

# A MULTISPECIES CALOGERO-SUTHERLAND MODEL

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## Abstract

Motivated by the concept of ideal mutual statistics, we study a multi-species Calogero-Sutherland model in which the interaction parameters and masses satisfy some specific relations. The ground state is exactly solvable if those relations hold, both on a circle and on a line with a simple harmonic potential. In the latter case, the one-particle densities can be obtained using a generalization of the Thomas-Fermi method. We calculate the second virial coefficients in the high temperature expansion for the pressure. We show that the low-energy excitations are the same as those of a Gaussian conformal field theory. Finally, we discuss similar relations between the statistics parameters and charges for a multispecies anyon model in a magnetic field.

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## 1. Introduction

The Calogero-Sutherland (CS) model [1, 2] consists of a system of identical particles in one dimension which interact with each other via a two-body inverse square potential. The model has been studied for many years from several points of view. It is exactly solvable and integrable [3, 4], it has interesting connections with random matrix theory [5], and it provides an example of exclusion statistics [6].

The concept of exclusion statistics introduced by Haldane has attracted considerable attention recently [7-15]. The examples of *ideal* exclusion statistics known so far include the Calogero-Sutherland (CS) model, spinons in the Haldane-Shastry model [16, 3], and anyons residing in the lowest Landau level in a strong magnetic field [7-10]. The term *ideal* means that the only interactions between particles are statistical, and that these interactions are scale invariant, i.e., independent of the momentum or energy scale [14].

The idea of exclusion statistics can be generalized from systems with a single species of indistinguishable particles to systems with several species. In the latter case, the concept of *mutual statistics* has proved to be very useful [10-12,14]. An example of mutual statistics is a multianyon system in two dimensions [8, 10]. It seems interesting to ask whether there is a one-dimensional model which can realize mutual statistics. With this and other motivations, a multispecies CS model was examined in Ref. [17]; however the model involved both two-body and three-body interactions. In this paper, we ask whether it may be possible to realize mutual statistics in a CS model with only two-body interactions. Somewhat unexpectedly, we discover that ideal mutual statistics cannot be realized unless the interaction parameters and masses of the different species satisfy certain relations. However we are not able to prove that the model does exhibit mutual statistics if those relations are satisfied.

The outline of our paper is as follows. In Sec. 2, we present a general analysis of a model satisfying ideal mutual statistics on a line. We assume that the mutual statistics is described by a *symmetric* matrix. (This assumption is motivated by a CS model). We find that the masses and interaction parameters must satisfy certain relations in order that the asymptotic Bethe ansatz be applicable to the ground state. In Sec. 3, we consider a CS model which satisfies those relations. It is known that the ground state is exactly solvable for the model on a circle; it is also exactly solvable if the CS model

is placed on a line in an external simple harmonic potential, with the same harmonic frequency for all species [18]. We discuss a class of exactly solvable excited states, including some subtleties involving the centre of mass excitations on a circle. In Sec. 4, we compute the one-particle densities in the harmonic potential problem using a generalization of the Thomas-Fermi method. In Sec. 5, we study the CS model at high temperature, and calculate the second virial coefficients in a high temperature expansion of the equation of state. In Sec. 6, we study the low-energy spectrum and eigenstates to leading order in  $1/L$ . We find that the spectrum is the same as that of a  $c = 1$  Gaussian conformal field theory. We derive the leading terms in the specific heat and pressure at low temperature. Finally, in Sec. 7, we consider a multispecies anyon model in an uniform magnetic field at zero temperature and show that it has some desirable properties if the *charges* and statistics parameters satisfy the same kinds of relations [8].

## 2. Mutual Exclusion Statistics in One Dimension

Given a many-body quantum system with a finite number of states and  $A$  species of particles, let  $N_a$  denote the number of particles of species  $a$  ( $a = 1, 2, \dots, A$ ). Let  $D_a$  be the number of states which are available to the next  $a$  particle which may be added to the system. The  $D_a$ 's are functions of the numbers  $N_a$ . The mutual statistics parameters  $g_{ab}$  are defined as

$$\Delta D_a = - \sum_b g_{ab} \Delta N_b . \quad (1)$$

If the diagonal matrix element  $g_{aa} = 0$  (or 1), the  $a$  particles are bosons (or fermions). We will henceforth assume that all  $g_{ab} \geq 0$ . In the literature, the *off-diagonal* element  $g_{ab}$  is referred to as the ‘mutual’ statistics between species  $a$  and  $b$ .

For an ideal system with a single species such as a CS model, the parameter  $g = g_{aa}$  can be interpreted in terms of the phase shift which appears in a two-particle scattering [11]. Namely, the scattering phase shift is given by  $\delta(p) = \pi g \theta(p)$ , where  $p$  is the relative momentum of the two particles, and  $\theta(p) = p/|p|$  for  $p \neq 0$ . (Thus the wave function of two bosons is symmetric with  $g = 0$ , and that of two fermions is antisymmetric with  $g = 1$ ). At zero

temperature, the particles occupy momenta satisfying

$$p_i = \frac{\pi \hbar g}{L} \sum_{j \neq i} \theta(p_i - p_j) , \quad (2)$$

where  $L$  is the length of the system. Eq. (2) may be derived by an asymptotic Bethe ansatz analysis of the CS model on a circle of length  $L$  [2, 3, 19]. Thus momenta from  $-p_F$  to  $p_F$  are occupied with a spacing of  $2\pi\hbar g/L$ . If  $N$  is the number of particles, we have

$$\int_{-p_F}^{p_F} dp \frac{L}{2\pi\hbar g} = N . \quad (3)$$

We are eventually interested in the thermodynamic limit  $N, L \rightarrow \infty$  keeping the particle density  $\rho = N/L$  fixed. Then

$$p_F = \pi \hbar g \rho , \quad (4)$$

and the total energy

$$E = \int_{-p_F}^{p_F} dp \frac{L}{2\pi\hbar g} \frac{p^2}{2m} = \frac{\pi^2 \hbar^2 g^2}{6m} L \rho^3 . \quad (5)$$

We may now generalize the phase shift argument to a multispecies model. Let us assume that a scattering between two particles  $i$  and  $j$  produces a phase shift  $\pi g_{ij} \theta(v_i - v_j)$ , where  $v = p/m$  is the velocity,  $m_i m_j / (m_i + m_j)$  is the reduced mass, and  $(v_i - v_j) m_i m_j / (m_i + m_j)$  is the relative momentum of the two particles. The two particles may or may not belong to the same species;  $g_{ij} = g_{ab}$ ,  $m_i = m_a$  and  $m_j = m_b$  if  $i$  and  $j$  belong to species  $a$  and  $b$  respectively. (It is clear that  $g_{ab} = g_{ba}$  in such a model). If we assume the asymptotic Bethe ansatz to be valid for the ground state of this model, we obtain

$$m_i v_i = \frac{\pi \hbar}{L} \sum_{j \neq i} g_{ij} \theta(v_i - v_j) . \quad (6)$$

We assume that no two particles have the same velocity; this will be verified in Eq. (9) below.

