Boson-fermion duality in $SU(m|n)$ supersymmetric Haldane-Shastry spin chain

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Abstract

By using the $Y(gl(m|n))$ super Yangian symmetry of the $SU(m|n)$ supersymmetric Haldane-Shastry spin chain, we show that the partition function of this model satisfies a duality relation under the exchange of bosonic and fermionic spin degrees of freedom. As a byproduct of this study of the duality relation, we find a novel combinatorial formula for the super Schur polynomials associated with some irreducible representations of the $Y(gl(m|n))$ Yangian algebra. Finally, we reveal an intimate connection between the global $SU(m|n)$ symmetry of a spin chain and the boson-fermion duality relation.

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1 Introduction

The appearance of boson-fermion duality in lower dimensional quantum field theoretical models and many-particle systems has attracted a lot of attention in recent years [1-8]. The equivalence of bosonic sine-Gordon model with fermionic massive Thirring model is a classic example of boson-fermion duality in the context of one-dimensional field theoretical models [1]. Such a duality is a consequence of the fact that spin and statistics become essentially irrelevant notions in one spatial dimension, and thus the bosonic and fermionic theories can be related to each other through a duality transformation. In the context of many-particle systems in one dimension, it has been found that a bosonic and a fermionic model, with distinct point interactions and related by coupling constant inversion, share the same spectrum [2,3]. The signature of boson-fermion duality has also been observed in the setting of quantum many-body systems like the Tomonaga-Luttinger liquid theory of one-dimensional systems of interacting fermions, where the low-lying excitations are describable through bosonic degrees of freedom [4-6]. Recently, an exact bosonization method has been applied to study this problem in the non-interacting case even beyond the regime of validity of the low-energy approximation [7].

Duality relation has also been explored in the context of supersymmetric quantum integrable spin models, where bosonic and fermionic spin degrees of freedom appear simultaneously. It may be noted that, such exactly solvable one dimensional quantum spin chains have a close relation with correlated systems in condensed matter physics, where holes moving in the dynamical background of spins behave as bosons and spin$-\frac{1}{2}$ electrons behave as fermions [9,10]. Recent studies also reveal some interesting connection of these supersymmetric spin chains with loop models [11]. The Haldane-Shastry (HS) spin chain and the Polychronakos spin chain are two well known examples of quantum integrable models with long range interaction, for which the exact spectra can be computed analytically even for finite number of lattice sites [12-16]. Supersymmetric extensions of these spin chains and related exactly solvable models have also been studied intensively [17-24]. In particular, by using the freezing trick [13,19], the exact partition function of the $SU(m|n)$ supersymmetric Polychronakos spin chain has been derived in a simple form [21,22]. Furthermore, it has been shown analytically that, the partition function of this supersymmetric Polychronakos spin chain satisfies a duality transformation under the exchange of bosonic and fermionic spin degrees of freedom. So it is natural to enquire whether a similar boson-fermion duality relation exists in the case of the $SU(m|n)$ supersymmetric HS spin chain.

The Hamiltonian of the $SU(m|n)$ HS model, with N number of lattice sites uniformly distributed on a circle, is given by

$$H^{(m|n)} = \frac{1}{2} \sum_{1\leq j<k\leq N} \frac{(1+\hat{P}_{jk}^{(m|n)})}{\sin^2(\xi_j - \xi_k)} \,, \tag{1.1}$$

where $\xi_j = j\pi/N$, and $\hat{P}_{jk}^{(m|n)}$ is the supersymmetric exchange operator (its definition is given in Section 2) which interchanges the 'spins' on the j-th and k-th lattice sites. In analogy with the nonsupersymmetric case [25], the exact partition function of the $SU(m|n)$ HS spin chain (1.1) has been computed recently by applying the freezing trick [23]. It has also been conjectured that this partition function, which is denoted by $Z_N^{(m|n)}(q)$, satisfies a duality relation of the form

$$Z_N^{(m|n)}(q) = q^{\frac{N(N^2-1)}{6}} Z_N^{(n|m)}(q^{-1}) \,, \tag{1.2}$$

where $q \equiv e^{-\frac{1}{k_BT}}$. With the help of a symbolic software package like Mathematica, one can easily check the validity of this conjecture for a wide range of values of m, n and N. However, an analytical proof of this conjecture for all possible values of m, n and N has been lacking till now. The boson-fermion duality for the supersymmetric HS spin chain can also be studied at the level of the corresponding spectrum. Comparing the coefficients of the same powers of q on the two sides of Eqn. (1.2), one finds that the spectrum of $H^{(m|n)}$ (1.1) is related to that of $H^{(n|m)}$ through an inversion and an overall shift of all energy levels. Such a relation between the spectra of the $SU(m|n)$ and $SU(n|m)$ HS spin chains was first empirically observed by Haldane on the basis of results obtained by numerical diagonalization [17].

In this article, we aim to provide an analytical proof for the duality relation (1.2). To this end, it may be noted that the $SU(m|n)$ supersymmetric spin Calogero-Sutherland (CS) model has a $Y(gl(m|n))$ super Yangian symmetry [26]. Since the $SU(m|n)$ HS spin chain can be obtained by taking the freezing limit of the $SU(m|n)$ spin CS model, the Hamiltonian (1.1) also exhibits the $Y(gl(m|n))$ super Yangian symmetry [17]. It is well known that, a family of irreducible representations of this $Y(gl(m|n))$ quantum group can be labeled by some super skew Young diagrams and the corresponding Schur polynomials. Interestingly, such super Schur polynomials obey a duality relation under the exchange of bosonic and fermionic variables [27,28]. This duality relation will play a key role in our approach for proving the boson-fermion duality relation in the case of the $SU(m|n)$ HS spin chain. In Sec. 2 of this article, we briefly review the super Yangian symmetry for the $SU(m|n)$ HS spin chain and also give a simple alternative proof of the duality relation satisfied by the super Schur polynomials. In Sec. 3, we find a novel combinatorial formula for these super Schur polynomials, which allows us to establish a connection between these polynomials and the partition function of the $SU(m|n)$ HS spin chain. By exploiting the above mentioned connection, we give an analytical proof of the boson-fermion duality relation (1.2) in Sec. 4. In Sec. 5, we explore the possibility of constructing a class of quantum integrable as well as nonintegrable spin chains which would satisfy the boson-fermion duality relation. Sec. 6 is the concluding section.

2 $Y(gl(m|n))$ super Yangian symmetry of $SU(m|n)$ HS spin chain

For the purpose of defining the super exchange operator in the Hamiltonian (1.1) of the $SU(m|n)$ HS spin chain, let us consider a set of operators like $C^{\dagger}_{j\alpha}(C_{j\alpha})$ which creates (annihilates) a particle of species α on the j-th lattice site. These creation (annihilation) operators are assumed to be bosonic when $\alpha \in \{1, 2, ..., m\}$ and fermionic when $\alpha \in \{m+1, m+2, ..., m+n\}$. Thus, the parity of $C^{\dagger}_{j\alpha}(C_{j\alpha})$ is defined as

$$\begin{aligned} p(C_{j\alpha}) = p(C^{\dagger}_{j\alpha}) = 0 \ for \ \alpha \in \{1, 2, ..., m\} \, , \\ p(C_{j\alpha}) = p(C^{\dagger}_{j\alpha}) = 1 \ for \ \alpha \in \{m+1, m+2, ..., m+n\} \, . \end{aligned}$$

These operators satisfy (anti-) commutation relations like

$$[C_{j\alpha}, C_{k\beta}]_{\pm} = 0 \, , \ [C^{\dagger}_{j\alpha}, C^{\dagger}_{k\beta}]_{\pm} = 0 \, , \ [C_{j\alpha}, C^{\dagger}_{k\beta}]_{\pm} = \delta_{jk}\delta_{\alpha\beta} \, , \tag{2.1}$$

where $[A, B]_{\pm} \equiv AB - (-1)^{p(A)p(B)}BA$. Next, we focus our attention on a subspace of the related Fock space, for which the total number of particles per site is always one:

$$\sum_{\alpha=1}^{m+n} C^{\dagger}_{j\alpha} C_{j\alpha} = 1, \tag{2.2}$$

for all j. On the above mentioned subspace, one can define the supersymmetric exchange operators as

$$\hat{P}^{(m|n)}_{jk} \equiv \sum_{\alpha,\beta=1}^{m+n} C^{\dagger}_{j\alpha} C^{\dagger}_{k\beta} C_{j\beta} C_{k\alpha} \, , \tag{2.3}$$

where $1 \leq j < k \leq N$. Inserting these exchange operators in (1.1), one obtains the Hamiltonian of the $SU(m|n)$ HS spin chain.

