\vec{\alpha}) = \sum_{j=1}^{n} \Lambda_{ij} A_j(\vec{x}_s + \underline{\Lambda}\vec{\alpha}) , \quad \tilde{g}_i(\vec{\alpha}) \equiv \tilde{A}_i(\vec{\alpha})/\Delta U . \tag{2.19}$$

Now, all curvatures sharpen as $\epsilon\to 0$, so we scale as

$$\vec{\alpha}' \equiv \vec{\alpha}/\epsilon^{1/2} , \tag{2.20}$$

where scaled variables are denoted by a prime. Expanding in $\epsilon$, the leading contributions are

$$\tilde{d}_{ij}(\epsilon^{1/2}\vec{\alpha}') \simeq \tilde{d}_{ij}(0) = \sum_{k,l=1}^{n} \Lambda_{ik}\Lambda_{jl} d_{kl}(\vec{x}_s) \tag{2.21a}$$

and, using the fact that the saddle point is a steady state,

$$\tilde{g}_i(\epsilon^{1/2}\vec{\alpha}') \simeq \epsilon^{1/2}\vec{\rho}'\cdot\vec{b}_i^{(\perp)} + \epsilon^{1/2} z' b_i^{(\parallel)} , \tag{2.21b}$$

where

$$\vec{b}_i^{(\perp)} \equiv \vec{\nabla}_{\rho}\tilde{g}_i(\vec{\rho},0)\,|_{\vec{\rho}=\vec{0}} , \quad b_i^{(\parallel)} \equiv \frac{\partial}{\partial z}\tilde{g}_i(\vec{0},z)\,|_{z=0} . \tag{2.21c}$$

Now, close to the saddle point, at points on the $z$ ($\rho_i$) axis, the direction of the deterministic force is along the $z$ ($\rho_i$) axis itself, by the choice of the preferred frame. Thus, from (2.21c) $b_i^{(\parallel)}$ ($b_i^{(\perp)}$) is in the $z'$ ($\rho_i'$) direction, other components being zero. Then (2.17) becomes

$$\left[\sum_{i,j=1}^{n} \tilde{d}_{ij}(0) \frac{\partial^2}{\partial\alpha_i' \partial\alpha_j'} - \sum_{i=1}^{n-1} \vec{\rho}'\cdot\vec{b}_i^{(\perp)} \frac{\partial}{\partial\rho_i'} - z' b_n^{(\parallel)} \frac{\partial}{\partial z'}\right] v^{(0)}(\vec{\alpha}') = 0 , \tag{2.22}$$

where now $v^{(0)}$ is assumed to depend on $\vec{\alpha}'$ only. Note that the "steepest ascent" line as well as the steepest descent line enters into and could affect the final result. It is the boundary condition that makes only the steepest descent variable relevant, as seen below (Ref. 10 considers the case where only $z$ needs scaling).

The boundary $\partial\Omega$ can be defined by the parametrization

$$z' = \sigma(\vec{\rho}') . \tag{2.23}$$

One can eliminate $z'$ in favor of the new variable $y' = z' - \sigma(\vec{\rho}')$ in terms of which the boundary condition (2.9) can be written as $v(\vec{\rho}',y'=0)=0$, so we can write $v(\vec{\rho}',y') = y' w(\vec{\rho}',y')$. Since the integral of (2.12) is taken along $y'=0$, this means $T_p(\infty)$ is determined by $[\partial v/\partial y']_{y'=0} = w(\vec{\rho}',y'=0)$. Rewriting (2.22) in terms of $y'$ instead of $z'$, one finds that a possible solution is

$w(\vec{\rho}\,',y')\simeq w(y')$ independent of $\vec{\rho}\,'$. This corresponds to suppressing the $\vec{\rho}\,'$ dependence of (2.22), so that a one-dimensional ordinary differential equation results:

$$\left[\tilde{d}_{nn}(0)\frac{d^2}{dz'^2}-z'b_n^{(\|)}\frac{d}{dz'}\right]v^{(0)}(z')\simeq 0 \; . \tag{2.24}$$

This has the solution, in terms of $z'$,

$$v^{(0)}(z')=\frac{\int_0^{z'}dz'e^{-[\,|b_n^{(\|)}/\tilde{d}_{nn}(0)|\,]z'^2/2}}{\int_0^{\infty}dz'e^{-[\,|b_n^{(\|)}/\tilde{d}_{nn}(0)|\,]z'^2/2}} \; , \tag{2.25}$$

where both conditions (2.9) and (2.10) hold.

With the use of (2.25) in (2.12), and evaluating integrals by sharp peaking arguments,

$$T_p(\infty)=\frac{\pi e^{\Delta\Phi}}{\left[\left|\frac{\partial\tilde{A}_n(\vec{0},z)}{\partial z}\right|_{z=0}\tilde{d}_{nn}(0)\right]^{1/2}\mathscr{H}_n^{1/2}(\vec{x}_m)\mathscr{H}_{n-1}^{-1/2}(\vec{x}_s)} \; , \tag{2.26}$$

where

$$\mathscr{H}_n(\vec{x})=\det\left|\left|\left[\frac{1}{n}\sum_{r=1}^{n}D_{rr}(\vec{x}_s)\right]\frac{\partial^2\Phi(\vec{x})}{\partial x_i\partial x_j}\right|\right| \tag{2.27}$$

is an $n\times n$ Hessian evaluated at $\vec{x}$.

Turning to statistics, the $r$th moment of the FPT

$$T_p^{(r)}(\vec{x}_0)\equiv\langle\tau^r\rangle \tag{2.27'}$$

obeys the equation[6,19]

$$L_{\vec{x}}^{\dagger}T_p^{(r)}(\vec{x})\equiv\left[\sum_{i,j=1}^{n}D_{ij}(\vec{x})\frac{\partial^2}{\partial x_i\partial x_j}-\sum_{i=1}^{n}A_i(\vec{x})\frac{\partial}{\partial x_i}\right]T_p^{(r)}(\vec{x})=-rT_p^{(r-1)}(\vec{x}) \; . \tag{2.28}$$

