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# Thomas-Fermi Method For Particles Obeying Generalized Exclusion Statistics

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**Abstract**

We use the Thomas-Fermi method to examine the thermodynamics of particles obeying Haldane exclusion statistics. Specifically, we study Calogero-Sutherland particles placed in a given external potential in one dimension. For the case of a simple harmonic potential (constant density of states), we obtain the exact one-particle spatial density and a *closed* form for the equation of state at finite temperature, which are both new results. We then solve the problem of particles in a $x^{2/3}$ potential (linear density of states) and show that Bose-Einstein condensation does not occur for any statistics other than bosons.

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A recent definition of generalized exclusion statistics by Haldane has aroused considerable interest [1]. The definition is based on the rate at which the number of available states in a system of fixed size decreases as more and more particles are added to it. We introduce a statistics parameter $\alpha$ and assign the values $\alpha = 0$ and 1 to bosons and fermions respectively, because the addition of one particle reduces the number of available states by $\alpha$. Some examples of particles obeying such generalized statistics are spinons in an antiferromagnetic spin chain with inverse square exchange [1,2] and two-dimensional anyons residing in the lowest Landau level in a strong magnetic field [3-5].

More recently, the statistical distribution function for such systems has been obtained [5,6]. If $\mu$ is the chemical potential , $\epsilon$ is the energy of a single-particle state and $T$ is the temperature, then $n(\epsilon, \mu, T)$ satisfies

$$(n^{-1} \; - \; \alpha)^{\alpha} \; (n^{-1} \; + \; 1 \; - \; \alpha)^{1-\alpha} \; = \; \exp[(\epsilon \; - \; \mu)/T \;]. \tag{1}$$

(We set Planck's and Boltzmann's constants equal to one). Let $g(\epsilon)$ be the density of states so that the total number of particles and the energy are given by

$$\begin{aligned} N \; &= \; \int \, d\epsilon \; g(\epsilon) \; n(\epsilon) \\ \text{and} \quad E \; &= \; \int \, d\epsilon \; g(\epsilon) \; n(\epsilon) \; \epsilon. \end{aligned} \tag{2}$$

Using Eq. (1), the thermodynamic potential $\Omega(\mu, T)$ satisfying $(\partial\Omega/\partial\mu)_T \; = \; - \, N$ can be shown to be given by

$$\Omega \; = \; - \, T \int \, d\epsilon \; g(\epsilon) \;\; \log[1 \; + \; \exp(- \, \tilde{\epsilon}/T) \;], \tag{3}$$

where $\tilde{\epsilon}(\epsilon, \mu, T)$ is defined by the relation

$$\tilde{\epsilon} \; + \; T \; (1 \; - \; \alpha) \;\; \log[1 \; + \; \exp(- \, \tilde{\epsilon}/T) \;] \; = \; \epsilon \; - \; \mu. \tag{4}$$

In terms of $\tilde{\epsilon}$, $n$ is given by

$$n \; = \; \frac{1}{\exp(\tilde{\epsilon}/T) \; + \; \alpha}. \tag{5}$$

If we keep $T$ fixed and let $\mu \rightarrow \ -\infty$, we find that $\tilde{\epsilon} \rightarrow \epsilon - \mu$ and $n(\epsilon) \rightarrow \ \exp[(\mu - \epsilon)/T]$ independent of $\alpha$. In this limit, therefore, $N$, $\Omega$ and $E$ all go to zero as $\exp(\mu/T)$. We now use the identity

$$\frac{\partial n}{\partial \epsilon} \ + \ \frac{\partial n}{\partial \mu} \ = \ 0, \tag{6}$$

which follows from Eq. (1) if T is held fixed. Now suppose that $g(\epsilon) \ \sim \ \epsilon^a$ . Inserting (6) in (2), we obtain

$$\left( \frac{\partial E}{\partial \mu} \right)_T \ = \ (a \ + \ 1) \ N. \tag{7}$$

Hence $E + (a+1)\Omega$ is independent of $\mu$. Since $E + (a+1)\Omega$ is zero at $\mu = -\infty$, it must be zero for all $\mu$ and $T$. This relation will be used below.