Next, we shall briefly review the super Yangian symmetry of this $SU(m|n)$ HS spin chain. Let $\mathbf{V}$ be an $(m+n)$-dimensional auxiliary graded vector space, through which the graded Yang–Baxter equation will be defined. We set $\mathbf{B} = \mathbf{B}_+ \sqcup \mathbf{B}_-$, where

$$\mathbf{B}_+ = \{\epsilon_1, \ldots, \epsilon_m\}\,, \qquad \mathbf{B}_- = \{\epsilon_{m+1}, \ldots, \epsilon_{m+n}\}. \tag{2.4}$$

The generators $\mathbf{E}^{\alpha\beta}$ of the $gl(m|n)$ Lie algebra satisfy (anti-)commutation relations given by

$$[\mathbf{E}^{\alpha\beta}, \mathbf{E}^{\gamma\delta}]_\pm = \delta^{\beta\gamma}\mathbf{E}^{\alpha\delta} - (-1)^{(p(\alpha)+p(\beta))(p(\gamma)+p(\delta))}\delta^{\alpha\delta}\mathbf{E}^{\gamma\beta}\,,$$

where $p(\alpha) = 0$ (resp. $p(\alpha) = 1$) if $\epsilon_\alpha \in \mathbf{B}_+$ (resp. $\epsilon_\alpha \in \mathbf{B}_-$). With these generators, we define the graded permutation operator as

$$\mathbf{P} = \sum_{\alpha,\beta=1}^{m+n} (-1)^{p(\beta)}\mathbf{E}^{\alpha\beta} \otimes \mathbf{E}^{\beta\alpha}\,. \tag{2.5}$$

Let $\{e^\alpha\}$ be a set of basis vectors of the auxiliary vector space $\mathbf{V}$. The generators $\mathbf{E}^{\alpha\beta}$ and the graded permutation operator $\mathbf{P}$ act on such basis vectors and their direct products as

$$\mathbf{E}^{\alpha\beta}e^\gamma = \delta^{\beta\gamma}e^\alpha\,,$$
$$\mathbf{P}\, e^\alpha \otimes e^\beta = (-1)^{p(\alpha)p(\beta)}e^\beta \otimes e^\alpha\,.$$

We have the rational solution of the graded Yang-Baxter equation given by

$$R(u) = u - \hbar\,\mathbf{P}\,, \tag{2.6}$$

where u is the spectral parameter. The $Y(gl(m|n))$ super Yangian [29] is associated to this R-matrix. Namely,

$$R(u-v)\,\overset{1}{\mathbf{T}}(u)\,\overset{2}{\mathbf{T}}(v) = \overset{2}{\mathbf{T}}(v)\,\overset{1}{\mathbf{T}}(u)\,R(u-v)\,, \tag{2.7}$$

where $\overset{1}{\mathbf{T}}(u) \equiv \mathbf{T}(u) \otimes \mathbb{1}$, $\overset{2}{\mathbf{T}}(v) \equiv \mathbb{1} \otimes \mathbf{T}(v)$ and $\mathbf{T}(u)$ is defined as

$$\mathbf{T}(u) = \sum_{\alpha,\beta=1}^{m+n} (-1)^{p(\alpha)}\, T^{\alpha\beta}(u)\,\mathbf{E}^{\alpha\beta}\,.$$

Computing the tensor products through the rule

$$(a_1 \otimes b_1)\ (a_2 \otimes b_2) = (-1)^{p(a_2)p(b_1)}\, a_1\, a_2 \otimes b_1\, b_2\,,$$

one can express Eqn. (2.7) as

$$\begin{aligned}&\left[T^{\alpha\beta}(u), T^{\gamma\delta}(v)\right]_\pm \\ &\quad = \frac{\hbar}{u-v}\,(-1)^{p(\alpha)p(\beta)+p(\gamma)p(\beta)+p(\alpha)p(\gamma)}\,\bigl(T^{\gamma\beta}(u)\,T^{\alpha\delta}(v) - T^{\gamma\beta}(v)\,T^{\alpha\delta}(u)\bigr)\,.\end{aligned} \tag{2.8}$$

The Yangian currents $T^{\alpha\beta}(u)$ may be expanded in powers of the spectral parameter as

$$T^{\alpha\beta}(u) = \delta^{\alpha\beta} + \hbar \sum_{n=0}^{\infty} (-1)^{p(\alpha)} \frac{T_n^{\alpha\beta}}{u^{n+1}} , \tag{2.9}$$

and the $Y(gl(m|n))$ algebra (2.8) can also be expressed in a spectral parameter independent way through the generators $T_n^{\alpha\beta}$.

The $Y(gl(m|n))$ super Yangian symmetry has been realized explicitly in the case of the $SU(m|n)$ HS spin chain [17,30]. Suitable combinations of the generators $T_0^{\alpha\beta}$ and $T_1^{\alpha\beta}$ yield conserved quantities of the Hamiltonian (1.1) in the form

$$Q_0^{\alpha\beta} = \sum_{j=1}^{N} \left(C_{j\alpha}^{\dagger} C_{j\beta} - \frac{1}{m+n} \delta_{\alpha\beta} \right) , \tag{2.10a}$$

$$Q_1^{\alpha\beta} = \frac{1}{2} \sum_{j \neq k} \sum_{\gamma=1}^{m+n} \cot \left(\frac{j-k}{N} \pi \right) C_{j\alpha}^{\dagger} C_{k\gamma}^{\dagger} C_{j\gamma} C_{k\beta} . \tag{2.10b}$$

It is well known that the $Y(gl(m|n))$ Yangian algebra is effectively generated by the lowest two generators $T_0^{\alpha\beta}$ and $T_1^{\alpha\beta}$. Consequently, by using the commutation relations among conserved quantities like $Q_0^{\alpha\beta}$ and $Q_1^{\alpha\beta}$, one can obtain the complete $Y(gl(m|n))$ Yangian symmetry of the $SU(m|n)$ HS spin chain.

Next, we shall prepare the super Schur polynomials (see e.g. [31]) which are closely related to a class of irreducible representations of $Y(gl(m|n))$ Yangian algebra. For $\mathbf{B}$, we set a usual ordering as

$$\epsilon_1 \prec \epsilon_2 \prec \cdots \prec \epsilon_{m+n} .$$

The Young tableaux T is obtained by filling the numbers $1, 2, \ldots, m+n$ in a given Young diagram λ by the rules:

- Entries in each row are increasing, allowing the repetition of elements in $\{i|\epsilon_i \in \mathbf{B}_+\}$, but not permitting the repetition of elements in $\{i|\epsilon_i \in \mathbf{B}_-\}$,
- Entries in each column are increasing, allowing the repetition of elements in $\{i|\epsilon_i \in \mathbf{B}_-\}$, but not permitting the repetition of elements in $\{i|\epsilon_i \in \mathbf{B}_+\}$.

The super Schur polynomial corresponding to the Young diagram λ is then defined as

$$S_\lambda(x, y) = \sum_{\text{tableaux } T \text{ of shape } \lambda} e^{\mathrm{wt}(T)} . \tag{2.11}$$

Here the weight $\mathrm{wt}(T)$ of the Young tableaux T is given by

$$\mathrm{wt}(T) = \sum_{\alpha} m_\alpha \, \epsilon_\alpha ,$$

where m_α denotes the number of α in T, and we use the notations

$$x_i \equiv e^{\epsilon_i} \text{ for } \epsilon_i \in \mathbf{B}_+ , \qquad\qquad y_i \equiv e^{\epsilon_{m+i}} \text{ for } \epsilon_{m+i} \in \mathbf{B}_- ,$$

along with $x \equiv \{x_1, \ldots, x_m\}$, $y \equiv \{y_1, \ldots, y_n\}$.

A skew Young diagram λ/μ is obtained by removing a smaller Young diagram μ from a larger one λ that contains it [32]. The super Schur polynomial $S_{\lambda/\mu}(x, y)$ corresponding to such skew Young diagram λ/μ can also be defined combinatorially as in (2.11). Let λ' denote the conjugate of the Young diagram λ (the conjugate of a Young diagram is obtained by flipping it over its main diagonal). It is evident that the rows of a conjugate diagram are mapped to the columns of the original diagram and vice versa. It is worth noting that the rule for filling up a *row* of a super Young tableaux, as stated before Eqn. (2.11), is transformed to the rule for filling up a *column* of a super Young tableaux (and vice versa) provided we substitute the elements $\{i|\epsilon_i \in \mathbf{B}_+\}$ in place of elements $\{i|\epsilon_i \in \mathbf{B}_-\}$ (and vice versa). Due to such a *duality* of the rules for filling numbers in the case of a skew Young tableaux, we easily obtain

$$S_{\lambda/\mu}(x, y) = S_{\lambda'/\mu'}(y, x) . \tag{2.12}$$

This duality relation between two super Schur polynomials associated with a skew Young diagram and its conjugate diagram has also been found earlier through a different approach [27,28].

Hereafter we shall consider only connected super skew Young diagrams which do not contain any 2×2 square box. Such a skew Young diagram is also called a 'border strip' and may be denoted by $\langle m_1, m_2, \ldots, m_r\rangle$:
These border strips will play a key role for our purpose due to their connection with 'motifs' [30,32,33], which represent irreducible representations of Yangian algebra and span the Fock space of Yangian invariant spin systems. The motif δ for an N-site super spin chain is given by an $N-1$ sequence of 0's and 1's, $\delta = (\delta_1, \delta_2, \ldots, \delta_{N-1})$ with $\delta_j \in \{0, 1\}$. There exists a one-to-one map from a motif δ to the border strip $\langle m_1, m_2, \ldots, m_r\rangle$; we read a motif $\delta = (\delta_1, \delta_2, \ldots)$ from the left, and add a box under (resp. left) the box when we encounter $\delta_j = 1$ (resp. $\delta_j = 0$). For example, the motif (10110) leads to the border strip $\langle 2, 3, 1\rangle$. The inverse mapping from a border strip to a motif can also be defined in a straightforward way.