In the one-variable case, this can be written as

$$\frac{\partial}{\partial x}e^{-\Phi(x)}\frac{\partial T_p^{(r)}(x)}{\partial x}=-r\frac{e^{-\Phi(x)}}{D(x)}T_p^{(r-1)}(x) \; , \tag{2.29a}$$

where $\Phi$ is the analog of (2.3). This can be written in an integral form

$$T_p^{(r)}(x)=r\int_x^{x_s}dx'[P_0(x)]^{-1}\times\int_{-\infty}^{x'}dx''T_p^{(r-1)}(x'')\frac{P_0(x'')}{D(x'')} \; . \tag{2.29b}$$

For the case of high barriers and low noise, $P_0(x)=e^{-\Phi(x)}$ is sharply peaked at $x_m$, and if we assume that $T_p^{(r-1)}(x)$ is slowly varying, then (2.29b) leads to

$$T_p^{(r)}(\infty)\simeq rT_p^{(r-1)}(\infty)T_p(\infty)\Longrightarrow T_p^{(r)}(\infty)=r![T_p(\infty)]^r \; . \tag{2.30}$$

Equation (2.30) corresponds to the following distribution of the first-passage times:

$$P(\tau)=T_p^{-1}e^{-\tau/T_p} \; . \tag{2.31}$$

This distribution was first obtained by Roy *et al.*[3] in the context of the ring laser, through a detailed disentangling of nested integrals.

In the $n$-variable case, following (2.8)

$$\tau_p^{(r)}(\vec{x})=\lambda_{(r)}v_{(r)}(\vec{x};\epsilon)e^{rK/\epsilon} \; , \tag{2.32}$$

where $v_{(r)}$ vanishes on the boundary. As in (2.12), the asymptotic limit is

$$\tau_p^{(r)}(\infty)\equiv\lambda_{(r)}e^{rK/\epsilon}=-\frac{\int_\Omega d^nx\,P_0(\vec{x})r\tau_p^{(r-1)}(\vec{x})}{\epsilon\int_{\partial\Omega}d^{n-1}x\,P_0(\vec{x})\sum_{i,j}^{n}\hat{v}_i(\vec{x})d_{ij}(\vec{x})\frac{\partial v_{(r)}^{(0)}(\vec{x})}{\partial x_j}} \; . \tag{2.33}$$

Neglecting terms $\sim re^{-K/\epsilon}$ after inserting (2.32) into (2.28), one gets the same equation as (2.14) for $v(\vec{x};\epsilon)$, with the same boundary conditions. Thus to this order, $v_{(r)}^{(0)}(\vec{x})\simeq v^{(0)}(\vec{x})$. As $P_0(\vec{x})$ is sharply peaked at $\vec{x}_m$, $\tau_p^{(r-1)}(\vec{x})$ is approximated by its asymptotic value, and using (2.12), one gets (2.30) holding even in the $n$-variable case. The first-passage-time statistics are seen to be independent of dimension in the high-barrier, weak-noise limit.

For completeness we also mention some results on the FPT statistics in one dimension, with the control parameter $\mu$ close to the limit of metastability $\mu_{c1}$, of the well at $x_m(\mu)$. For simplicity, a constant diffusion term $D$ is taken. Then one is in the low-barrier, high-noise limit and from (2.29b) with $x_{c1}=x_m^{(1)}(\mu_{c1})$,

$$T_p^{(r)}(x_m)\simeq(x_m-x_{c1})\chi^{(r)} \; , \tag{2.34}$$

where $\chi^{(1)}$ has been evaluated[7] and, for $r>1$,

$$\chi^{(r)}\equiv -r\int_{-\infty}^{x_{c1}}dx'(x'-x_{c1})e^{[\Phi(x_{c1})-\Phi(x)]/D}\chi^{(r-1)} \; . \tag{2.35}$$

Expanding $\Phi(x,\mu_{c1})$ about $x_{c1}$, with $\Phi''(x_{c1},\mu_{c1})=0$ (primes are $x$ derivatives),

$$\chi^{(r)}\approx 0.29\frac{\pi r}{D}\left[\frac{2D}{|\Phi'''|}\right]^{2/3}\chi^{(r-1)} \quad (r>1) \; . \tag{2.36}$$

With the use of

$$\chi^{(1)}=-0.41\frac{\pi}{D}\left[\frac{2D}{|\Phi'''|}\right]^{1/3} \tag{2.37}$$

it follows that

$$T_p^{(r)}(x_m)=1.41\left[\frac{2D}{|\Phi'''|}\right]^{-1/3}(r!)\nu^r , \tag{2.38a}$$

where

$$\nu\equiv 0.41\frac{\pi}{D}\left[\frac{2D}{|\Phi'''|}\right]^{2/3} . \tag{2.38b}$$

Using the general expansion for a probability distribution in terms of its moments[20] one finds from (2.38) a distribution $e^{-\tau/\nu}$ similar to (2.31), but with a delta-function term that does not contribute to the moments.

## III. THE HYSTERESIS WINDOW IN $n$ VARIABLES

The two wells of $\Phi(\vec{x})$, at $\vec{x}_m^{(1)}$ and $\vec{x}_m^{(2)}$, move relatively up and down as the drive parameter $\mu$ is varied between the limits of metastability $\mu_{c1}>\mu>\mu_{c2}$ of wells 1 and 2. Since the probability is $P_0(\vec{x})=e^{-\Phi(\vec{x})}$, thermodynamic considerations dictate a first-order transition to the more probable global minimum, as soon as $\mu$ crosses $\mu_M$, where the well depths are equal: $\Phi(\vec{x}_m^{(1)},\mu_M)=\Phi(\vec{x}_m^{(2)},\mu_M)$. In fact, hysteresis can occur, and the system can jump at some $\mu_J>\mu_M$, from the metastable to the stable well. As noted elsewhere[17,21] the question of when the jump actually occurs is a question of the relative size of three time scales.

We have in mind a sawtooth time variation of $\mu(t)$ between limits $\mu_{c1}$ and $\mu_{c2}$, so $\dot{\mu}$ is a constant, within a sweep. The relevant rates are then as follows:

(i) $\dot{\mu}$ the rate of change of the control parameter,
(ii) $T_p^{-1}$ the "hop-over" or first-passage rate, and
(iii) $T_r^{-1}$ the "roll-back" or relaxation rate in the metastable well.

The basic idea is that $\dot{\mu}$ must raise the metastable well too fast for a hop-over to occur, but slow enough so that the system sits near the moving well minimum (adiabatic following). These two requirements define the brackets of a hysteresis window for $\dot{\mu}$.