The simplest example of particles obeying the generalized exclusion statistics is provided by the Calogero-Sutherland (CS) model [7,8]. This one-dimensional model has the Hamiltonian

$$H \ = \ \frac{1}{2m} \sum_i \ p_i^2 \ + \ \frac{\alpha(\alpha-1)}{m} \sum_{i<j} \frac{1}{(x_i \ - \ x_j)^2}. \tag{8}$$

Here $\alpha \geq 0$ with the understanding that the wave function vanishes as $|x_i \ - \ x_j \ |^\alpha$ whenever the particles i and j approach each other. Note that $\alpha = 0$ and 1 yield free particles (bosons and fermions repectively) while the maximally attractive potential occurs for $\alpha = 1/2$ (semions). Using the thermodynamic Bethe ansatz [9] and other means, it has been argued that the CS system can be thought of as an ideal gas in the sense that the particles only have statistical interactions amongst each other [6,10,11].

While the CS model has been studied in great detail over the years, we would like to examine it here from the view point of exclusion statistics. The CS model has so far been exactly solved only in an external simple harmonic potential $V(x) \ = \ m\omega^2 x^2/2$ . However, once we realise that the particles only undergo statistical interactions amongst each other, we can develop other techniques to study the thermodynamic of the CS system placed in *any* external potential . One such technique is the Thomas-Fermi

(TF) method which has been used to study fermionic systems for a long time [12]. In addition to being exact in the large-$N$ limit, the TF method does not require a knowledge of the energy spectrum and wave function of the system. Here we modify the method by using the distribution in (1) rather than the Fermi-Dirac distribution while continuing to use the usual phase space description of quantum systems. We show below that this procedure leads to the well-known expressions for the one-particle spatial density and the ground state energy for the many-body system in a simple harmonic potential at zero temperature [7,8]. Encouraged by this, we analyse the situation at finite temperaure and obtain the one-particle density which is a new result. We also find closed forms for the thermodynamic quantities $\mu$ and $\Omega$, in contrast to previous expressions in the literature which are given as high-temperature series. Finally, to demonstrate the power of the TF method, we study the CS model placed in a $x^{2/3}$ potential which has $g(\epsilon) \sim \epsilon$. Once again, we derive the thermodynamic quantities for all $\alpha$ and $T$. Since the $g(\epsilon)$ is the same as for free particles in four dimensions, it is natural ask whether Bose-Einstein (BE) condensation occurs at low temperature for a *finite* range of $\alpha$. It is shown that condensation occurs *only* for bosons ($\alpha = 0$) and not for any positive $\alpha$.

The TF density of states is obtained from the integral

$$\begin{aligned} g(\epsilon) &= \int \frac{dxdp}{2\pi} \, \delta(\epsilon - \frac{p^2}{2m} - V(x)) \\ &= \int \frac{dx}{\pi} \, \frac{\theta(\epsilon - V(x))}{\sqrt{2(\epsilon - V(x))/m}}, \end{aligned} \tag{9}$$

where $\theta(y) = 1$ if $y > 0$ and 0 if $y < 0$. The TF expression for the one-particle density is

$$\rho(x,T) = \int \frac{dp}{2\pi} \, n(\epsilon, T), \tag{10}$$

where $\epsilon = p^2/2m + V(x)$ and $\int dx \, \rho(x,T) = N$.

At $T = 0$, Eq. (1) implies that there is a Fermi energy $\mu(0)$ such that

$$\begin{aligned} n(\epsilon) &= 1/\alpha \quad \text{if} \quad \epsilon < \mu(0), \quad \text{and} \\ &= 0 \quad \text{if} \quad \epsilon > \mu(0). \end{aligned} \tag{11}$$

Thus $\alpha < 1$ (or $> 1$) corresponds to less (or more) exclusion than fermions. We now study the harmonic potential problem. From (9), $g(\epsilon) = 1/\omega$ . Therefore the Fermi and ground state energies are given by

$$\begin{aligned} \mu(0) &= \alpha\omega N \\ \text{and} \quad E(0) &= \frac{1}{2}\, N\mu(0). \end{aligned} \tag{12}$$

Eqs. (10, 11) yield the well-known semicircle law [8,13]

$$\rho(x, 0) = \frac{1}{\pi\alpha}\,(2m\mu(0) - m^2\omega^2x^2\,)^{1/2} \tag{13}$$

for $x^2 \leq x_o^2 = 2\mu(0)/m\omega^2$ . The thermodynamic limit for this system is obtained by taking $N \to \infty$ holding $\mu(0)$ and therefore $\omega N$ fixed. (This is analogous to keeping the density in a box fixed with $1/\omega$ playing the role of the volume).