Finally we discuss a convenient way of expressing the super Schur polynomials associated with border strips through the supersymmetric elementary functions. Let us assume that the polynomial $e_\ell(x)$ represents the sum of all monomials $x_{i_1} x_{i_2} \cdots x_{i_\ell}$ for all strictly increasing sequences $1 \leq i_1 < i_2 < \cdots < i_\ell \leq m$, while the polynomial

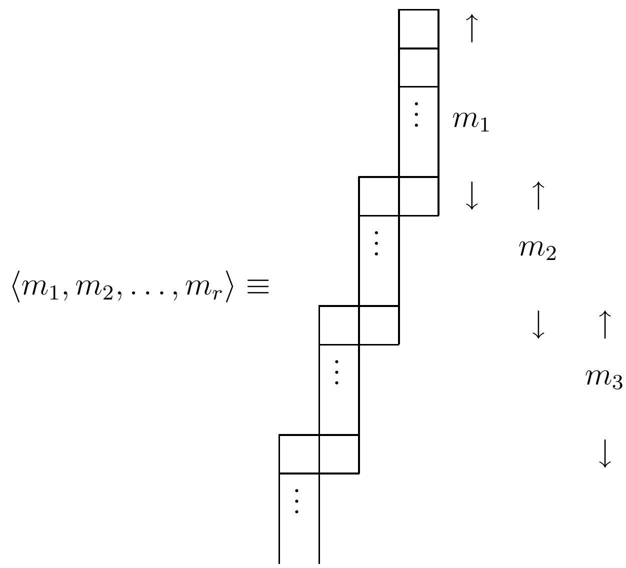


Figure 1: Shape of the border strip $\langle m_1, m_2, \ldots, m_r \rangle$

$h_\ell(y)$ represents the sum of all distinct monomials of degree ℓ in the variables y. We have the following generating functions for polynomials $e_\ell(x)$ and $h_\ell(y)$:

$$\prod_{i=1}^{m}(1 + t\,x_i) = \sum_{\ell=0}^{m} e_\ell(x)\,t^\ell\,, \qquad \prod_{i=1}^{n}\frac{1}{1 - t\,y_i} = \sum_{\ell=0}^{\infty} h_\ell(y)\,t^\ell\,. \tag{2.13a, b}$$

The supersymmetric elementary function $E_j(x, y)$ ($\equiv S_{[1^j]}(x, y)$) may be written through the polynomials $e_\ell(x)$ and $h_\ell(y)$ as [22,34]

$$E_j(x, y) \;=\; \sum_{\ell=0}^{j} e_\ell(x)\,h_{j-\ell}(y)\,. \tag{2.14}$$

By using these supersymmetric elementary functions, one can express the super Schur polynomial corresponding to the border strip $\langle m_1, m_2, \ldots, m_r \rangle$ in the form of a determinant given by [22,32]

$$S_{\langle m_1, m_2, \ldots, m_r \rangle}(x, y) = \begin{vmatrix} E_{m_r} & E_{m_r+m_{r-1}} & \ldots & \ldots & E_{m_r+\cdots+m_1} \\ 1 & E_{m_{r-1}} & E_{m_{r-1}+m_{r-2}} & \ldots & E_{m_{r-1}+\cdots+m_1} \\ 0 & 1 & E_{m_{r-2}} & \ldots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & E_{m_1} \end{vmatrix}. \tag{2.15}$$

Expansion of the above determinant along its first row yields a recursion relation for the super Schur polynomials as

$$S_{\langle m_1, m_2, \ldots, m_r \rangle}(x, y) = \sum_{s=1}^{r} (-1)^{s+1} E_{m_r+m_{r-1}+\cdots+m_{r-s+1}}(x, y) \cdot S_{\langle m_1, m_2, \ldots, m_{r-s} \rangle}(x, y)\,, \tag{2.16}$$

where $S_{\langle 0\rangle}(x,y)=1$. This recursion relation will play an important role in our analysis in the next section.

3 Partition function of $SU(m|n)$ HS spin chain and super Schur polynomials

Here our aim is to make a connection between the partition function of the $SU(m|n)$ supersymmetric HS spin chain and the super Schur polynomials associated with the border strips. The partition function for the Hamiltonian (1.1) of the $SU(m|n)$ HS spin chain is found to be [23]

$$Z_N^{(m|n)}(q)=\sum_{r=1}^{N}\ \sum_{\{m_1,m_2,\ldots,m_r\}\in\mathcal{P}_N(r)}\Big(\prod_{i=1}^{r}d_{m_i}^{(m|n)}\Big)\,F_{m_1,m_2,\ldots,m_r}(q)\,, \tag{3.1}$$

where $q\equiv e^{-1/k_BT}$, $d_{m_i}^{(m|n)}$ is given by

$$d_{m_i}^{(m|n)}=\sum_{k=0}^{\min(m_i,\,m)}C_k^m\,C_{m_i-k}^{m_i-k+n-1}\,, \tag{3.2}$$

with $C_k^m=\frac{m!}{k!(m-k)!}$, $F_{m_1,m_2,\ldots,m_r}(q)$ is a polynomial of q which is defined in the following, and $\mathcal{P}_N(r)$ denotes the set of all partitions (taking care of ordering) of N with length r. For example, the set $\mathcal{P}_4(2)$ is given by $\big\{\{3,1\},\{1,3\},\{2,2\}\big\}$. Let us introduce the partial sums corresponding to the partition $\{m_1,m_2,\ldots,m_r\}\in\mathcal{P}_N(r)$ as

$$M_j=\sum_{i=1}^{j}m_i, \tag{3.3}$$

where $j\in\{1,2,\ldots,r\}$. It may be noted that, $1\le M_1<M_2<\cdots<M_{r-1}\le N-1$ and $M_r=N$. The complementary partial sums corresponding to the partition $\{m_1,m_2,\ldots,m_r\}$ are denoted by M_j with $j\in\{r+1,r+2,\ldots,N\}$, and they are defined through the relation

$$\{M_{r+1},M_{r+2},\ldots,M_N\}\equiv\{1,2,\ldots,N\}-\{M_1,M_2,\ldots,M_r\}. \tag{3.4}$$

In contrast to the case of partial sums, there exists no natural ordering among these complementary partial sums and they can be ordered in an arbitrary way. The 'energy function' corresponding to a partial sum or complementary partial sum is defined as

$$\mathcal{E}(M_j)=M_j(N-M_j), \tag{3.5}$$

where $j \in \{1, 2, \ldots, N\}$. The polynomial $F_{m_1,m_2,\ldots,m_r}(q)$ in Eqn. (3.1) is expressed through these energy functions as

$$F_{m_1,m_2,\ldots,m_r}(q) = q^{\sum\limits_{j=1}^{r-1} \mathcal{E}(M_j)} \prod_{j=r+1}^{N} (1 - q^{\mathcal{E}(M_j)}) \,. \tag{3.6}$$

It should be noted that $Z_N^{(m|n)}(q)$ in Eqn. (3.1) does not depend on the parameters x and y, which are present in the expression of $S_{\langle m_1,m_2,\ldots,m_r \rangle}(x, y)$ in Eqn. (2.15). So, for making a connection with the super Schur polynomials, we introduce a 'generalized' partition function for the $SU(m|n)$ HS spin chain:

$$Z_N^{(m|n)}(q; x, y) = \sum_{r=1}^{N} \sum_{\{m_1,m_2,\ldots,m_r\} \in \mathcal{P}_N(r)} \Big(\prod_{i=1}^{r} E_{m_i}(x, y) \Big) F_{m_1,m_2,\ldots,m_r}(q), \tag{3.7}$$

$E_{m_i}(x, y)$ being a supersymmetric elementary function which is defined in Eqn. (2.14). Setting all x_i's and y_i's equal to 1, and then equating the coefficients of the same powers of t from both sides of Eqn. (2.13a) or Eqn. (2.13b), we find that

$$e_k(x)\Big|_{x=1} = C_k^m, \qquad h_{m_i-k}(y)\Big|_{y=1} = C_{m_i-k}^{m_i-k+n-1}.$$

Substituting these relations in Eqn. (2.14) and comparing it with Eqn. (3.2), it is easy to see that

$$E_{m_i}(x, y)\Big|_{x=1,y=1} = d_{m_i}^{(m|n)}. \tag{3.8}$$

Consequently, by inserting $x = y = 1$ in Eqn. (3.7) and comparing it with Eqn. (3.1), we find that

$$Z_N^{(m|n)}(q; x, y)\Big|_{x=1,y=1} = Z_N^{(m|n)}(q). \tag{3.9}$$

Next, we note that $F_{m_1,m_2,\ldots,m_r}(q)$ in Eqn. (3.6) can be explicitly written in the form of a polynomial as

$$F_{m_1,m_2,\ldots,m_r}(q) = q^{\sum\limits_{j=1}^{r-1} \mathcal{E}(M_j)} \sum_{\alpha_{r+1}=0}^{1} \sum_{\alpha_{r+2}=0}^{1} \cdots \sum_{\alpha_N=0}^{1} (-1)^{\sum\limits_{i=r+1}^{N} \alpha_i} q^{\sum\limits_{i=r+1}^{N} \alpha_i \, \mathcal{E}(M_i)} . \tag{3.10}$$