In $n$ variables, the $n\times n$ relaxation matrix is obtained from linearizing the deterministic equation about the (metastable) steady state $\vec{x}_m^{(1)}$.

$$\frac{dx_i(t)}{dt}=-A_i(\vec{x},\mu)\simeq-\sum_{j=1}^{n}\frac{\partial A_i(\vec{x}_m^{(1)},\mu)}{\partial x_{mj}^{(1)}}(x_j-x_{mj}^{(1)}) . \tag{3.1}$$

In the representation $\{y_i\}$ in which the relaxation matrix is diagonal, with

$$[\tilde{y}_i(t,\mu(t))\equiv y_i(t)-y_{mi}(\mu(t))] ,$$
$$\frac{d\tilde{y}_i(t,\mu(t))}{dt}=-\frac{\partial y_{mi}}{\partial\mu}\dot{\mu}-\left[\frac{1}{T_r}\right]_{ii}\tilde{y}_i(t,\mu(t)) , \tag{3.2}$$

where we have allowed the drive $\mu(t)$ to vary with time. For adiabatic following of the moving minimum, the deviation $\tilde{y}_i/y_{mi}\ll 1$ is small, and remains so, $d\tilde{y}_i/dt=0$ for all components $i$. This gives the sufficient condition (absolute values implicit on all quantities)

$$|\dot{\mu}|\ll\min\left[\left[\frac{1}{T_r}\right]_{ii}\left[\frac{\partial}{\partial\mu}\ln y_{mi}\right]^{-1}\right]$$
$$=\min\left[\left[\frac{1}{T_r}\right]_{ii}^2\frac{y_{mi}}{\dfrac{\partial A_i(\vec{y}_m,\mu)}{\partial\mu}}\right] , \tag{3.3}$$

where $\vec{A}(\vec{y}_m)=\vec{0}$ has been used. This is a condition that the hysteresis state should have the same simple (potential-minimum) description as a true steady state. If the drive is varied too fast and the inequality is violated, the fluctuation states away from the minimum will be explored.

The other bracket of the hysteresis window says that the drive is varied faster than the net depletion time of the population in the metastable well. We generalize this bound[17] to cover experimental situations where the wells are almost of equal depth.

A probability distribution may evolve either due to jumps to the more stable well, for fixed $\mu$ (decay), or due to parametric time variation $\mu(t)$ of the initial distribution (hysteresis). In a one-variable situation, with a plot of $P(x;\mu;t)$ ($x$ being the vertical axis, $\mu$ the horizontal axis, and $P$ out of the paper) this corresponds to a probability current flowing either vertically (decay) or horizontally (hysteresis).

Consider a sweep along branch 1 from $\mu=\mu_{c2}$ at $t=0$ to $\mu=\mu_{c1}$ at $t=(\mu_{c1}-\mu_{c2})/\dot{\mu}$. The fraction of the population in well 1 is

$$n_1(\mu)=\int_{\text{well 1}}P(\vec{x};\mu(t),t) \tag{3.4}$$

with the probability density normalized over all space (well 1 and well 2).

At $t=0$, $\mu=\mu_{c2}$, the distribution is sharply peaked about $\vec{x}_m^{(1)}(\mu_{c2})$, the branch 1 most probable value, just at the edge of the bistable region. If hysteresis occurs, this distribution is swept along at a rate $T_{\text{hyst}}^{-1}$ given by

$$T_{\text{hyst}}^{-1}\simeq\left|\frac{\partial}{\partial t}\ln P_0(\vec{x}_m^{(1)}(\mu),\mu)\right|$$
$$\simeq\left|\dot{\mu}\frac{\partial}{\partial\mu}\Phi(\vec{x}_m^{(1)}(\mu),\mu)\right| . \tag{3.5}$$

For hysteresis to occur, the rate $T_{\text{hyst}}^{-1}$ must exceed the decay rate of $n_1$ across the shrinking barrier. As barrier shrinkage can only increase the decay rate, a lower bound will be the time-independent decay rate $\dot{n}_1/n_1$ at that instantaneous value of $\mu=\mu(t)$. This fixed-$\mu$ decay rate depends on the initial conditions and will be largest for a large asymmetry of the initial population in the metastable well. A lower bound, in turn, will therefore be a $(\frac{1}{2},\frac{1}{2})$ initial distribution of $(n_1,n_2)$. The necessary condition for hysteresis is therefore

$$T_{\text{hyst}}^{-1}>\left|\frac{\dot{n}_1^{(1/2,1/2)}}{n_1^{(1/2,1/2)}}\right| , \tag{3.6}$$

where the superscript refers to the initial conditions.

For fixed $\mu$, i.e., time-independent decay rates, the kinetic equations are[22]

$$\dot{n}_1 = -\frac{1}{T_{p\downarrow}} n_1 + \frac{1}{T_{p\uparrow}} n_2 \ , \quad n_1 + n_2 = 1 \tag{3.7}$$

where $T_{p\downarrow}$ ($T_{p\uparrow}$) are the first-passage times for 1 to 2 (2 to 1); $T_{p\downarrow} = T_p(\vec{x}_m^{(1)}, \mu)$ $[T_{p\uparrow} = T_p(\vec{x}_m^{(2)}, \mu)]$. From (3.7)

$$n_1^{(1/2,1/2)}(t) = \frac{T_{p\uparrow}^{-1}}{T_{p\uparrow}^{-1} + T_{p\downarrow}^{-1}} + \frac{T_{p\downarrow}^{-1} - T_{p\uparrow}^{-1}}{2(T_{p\uparrow}^{-1} + T_{p\downarrow}^{-1})} e^{-(T_{p\uparrow}^{-1} + T_{p\downarrow}^{-1})t} \ . \tag{3.8}$$

The initial decay rate is defined by the initial ($t \to 0$) relative change in the well 1 population,

$$\dot{n}_1^{(1/2,1/2)} / n_1^{(1/2,1/2)} \simeq -T_{p\downarrow}^{-1} + T_{p\uparrow}^{-1} \ . \tag{3.9}$$

From (3.5), (3.6), and (3.9) we get the inequality

$$|\dot{\mu}| > [T_p^{-1}(\vec{x}_m^{(1)}, \mu) - T_p^{-1}(\vec{x}_m^{(2)}, \mu)] \left| \frac{\partial \Phi(\vec{x}_m^{(1)}, \mu)}{\partial \mu} \right|^{-1} \tag{3.10}$$

for hysteresis along branch 1. Similar considerations apply along branch 2, and the window is not, in general, symmetrical. The jump occurs when the FPT drops towards zero, and the inequality (3.8) is violated. For $\mu$ close to limit of metastability, return jumps are unlikely, $T_{p\downarrow} \ll T_{p\uparrow}$ and the previous[17] bound is recovered.