If  $N = \sum_a N_a$ , Eq. (6) has  $N! / \prod_a N_a!$  distinct solutions; this is because the particles  $1, 2, \dots, N$  can be ordered with increasing velocities in

that many different ways. (Recall that different orderings of identical particles are physically indistinguishable). The asymptotic Bethe ansatz ground state wave function will then be a superposition of that many different plane waves. For this idea to work, it is clearly essential that the total energy  $E = \sum_i m_i v_i^2 / 2$  be the same for all the different solutions of (6). We find that this is true if and only if

$$\frac{g_{ij}g_{ik}}{m_i} = \frac{g_{ij}g_{jk}}{m_j} \quad (7)$$

for any three different particles  $i, j$  and  $k$ ; they may or may not belong to the same species. The non-trivial solution of (7) is given by

$$g_{ij} = \alpha m_i m_j . \quad (8)$$

Eq. (8) is one of our main results. Note that the  $A \times A$  matrix  $g_{ab}$  then has only one non-zero eigenvalue.

In deriving (8) from (7), we have assumed that all the off-diagonal elements  $g_{ab} > 0$ . There are other solutions of (7) in which an off-diagonal  $g_{ab}$  vanishes. However in that case, (7) implies that  $g_{ac}g_{bc} = 0$  for all  $c$ . We can then prove, by induction in the number  $A$ , that the system consists of two or more disjoint groups of species, such that  $g_{ab} = 0$  whenever  $a$  belongs to one group and  $b$  belongs to another. We can therefore study each group separately. We will not consider such cases any further, and will assume henceforth that all the  $g_{ab}$  satisfy (8).

It is rather surprising and novel that *mass-dependent* conditions such as (8) appear to be necessary for mutual statistics. Such conditions may be peculiar to models in which the two-particle scattering phase shift does not depend on the *magnitude* of  $v_i - v_j$ . We do not know of any physical system in which such relations might arise in a natural way.

Using (8), we can solve (6) explicitly. If the particles are ordered so that  $v_1 < v_2 < \dots < v_N$ , we get

$$v_i = \frac{\pi \hbar \alpha}{L} \sum_{j \neq i} m_j \theta(i - j) . \quad (9)$$

From Eqs. (6) and (8), we deduce the ground state energy to be

$$E_o = \frac{\pi^2 \hbar^2 \alpha^2}{6L^2} \left[ \left( \sum_a m_a N_a \right)^3 - \sum_a m_a^3 N_a \right] . \quad (10)$$

The Fermi momentum for species  $a$  may be obtained in two different ways. Firstly, for an ideal system at zero temperature, the chemical potential  $\mu_a = p_{Fa}^2/2m_a$ . Further,  $\partial E_o/\partial N_a = \mu_a$ . In the thermodynamic limit, the Fermi momentum for species  $a$  is therefore given by

$$p_{Fa} = \pi\hbar \sum_b g_{ab} \rho_b , \quad (11)$$

where  $\rho_b = N_b/L$ . We observe that the Fermi velocity

$$v_F = \frac{p_{Fa}}{m_a} = \pi\hbar\alpha \sum_b m_b \rho_b \quad (12)$$

is the same for all species. The second way of deriving  $v_F$  is to examine (6); amongst all the solutions of that equation, the one with the largest possible value of  $v_a$  is clearly given by (12). We note that the ground state pressure and compressibility are given by

$$\begin{aligned} P &= - \frac{\partial E_o}{\partial L} = \frac{v_F^3}{3\pi\hbar\alpha} , \\ \kappa &= - \left( L \frac{\partial P}{\partial L} \right)^{-1} = \frac{\pi\hbar\alpha}{v_F^3} . \end{aligned} \quad (13)$$

We can understand Eq. (1) semiclassically by identifying the number of states with the ‘volume’ of phase space. The volume of phase space occupied in the ground state by particles of species  $a$  is given by  $\int(dxdp/2\pi\hbar)[1+\theta(\mu_a-h_a)]/2$ , where  $h_a = p^2/2m_a$  is the single-particle Hamiltonian. The volume available to the next  $a$  particle is then given by the ‘remaining’ amount

$$D_a(\epsilon) = \int \frac{dxdp}{2\pi\hbar} \frac{1}{2}[1 - \theta(\mu_a - h_a)] \exp(-\epsilon h_a) , \quad (14)$$

where we have introduced a cutoff  $\epsilon$  to make sense of the divergent integral; we will let  $\epsilon \rightarrow 0^+$  at the end of the calculation. (The final result should not depend on the cutoff procedure [9]). From Eq. (11), the change in Fermi momentum is given by

$$\Delta p_{Fa} = \frac{\pi\hbar}{L} \sum_b g_{ab} \Delta N_b . \quad (15)$$

Combining Eqs. (14-15), we find that  $\Delta D_a \equiv \lim_{\epsilon \rightarrow 0} \Delta D_a(\epsilon)$  is given by Eq. (1).

In the entire discussion above, we have implicitly assumed the system to be homogeneous, that is, the densities  $\rho_a$  do not depend on the position  $x$ . This is a non-trivial assumption for the ground state of a multispecies model. For instance, consider a two-species model in which  $g_{12}$  is much larger than both  $g_{11}$  and  $g_{22}$ . Then it is intuitively clear that the ground state will exhibit phase separation, i.e., the particles belonging to species 1 and 2 will prefer to reside in two different regions of space. One would then use Eq. (4) in the two regions separately, where  $\rho_a$  is the *local* density of the species appropriate to a particular region; it would be incorrect to use the multispecies Eq. (11) where  $\rho_a$  is the *average* density defined over the entire system.

One can derive the *leading* term in Eq. (10) in a different way using the result (5) for the case of a single species. For a CS model, the particles have such a singular two-particle interaction that no two of them can cross each other. (This will be explained in more detail in the next section). The Hilbert space thus breaks up into a number of different sectors corresponding to different orderings of the particles on the line. It then seems reasonable to demand that a thermodynamic quantity like the ground state energy should only depend on the numbers  $N_a$ , and not on which sector we are working in. This is equivalent to requiring that the ground state should not show phase separation. Let us therefore calculate the ground state energy in the sector where the different species are completely segregated. The  $N_a$  particles of species  $a$  occupy a length  $L_a$ , so that  $\sum_a L_a = L$ . For a single species, we know that the ground state energy  $E_{oa} = \pi^2 \hbar^2 g_{aa}^2 N_a^3 / (6L_a^2 m_a)$ . Now we minimize  $E_o = \sum_a E_{oa}$  as a function of the  $L_a$  keeping the sum of the  $L_a$  fixed; this is equivalent to saying that the pressures exerted by the different species must be equal. (In the thermodynamic limit, the contributions to the energy coming from the regions separating two species can be neglected). This calculation leads to the expression in (10). For later use, we note that the product  $m_a N_a / L_a$  is the same for all  $a$  in this sector. In fact, the mass density  $\sum_a m_a \rho_a$  must have the same value at all positions and in *all* sectors due to Eqs. (12-13), because the pressure must be the same throughout the system. However this sum rule does not fix the *individual* densities  $\rho_a$  which depend on the sector (see Sec. 4).

Since particles cannot exchange their positions in the CS model, two species are indistinguishable if their masses are equal. (Actually, this is only

true for the model on a line, not on a circle. For instance, for two particles on a circle, we will show below that the energy spectrum is different for identical and non-identical particles). We will assume that all the masses are unequal; if some of them are equal, the CS model on a line reduces to one with a smaller number of species.