The lowest power of q in this polynomial is given by $\sum_{j=1}^{r-1} \mathcal{E}(M_j)$, which is obtained by choosing $\alpha_{r+1} = \alpha_{r+2} = \cdots = \alpha_N = 0$ in the r.h.s. of the above equation. On the other hand, for the choice $\alpha_i = 1$ when $i \in \{l_1, l_2, \ldots, l_k\}$ (where $\{l_1, l_2, \ldots, l_k\} \subseteq \{r + 1, r + 2, \ldots, N\}$) and $\alpha_i = 0$ when $i \notin \{l_1, l_2, \ldots, l_k\}$, a term with a higher power of q given by $\sum_{j=1}^{r-1} \mathcal{E}(M_j) + \sum_{i=1}^{k} \mathcal{E}(M_{l_i})$ will appear in this polynomial. It is easy to check that this higher power of q coincides with the lowest power of q appearing in another polynomial

$F_{m'_1,m'_2,\ldots,m'_{r+k}}(q)$ associated with the partition $\{m'_1, m'_2, \ldots, m'_{r+k}\} \in \mathcal{P}_N(r+k)$, for which the partial sums form a set given by

$$\{M_1, M_2, \ldots, M_{r-1}, M_r\} \cup \{M_{l_1}, M_{l_2}, \ldots, M_{l_k}\}\,.$$

In this way, any higher power of q appearing in the polynomial $F_{m_1,m_2,\ldots,m_r}(q)$ would coincide with the lowest power of q appearing in some other polynomial $F_{m'_1,m'_2,\ldots,m'_{r+k}}(q)$. Consequently, the lowest order terms of all possible polynomials like $F_{m_1,m_2,\ldots,m_r}(q)$ (associated with all possible partitions of N) form a 'complete set', through which $Z_N^{(m|n)}(q;x,y)$ in Eqn. (3.7) can be expressed as a polynomial in q as

$$Z_N^{(m|n)}(q;x,y) = \sum_{r=1}^{N} \sum_{\{m_1,m_2,\ldots,m_r\}\in\mathcal{P}_N(r)} q^{\sum_{j=1}^{r-1}\mathcal{E}(M_j)}\, \tilde{S}_{\langle m_1,m_2,\ldots,m_r\rangle}(x,y), \tag{3.11}$$

where $\tilde{S}_{\langle m_1,m_2,\ldots,m_r\rangle}(x,y)$ are some unknown functions of x and y, which will be determined in the following.

Comparing the r.h.s. of Eqns. (3.7) and (3.11), and also using (3.10), we find that

$$\begin{aligned}&\sum_{r=1}^{N} \sum_{\{m_1,m_2,\ldots,m_r\}\in\mathcal{P}_N(r)} q^{\sum_{j=1}^{r-1}\mathcal{E}(M_j)}\, \tilde{S}_{\langle m_1,m_2,\ldots,m_r\rangle}(x,y)\\ &= \sum_{k=1}^{N} \sum_{\{m'_1,m'_2,\ldots,m'_k\}\in\mathcal{P}_N(k)} \left(\prod_{i=1}^{k} E_{m'_i}(x,y)\right) q^{\sum_{j=1}^{k-1}\mathcal{E}(M'_j)} \sum_{\alpha_{k+1},\cdots,\alpha_N=0}^{1} (-1)^{\sum_{i=k+1}^{N}\alpha_i}\, q^{\sum_{i=k+1}^{N}\alpha_i\mathcal{E}(M'_i)},\end{aligned} \tag{3.12}$$

where M'_j's denote the partial sums and complementary sums corresponding to the partition $\{m'_1, m'_2, \ldots, m'_k\}$. Note that corresponding to each partition $\{m'_1, m'_2, \ldots, m'_k\} \in \mathcal{P}_N(k)$, many terms with different powers of q in general appear in the r.h.s. of the above equation. Let us first try to find out the necessary condition for which a partition yields at least one term with the power of q being given by $\sum_{j=1}^{r-1}\mathcal{E}(M_j)$. It should be observed that $q^{\sum_{j=1}^{k-1}\mathcal{E}(M'_j)}$ is the common factor of all terms generated by the partition $\{m'_1, m'_2, \ldots, m'_k\}$ in the r.h.s. of Eqn. (3.12). Consequently, the term $q^{\sum_{j=1}^{r-1}\mathcal{E}(M_j)}$ can be generated through the partition $\{m'_1, m'_2, \ldots, m'_k\}$ only if $k \leq r$ and the corresponding partial sums satisfy the condition

$$\{M'_1, M'_2, \cdots, M'_k\} \subseteq \{M_1, M_2, \cdots, M_r\}. \tag{3.13}$$

Hence, we can write these partial sums as

$$M'_i = M_{L_i}, \tag{3.14}$$

where $i \in \{1, 2, \ldots, k\}$, and the indices $L_1, L_2, \ldots, L_k$ satisfy the condition

$$1 \leq L_1 < L_2 \cdots < L_k = r. \tag{3.15}$$

Let us consider another set of indices like $L_{k+1}, L_{k+2}, \ldots, L_r$, and define the corresponding set as

$$\{L_{k+1}, L_{k+2}, \ldots, L_r\} \equiv \{1, 2, \ldots, r\} - \{L_1, L_2, \ldots, L_k\}. \tag{3.16}$$

Using Eqns. (3.14) and (3.16) we obtain

$$\begin{aligned} \{M_1, M_2, \ldots, M_r\} - \{M'_1, M'_2, \ldots, M'_k\} &= \{M_1, M_2, \ldots, M_r\} - \{M_{L_1}, M_{L_2}, \ldots, M_{L_k}\} \\ &= \{M_{L_{k+1}}, M_{L_{k+2}}, \ldots, M_{L_r}\}. \end{aligned} \tag{3.17}$$

With the help of the embedding condition $\{M'_1, M'_2, \cdots, M'_k\} \subseteq \{M_1, M_2, \cdots, M_r\} \subseteq \{1, 2, \cdots, N\}$, the set of complementary partial sums associated with the partition $\{m'_1, m'_2, \ldots, m'_k\}$ can be written as

$$\begin{aligned} &\{M'_{k+1}, M'_{k+2}, \ldots, M'_N\} \\ &= \{1, 2, \ldots, N\} - \{M'_1, M'_2, \ldots, M'_k\} \\ &= \big(\{1, 2, \ldots, N\} - \{M_1, M_2, \ldots, M_r\}\big) \cup \big(\{M_1, M_2, \ldots, M_r\} - \{M'_1, M'_2, \ldots, M'_k\}\big). \end{aligned} \tag{3.18}$$

Using the relation (3.18) along with (3.4) and (3.17), we find that

$$\{M'_{k+1}, M'_{k+2}, \ldots, M'_N\} = \{M_{r+1}, M_{r+2}, \ldots, M_N\} \cup \{M_{L_{k+1}}, M_{L_{k+2}}, \ldots, M_{L_r}\}. \tag{3.19}$$

Consequently, we can express the complementary partial sums associated with the partition $\{m'_1, m'_2, \ldots, m'_k\}$ as

$$M'_j = M_{L_j}, \tag{3.20}$$

where the L_j's are defined through Eqn. (3.16) when $j \in \{k+1, k+2, \cdots, r\}$ (the ordering of indices $L_{k+1}, L_{k+2}, \ldots, L_r$ is not important for our purpose), and $L_j = j$ when $j \in \{r+1, r+2, \ldots, N\}$.

With the help of Eqns. (3.17) and (3.20) we find that, for the choice of summation variables like $\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_r = 1$ and $\alpha_{r+1} = \alpha_{r+2} = \cdots = \alpha_N = 0$, one term with the power of q given by $\sum_{j=1}^{r-1} \mathcal{E}(M_j)$ is generated through the partition $\{m'_1, m'_2, \ldots, m'_k\}$ in the r.h.s. of Eqn. (3.12). Moreover, the coefficient of $q^{\sum_{j=1}^{r-1} \mathcal{E}(M_j)}$ in the above mentioned term is obtained as

$$\mathcal{C}_{\{m'_1, m'_2, \ldots, m'_k\}} = (-1)^{r-k} \prod_{i=1}^{k} E_{m'_i}(x, y) = (-1)^{r-k} \prod_{i=1}^{k} E_{M'_i - M'_{i-1}}(x, y)\,, \tag{3.21}$$

where it is assumed that $M'_0 = 0$. Using Eqn. (3.14), these coefficients can also be written as

$$\mathcal{C}_{\{m'_1, m'_2, \ldots, m'_k\}} = (-1)^{r-k} \prod_{i=1}^{k} E_{M_{L_i} - M_{L_{i-1}}}(x, y)\,, \tag{3.22}$$

where the indices $L_1, L_2, \ldots, L_k$ satisfy the condition (3.15) and it is assumed that $M_{L_0} = 0$.