Equations (3.10) and (3.3) constitute a generalization of the hysteresis window idea to $n$ dimensions, with $\vec{x}$-dependent, nondiagonal diffusion terms. For the potential conditions[18] holding, $\Phi$ is given by (2.3), otherwise it must itself be calculated by an $\epsilon$ expansion,[23] and the FPT also evaluated.[9]

For a S-shaped hysteresis curve, $T_r^{-1} \sim |\mu - \mu_{c1,c2}|^{1/2}$ and $T_p^{-1} \sim |\mu - \mu_{c1,c2}|^{-1/2}$ near the limits of metastability. At $\mu \simeq \mu_M$, $T_{p\downarrow}^{-1} - T_{p\uparrow}^{-1} \sim |\mu - \mu_M|$ in general. Thus

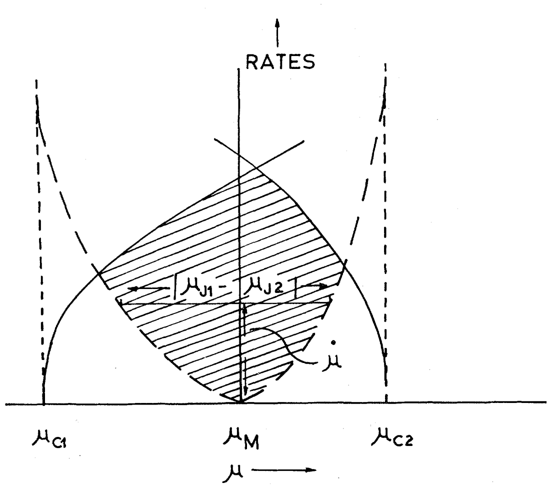


FIG. 2. Hysteresis window bounds (shaded area) on the sweep rate $\dot{\mu}$ and hysteresis width $|\mu_{J1} - \mu_{J2}|$ for jumps between branches. Heavy dashed line is the decay bound or no-jump condition. Solid line is the relaxation bound or small-fluctuation condition.

if the bounds are plotted vertically, and $\mu$ horizontally, the hysteresis window is the shaded area in Fig. 2. For a given sweep rate, suppose an average over many cycles is made of the points $\mu_{J1}, \mu_{J2}$ on branches 1,2, where the jumps actually take place. Then a horizontal bar of width $|\mu_{J1} - \mu_{J2}|$ (extent of hysteresis) at a height $\dot{\mu}$ (sweep rate), will fall inside the hysteresis window.

Notice that from Fig. 2, the adiabatic following condition has to be violated if we want to approach $\mu_{c1,c2}$ very closely. The approach to the "spinodal line," defined by the locus of points $[\vec{x}_m^{(1)}(\mu_{c1}), \mu_{c1}]$ thus cannot be described in terms of states close to $\vec{x}_m(\mu)$ alone. Fluctuations become important. Of course, in practice the stochastic switching region near the spinodal line may be narrow, so mean-field spinodal exponent behavior[24] could still be seen.

## IV. THE TWO-MODE RING LASER AND THE PREFERRED FRAME

The two-mode ring laser[14] is a useful system for application of the ideas of Secs. II and III. The model is theoretically simple, leading to tractable equations. On the other hand, the intrinsic time scales ($\sim$msec) are such that first-passage-time scales and statistics are experimentally measurable.[3]

The ring laser has two possible modes of excitation representing traveling waves in opposite directions. The coherent photon amplitudes are

$$E_{1,2} = r_{1,2} e^{i\theta_{1,2}} \equiv \sqrt{I_{1,2}}\, e^{i\theta_{1,2}} \tag{4.1}$$

with the Langevin equations (all times dimensionless)

$$\dot{E}_{1,2} = (a_{1,2} - |E_{1,2}|^2 - \xi |E_{2,1}|^2) E_{1,2} + f_{1,2}(t) \ . \tag{4.2}$$

Here $a_1, a_2$ are the pump parameters for each mode, $\xi$ is the mode coupling parameter, $\xi > 1$ for inhomogeneous broadening, and the random forces are $\langle f_i^*(t) f_j(0) \rangle = 2\delta_{ij}\delta(t)$ representing a constant diagonal diffusion matrix, scaled to be unity.

It is convenient to define

$$a \equiv \tfrac{1}{2}(a_1 + a_2) \ , \quad \Delta a \equiv a_1 - a_2 \ . \tag{4.3}$$

Clearly, with real and imaginary parts defined as

$$E_1 \equiv x_1 + i x_2 \ , \quad E_2 \equiv x_3 + i x_4 \ , \tag{4.4}$$

this is a four-dimensional problem.

The stationary solution of the Fokker-Planck equation is $P_0(\vec{x}) = e^{-U(\vec{x})}$ where the potential depends only on the intensities:

$$U(I_1, I_2) = -\tfrac{1}{2} a_1 I_1 - \tfrac{1}{2} a_2 I_2 + \tfrac{1}{4} I_1^2 + \tfrac{1}{4} I_2^2 + \tfrac{1}{2} \xi I_1 I_2 \ . \tag{4.5}$$

For $\xi > 1$, this has, as steady states, two minima

$$\vec{I}_m^{(1)} = (a_1, 0) = (a + \tfrac{1}{2}\Delta a, 0) \ ,$$
$$\vec{I}_m^{(2)} = (0, a_2) = (0, a - \tfrac{1}{2}\Delta a) \ , \tag{4.6a}$$

and a saddle point

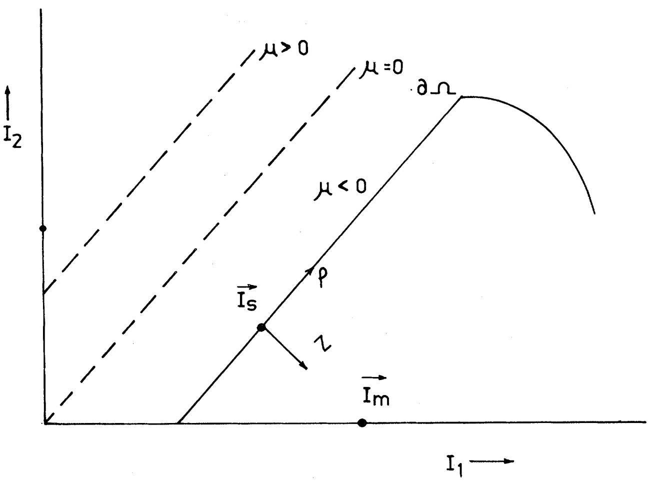


FIG. 3. Two intensities $I_1,I_2$ of the two-mode laser, showing minima, saddle points, boundary $\partial\Omega$, and preferred axes.