### 3. A Multispecies Calogero-Sutherland Model

A CS model satisfying the relations in Eq. (8) has another attractive feature, namely, that its ground state can be solved exactly [18]. We first present the Hamiltonian for a general CS model defined on a circle of perimeter  $L$ .

$$H = \sum_i \frac{p_i^2}{2m_i} + \frac{\pi^2 \hbar^2}{L^2} \sum_{i < j} g_{ij}(g_{ij} - 1) \frac{m_i + m_j}{2m_i m_j} \frac{1}{\sin^2[\frac{\pi}{L}(x_i - x_j)]}, \quad (16)$$

where the sum over  $i, j$  runs over all the  $N = \sum_a N_a$  particles. Since the Hamiltonian does not distinguish between  $g_{ij}$  and  $1 - g_{ij}$ , one needs to specify the boundary condition on the allowed wave functions, namely, that  $\Psi \sim |x_i - x_j|^{g_{ij}}$  when  $x_i - x_j \rightarrow 0$ . Since the wave functions vanish at coincident points, it is clear that there are a number of disjoint sectors which correspond to different orderings of distinguishable particles as we go anticlockwise around the circle [1]. Under time evolution, a configuration of particles cannot change from one sector to another. To illustrate the idea of sectors, consider a system with two particles of species 1 and two of species 2. Then there are two sectors, namely, (1122) and (1212).

We will work in a *bosonic* basis; if all the  $N$  particles belong to the same species, we demand that

$$\Psi(0, x_1, x_2, \dots, x_{N-1}) = \Psi(x_1, x_2, \dots, x_{N-1}, L), \quad (17)$$

where  $0 < x_1 < x_2 < \dots < x_{N-1} < L$ . (In a fermionic basis, the two sides of (17) would differ by the phase  $(-1)^{N-1}$ ). On the other hand, if all the particles are distinguishable, we demand that the wave function should be periodic in each of the particles separately; namely,

$$\Psi(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_N) = \Psi(x_1, \dots, x_{i-1}, L, x_{i+1}, \dots, x_N) \quad (18)$$

whenever  $0 < x_{i+1} < \dots < x_N < x_1 < \dots < x_{i-1} < L$ . We can similarly work out the appropriate boundary conditions if some of the particles are

distinguishable and some are not. In every case, we see that it is sufficient to specify the wave function in the domain  $0 \leq x_1 < x_2 < \dots < x_N \leq L$ ; we will work in this particular domain from now on. The wave function in all other domains can be obtained by using the appropriate boundary conditions. It is important to distinguish between the concepts of sectors and domains. For instance, in the above example with four particles, the sector (1122) has four domains while the sector (1212) has two domains. Under time evolution, particles can move from one domain to another. For any  $N$  and any sector, it can be shown that  $N$  is exactly divisible by the number of domains  $D$ . Further, a sector can be specified by repeating a pattern of  $D$  integers  $N/D$  times; the  $D$  integers are chosen from the set  $\{1, 2, \dots, A\}$ . For a single species CS model, there is only one sector and one domain.

For  $N = 2$ , there is only one sector; the number of domains is two and one if the particles are distinguishable and indistinguishable respectively. The energy spectrum can be found exactly by separating the wave function into functions of the centre of mass and relative coordinates; however we will see that there is an interesting selection rule if the particles are indistinguishable. The wave function  $\Psi$  is labelled by two integers  $n_1$ , the centre of mass quantum number which runs over all integers, and  $n_2$ , the relative coordinate number which must be zero or positive. In terms of the coordinates  $X = (m_1 x_1 + m_2 x_2)/(m_1 + m_2)$  and  $x = x_2 - x_1$ , we have

$$\Psi_{n_1 n_2} = \exp \left( i 2\pi n_1 \frac{X}{L} \right) \mid \sin \left( \frac{\pi x}{L} \right) |^{g_{12}} Q_{n_2} \left( \cos \left( \frac{\pi x}{L} \right) \right), \quad (19)$$

where  $Q_{n_2}$  is a polynomial of degree  $n_2$ ; it is even or odd depending on  $n_2$ . The energy is

$$E_{n_1 n_2} = \frac{\pi^2 \hbar^2}{L^2} \left[ \frac{2n_1^2}{m_1 + m_2} + (n_2 + g_{12})^2 \frac{m_1 + m_2}{2m_1 m_2} \right]. \quad (20)$$

For indistinguishable particles, we set  $m_1 = m_2$  and  $g_{12} = g_{11}$  in Eqs. (19-20); in addition, we must restrict  $n_1 + n_2$  to be even (or odd, depending on the choice of basis). To see this, we note that  $\Psi(0, y) = (-1)^{n_1 + n_2} \Psi(y, L)$ . In a bosonic basis,  $n_1 + n_2$  must be even. In a fermionic basis,  $n_1 + n_2$  must be odd. Note that the statement that the two particles are bosons (or fermions) if  $g_{11} = 0$  (or 1) is independent of the choice of basis.

For  $N \geq 3$ , the ground state of Eq. (16) cannot be found exactly in general. It is also not clear that the ground state energy will be the same

in all sectors. However the situation simplifies dramatically if we impose the relations (8). The exact ground state wave function is then given by

$$\Psi_o = \prod_{i < j} | \sin [ \frac{\pi}{L} (x_i - x_j) ] |^{g_{ij}} . \quad (21)$$

This is the ground state since it has no nodes for  $x_i \neq x_j$ . The energy is exactly the same as in (10). It is clear that the energy is the same in all sectors.

If we write an eigenstate of Eq. (16) in the form  $\Psi = \Phi \Psi_o$ , then  $\Phi$  must satisfy the eigenvalue equation

$$- [ \sum_i \frac{\hbar^2}{2m_i} \partial_i^2 + \frac{\pi \hbar^2 \alpha}{L} \sum_{i < j} \cot[\frac{\pi}{L}(x_i - x_j)] (m_j \partial_i - m_i \partial_j) ] \Phi = (E - E_o) \Phi , \quad (22)$$

where  $\partial_i = \partial/\partial x_i$ . A class of exactly solvable eigenstates is given by the excitations of the centre of mass; these are discussed in detail in the next paragraph. We have not been able to find any other excited states exactly; however we will discuss some approximate low-energy eigenstates in Sec. 6.