From the above discussion it is evident that, Eqn. (3.13) represents not only the *necessary* but also the *sufficient* condition for which the partition $\{m'_1, m'_2, \ldots, m'_k\}$ yields one term with the power of q being given by $\sum_{j=1}^{r-1} \mathcal{E}(M_j)$. By summing up the coefficients of $q^{\sum_{j=1}^{r-1} \mathcal{E}(M_j)}$ associated with all such partitions, we can determine $\tilde{S}_{\langle m_1, m_2, \ldots, m_r \rangle}(x, y)$ appearing in Eqn. (3.12). Thus, by using Eqn. (3.22), we find that

$$\tilde{S}_{\langle m_1, m_2, \ldots, m_r \rangle}(x, y) = \sum_{k=1}^{r} \sum_{1 \leq L_1 < \cdots < L_k = r} (-1)^{r-k} \prod_{i=1}^{k} E_{M_{L_i} - M_{L_{i-1}}}(x, y)\,. \tag{3.23}$$

Since the L_i's appearing in the above equation satisfy the condition (3.15), they can be written as

$$L_i = \sum_{s=1}^{i} \ell_s\,, \tag{3.24}$$

where the ℓ_s's are k number of positive integers such that $\{\ell_1, \ell_2, \ldots, \ell_k\} \in \mathcal{P}_r(k)$. Consequently, $\tilde{S}_{\langle m_1, m_2, \ldots, m_r \rangle}(x, y)$ in Eqn. (3.23) can also be expressed in the form

$$\tilde{S}_{\langle m_1, m_2, \ldots, m_r \rangle}(x, y) = \sum_{k=1}^{r} \sum_{\{\ell_1, \ell_2, \ldots, \ell_k\} \in \mathcal{P}_r(k)} (-1)^{r-k} \prod_{i=1}^{k} E_{M_{L_i} - M_{L_{i-1}}}(x, y)\,, \tag{3.25}$$

where the L_i's are related to the ℓ_s's through Eqn. (3.24).

Next, we find that $\tilde{S}_{\langle m_1, m_2, \ldots, m_r \rangle}(x, y)$ given by Eqn. (3.25) satisfies the following recursion relation:

$$\tilde{S}_{\langle m_1, m_2, \ldots, m_r \rangle}(x, y) = \sum_{s=1}^{r} (-1)^{s+1} E_{m_r + m_{r-1} + \cdots + m_{r-s+1}}(x, y) \cdot \tilde{S}_{\langle m_1, m_2, \ldots, m_{r-s} \rangle}(x, y)\,, \tag{3.26}$$

where $\tilde{S}_{\langle 0 \rangle}(x, y) = 1$. The derivation of this recursion relation is presented in Appendix A. It is interesting to observe that, the above recursion relation is exactly the same in form as the recursion relation (2.16), which is satisfied by the super Schur polynomials associated with the border strips. Consequently, the function $\tilde{S}_{\langle m_1, m_2, \ldots, m_r \rangle}(x, y)$ coincides with the super Schur polynomial $S_{\langle m_1, m_2, \ldots, m_r \rangle}(x, y)$. Thus, Eqn. (3.25) gives us a novel combinatorial formula for the super Schur polynomial corresponding to the border strip $\langle m_1, m_2, \ldots, m_r \rangle$, which is usually defined through Eqn. (2.15). Substituting

$\tilde{S}_{\langle m_1, m_2, \ldots, m_r \rangle}(x,y)$ by $S_{\langle m_1, m_2, \ldots, m_r \rangle}(x,y)$ in Eqn. (3.11), we can express the generalized partition function of the $SU(m|n)$ HS spin through the super Schur polynomials as

$$Z_N^{(m|n)}(q;x,y) = \sum_{r=1}^{N} \sum_{\{m_1,m_2,\ldots,m_r\} \in \mathcal{P}_N(r)} q^{\sum_{j=1}^{r-1} \mathcal{E}(M_j)} S_{\langle m_1, m_2, \ldots, m_r \rangle}(x,y) \, . \tag{3.27}$$

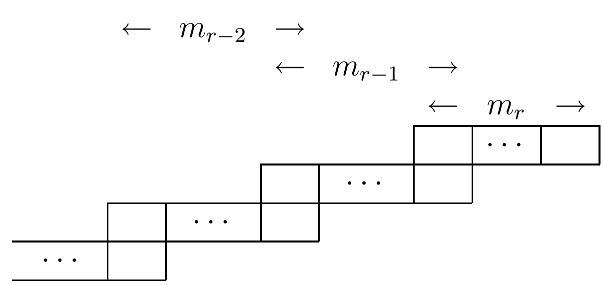
In the limit $x = y = 1$, this generalized partition function clearly reduces to the standard partition function $Z_N^{(m|n)}(q)$ given in Eqn. (3.1).

It should be noted that, the partition function of the Polychronakos model can also written in a form similar to (3.27) by using the corresponding energy function [32,22]. This is due to the fact that both models, the HS model and the Polychronakos model, share the same Yangian symmetry. See Ref. 35 for the conformal field theoretic construction of conserved quantities leading to the Yangian symmetry.

4 Duality relation for supersymmetric HS spin chain

In this section, our aim is to give an analytical proof for the duality relation (1.2) involving the partition functions of the $SU(m|n)$ and $SU(n|m)$ HS spin chains. A central role in this proof will be played by the duality relation (2.12) for the super Schur polynomials. Let us assume that λ/μ in (2.12) represents a border strip: $\lambda/\mu \equiv \langle m_1, m_2, \ldots, m_r \rangle$, where $\{m_1, m_2, \ldots, m_r\} \in \mathcal{P}_N(r)$, and denote the conjugate of this border strip as $\langle m_1, m_2, \ldots, m_r \rangle'$. Applying the rule for obtaining a conjugate diagram to Fig. 1, we find that

$\langle m_1, m_2, \ldots, m_r \rangle' \equiv$



.

Figure 2: Shape of the border strip conjugate to $\langle m_1, m_2, \ldots, m_r \rangle$

It may be observed that, the vertical length of the border strip $\langle m_1, m_2, \ldots, m_r \rangle$ drawn in Fig. 1 is given by $(N - r + 1)$. Since this vertical length coincides with the horizontal length of the conjugate border strip $\langle m_1, m_2, \ldots, m_r\rangle'$ (as drawn in Fig. 2), it can also be expressed as

$$\langle m_1, m_2, \ldots, m_r\rangle' \equiv \langle m'_1, m'_2, \ldots, m'_{N-r+1}\rangle,$$

where m'_j's are some functions of m_j's and $\{m'_1, m'_2, \ldots, m'_{N-r+1}\} \in \mathcal{P}_N(N-r+1)$. A relation between m_j's and m'_j's will be established in the following by exploiting the motif representations corresponding to the border strips.

Using the rule for mapping a motif to a border strip as discussed in Sec. 2, and observing Fig. 1, it is easy to check that

$$(\underbrace{1, \ldots, 1}_{m_1-1}, 0, \underbrace{1, \ldots, 1}_{m_2-1}, 0, \ldots\ldots, 0, \underbrace{1, \ldots, 1}_{m_r-1}) \implies \langle m_1, m_2, \ldots, m_r\rangle \,. \tag{4.1}$$

Let us denote the set formed by the positions of 0's (resp. 1's) in the motif associated with the border strip $\langle m_1, m_2, \ldots, m_r\rangle$ as $Q_{\langle m_1,m_2,\ldots,m_r\rangle}(0)$ (resp. $Q_{\langle m_1,m_2,\ldots,m_r\rangle}(1)$). For example, since the motif (10110) leads to the border strip $\langle 2, 3, 1\rangle$, $Q_{\langle 2,3,1\rangle}(0) = \{2, 5\}$ and $Q_{\langle 2,3,1\rangle}(1) = \{1, 3, 4\}$. Observing the l.h.s. of (4.1) we find that, the set $Q_{\langle m_1,m_2,\ldots,m_r\rangle}(0)$ can be expressed through the partial sums corresponding to the partition $\{m_1, m_2, \ldots, m_r\}$ as

$$Q_{\langle m_1,m_2,\ldots,m_r\rangle}(0) = \{M_1, M_2, \cdots, M_{r-1}\} \,. \tag{4.2}$$

Using (4.2) along with the relation

$$Q_{\langle m_1,m_2,\ldots,m_r\rangle}(1) = \{1, 2, \cdots, N-1\} - Q_{\langle m_1,m_2,\ldots,m_r\rangle}(0) \,,$$

it is easy to express the set $Q_{\langle m_1,m_2,\ldots,m_r\rangle}(1)$ through the complementary partial sums corresponding to the partition $\{m_1, m_2, \ldots, m_r\}$ as

$$Q_{\langle m_1,m_2,\ldots,m_r\rangle}(1) = \{M_{r+1}, M_{r+2}, \cdots, M_N\} \,. \tag{4.3}$$

Applying the rule for mapping a motif to a border strip and observing Fig. 2, it is easy to check that

$$(\underbrace{0, \ldots, 0}_{m_r-1}, 1, \underbrace{0, \ldots, 0}_{m_{r-1}-1}, 1, \ldots\ldots, 1, \underbrace{0, \ldots, 0}_{m_1-1}) \implies \langle m'_1, m'_2, \ldots, m'_{N-r+1}\rangle \,. \tag{4.4}$$

Comparing the l.h.s. of (4.1) and (4.4) we find interestingly that the conjugate motif can be obtained from the original motif by applying the following two rules:

- Replacing 0's with 1's and vice versa,

- Rewriting all binary digits in the reverse order.