$$\vec{I}_s=\left[\frac{\xi a_2-a_1}{\xi^2-1},\frac{\xi a_1-a_2}{\xi^2-1}\right]$$
$$=\left[\frac{a}{\xi+1}-\frac{\frac{1}{2}\Delta a}{\xi-1},\frac{a}{\xi+1}+\frac{\frac{1}{2}\Delta a}{\xi-1}\right] \tag{4.6b}$$

thus exhibiting bistability, provided the drive parameter

$$\mu\equiv\Delta a/2a \tag{4.7}$$

is between two limits

$$(\mu_{c2}=\mu_c)>\mu>(\mu_{c1}=-\mu_c)\ ,$$
$$\mu_c\equiv(\xi-1)/(\xi+1)\ . \tag{4.8}$$

The steady states (4.6) are shown as points on an $I_1$ vs $I_2$ plot in Fig. 3, while the hysteresis curve as seen in terms of the well $I_1$ is plotted in Fig. 4. The two branches are $I_1=a(1+\mu)$ (branch 1) and $I_1=0$ (branch 2) while the intervening barrier is given by $I_{s1}=a(1-\mu/\mu_c)/(\xi+1)$. The choice of labels $\mu_{c1,c2}$ in (4.8) is such that well 1 (2) disappears at $\mu_{c1}$ ($\mu_{c2}$). The slope of the stable branch at $\mu_{c1}$ is not infinite; this has consequences.

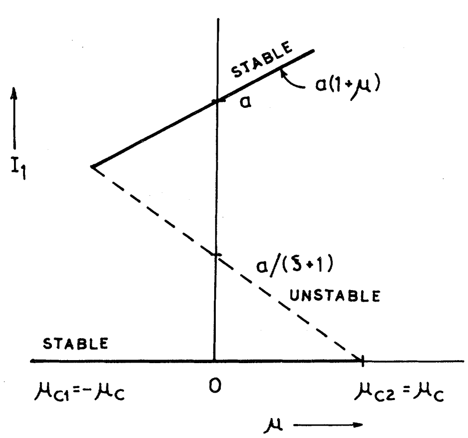


FIG. 4. Hysteresis curve projected on $I_1$ axis for two-mode laser showing limits of metastability $\mu_{c1,c2}=\mp\mu_c=\mp(\xi-1)/(\xi+1)$ where well 1,2 disappears.

In terms of the drive parameter (4.8) the barrier for jumping out of the well at $\vec{I}_m^{(1)}$ is

$$\Delta U\equiv U(\vec{I}_s)-U(\vec{I}_m^{(1)})=\tfrac{1}{4}a^2\mu_c(1+\mu/\mu_c)^2 \tag{4.9}$$

and vanishes, as expected, at $\mu=\mu_{c1}$. This corresponds to the ridge of Fig. 3 (dashed line), containing the saddle point $\vec{I}_s$, rolling down to annihilate the minimum at $\vec{I}_m^{(1)}$ ($\vec{I}_m^{(2)}$) for $\mu\to\mu_{c1}$.

The two-mode laser problem is considered elsewhere[14,15] in units where the diffusion constant is unity. We scale in the barrier height $\Delta U$, as in (2.6). However, it is convenient to further scale the variables $\vec{x}$ and pump parameters $a_1,a_2$ to absorb the $\Delta U$ factor so that $g(x,a)=A(x,a)/\Delta U=A(\bar{x},\bar{a})$, where the overbar denotes scaled variables. One then has, with (4.2),

$$(\Delta U)^{-1}=\epsilon\ ,\quad I_{1,2}=\epsilon^{-1/2}\bar{I}_{1,2}\ ,$$
$$a_{1,2}=\epsilon^{-1/2}\bar{a}_{1,2}\ ,\quad T_p=\epsilon^{1/2}\bar{T}_p \tag{4.10}$$

with $\xi$ unchanged. In terms of the new variables the barrier height is unity. Explicitly, from (4.9)

$$\epsilon^{-1}=\tfrac{1}{4}a^2\mu_c(1+\mu/\mu_c)^2=(a/\bar{a})^2\ . \tag{4.11}$$

Similar results hold for jumping out of the well at $I_m^{(2)}$. We now drop overbars on $\bar{I}_{1,2}$, $\bar{a}_{1,2}$, and $\bar{T}_p$, reintroducing them in (5.12) below.

The potential, when seen from a frame centered at the saddle point

$$I_{1,2}=I_{s1,s2}+\tilde{I}_{1,2} \tag{4.12}$$

has no linear terms:

$$U(\tilde{I}_1,\tilde{I}_2)=-\frac{1}{2}\frac{a^2}{\xi+1}\left[1-\frac{\mu^2}{\mu_c}\right]+\tfrac{1}{4}\tilde{I}_1^2+\tfrac{1}{4}\tilde{I}_2^2+\tfrac{1}{2}\xi\tilde{I}_1\tilde{I}_2\ . \tag{4.13}$$

Diagonalizing the $2\times2$ matrix, the preferred axis variables are $\rho,z$;

$$\tilde{I}_i=\sum_{j=1,2}\Lambda_{ij}\alpha_j\ ,\quad \alpha_1=\rho\ ,\quad \alpha_2=z \tag{4.14a}$$

with $\underline{\Lambda}$ corresponding to a rotation of the original matrix by 45° plus a reflection of the $z$ axis to point inwards:

$$\underline{\Lambda}=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\ . \tag{4.14b}$$

This is an explicit illustration of (2.16) and

$$U(\rho,z)=-\frac{1}{2}\frac{a^2}{\xi+1}\left[1-\frac{\mu^2}{\mu_c}\right]+\tfrac{1}{4}(\xi+1)(\rho^2-\mu_c z^2)\ , \tag{4.15}$$

clearly displaying the saddle-point structure around $\vec{I}_s$, i.e., around $(\rho,z)=(0,0)$, with $z$ along the line of steepest descent. (As $I_1,I_2$ are restricted to be positive, and the triangles defined by $|z|>\mu_c^{-1/2}|\rho|$ intersect the axes, there are no convergence difficulties in normalization factors.) Because of the simplicity of the model, this form is exact, and not an expansion as would occur in the general case. But the special property that $U$ is quartic in $r_1,r_2$ needs a special analysis later.