We will now study the centre of mass excitations assuming that the relative coordinates problem is in its ground state (21). In our preferred domain  $0 \leq x_1 < x_2 < \dots < x_N \leq L$ , the centre of mass excitations have the wave function

$$\Phi = \exp (i \frac{2\pi n}{LM} \sum_i m_i x_i) , \quad (23)$$

where  $M = \sum_i m_i$  is the total mass. We now have to find the allowed values of  $n$ . To do that, we have to consider all the domains and boundary conditions together. For a given sector, let us label the different domains by an integer  $d = 1, 2, \dots, D$ , where  $d = 1$  denotes our preferred domain. The functional form of the wave function in any domain  $d$  is given by

$$\Phi_d = \Phi_1 \exp (i\theta_d) , \quad (24)$$

where  $\Phi_1$  is defined in (23) and  $\theta_1 = 0$ . The boundary conditions now imply that if we go from the domain 1 to a domain 2 by moving only one particle  $i$  across the point  $x = 0$  from  $L^-$  to  $0^+$ , then

$$\theta_2 - \theta_1 = 2\pi n \frac{m_i}{M} \quad (25)$$

(There is no extra phase arising from the relative coordinates wave function (21) since it remains unchanged if a coordinate  $x_i$  changes by  $L$ ). We can then go to a domain 3 by moving another particle across  $x = 0$ ; if we repeat this process  $D$  times, we come back to domain 1 with the wave function  $\Phi_1$  times a phase. We can show that the total phase accumulated is  $2\pi nD/N$ . Hence  $nD/N$  must be an integer. We therefore have the important result that the spectrum of centre of mass excitations

$$\Delta E_n = \left(\frac{2\pi\hbar}{L}\right)^2 \frac{n^2}{2M} \quad (26)$$

depends on the sector through the number of domains  $D$ . We would like to emphasize that this discussion of the centre of mass excitations holds for a *general* multispecies CS model in which the interaction parameters may not satisfy (8); this is because the dynamics of the centre of mass does not depend on the  $g_{ij}$ 's.

The ground state of the multispecies CS model is exactly solvable if we retain (8) but consider the problem on a line in the presence of a simple harmonic potential, with the *same* harmonic frequency  $\omega$  for all the particles [18]. The Hamiltonian is

$$H = \sum_i \frac{p_i^2}{2m_i} + \sum_{i < j} g_{ij}(g_{ij}-1) \frac{m_i + m_j}{2m_i m_j} \frac{\hbar^2}{(x_i - x_j)^2} + \frac{\omega^2}{2} \sum_i m_i x_i^2. \quad (27)$$

The number of different sectors now is  $N! / (\prod_a N_a! )$ . In any sector, the ground state wave function is

$$\Psi_o = \prod_{i < j} |x_i - x_j|^{g_{ij}} \exp \left[ -\frac{\omega}{2\hbar} \sum_i m_i x_i^2 \right], \quad (28)$$

and the energy is

$$E_o = \frac{\hbar\omega\alpha}{2} \left( \sum_a m_a N_a \right)^2 + \frac{\hbar\omega}{2} \sum_a (1 - \alpha m_a^2) N_a. \quad (29)$$

We can find an infinite number of excited states (but not all the excited states) as follows. First of all, the Hamiltonian separates into a centre of mass problem with coordinate  $X = \sum_i m_i x_i /M$ , and a relative coordinate problem. The latter further separates into one 'radial' coordinate  $r$  where

$$Mr^2 = \frac{1}{M} \sum_{i < j} m_i m_j (x_i - x_j)^2, \quad (30)$$

(this denotes the moment of inertia about the centre of mass) and  $N - 2$  ‘angular’ coordinates collectively denoted by  $\Omega_i$ . The term inside the exponential in Eq. (28) is  $\sum_i m_i x_i^2 = M(X^2 + r^2)$ . The Hamiltonian can be written as

$$H = - \frac{\hbar^2}{2M} \frac{\partial^2}{\partial X^2} + \frac{1}{2} M\omega^2 X^2 - \frac{\hbar^2}{2M} \left[ \frac{\partial^2}{\partial r^2} + \frac{N-2}{r} \frac{\partial}{\partial r} \right] + \frac{1}{2} M\omega^2 r^2 + \frac{\hbar^2}{2Mr^2} \mathcal{L}, \quad (31)$$

where  $\mathcal{L}$  is a differential operator which only acts on functions of  $\Omega_i$ . The eigenvalues of the centre of mass Hamiltonian are given by  $\hbar\omega(n_1 + 1/2)$ ; the corresponding wave functions take the form

$$\Psi_{n_1}(X) = P_{n_1}(\sqrt{\frac{M\omega}{\hbar}}X) \exp\left(-\frac{M\omega}{2\hbar}X^2\right), \quad (32)$$

where  $P_{n_1}$  denotes a polynomial of degree  $n_1$ . The entire difficulty in finding the spectrum of (31) lies in finding eigenstates  $Y(\Omega_i)$  of  $\mathcal{L}$  which satisfy the boundary conditions at coincident points and the symmetry between particles of the same species. For any eigenstate of  $\mathcal{L}$  with eigenvalue  $\lambda(\lambda + N - 3)$ , where  $\lambda > 0$ , we get an infinite tower of eigenvalues of the relative coordinate Hamiltonian of the form  $\hbar\omega(2n_2 + \lambda + (N - 1)/2)$ . The  $r$ -dependence of the corresponding wave function takes the form

$$\Psi_{n_2}(r) = r^\lambda Q_{n_2}(\frac{M\omega}{\hbar}r^2) \exp\left(-\frac{M\omega}{2\hbar}r^2\right), \quad (33)$$

where  $Q_{n_2}$  is a polynomial of degree  $n_2$ ; it has  $n_2$  nodes. The excitations corresponding to  $Q_{n_2}$  are called breathing modes. In particular, a set of *exact* eigenstates of (31) is given by

$$\Psi_{n_1 n_2} = P_{n_1}(\sqrt{\frac{M\omega}{\hbar}}X) Q_{n_2}(\frac{M\omega}{\hbar}r^2) \Psi_o, \quad (34)$$

and the energy is  $E = E_o + \hbar\omega(n_1 + 2n_2)$ .

The Hamiltonians (16) and (27) can be written as

$$H = \sum_i Q_i^\dagger Q_i + E_o, \quad (35)$$

where

$$Q_i = \frac{1}{\sqrt{2m_i}} [ p_i + \frac{i\pi\hbar}{L} \sum_{j \neq i} g_{ij} \cot \frac{\pi}{L} (x_i - x_j) ] \quad (36)$$

in (16), and

$$Q_i = \frac{1}{\sqrt{2m_i}} [ p_i + i\hbar \sum_{j \neq i} \frac{g_{ij}}{x_i - x_j} - im_i\omega x_i ] \quad (37)$$

in (27). The states (21) and (28) are annihilated by all the  $Q_i$  in (36) and (37) respectively.

Incidentally, one can check that some of the assumptions made in Sec. 2 are indeed valid for the CS model discussed here. Firstly, for two particles on a line, with the Hamiltonian being given by (27) but without the simple harmonic confinement, we can solve for the scattering problem after going to relative coordinates. We find that the scattering phase shift is given by  $\pi g_{ij}\theta(v_i - v_j)$ . Secondly, if all the parameters  $g_{ij}$  are integers, we can expand the ground state wave function (21); it consists of a superposition of plane waves with different values of the particle momenta. We then discover that all the solutions of Eq. (6) appear in the superposition. There are also terms in the superposition which do not correspond to solutions of (6); however this is true even for the single species CS model.

#### 4. Thomas-Fermi Calculation of the One-Particle Densities

In this section, we will use a generalized Thomas-Fermi (TF) method [20] to obtain the one-particle densities  $\rho_a(x)$  in the ground state of the CS model in an external harmonic potential. The advantage of using the TF method is that one does not need to know the exact wave function; for the single species CS model, the semicircle law comes out quite easily. For the multispecies model, we will discover that the particle densities depend on the sector.