For example, the conjugate of the motif (10110) is obtained as (10110) → (01001) → (10010). Using the above mentioned rules along with Eqn. (4.3), it is found that

$$Q_{\langle m'_1, m'_2, \ldots, m'_{N-r+1}\rangle}(0) = \{N - M_{r+1}, N - M_{r+2}, \cdots, N - M_N\}\,. \tag{4.5}$$

On the other hand, using Eqn. (4.2) for the case of border strip $\langle m'_1, m'_2, \ldots, m'_{N-r+1}\rangle$, we obtain

$$Q_{\langle m'_1, m'_2, \ldots, m'_{N-r+1}\rangle}(0) = \{M'_1, M'_2, \cdots, M'_{N-r}\}\,, \tag{4.6}$$

where M'_i's denote the first $(N-r)$ number of partial sums corresponding to the partition $\{m'_1, m'_2, \ldots, m'_{N-r+1}\}$. Comparing the r.h.s. of (4.5) with (4.6) we find that

$$\{M'_1, M'_2, \cdots, M'_{N-r}\} = \{N - M_{r+1}, N - M_{r+2}, \cdots, N - M_N\}\,. \tag{4.7}$$

This relation between the partial sums associated with the conjugate border strip $\langle m'_1, m'_2, \ldots, m'_{N-r+1}\rangle$ and complementary partial sums associated with the border strip $\langle m_1, m_2, \ldots, m_r\rangle$ will be used shortly for proving the duality relation (1.2).

With the help of Eqn. (2.12), we express the generalized partition function (3.27) as

$$Z_N^{(m|n)}(q; x, y) = \sum_{r=1}^{N} \sum_{\{m_1, m_2, \ldots, m_r\} \in \mathcal{P}_N(r)} q^{\sum_{j=1}^{r-1} \mathcal{E}(M_j)} S_{\langle m'_1, m'_2, \ldots, m'_{N-r+1}\rangle}(y, x)\,. \tag{4.8}$$

Due to Eqn. (3.4), it follows that $\{M_1, M_2, \ldots, M_N\} = \{1, 2, \ldots N\}$. So we obtain

$$\sum_{j=1}^{N} \mathcal{E}(M_j) = \sum_{j=1}^{N} j(N-j) = \frac{N(N^2-1)}{6}.$$

Using this relation along with the fact that $\mathcal{E}(M_r) = 0$, one can rewrite Eqn. (4.8) as

$$Z_N^{(m|n)}(q; x, y) = q^{\frac{N(N^2-1)}{6}} \sum_{r=1}^{N} \sum_{\{m_1, m_2, \ldots, m_r\} \in \mathcal{P}_N(r)} q^{-\sum_{j=r+1}^{N} \mathcal{E}(M_j)} S_{\langle m'_1, m'_2, \ldots, m'_{N-r+1}\rangle}(y, x)\,. \tag{4.9}$$

From the definition of $\mathcal{E}(M_j)$ in Eqn. (3.5), it is evident that

$$\mathcal{E}(M_j) = \mathcal{E}(N - M_j).$$

Applying this relation along with Eqn. (4.7), we obtain

$$\sum_{j=r+1}^{N} \mathcal{E}(M_j) = \sum_{j=r+1}^{N} \mathcal{E}(N - M_j) = \sum_{\ell=1}^{N-r} \mathcal{E}(M'_\ell).$$

Using the above relation and rearranging the summation variables, $Z_N^{(m|n)}(q;x,y)$ in Eqn. (4.9) may be written as

$$Z_N^{(m|n)}(q;x,y) = q^{\frac{N(N^2-1)}{6}} \sum_{r=1}^{N} \sum_{\{m_1,m_2,\ldots,m_r\}\in\mathcal{P}_N(r)} q^{-\sum_{\ell=1}^{N-r}\mathcal{E}(M'_\ell)} S_{\langle m'_1,m'_2,\ldots,m'_{N-r+1}\rangle}(y,x)$$
$$= q^{\frac{N(N^2-1)}{6}} \sum_{s=1}^{N} \sum_{\{m'_1,m'_2,\ldots,m'_s\}\in\mathcal{P}_N(s)} q^{-\sum_{\ell=1}^{s-1}\mathcal{E}(M'_\ell)} S_{\langle m'_1,m'_2,\ldots,m'_s\rangle}(y,x)\,, \quad (4.10)$$

where we have used the notation $s \equiv N-r+1$. With the help of (3.27), Eqn. (4.10) is finally expressed as

$$Z_N^{(m|n)}(q;x,y) = q^{\frac{N(N^2-1)}{6}} Z_N^{(n|m)}(q^{-1};y,x). \quad (4.11)$$

Thus we are able to prove a duality relation between the generalized partition functions of the $SU(m|n)$ and $SU(n|m)$ HS spin chains under the exchange of bosonic and fermionic degrees of freedom. In the limit $x=y=1$, this duality relation reduces to the duality relation (1.2) for the partition functions of supersymmetric HS spin chains.

5 Duality in spin models with global $SU(m|n)$ symmetry

The super Yangian symmetry of the $SU(m|n)$ HS spin chain has clearly played a key role in the previous section for establishing the duality relation (1.2). However, such a Yangian symmetry is found to exist only in very few quantum integrable spin chains. So it is natural to ask whether nonintegrable spin models can also exhibit duality relation under the exchange of bosonic and fermionic spin degrees of freedom. In the following, we shall try to answer this question by using a rather different approach.

Let us consider a Hamiltonian of the form

$$\mathcal{H}^{(m|n)} = \omega_0 + \sum_{1\le j<k\le N} \omega_{jk}\, \hat{P}_{jk}^{(m|n)}\,, \quad (5.1)$$

where ω_0 and ω_{jk}'s are arbitrary constant parameters and $\hat{P}_{jk}^{(m|n)}$ is the supersymmetric exchange operator defined in Eqn. (2.3). Similar to the case of the $SU(m|n)$ supersymmetric HS spin chain, the action of Hamiltonian (5.1) is restricted to the state vectors which satisfy the condition (2.2). Due to this condition, the supersymmetric exchange operator $\hat{P}_{jk}^{(m|n)}$ becomes equivalent to an 'anyon like' representation of the permutation algebra [20]. The vector space corresponding to this anyon like representation of the permutation algebra is a direct product of N number of $(m+n)$-dimensional spin spaces, and it contains orthonormal basis vectors like $|\alpha_1\alpha_2\cdots\alpha_j\cdots\alpha_N\rangle_{m,n}$, where $\alpha_j \in \{1,2,\ldots,m+n\}$. Let us denote this space as $V_{(m|n)}$. For the sake of convenience, we assign a 'parity' $p(\alpha_j)$ to each spin component α_j. Moreover, we call α_j a 'bosonic' spin with $p(\alpha_j)=0$ when $\alpha_j \in \{1,2,\ldots,m\}$ and a 'fermionic' spin with $p(\alpha_j)=1$ when $\alpha_j \in \{m+1,m+2,\ldots,m+n\}$. The symbol $\rho_{jk}^{\alpha}(f)$ denotes the total number of fermionic spins lying in between the j-th and k-th lattice sites for the case of state vector $|\alpha_1\alpha_2\cdots\alpha_j\cdots\alpha_k\cdots\alpha_N\rangle_{m,n}$:

$$\rho_{jk}^{\alpha}(f) \equiv \sum_{\ell=j+1}^{k-1} p(\alpha_\ell)\,. \tag{5.2}$$

If we define an anyon like representation $\tilde{P}_{jk}^{(m|n)}$ on the space $V_{(m|n)}$ as

$$\begin{aligned}&\tilde{P}_{jk}^{(m|n)}|\alpha_1\alpha_2\cdots\alpha_j\cdots\alpha_k\cdots\alpha_N\rangle_{m,n}\\&\qquad=(-1)^{p(\alpha_j)p(\alpha_k)+\,\{p(\alpha_j)+p(\alpha_k)\}\,\rho_{jk}^{\alpha}(f)}\,|\alpha_1\alpha_2\cdots\alpha_k\cdots\alpha_j\cdots\alpha_N\rangle_{m,n}\,,\end{aligned} \tag{5.3}$$

that will be equivalent to the supersymmetric exchange operator $\hat{P}_{jk}^{(m|n)}$ given in Eqn. (2.3) [20,23]. The relation (5.3) implies that the exchange of two bosonic (resp. fermionic) spins produces a phase factor of 1 (resp. -1) irrespective of the nature of the spins situated in between the j-th and k-th lattice sites. However, if we exchange one bosonic spin with one fermionic spin, then the phase factor becomes 1 (resp. -1) if there exist even (resp. odd) number of fermionic spins situated in between the j-th and k-th lattice sites. Due to the above mentioned equivalence between the supersymmetric exchange operator $\hat{P}_{jk}^{(m|n)}$ and the anyon like representation $\tilde{P}_{jk}^{(m|n)}$, the Hamiltonian $\mathcal{H}^{(m|n)}$ in Eqn. (5.1) is equivalently expressed as

$$\mathcal{H}^{(m|n)} = \omega_0 + \sum_{1\le j<k\le N} \omega_{jk}\,\tilde{P}_{jk}^{(m|n)}\,. \tag{5.4}$$