As shown in Fig. 3, the natural boundary $\partial\Omega$ for defining escape from well 1 is along the ridge $z=0$ containing the saddle point, with, say, a far-off circular section closing the area. Thus $\partial\Omega$ is particularly simple, and in terms of (2.23) is defined by $y'=z'=0$ (cf. Fig. 1).

## V. THE FIRST-PASSAGE TIME FOR THE TWO-MODE LASER

The FPT equation is

$$L^{\dagger}T_p=\sum_{i=1}^{4}\left[\epsilon\frac{\partial^2}{\partial x_i^2}-A_i(x)\frac{\partial}{\partial x_i}\right]T_p=-1\ , \tag{5.1}$$

where $\vec{x}=(x_1,x_2,x_3,x_4)$ as defined in (4.4) and

$$A_i(\vec{x})=\frac{\partial U(\vec{I})}{\partial x_i}\ . \tag{5.2}$$

In polar coordinates

$$\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}=\frac{\partial^2}{\partial r_1^2}+\frac{1}{r_1}\frac{\partial}{\partial r_1}+\frac{1}{r_1^2}\frac{\partial^2}{\partial\theta_1^2}\ , \tag{5.3}$$

where the last term is dropped as the potential (4.5) over which the passage occurs, and hence the passage time, are both independent of $\theta_1,\theta_2$. In terms of intensities

$$\frac{1}{r_1}\frac{\partial}{\partial r_1}=2\frac{\partial}{\partial I_1}\ ,\quad \frac{\partial^2}{\partial r_1^2}=2\frac{\partial}{\partial I_1}+4I_1\frac{\partial^2}{\partial I_1^2}\ , \tag{5.4}$$

we have the equation

$$L^{\dagger}T_p(\vec{x})=\sum_{i=1}^{4}\left[4\epsilon I_i\frac{\partial^2}{\partial I_i^2}+\left[4\epsilon-4I_i\frac{\partial U}{\partial I_i}\right]\frac{\partial}{\partial I_i}\right]T_p(\vec{I})$$
$$=-1\ . \tag{5.5}$$

The $n=4$ problem with an $I$-independent diffusion term becomes an $n=2$ problem with a diffusion term[15] $D_{ij}(\vec{I})=4\epsilon I_i\delta_{ij}$ dependent on $\vec{I}$. Physically this can be understood: As all angles are equally allowed, the phase space for diffusion for a given $r_1$ is increased by a factor of $\sim\pi r_1^2$. Singh and Mandel[15] have substituted $D_{ij}(\vec{I})$ into a one-dimensional formula to estimate $T_p$. Here we can use the general $n$-variable ideas of Secs. II and III to evaluate $T_p$.

In the preferred frame, from (4.14) and (5.5),

$$L^{\dagger}T_p=\epsilon\left[\frac{4a}{\xi+1}+2\sqrt{2}\rho\right]\left[\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial z^2}\right]T_p+\epsilon\left[\frac{-4\Delta a}{\xi-1}+4\sqrt{2}z\right]\frac{\partial^2 T_p}{\partial\rho\partial z}$$
$$+[4\sqrt{2}\epsilon-2a\rho-\Delta a\,z-\sqrt{2}(\xi+1)\rho^2+\sqrt{2}(\xi-1)z^2]\frac{\partial T_p}{\partial\rho}+\left[\frac{\Delta a}{\mu_c}\rho+2\mu_c az-2\sqrt{2}\rho z\right]\frac{\partial T_p}{\partial z}=-1 \tag{5.6}$$

and using (2.8)

$$L^{\dagger}v^{(0)}(\rho,z)=0 \tag{5.7}$$

as in (2.17) with $\tilde{d}_{ij}$ and $\tilde{g}_i$ now known explicitly.

Scaling in $\epsilon^{1/2}$, as in (2.20), and with $\epsilon\to 0$, i.e., $a\to\infty$ along lines $G<a_2/a_1<G^{-1}$, where $G\equiv(1-2\mu_c)/(1+2\mu_c)$,

$$\left[\frac{\partial^2}{\partial\rho'^2}+\frac{\partial^2}{\partial z'^2}\right]v^{(0)}(\rho',z')-\frac{2\mu}{\mu_c}\frac{\partial^2 v^{(0)}}{\partial\rho'\partial z'}-\tfrac{1}{2}(\xi+1)(\rho'+\mu z')\frac{\partial v^{(0)}}{\partial\rho'}+\tfrac{1}{2}(\xi-1)\left[z'+\frac{\mu}{\mu_c^2}\rho'\right]\frac{\partial v^{(0)}}{\partial z'}=0\ . \tag{5.8}$$

Comparing with (2.21)

$$\tilde{d}_{ij}(0)=\sum_{k=1}^{2}\Lambda_{ik}\Lambda_{jk}4I_{sk} \tag{5.9a}$$

with $\underline{\Lambda}$ given by (4.14), $\vec{I}_s$ being the saddle point of (4.6b), and

$$b_1^{(\perp)}=-\tfrac{1}{2}(\xi+1)\ ,\quad b_2^{(\perp)}=\tfrac{1}{2}(\xi-1)(\mu/\mu_c^2)\ ,$$
$$b_1^{(\parallel)}=-\tfrac{1}{2}(\xi+1)\mu\ ,\quad b_2^{(\parallel)}=\tfrac{1}{2}(\xi-1)\ . \tag{5.9b}$$

Notice that because the potential is quadratic in intensity variables (quartic in amplitudes) the drift term in (5.5) contains phase-space factors of intensity and is not just a vector (gradient) in intensity variables. Therefore cross terms do appear and the argument below (2.21) for the $\rho$ independence of $[\partial v(\rho,z)/\partial z]_{z=0}$ cannot be taken over as it stands. However, the same result can be obtained, by similar arguments, directly.