In the absence of an external potential, Eq. (11) says that

$$\frac{p_{Fa}}{m_a} = \pi\hbar\alpha \sum_b m_b \rho_b \quad (38)$$

is independent of both  $a$  and  $x$  in all sectors. In the presence of a potential, the TF method uses Eq. (38) in a local sense, i.e.,  $p_{Fa}$  and  $\rho_a$  become

functions of  $x$ . The most energetic particle of species  $a$  at position  $x$  must satisfy

$$\frac{p_{Fa}^2}{2m_a} + \frac{1}{2}m_a\omega^2x^2 = \mu_a, \quad (39)$$

where  $\mu_a$  is the global Fermi energy for  $a$  particles. ( $\mu_a$  must be independent of  $x$ , otherwise one could move an  $a$  particle from a position with a higher value of  $\mu_a$  to a position with lower  $\mu_a$ . This would give us a different sector with a lower energy which is not possible). We thus see that

$$\pi\hbar\alpha \sum_b m_b \rho_b(x) = \left( \frac{2\mu_a}{m_a} - \omega^2 x^2 \right)^{1/2} \quad (40)$$

must be independent of  $a$ . Clearly, the particle densities must vanish outside an interval  $[-x_o, x_o]$ , where

$$x_o^2 = \frac{2\mu_a}{m_a\omega^2} = \frac{2\hbar\alpha}{\omega} \sum_b m_b N_b. \quad (41)$$

Eq. (40) only gives us a weighted sum of the particle densities, not the individual values of  $\rho_a$ . This is because the individual densities on the left hand side of (40) depend on the ordering of the particles, although the right hand side is the same for all orderings.

If we are given a particular ordering, we can compute the densities as follows. Introduce a parameter  $\tau$ , where  $0 \leq \tau \leq 1$ , and functions  $f_a(\tau)$  such that the first  $\tau N$  particles from the left contain  $f_a(\tau)N_a$  particles of species  $a$ . ( $N$  is the total number of particles). We assume that  $f_a(\tau)$  varies smoothly with  $\tau$ . Clearly,  $f_a(0) = 0$ ,  $f_a(1) = 1$ , and  $f'_a(\tau) \geq 0$ . We emphasize that the functions  $f_a$  are fixed once the ordering of the particles (i.e., a sector) is specified. Now we want to express  $\tau$  as a function of  $x$ . At  $x = -x_o$  and  $x_o$ ,  $\tau = 0$  and 1 respectively. In the interval  $x$  to  $x + dx$  or, equivalently,  $\tau$  to  $\tau + d\tau$ , the number of  $a$  particles is equal to

$$\rho_a(x) dx = N_a f'_a(\tau) d\tau. \quad (42)$$

From Eqs. (40-42), we deduce that

$$\omega \int_{-x_o}^x dy (x_o^2 - y^2)^{1/2} = \pi\hbar\alpha \sum_a m_a N_a f_a(\tau), \quad (43)$$

so that  $\tau$  is implicitly a function of  $x$ . The densities then follow from Eq. (42),  $\rho_a(x) = N_a f'_a(\tau) d\tau/dx$ .

It is interesting to observe that Eq. (1) is valid semiclassically even in the presence of a simple harmonic potential. We simply use Eq. (14) with  $h_a = p^2/2m_a + m_a\omega^2 x^2/2$ , and let  $\epsilon \rightarrow 0$  after calculating  $\Delta D_a(\epsilon)$ .

We can use the TF method to calculate the one-particle densities even if the confining potential is not simple harmonic and the ground state is not exactly solvable. If  $V_a(x)$  is the one-body potential felt by a particle of species  $a$ , we can find the condition under which the ground state energy will be the same in all sectors. As in Eq. (39), we want

$$\frac{p_{Fa}^2}{2m_a} + V_a(x) = \mu_a \quad (44)$$

with  $\mu_a$  independent of  $x$ . From (38), this implies that  $(\mu_a - V_a(x))/m_a$  should be the same for all species. We conclude that  $d(V_a(x)/m_a)/dx$  (i.e., the classical acceleration) should be independent of  $a$ .

## 5. High Temperature

The entire analysis so far has been at zero temperature. We now study the CS model at finite temperature. Two things are particularly important to check.

- (a) Are thermodynamic properties like the equation of state the same in all sectors (as is true at zero temperature)? We will argue that the properties probably depend on the sector at high temperature.
- (b) Does the model show mutual statistics, if not exactly, then at least approximately in some range of temperatures?

In this section, we consider the equation of state of the system at high temperature. The pressure  $P$  can then be expanded in terms of the densities  $\rho_a = N_a/L$ . We will only be interested in the expansion upto quadratic order, i.e., in the second virial coefficients. The grand canonical partition function  $Z$  has an expansion of the form

$$Z = 1 + \sum_a \zeta_a Z_a + \sum_a \zeta_a^2 Z_{aa} + \sum_{a < b} \zeta_a \zeta_b Z_{ab} + \dots, \quad (45)$$

where  $\zeta_a$  denotes the fugacity of species  $a$ ,  $Z_a$  is the partition function of a single particle of species  $a$  on a circle of length  $L$ , and  $Z_{ab} = Z_{ba}$  is the

partition function of two particles of species  $a$  and  $b$ . Next, the densities satisfy

$$\rho_a = \frac{\zeta_a}{L} \frac{\partial \log Z}{\partial \zeta_a} . \quad (46)$$

We then obtain

$$\beta P = \sum_a \rho_a + \sum_a A_{aa} \rho_a^2 + \sum_{a < b} A_{ab} \rho_a \rho_b + \dots , \quad (47)$$

where  $\beta = 1/k_B T$ , and the second virial coefficients are given by

$$\begin{aligned} A_{aa} &= \lim_{L \rightarrow \infty} L \left( \frac{1}{2} - \frac{Z_{aa}}{Z_a^2} \right) , \\ A_{ab} &= \lim_{L \rightarrow \infty} L \left( 1 - \frac{Z_{ab}}{Z_a Z_b} \right) . \end{aligned} \quad (48)$$

In the limit  $L \rightarrow \infty$ , we can use the Poisson resummation formula

$$\sum_{n=-\infty}^{\infty} \exp(-\pi n^2/L^2) = L \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 L^2) \quad (49)$$

to show that  $Z_a = L/\lambda_a$  plus exponentially small terms;  $\lambda_a = (2\pi\hbar^2\beta/m_a)^{1/2}$  is called the thermal wavelength. Using Eq. (20), we can calculate  $Z_{aa}$  and  $Z_{ab}$ . We finally obtain

$$\begin{aligned} A_{aa} &= (g_{aa} - \frac{1}{2}) \frac{\lambda_a}{\sqrt{2}} , \\ A_{ab} &= (g_{ab} - \frac{1}{2}) \sqrt{\lambda_a^2 + \lambda_b^2} . \end{aligned} \quad (50)$$

As mentioned earlier, there is only one sector for the two-particle problem on a circle, but there is an exponentially large number of sectors in the thermodynamic limit  $N \rightarrow \infty$ . The virial coefficients calculated above must therefore be some kind of an average over the different sectors, although we do not know at this point if all sectors contribute to the average with equal weight.