Let us now define another set of orthonormal basis vectors for the space $V_{(m|n)}$ by multiplying the states like $|\alpha_1\alpha_2\cdots\alpha_j\cdots\alpha_N\rangle_{m,n}$ through a phase factor, which takes the value $+1$ (resp. -1) when the total number of fermionic spins sitting on all odd numbered lattices sites is even (resp. odd):

$$|\alpha_1\alpha_2\cdots\alpha_j\cdots\alpha_N\rangle_{m,n}^{*} = (-1)^{\sum\limits_{\ell=1}^{N}\ell\,p(\alpha_\ell)}\,|\alpha_1\alpha_2\cdots\alpha_j\cdots\alpha_N\rangle_{m,n}\,. \tag{5.5}$$

By using Eqns. (5.3) and (5.5) we find that the action of $\tilde{P}_{jk}^{(m|n)}$ on these new basis vectors is given by

$$
\begin{aligned}
&\tilde{P}_{jk}^{(m|n)}|\alpha_1\alpha_2\cdots\alpha_j\cdots\alpha_k\cdots\alpha_N\rangle_{m,n}^* \\
&= (-1)^{\sum\limits_{\ell=1}^{N}\ell\, p(\alpha_\ell)}\tilde{P}_{jk}^{(m|n)}|\alpha_1\alpha_2\cdots\alpha_j\cdots\alpha_k\cdots\alpha_N\rangle_{m,n} \\
&= (-1)^{\sum\limits_{\ell=1}^{N}\ell\, p(\alpha_\ell)}(-1)^{p(\alpha_j)p(\alpha_k)+\,\{p(\alpha_j)+p(\alpha_k)\}\,\rho_{jk}^{\alpha}(f)}\,|\alpha_1\alpha_2\cdots\alpha_k\cdots\alpha_j\cdots\alpha_N\rangle_{m,n} \\
&= (-1)^{p(\alpha_j)p(\alpha_k)+\,\{p(\alpha_j)+p(\alpha_k)\}\left(\rho_{jk}^{\alpha}(f)+j+k\right)}\,|\alpha_1\alpha_2\cdots\alpha_k\cdots\alpha_j\cdots\alpha_N\rangle_{m,n}^* \,. \qquad (5.6)
\end{aligned}
$$

Let us now consider the vector space $V_{(n|m)}$, which might be spanned through orthonormal basis vectors like $|\beta_1\beta_2\cdots\beta_j\cdots\beta_N\rangle_{n,m}$, where $\beta_j \in \{1,2,\ldots,m+n\}$. In this case, we call β_j a 'bosonic' spin with $p(\beta_j)=0$ when $\beta_j\in\{1,2,\ldots,n\}$ and a 'fermionic' spin with $p(\beta_j)=1$ when $\beta_j\in\{n+1,n+2,\ldots,n+m\}$. In analogy with Eqn. (5.3), we can express the action of $\tilde{P}_{jk}^{(n|m)}$ on the space $V_{(n|m)}$ as

$$
\begin{aligned}
&\tilde{P}_{jk}^{(n|m)}\,|\beta_1\beta_2\cdots\beta_j\cdots\beta_k\cdots\beta_N\rangle_{n,m} \\
&\qquad = (-1)^{p(\beta_j)p(\beta_k)+\,\{p(\beta_j)+p(\beta_k)\}\,\rho_{jk}^{\beta}(f)}\,|\beta_1\beta_2\cdots\beta_k\cdots\beta_j\cdots\beta_N\rangle_{n,m}\,, \qquad (5.7)
\end{aligned}
$$

where $\rho_{jk}^{\beta}(f)$ denotes the total number of fermionic spins lying in between the j-th and k-th lattice sites in the case of the state vector $|\beta_1\beta_2\cdots\beta_j\cdots\beta_k\cdots\beta_N\rangle_{n,m}$:

$$
\rho_{jk}^{\beta}(f) \equiv \sum_{\ell=j+1}^{k-1} p(\beta_\ell)\,. \qquad (5.8)
$$

Let U be a permutation of the set $\{1,2,\ldots,m+n\}$, which satisfies the relations

$$
\begin{aligned}
&\{\,U(1),U(2),\ldots,U(m)\,\} = \{n+1,n+2,\ldots,n+m\}\,, \\
&\{\,U(m+1),U(m+2),\ldots,U(m+n)\,\} = \{1,2,\ldots,n\}\,. \qquad (5.9)
\end{aligned}
$$

With the help of this permutation, we define an unitary operator ($\mathcal{U}$) which maps the vectors of $V_{(m|n)}$ to the vectors of $V_{(n|m)}$ as

$$
\mathcal{U}|\alpha_1\alpha_2\cdots\alpha_j\cdots\alpha_N\rangle_{m,n}^* = |\beta_1\beta_2\cdots\beta_j\cdots\beta_N\rangle_{n,m}\,, \qquad (5.10)
$$

where $\beta_j\equiv U(\alpha_j)$. If α_j represents a bosonic (resp. fermionic) spin in the space $V_{(m|n)}$, then, due to Eqn. (5.9), β_j would represent a fermionic (resp. bosonic) spin in the space $V_{(n|m)}$. Hence we can write

$$
p(\beta_j) = 1 - p(\alpha_j)\,. \qquad (5.11)
$$

Using the relations (5.2), (5.8) and (5.11), we also obtain

$$\rho^{\beta}_{jk}(f) = (j-k-1) - \rho^{\alpha}_{jk}(f)\,. \tag{5.12}$$

With the help of Eqns. (5.10), (5.11) and (5.12), one can easily express equation (5.7) in the form

$$\begin{aligned}&\mathcal{U}^{\dagger}\,\tilde{P}^{(n|m)}_{jk}\,\mathcal{U}\,|\alpha_1\alpha_2\cdots\alpha_j\cdots\alpha_k\cdots\alpha_N\rangle^{*}_{m,n}\\ &\quad= -(-1)^{p(\alpha_j)p(\alpha_k)+\,\{p(\alpha_j)+p(\alpha_k)\}\,\left(\rho^{\alpha}_{jk}(f)+j+k\right)}\,|\alpha_1\alpha_2\cdots\alpha_k\cdots\alpha_j\cdots\alpha_N\rangle^{*}_{m,n}\,.\end{aligned} \tag{5.13}$$

Comparison of Eqn. (5.6) with Eqn. (5.13) implies that

$$\mathcal{U}\tilde{P}^{(m|n)}_{jk}\mathcal{U}^{\dagger} = -\tilde{P}^{(n|m)}_{jk}\,, \tag{5.14}$$

for all possible values of j and k. Using Eqns. (5.4) and (5.14), we find that

$$\mathcal{U}\mathcal{H}^{(m|n)}\mathcal{U}^{\dagger} = 2\omega_0 - \mathcal{H}^{(n|m)}\,. \tag{5.15}$$

Hence the spectrum of the Hamiltonian $\mathcal{H}^{(m|n)}$ can be obtained by inverting the spectrum of its dual Hamiltonian $\mathcal{H}^{(n|m)}$ and giving this inverted spectrum an overall shift of amount $2\omega_0$. If we define the partition function corresponding to the Hamiltonian $\mathcal{H}^{(m|n)}$ as $\mathcal{Z}^{(m|n)}_N(q) \equiv tr\big(q^{\mathcal{H}^{(m|n)}}\big)$, then by using Eqn. (5.15) we obtain a duality relation at the level of the partition function as

$$\mathcal{Z}^{(m|n)}_N(q) = q^{2\omega_0}\,\mathcal{Z}^{(n|m)}_N(q^{-1})\,. \tag{5.16}$$

It should be observed that, while proving the above duality relation, we have kept the coupling constants ω_0 and ω_{jk} in the Hamiltonian $\mathcal{H}^{(m|n)}$ (5.1) as completely free parameters. By properly choosing these free parameters, one can generate many quantum integrable models like $SU(m|n)$ supersymmetric versions of the Haldane-Shastry spin chain, the Polychronakos spin chain, and the isotropic Heisenberg spin chain. For example, by choosing $\omega_0 = \sum_{1\le j<k\le N}\frac{1}{\sin^2(\xi_j-\xi_k)} = \frac{N(N^2-1)}{12}$ and $\omega_{jk} = \frac{1}{\sin^2(\xi_j-\xi_k)}$, with $\xi_j = \frac{j\pi}{N}$, in Eqn. (5.1), we recover the Hamiltonian (1.1) of the $SU(m|n)$ supersymmetric HS spin chain. The partition functions for all of the above mentioned quantum integrable models will naturally satisfy the duality relation (5.16). However, since the spin chain Hamiltonian (5.1) is nonintegrable for generic choice of ω_{jk}, it is obvious that integrability or quantum group symmetry is not a necessary requirement for the existence of the duality relation (5.16).