Note that (5.8) is invariant under the following generalized parity transformations:

(i) $\rho',z'\to-\rho',-z'$, $\mu$ fixed;
(ii) $\mu,z'\to-\mu,-z'$, $\rho'$ fixed;
(iii) $\rho',\mu\to-\rho',-\mu$, $z'$ fixed.

Thus the solution $v^{(0)}(\rho',z';\mu)$ can be chosen to be even or odd under (i), (ii), and (iii). From (4.11), the $\epsilon\ll 1$ limit corresponds, in terms of the original, unbarred variables to large $a\gg 1$, i.e., $\mu=\Delta a/2a\ll 1$ for fixed $\Delta a$. Thus a power-series solution in $\mu$ is

$$v^{(0)}(\rho',z';\mu)=f_0(\rho',z')+\mu f_1(\rho',z')+\cdots\ . \tag{5.10}$$

Setting $\mu=0$ in (5.8) the zeroth-order solution $f_0$ is found to be independent of $\rho'$:

$$f_0(z')=\left[\frac{\pi}{\xi-1}\right]^{-1/2}\int_0^{z'}dz'e^{-(1/4)(\xi-1)z'^2}\ . \tag{5.11}$$

This is odd under (i) and (ii), but even under (iii) and $\mu f_1(\rho',z',\mu)$ must have the same parity. Trying $f_1=\rho' h_1(z'^2)$ the correction term is found to be $\sim\mu\rho' z'^2$ for small $z'$, and does not contribute to the $z'$ derivative of $v^{(0)}(\rho',z')$ for $z'=0$, of (2.12). The relevant contribution (5.11) is just what we would get from solving (5.8) after suppressing all the $\rho'$ dependence. From (2.26) the mean FPT is

$$\bar{T}_p(\infty)=\frac{2\sqrt{\pi}\epsilon^{1/2}}{(\bar{I}_{s1}+\bar{I}_{s2})\bar{I}_{s2}(\xi^2-1)}\left[\frac{\xi+1}{\xi-1}\right]^{1/2}e^{-1/\epsilon}, \quad (5.12)$$

where the bars on the scaled variables have been restored. Going back, through (4.10)

$$T_p(\infty)=\frac{4(\xi+1)}{(1+I_{s2}/I_{s1})}\frac{I_{s1}}{I_{s2}}\left[\frac{\sqrt{\pi}}{2}\frac{e^{\Delta U}}{(\xi^2-1)^{3/2}I_{s1}^2}\right]. \quad (5.13)$$

Here the expression in large parentheses is the Singh-Mandel one-dimensional estimate[15] and $I_{s1}$, $I_{s2}$, and $\Delta U$ are given in (4.6) and (4.9). In terms of $\mu$

$$T_p(\infty)=\frac{\sqrt{\pi}}{a^2}\frac{e^{(1/4)a^2\mu_c(1+\mu/\mu_c)^2}}{\mu_c^{3/2}(1+\mu/\mu_c)}. \quad (5.14)$$

Experimentally,[13] $\ln T_p$ has been found to vary linearly in $a$ for $a\gg1$, with the disagreement with $\sim a^2$ dependence attributed to backscattering.

The statistics of the FPT have been measured and found to obey the exponential distribution of (2.31) first obtained in the $n=1$ case. From the generalization of Sec. II, the FPT statistics are independent of dimension, accounting for this behavior.

It is also of interest to explicitly find the mean FPT near the limits of metastability $\mu_{c1,c2}$. From (4.11) this corresponds to $\epsilon\gg1$, a low-barrier, high-noise regime. Here $a_2/a_1\to(1+2\mu_c)/(1-2\mu_c)$ while $a_1+a_2=a$ is large and fixed. From (4.6) the small parameter is the separation between the saddle point and the minimum $\propto(1+\mu/\mu_c)\propto\epsilon^{-1/2}$. From (5.6),

$$\left[1+\frac{(\xi+1)\rho}{\sqrt{2}a\epsilon^{1/2}}\right]\left[\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial z^2}\right]v+2\left[-\frac{\mu}{\mu_c}+\frac{(\xi+1)z}{\sqrt{2}a\epsilon^{1/2}}\right]\frac{\partial^2 v}{\partial\rho\partial z}$$
$$+\frac{(\xi+1)}{4a\epsilon^{1/2}}\left[4\sqrt{2}-\frac{2a(\rho-\mu z)}{\epsilon^{1/2}}-\frac{\sqrt{2}(\xi+1)(\rho^2+\mu_c z^2)}{\epsilon}\right]\frac{\partial v}{\partial\rho}+\frac{(\xi+1)}{4a\epsilon^{1/2}}\left[\frac{2a(\mu z-\rho)}{\epsilon^{1/2}}-\frac{2\sqrt{2}\rho z}{\epsilon}\right]\frac{\partial v}{\partial z}$$
$$=-\frac{(\xi+1)}{4a\epsilon^{3/2}T_p(\vec{\alpha}_m)}, \quad (5.15)$$

where variables are again implicitly overbarred as in (4.10), and $T_p(\vec{\alpha})=T_p(\vec{\alpha}_m)v(\vec{\alpha};\epsilon^{-1/2})$ with $v\simeq v^{(0)}(\vec{\alpha})+\epsilon^{-1/2}v^{(1)}(\vec{\alpha})+\cdots$. Here $v$ vanishes at $z=0$ and is unity at $\vec{\alpha}_m=(\rho_m,z_m)=\sqrt{2}(\mu_c^{1/2},\mu_c^{-1/2})$. In these overbarred variables, the minimum is independent of $\epsilon^{-1/2}$, and the transition is forced by the divergence of the noise. For $\epsilon^{-1/2}\to0$, $\sqrt{\xi+1}/a\ll1$ and $(\xi-1)/16\ll1$, the diffusion terms dominate, and $v^{(0)}\simeq z/z_m$. $T_p(\vec{\alpha}_m)$ is found from the analog of (2.12). Restoring overbars, the result is

$$\bar{T}_p(\vec{\alpha}_m)=\frac{\bar{z}_0\bar{z}_m}{\epsilon\left[\frac{4\bar{a}}{\xi+1}+2\sqrt{2}\langle\bar{\rho}\rangle_{\partial\Omega}\right]}, \quad (5.16)$$

where the diffusion constant in the denominator is averaged over the boundary. Here, since $I_{1,2}>0$, $\rho>-\bar{\rho}_0$, and $\bar{z}_0>\bar{z}$ with $\bar{z}_0=-\bar{\rho}_0=2\sqrt{2}/[(\xi+1)\mu_c^{1/2}]$ (see Fig. 3). In unscaled variables one has the result

$$T_p(\vec{\alpha}_m)=\frac{a(1+\mu/\mu_c)^2}{\{1+[(2/\pi)(\xi+1)]^{1/2}/a\}}. \quad (5.17)$$