Now there is one sector in which we can calculate many properties exactly; this is the sector defined at the end of Sec. 2 where the different species are segregated, with  $N_a$  particles of species  $a$  occupying a segment of length  $L_a$ . We may compute the second virial coefficients in this sector as follows. For a

single species, we know from the above equations that the pressure of species  $a$  is given by

$$\beta P_a = \tilde{\rho}_a + A_{aa} \tilde{\rho}_a^2 + \dots, \quad (51)$$

where  $\tilde{\rho}_a = N_a/L_a$ . At equilibrium, the pressures exerted by different species must be equal. This enables us to eliminate the  $L_a$  in terms of the total length  $L$  and the densities  $\rho_a$ . We thus get

$$\beta P = \sum_a \rho_a + \sum_a A_{aa} \rho_a^2 + \sum_{a < b} (A_{aa} + A_{bb}) \rho_a \rho_b + \dots. \quad (52)$$

We see that the virial coefficients  $A_{ab}$  in this sector do not agree with those in Eq. (50) since the masses are unequal. Thus the equation of state seems to depend on the sector. Since the second virial coefficients should be related to the statistics parameters [9, 15], we conclude that the model cannot realize mutual statistics in all sectors at high temperature.

We now observe that the particle density  $\sum_a \rho_a$  is independent of both the position  $x$  and the sector at  $T \rightarrow \infty$ , while the mass density  $\sum_a m_a \rho_a$  is independent of the position and sector at  $T \rightarrow 0$  (cf. Sec. 2). This implies that the individual densities  $\rho_a$  must vary with the temperature in at least some sectors. Such a variation may seem surprising in a model which is scale invariant, in the sense that the kinetic terms and the two-body interactions transform in the same way under an uniform scaling of all the coordinates. However the temperature-dependence might be understandable if the model is found to be non-integrable. We have therefore checked for integrability, both classically and quantum mechanically, using the methods given in Refs. [3, 4]; the model appears to be non-integrable. Hence the asymptotic Bethe ansatz is not expected to be valid for the excited states, except for the centre of mass excitations in which all the velocities are shifted by the same amount from their ground state values.

We recall that in the *single* species CS model, the exclusion statistics is ideal. This happens because the scattering phase shift is independent of the momenta of the two particles *and* the model is integrable [3, 4]. The second property implies that an  $N$ -particle scattering phase shift is given by a sum of two-particle phase shifts.

Although the CS model does not seem to exhibit mutual statistics in all sectors at high temperature, it may do so in an approximate way at low temperature. In the next section, we study the low-energy excitations in some detail.

## 6. Low Temperature

A general formalism for obtaining the distribution functions for a system with mutual statistics is given in Ref. [14]. It is therefore interesting to investigate, (a) whether the low-energy excitations are described by one or more  $c = 1$  Gaussian conformal field theories [14], and (b) whether the equation of state of the CS model at low temperature agrees with that of a system with mutual statistics. We will provide a complete answer to (a), but we are unable to answer question (b) by analytical methods.

The elementary excitations in a Gaussian theory satisfy the dispersion relation

$$\Delta E = v_F |\Delta p| \quad (53)$$

where  $\Delta E$  and  $\Delta p$  respectively denote the energy and momentum with respect to the ground state, and  $v_F$  is the Fermi velocity. From Eq. (12),  $v_F = \pi \hbar \alpha M / L$ , where  $M = \sum_i m_i$  is the total mass;  $v_F$  remains finite in the thermodynamic limit  $M, L \rightarrow \infty$ . (Actually, Eq. (53) describes the excitations of a Gaussian theory based on an *uncompactified* scalar field. For a compactified scalar field, there are some additional terms in the energy which will be discussed in Eq. (66) below).

We can now find a right-moving elementary excitation approximately by using the Sutherland ansatz for the wave function [2]. Let us first define

$$z_i = \exp(i \frac{2\pi}{L} x_i) . \quad (54)$$

Following Eq. (22), we write  $\Psi_1 = \Phi_1 \Psi_o$  where

$$\Phi_1 = \sum_i m_i z_i . \quad (55)$$

This has  $\Delta p = 2\pi\hbar/L$  since the momentum operator is  $p = \sum_i p_i$ . We can show that the expectation value

$$\Delta E = \frac{\langle \Psi_1 | H - E_o | \Psi_1 \rangle}{\langle \Psi_1 | \Psi_1 \rangle} = v_F \Delta p + O(\frac{\Delta p}{L}) . \quad (56)$$

The corrections of order  $\Delta p/L$  arise due to the approximate nature of the wave function, and they go to zero as  $L, N \rightarrow \infty$ ; we then recover  $\Delta E/\Delta p = v_F$ . Similarly, a left-moving elementary excitation with  $\Delta p = -2\pi\hbar/L$  is

given by  $\Psi_{-1} = \Phi_{-1}\Psi_o$  where  $\Phi_{-1} = \Phi_1^*$ . It is important to note that the leading term in  $\Delta E/\Delta p$  is independent of the sector.

The first term in  $\Delta E/\Delta p$  in Eq. (56), which survives as  $L \rightarrow \infty$ , arises entirely from the *linear* differential operators in Eq. (22). On the other hand, the corrections of order  $1/L$  arise from both the linear and the quadratic differential operators in (22). It is very useful to know that the leading term in  $\Delta E$  arises from a linear operator; it implies that the energy eigenvalues will add up when we multiply two eigenstates together. Thus a family of composite excitations with  $\Delta p = 2\pi\hbar(n_1 - n_2)/L$  is given by

$$\Psi_{n_1, n_2} = \Phi_1^{n_1} \Phi_{-1}^{n_2} \Psi_o , \quad (57)$$

where  $n_1, n_2 \geq 0$ ; this state has  $\Delta E = v_F 2\pi\hbar(n_1 + n_2)/L$ . We require that  $n_1, n_2 \ll N$  so that this is a *low-energy* excitation and the corrections of order  $n_i/L$  vanish as  $L \rightarrow \infty$ .

Following Ref. [2], we can show that the general excitation is specified by a set of non-zero integers  $(n_1, n_2, \dots, n_k)$ ; they may be positive or negative. (For this to describe a low-energy excitation, we again require that each of the magnitudes  $|n_i|$  as well as  $k$  should be much less than  $N$ ). The energy and momentum of such a state are given by

$$\begin{aligned} \Delta E &= v_F \frac{2\pi\hbar}{L} \sum_i |n_i| , \\ \Delta p &= \frac{2\pi\hbar}{L} \sum_i n_i . \end{aligned} \quad (58)$$

If all the integers  $n_i$  are positive, the ansatz wave function  $\Phi_{n_1, n_2, \dots, n_k}$  contains terms like  $(m_1 z_1)^{n_1} (m_2 z_2)^{n_2} \dots (m_k z_k)^{n_k}$  plus permutations, plus other terms which are obtained by squeezing (see below). If an integer  $n_i$  is negative,  $(m_i z_i)^{n_i}$  is replaced by  $(m_i z_i^*)^{-n_i}$ . The wave function  $\Phi_{n_1, n_2, \dots, n_k}$  is obtained by multiplying together the ansatz wave functions of a number of ‘elementary’ states; an elementary state is one in which all the integers  $n_i$  are equal to 1, or all of them are equal to  $-1$ .