By using the (anti-)commutation relations (2.1), it is easy to verify that the supersymmetric exchange operator $\hat{P}^{(m|n)}_{jk}$ (2.3) commutes with the set of operators $Q^{\alpha\beta}_0$

given in Eqn. (2.10a). Therefore, the spin chain Hamiltonian $\mathcal{H}^{(m|n)}$ (5.1) also commutes with the set of operators $Q_0^{\alpha\beta}$. Since commutation relations among the $Q_0^{\alpha\beta}$'s generate the $SU(m|n)$ algebra, it is clear that $\mathcal{H}^{(m|n)}$ has a global $SU(m|n)$ symmetry for any value of the parameters ω_0 and ω_{jk}. It may be noted that $\mathcal{H}^{(m|n)}$ in Eqn. (5.1) depends linearly on the supersymmetric exchange operators $\hat{P}_{jk}^{(m|n)}$. One can also construct more general spin chain Hamiltonians with a global $SU(m|n)$ symmetry by including the products of different exchange operators (corresponding to different lattice sites) with arbitrary coefficients. It is easy to see that such a Hamiltonian would satisfy the duality relation (5.16), provided we construct the dual Hamiltonian $\mathcal{H}^{(n|m)}$ from the original Hamiltonian $\mathcal{H}^{(m|n)}$ by keeping ω_0 as well as all coupling constants associated with the products of odd numbers of exchange operators unchanged, reversing the sign of all coupling constants associated with the products of even numbers of exchange operators, and finally replacing $\hat{P}_{jk}^{(m|n)}$ by $\hat{P}_{jk}^{(n|m)}$. Since any quantum integrable or nonintegrable spin chain with a global $SU(m|n)$ symmetry can be expressed as a polynomial function of the exchange operators $\hat{P}_{jk}^{(m|n)}$, it is evident that the partition functions associated with such spin chains would satisfy the duality relation (5.16).

6 Concluding remarks

We have provided here an analytical proof for the boson-fermion duality relation in the case of the $SU(m|n)$ supersymmetric HS spin chain. To this end, we utilize the $Y(gl(m|n))$ super Yangian symmetry of the $SU(m|n)$ HS spin chain in a crucial way. At first, we define a generalized partition function which reduces to the usual partition function of this spin chain in some limit of the related parameters. Subsequently, we express this generalized partition function in terms of the super Schur polynomials associated with border strips, which label a family of irreducible representations of the $Y(gl(m|n))$ quantum group. It is well known that such super Schur polynomials satisfy a duality relation under the exchange of bosonic and fermionic variables. Using this duality relation, we finally derive the boson-fermion duality (1.2) for the partition function of the $SU(m|n)$ HS spin chain. Apart from leading to a proof for this duality relation, our expression (3.27) for the generalized partition function of the $SU(m|n)$ HS spin chain through the super Schur polynomials might be useful in future for finding various correlation functions.

As a byproduct of our investigation of the duality relation of the $SU(m|n)$ HS spin chain, we obtain some results which are interesting from the mathematical point of view. For example, while expressing the partition function of the $SU(m|n)$ HS spin chain in terms of the super Schur polynomials, we find a novel combinatorial formula for the super Schur polynomials. Furthermore, while studying the transformation of a border strip under the conjugation operation, we give a new proof for the known duality relation of corresponding super Schur polynomials.

In this article we also explore the question whether nonintegrable quantum spin chains can also exhibit duality relation under the exchange of bosonic and fermionic spin degrees of freedom. By using a mapping between the anyon like representations of the graded exchange operators like $\hat{P}_{jk}^{(m|n)}$ and $\hat{P}_{jk}^{(n|m)}$, we are able to show that the partition function of any quantum spin chain with a global $SU(m|n)$ symmetry would satisfy such a duality relation (5.16). This duality relation implies that the spectrum of the $SU(m|n)$ model can be obtained by inverting the spectrum of the $SU(n|m)$ model and giving an overall shift to this inverted spectrum. Since a global $SU(m|n)$ symmetry can be found in a wide class of integrable as well as nonintegrable spin chains, this duality relation seems to be a useful probe to study their spectra.

Appendix A. Derivation of the recursion relation (3.26)

Let us rewrite $\tilde{S}_{\langle m_1,m_2,\ldots,m_r\rangle}(x,y)$ in Eqn. (3.25) as

$$\tilde{S}_{\langle m_1,m_2,\ldots,m_r\rangle}(x,y) = (-1)^{r-1}E_{M_r}(x,y) + \sum_{k=2}^{r} \sum_{\{\ell_1,\ell_2,\ldots,\ell_k\}\in\mathcal{P}_r(k)} (-1)^{r-k} \prod_{i=1}^{k} E_{M_{L_i}-M_{L_{i-1}}}(x,y)\,. \quad \text{(A-1)}$$

Since, $\{\ell_1,\ell_2,\ldots,\ell_k\}\in\mathcal{P}_r(k)$ and all these ℓ_s's are positive integers, it is evident that $\ell_k = r-(\ell_1+\ell_2+\cdots+\ell_{k-1}) \leq r-k+1$. Therefore, Eqn. (A-1) can also be expressed as

$$\tilde{S}_{\langle m_1,m_2,\ldots,m_r\rangle}(x,y) = (-1)^{r-1}E_{M_r}(x,y) + \sum_{k=2}^{r} \sum_{\ell_k=1}^{r-k+1} \sum_{\{\ell_1,\ell_2,\ldots,\ell_{k-1}\}\in\mathcal{P}_{r-\ell_k}(k-1)} (-1)^{r-k} \prod_{i=1}^{k} E_{M_{L_i}-M_{L_{i-1}}}(x,y).$$

By interchanging the order of the summation variables k and ℓ_k in the above equation, we obtain

$$\tilde{S}_{\langle m_1,m_2,\ldots,m_r\rangle}(x,y) = (-1)^{r-1}E_{M_r}(x,y) + \sum_{\ell_k=1}^{r-1} \sum_{k=2}^{r-\ell_k+1} \sum_{\{\ell_1,\ell_2,\ldots,\ell_{k-1}\}\in\mathcal{P}_{r-\ell_k}(k-1)} (-1)^{r-k} \prod_{i=1}^{k} E_{M_{L_i}-M_{L_{i-1}}}(x,y).$$

Now, using the fact that, $L_k = \sum_{s=1}^{k}\ell_s = r$ and $L_{k-1} = \sum_{s=1}^{k-1}\ell_s = r-\ell_k$, we can rewrite the above equation as

$$\tilde{S}_{\langle m_1,m_2,\ldots,m_r\rangle}(x,y) = (-1)^{r-1}E_{M_r}(x,y) + \Big[\sum_{\ell_k=1}^{r-1}(-1)^{\ell_k-1}E_{M_r-M_{r-\ell_k}}(x,y) \times \sum_{k=2}^{r-\ell_k+1} \sum_{\{\ell_1,\ell_2,\ldots,\ell_{k-1}\}\in\mathcal{P}_{r-\ell_k}(k-1)} (-1)^{\ell_1+\ell_2+\cdots+\ell_{k-1}-(k-1)} \prod_{i=1}^{k-1} E_{M_{L_i}-M_{L_{i-1}}}(x,y)\Big]. \quad \text{(A-2)}$$

Redefining $\ell_k \equiv s$ and $k-1\equiv t$, we obtain

$$\tilde{S}_{\langle m_1,m_2,\ldots,m_r\rangle}(x,y) = (-1)^{r-1}E_{M_r}(x,y) + \Big[\sum_{s=1}^{r-1}(-1)^{s-1}E_{M_r-M_{r-s}}(x,y) \times \sum_{t=1}^{r-s} \sum_{\{\ell_1,\ell_2,\ldots,\ell_t\}\in\mathcal{P}_{r-s}(t)} (-1)^{\ell_1+\ell_2+\cdots+\ell_t-t} \prod_{i=1}^{t} E_{M_{L_i}-M_{L_{i-1}}}(x,y)\Big]. \quad \text{(A-3)}$$

Finally, by using the above equation and the definition of $\tilde{S}_{\langle m_1,m_2,\ldots,m_{r-s}\rangle}(x,y)$ from Eqn. (3.25), we derive the recursion relation for $\tilde{S}_{\langle m_1,m_2,\ldots,m_r\rangle}(x,y)$ as

$$\begin{aligned}\tilde{S}_{\langle m_1,m_2,\ldots,m_r\rangle}(x,y) &= (-1)^{r+1}E_{M_r}(x,y) + \sum_{s=1}^{r-1}(-1)^{s+1}E_{M_r-M_{r-s}}(x,y)\cdot\tilde{S}_{\langle m_1,m_2,\ldots,m_{r-s}\rangle}(x,y)\\ &= \sum_{s=1}^{r}(-1)^{s+1}E_{m_r+m_{r-1}+\cdots+m_{r-s+1}}(x,y)\cdot\tilde{S}_{\langle m_1,m_2,\ldots,m_{r-s}\rangle}(x,y), \qquad \text{(A-4)}\end{aligned}$$

where $\tilde{S}_{\langle 0\rangle}(x,y)=1$.

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