## VI. THE HYSTERESIS WINDOW FOR THE TWO-MODE LASER

The relaxation-rate matrix is,[25] from (4.2) and (3.2),

$$(\underline{T}_r)^{-1}=\begin{bmatrix}-\frac{1}{2}a_1+|E_1|^2+\frac{1}{2}\xi|E_2|^2 & \frac{1}{2}\xi E_1^*E_2\\ \frac{1}{2}\xi E_1E_2^* & -\frac{1}{2}a_2+|E_2|^2+\frac{1}{2}\xi|E_1|^2\end{bmatrix}. \quad (6.1)$$

The relaxation matrix around $\vec{I}_m^{(1)}=(a_1,0)$ is already diagonal, with

$$[(\underline{T}_r)^{-1}]_{11}=\tfrac{1}{2}a(1+\mu),\quad [(\underline{T}_r)^{-1}]_{22}=\tfrac{1}{2}a(\xi-1)\left[1-\frac{\mu}{\mu_{c1}}\right]. \quad (6.2)$$

The smaller of these enters the inequality of (3.3). The $\sim(1-\mu/\mu_{c1})$ behavior comes from the linear $I_1(\mu)$ in Fig. 4.

The first-passage times of (5.14) and (5.16) are for jumps from metastable states to the stable states $T_{p\downarrow}$, in the notation of (3.7). $T_{p\uparrow}$ is obtained by $\mu\to-\mu$. Then

$$\left|\frac{1}{T_{p\downarrow}}-\frac{1}{T_{p\uparrow}}\right|=\frac{a^2\mu_c^{3/2}}{\sqrt{\pi}}\left|\left[1+\frac{\mu}{\mu_c}\right]\exp\left[-\tfrac{1}{4}a^2\mu_c\left[1+\frac{\mu}{\mu_c}\right]^2\right]-(\mu\to-\mu)\right|\equiv\kappa \tag{6.3}$$

in the high-barrier, low-noise limit valid near $\mu\simeq\mu_M$ and we have

$$\left|\frac{1}{T_{p\downarrow}}-\frac{1}{T_{p\uparrow}}\right|=\frac{4|\mu|\{1+[2(\xi+1)/\pi]^{1/2}/a\}}{a\mu_c[1-(\mu/\mu_c)^2]^2} \tag{6.4}$$

in the low-barrier, high-noise limit valid near $\mu\simeq\mu_{c1,c2}$. Since

$$\frac{\partial}{\partial\mu}U(\vec{I}_m^{(1)}(\mu),\mu)=\tfrac{1}{2}a^2(1+\mu) \tag{6.5}$$

and

$$\left|E_1\Big/\left[\frac{\partial A_1}{\partial\mu}\right]\right|=a^{-1}, \tag{6.6}$$

the hysteresis window estimate for branch 1 is

$$\min\left[\tfrac{1}{4}a(1+\mu)^2,\tfrac{1}{4}a(\xi-1)^2\left[1-\frac{\mu}{\mu_{c1}}\right]^2\right]\gg|\dot{\mu}|\gg\begin{cases}2\kappa/a^2(1+\mu)\,, & \text{for } |\mu|\ll 1\,,\\ \dfrac{8|\mu|\{1+[2(\xi+1)/\pi]^{1/2}/a\}}{a^3\mu_c(1+\mu)[1-(\mu/\mu_c)^2]^2}\,, & \text{for } 1-\dfrac{\mu}{\mu_{c1}}\ll 1\,.\end{cases} \tag{6.7}$$

If this is plotted as in Fig. 2 one would see linear decreases to zero of $T_r^{-1}\sim|\mu-\mu_{c1,c2}|$ and quadratic divergences $T_p^{-1}\sim|\mu-\mu_{c1,c2}|^{-2}$.

## VII. DISCUSSION

The principal features of this investigation are as follows. (1) The application of the Schuss-Matkowsky method to situations where the diffusion term is a matrix dependent on the system variables, (2) statistics of the first-passage time for a multivariate system (2.30), and (3) generalization of the hysteresis window [(3.3) and (3.10)] to many variables. There are also new predictions for the dynamics of the two-mode ring laser, including (4) new estimates [(5.14) and (5.17)] for the first-passage time based on the full Fokker-Planck equation and (5) the condition (6.7) for the observation of hysteresis within this model. Further, (6) the low-barrier, high-noise FPT is estimated.

The experiments done[3] have defined "passage" by the boundary $I_1=I_{s1}$ parallel to the $I_2$ axis, with $I_1>I_{s1}$ as "on" and $I_1<I_{s1}$ as "off." From Fig. 3, a more natural choice is the $z=0$ line. For $\Delta a=0$ this means a division into "state 1" as $I_1>I_2$ and "state 2" as $I_2>I_1$.

The results given here could be tested by the following methods.

(a) A measurement of the passage time $T_p$ as a function of $a$ and $\Delta a$, showing that $T_p$ vs $\mu=\Delta a/2a$ goes quadratically to zero at $\mu=\pm(\xi-1)/(\xi+1)$.

(b) Measurement of the relaxation-rate matrix, showing $(T_r^{-1})_{22}$ goes linearly to zero as $|\mu-\mu_{c1}|$.

(c) Rapid cycling of $\Delta a$ and hence $\mu$ ($\sim$100 Hz for the parameters of Ref. 3) to trace out the hysteresis window of Fig. 4.

(d) Recording of average jump points $\mu_{J1},\mu_{J2}$ to test the hysteresis window idea, seeing if the range for (well-defined) hysteresis is largest at some $\dot{\mu}$, where the window is widest. See if there is an increase in "noisiness" of the signal as $\dot{\mu}$ is increased so that $\mu_{c1,c2}$ are approached closely, violating (3.3).

The two-mode ring laser thus may have both pedagogic simplicity and experimental significance in understanding lifetimes and hysteresis. The usefulness and generality of the Schuss-Matkowsky methods for first-passage times in multivariate systems is clearly demonstrated in this context.


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