Let us first examine the wave function for an elementary state, say,  $(1, 1)$ . This is given by  $\Psi_{1,1} = \Phi_{1,1}\Psi_o$  where

$$\Phi_{1,1} = \sum_{i < j} f(x_i - x_j) m_i m_j z_i z_j . \quad (59)$$

The factor  $f(x_i - x_j)$  in (59) requires some explanation. If the particles  $i$  and  $j$  belong to the same species, we set  $f = 1$ . But if they belong to *different* species, we require that  $f$  should go to zero sufficiently fast if  $i$  and  $j$  approach each other, and should go to 1 if their separation is much larger than the inverse density  $L/N$ . For instance, we could define the function as

$$f(x_i - x_j) = \exp\left(-\frac{|g_{ii} - g_{jj}|}{N \sin[\frac{\pi}{L}(x_i - x_j)]}\right) \quad (60)$$

The reason for introducing such a function in (59) is that if we always set  $f = 1$ , the linear operator in Eq. (22) would act on (59) to produce a function which diverges when  $x_i \rightarrow x_j$ ; the expectation value  $\langle \Psi | H | \Psi \rangle$  would then be ill-defined if  $g_{ij}$  is not large enough. (We would like to emphasize that the function  $f$  is only required because we are working with *approximate* wave functions; such a short-distance cutoff is not present in the exact wave functions). Since  $f$  only differs from 1 in certain microscopic regions of space, its presence can only change the expectation value of  $\Delta E/\Delta p$  by an amount which vanishes as  $L \rightarrow \infty$ . We can now verify that the energy and momentum of the state (59) is given by (58).

We can write down the wave functions of other elementary states like  $(1, 1, 1)$  in a similar way. The wave functions of left-moving states such as  $(-1, -1)$  are obtained by complex conjugating  $\Phi_{1,1}$ . Finally, composite wave functions such as  $\Phi_{2,1,-1}$  can be written as a product of  $\Phi_{1,1}$ ,  $\Phi_1$  and  $\Phi_{-1}$ . The general rule for writing  $\Phi_{n_1, n_2, \dots, n_k}$  as a product of elementary wave functions can be expressed inductively as follows. Suppose that the numerically largest positive (or negative) integer  $n_i$  appears  $l$  times in the set  $(n_1, n_2, \dots, n_k)$ . For instance, let  $n_1 = n_2 = \dots = n_l$  be the largest positive integers. Then

$$\Phi_{n_1, n_2, \dots, n_l, n_{l+1}, \dots, n_k} = \Phi_{n_1-1, n_2-1, \dots, n_l-1, n_{l+1}, \dots, n_k} \Phi_{1,1,\dots,1}, \quad (61)$$

where the last  $\Phi$  has  $l$  1's. This factorization can be repeated till we only get elementary wave functions on the right hand side of Eq. (61). If  $n_1 = \dots = n_l$  is negative, we replace  $n_i-1$  by  $n_i+1$ , for  $1 \leq i \leq l$ , and  $\Phi_{1,1,\dots,1}$  by  $\Phi_{-1,-1,\dots,-1}$  on the right hand side of (61). An important point to note in all this is that a factor of  $m_i$  comes along with each  $z_i$  or  $z_i^*$ .

Finally, we observe that  $\Phi_{2,1,-1}$  contains within it other wave functions such as  $\Phi_{1,1,1,-1}$ ,  $\Phi_{1,1}$  and  $\Phi_2$ , because the sets  $(1, 1, 1, -1)$ ,  $(1, 1)$  and  $(2)$  can be obtained from the set  $(2, 1, -1)$  by a process called squeezing [2]. We say that a set  $\{n'_i\}$  is obtained from another set  $\{n_i\}$  by squeezing if, either

- (a) an integer  $n_i$  in the second set is equal to the sum of two integers  $n'_j$  and  $n'_k$  in the first set, where  $n_i, n'_j$  and  $n'_k$  all have the same sign, or
- (b) there are two integers  $n_i, n_l$  in the second set, and *upto* two integers  $n'_j$  and  $n'_k$  in the first set, which satisfy  $n_i + n_l = n'_j + n'_k$  and  $n_i > n'_j, n'_k > n_l$ . (One or both of the integers  $n'_j$  and  $n'_k$  may be zero, in which case it is omitted from the set  $\{n'_i\}$ ).

Following (58), we can think of the low-energy excitations in terms of a system of ‘bosons’ whose allowed energy levels are given by

$$e_n = v_F \frac{2\pi\hbar}{L} |n|, \quad (62)$$

where  $n$  can be any non-zero integer. These bosons have zero chemical potential. The partition function is therefore

$$Z = \exp(-\beta E_o) \left[ \prod_{n=1}^{\infty} (1 - \exp(-\beta e_n)) \right]^{-2}. \quad (63)$$

We can now calculate the leading terms in the specific heat and pressure at low temperature. The mean energy is given by  $E = -(\partial \ln Z / \partial \beta)_L$ , the specific heat is  $C_V = (\partial E / \partial T)_L$ , and the pressure is  $P = k_B T (\partial \ln Z / \partial L)_T$ . The leading terms in the specific heat and pressure are therefore given by

$$\begin{aligned} \frac{C_V}{L} &= \frac{\pi k_B^2 T}{3\hbar v_F}, \\ P &= \frac{v_F^3}{3\pi\hbar\alpha} + \frac{\pi k_B^2 T^2}{3\hbar v_F}. \end{aligned} \quad (64)$$

These leading terms are independent of the sector. However we are unable to obtain the next term of order  $T^2$  in  $C_V$  or order  $T^3$  in  $P$ , since that requires a knowledge of the  $1/L$  terms in the energy; we have ignored those terms because they do not seem to be analytically computable. The importance of the next term in  $C_V$  or  $P$  is that it contains information about the statistics parameters [15].

Thus the low-energy excitations match those of a  $c = 1$  Gaussian conformal field theory, rather than  $c = A$  (as would be the case if all the off-diagonal elements  $g_{ab}$  were zero). This raises the following question. Are the low temperature thermodynamic properties of the model the same as those of a *single* species CS model with some ‘effective’ particle density  $\rho$ , mass  $m$ ,

and interaction parameter  $g$ ? On comparing the specific heat and pressure in (64) with those of a single species model, we see that the three effective parameters must satisfy the two relations

$$\begin{aligned}\frac{\pi\hbar g\rho}{m} &= v_F, \\ \frac{g}{m^2} &= \alpha.\end{aligned}\tag{65}$$

Clearly, we need a third low-energy property (for instance, the next term in  $C_V$  or  $P$ ) to fix the individual values of  $\rho$ ,  $m$  and, most importantly, the exclusion statistics parameter  $g$ . As mentioned earlier, we also have to determine whether or not these values depend on the sector.

Before concluding this section, we note that the centre of mass excitations discussed around Eq. (26) can have energies which are much *lower* than the ones discussed above if the number of domains  $D$  in a given sector is comparable to  $N$ . If the number of domains  $D \ll N$  (as in a single species CS model where  $D = 1$ ), the centre of mass energies are of the same order as the ones discussed above. In any case, the centre of mass constitutes only a single degree of freedom, so it has no effect on thermodynamic properties.

For the sake of completeness, we should mention that the low-energy excitations of a single species CS model have some terms in the expressions for energy and pressure in addition to the ones given in (58). These are characterized by two integers,  $\Delta K$  equal to the number of particles transferred from the left Fermi momentum  $-p_F$  to the right Fermi momentum  $p_F$ , and  $\Delta N$  equal to the change in the number of particles. The contributions of these to the energy and momentum take the form [14]

$$\begin{aligned}\Delta E &= \mu\Delta N + v_F \frac{2\pi\hbar}{L} \left[ \frac{(\Delta K)^2}{g} + \frac{g}{4}(\Delta N)^2 \right], \\ \Delta p &= 2\pi\hbar\rho \Delta K + \frac{2\pi\hbar}{L} \Delta K \Delta N.\end{aligned}\tag{66}$$

We observe that  $\Delta K$  simply corresponds to a centre of mass excitation in which all the particle momenta are shifted by the same amount  $2\pi\hbar\Delta K/L$ ; in the notation of Eq. (26),  $\Delta K = n/N$ . Eq. (66) is essential in order to identify the radius of compactification of the scalar field in the Gaussian theory as  $R = 1/\sqrt{g}$ . In the multispecies model, we can certainly calculate terms similar to (66) using Eqs. (26) and (10); however we will not exhibit them

here since these terms have no bearing on the thermodynamics properties which were calculated above with fixed particle numbers  $N_a$ .

## 7. A Multispecies Anyon Model in Two Dimensions

In this section, we would like to point out that relations similar to (8) lead to some special properties for mutual statistics in two dimensions. Consider a collection of  $A$  species of anyons with mutual statistics parameters  $g_{ab} = g_{ba}$  and charges  $q_a$  [8]. Let us first assume that all the  $q_a$  have the same sign and all the  $g_{ab}$  are positive. We now apply an uniform magnetic field in a direction normal to the plane. At zero temperature, all the particles lie in the lowest Landau level (LLL). Let  $Bq_a > 0$ . Then in the symmetric gauge  $(A_x, A_y) = (-By/2, Bx/2)$ , all the wave functions in the LLL have the form

$$\Psi = f(z_i) \exp\left[-\frac{B}{4c\hbar} \sum_i q_i \vec{r}_i^2\right], \quad (67)$$

where  $z_i = x_i + iy_i$  and  $\vec{r}_i^2 = z_i z_i^*$ ;  $c$  is the speed of light. (Note that the particle masses play no role in the LLL). When anyon  $i$  is taken around a closed loop surrounding anyon  $j$ , the wave function should pick up a phase  $\exp(i2\pi g_{ij})$ . The simplest wave function of this kind is given by

$$f(z_i) = \prod_{i < j} (z_i - z_j)^{g_{ij}}. \quad (68)$$

This is multivalued if any of the  $g_{ij}$  is not equal to 0 or 1. Other wave functions in the LLL are obtained by multiplying (68) by any polynomial in the  $z_i$  which is symmetric between anyons of the same species. We will only consider the wave function (67-68) henceforth. Amongst all the LLL wave functions, this has the lowest angular momentum, i.e.,

$$L = \hbar \sum_i \left( z_i \frac{\partial}{\partial z_i} - z_i^* \frac{\partial}{\partial z_i^*} \right) \quad (69)$$

is equal to  $\hbar \sum_{i < j} g_{ij}$ . The particles with this wave function lie closest to the origin  $z_i = 0$ .

The probability density is given by

$$|\Psi|^2 = \prod_{i < j} |\vec{r}_i - \vec{r}_j|^{2g_{ij}} \exp \left[ -\frac{B}{2c\hbar} \sum_i q_i \vec{r}_i^2 \right] \quad (70)$$

We now note that the expression (70) may sometimes be valid even in a situation in which the wave function is *not* given by (67); in that case,  $q_i$  in (70) will denote the magnitude of the charge of species  $i$ , rather than the charge itself. This is in fact true for the example given at the end of this section.

We will calculate the one-particle densities  $\rho_a(\vec{r})$  using a plasma analogy [21]. Since (70) is circularly symmetric,  $\rho_a$  can only be a function of the magnitude  $r$ . Now  $-\ln|\Psi|^2$  can be interpreted as the potential energy of a system of point charges interacting with each other through a repulsive Coulomb potential (which is logarithmic in two dimensions) and with an uniform attractive background charge density equal to  $-B/\pi c\hbar$ . A particle of species  $a$  which is at a distance  $r$  from the origin will be at equilibrium if

$$\sum_b 2g_{ab} \frac{Q_b(r)}{r} = \frac{B}{c\hbar} q_a r, \quad (71)$$

where

$$Q_b(r) = 2\pi \int_0^r r' dr' \rho_b(r') \quad (72)$$

is the total charge of species  $b$  inside the circle of radius  $r$ . Eqs. (71-72) imply that

$$\pi \sum_b g_{ab} \rho_b(r) = \frac{B}{2c\hbar} q_a. \quad (73)$$

This is a system of  $A$  equations if all the densities  $\rho_a$  are non-zero at  $r$ ; if some of them are zero, we have to drop the corresponding equations from (73).

We now demand that the system should satisfy the following requirements. Firstly, since the right hand side of (73) does not depend on  $r$ , we would like the particle distribution to be homogeneous, that is,  $\rho_a$  should be independent of  $r$ . Secondly, if the numbers of particles  $N_a$  are large but finite, we demand that there should be a single radius  $R \gg \sqrt{c\hbar/Bq_a}$  such that *all* the densities  $\rho_a$  are non-zero inside  $R$  and zero outside  $R$ . These two requirements rule out the possibility of phase separation, namely, a situation in which some, but not all, of the densities vanish in some region. We then have

$$\sum_b g_{ab} N_b = \frac{BR^2}{2c\hbar} q_a. \quad (74)$$

Finally, we demand that the numbers  $N_a$  be independent variables; for instance,  $N_1$  should not get fixed if  $N_2, N_3, \dots, N_A$  are specified. We can show that these three requirements will be satisfied if and only if all the equations in (74) are linearly dependent on just one of them, i.e., if

$$g_{ab} = \alpha q_a q_b \quad (75)$$

for both  $a = b$  and  $a \neq b$ .

It is interesting to note that the relations in (75) arise quite naturally in the fractional quantum Hall effect. For instance, consider the situation when the filling fraction is close to but slightly less than  $1/m$ , where  $m$  is an odd integer like 3 or 5. In the first hierarchy, the ground state contains two species of particles with opposite charges, namely, the electron (species 1) and the quasi-hole (species 2). Laughlin's wave function then tells us that  $g_{11} = m$ ,  $g_{12} = 1$ ,  $g_{22} = 1/m$ , and the ratio of the magnitudes of the charges is  $q_2/q_1 = 1/m$  [21]; hence (75) is satisfied. Ref. [8] points out the interesting fact a fractional quantum Hall system can be described by a *single* Chern-Simons gauge field only if Eqs. (75) are satisfied.

## 8. Discussion

To conclude, we have argued that certain relations between the interaction parameters and masses are *necessary* for the ground state of a one-dimensional scale invariant multispecies model to satisfy the asymptotic Bethe ansatz. We have shown that a CS model with those relations has an exactly solvable ground state, and that the low-energy excitations are the same as those of a Gaussian conformal field theory. However we have not proved that the same relations are *sufficient* to give rise to either mutual statistics, or to a single species exclusion statistics. To address that question, we need to understand the spectrum of low-energy excitations more accurately.

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