At finite temperature, we find from Eqs. (2, 6) that

$$\omega \left(\frac{\partial N}{\partial \mu}\right)_T = n(0), \tag{14}$$

where $n(0)$ is the solution of (1) with $\epsilon = 0$. This has the solution

$$\mu(N, T) = \alpha\omega N + T\,\log[1 - \exp(-\omega N/T)]. \tag{15}$$

(Here we used the boundary condition that for fixed $T$, $N \to 0$ as $\mu \to -\infty$). Now

$$\left(\frac{\partial \Omega}{\partial N}\right)_T = -N\left(\frac{\partial \mu}{\partial N}\right)_T \tag{16}$$

implies that [14]

$$-\,\Omega = \frac{1}{2}\alpha\omega N^2 + \frac{T^2}{\omega}\int_0^{\omega N/T} dy\,\frac{y}{e^y - 1}. \tag{17}$$

Since $E = -\Omega$, the entropy $S$ (given by $\Omega = E - TS - \mu N$ ) is independent of $\alpha$. Eq. (17) has a high-temperature expansion in $\omega N/T$ of the form

$$-\,\Omega \;=\; NT \;+\; \frac{1}{2}\,\omega N^2\,(\alpha \;-\; \frac{1}{2}) \;+\; \ldots\,. \tag{18}$$

This resembles the virial expansion for the equation of state in a box and it has the remarkable property that only the second term depends on $\alpha$ [11]. However, this expansion only has a finite radius of convergence ($2\pi$). Eq. (17) has an alternative expansion in powers of $\exp(-N\omega/T\,)$ which is valid for *all* $T$. For instance, we may read off the low-temperature result

$$-\,\Omega \;=\; \frac{1}{2}\alpha\omega N^2 \;+\; \frac{\pi^2}{6}\frac{T^2}{\omega} \tag{19}$$

plus exponentially small terms.

We may now obtain the finite temperature density $\rho(x,T)$ from Eqs. (1) and (10). This is somewhat difficult to compute explicitly except for special values of $\alpha$. For instance,

$$\begin{aligned} n(\epsilon) \;&=\; \frac{2}{\sqrt{1+4\gamma^2}} \quad \text{for} \quad \alpha \;=\; 1/2 \quad \text{and} \\ &=\; \frac{1}{2} \;-\; \frac{1}{2}\sqrt{\frac{\gamma}{4+\gamma}} \quad \text{for} \quad \alpha \;=\; 2, \end{aligned} \tag{20}$$

where $\gamma = \exp[(\epsilon-\mu)/T]$ . In Fig. 1, we show $\rho(x,T)$ at $T = \omega N/2$ for $\alpha = 1/2,\ 1$ and 2. For convenience, we have rescaled $x \to x/x_o$ and $\rho \to\ \rho(x,T)/\rho(0,0)$ so that

$$\begin{aligned} \rho(x,0) \;&=\; \sqrt{1-x^2}\;\theta(1-x^2) \\ \text{and} \quad \int_0^\infty dx\;\rho(x,T) \;&=\; \pi/4 \end{aligned} \tag{21}$$

for all $\alpha$. Although $T$ is fixed in Fig. 1, its ratio to $\mu(0) = \alpha\omega N$ is greater for smaller $\alpha$. This explains why the curve for $\alpha = 1/2$ is spread out to the largest extent. We have also exhibited the $T = 0$ (or $\alpha \to \infty$) semicircle for comparison.

We now apply the TF idea to a CS system placed in a different potential which has not been studied before. An interesting and exactly solvable TF case is the potential

$$V(x) \;=\; \frac{1}{2}\left(9m\eta^2x^2\right)^{1/3}. \tag{22}$$

The TF density of states is given by $g(\epsilon) = \epsilon/\eta$ . Hence $E = -2\Omega$. At $T = 0$, we find that

$$\mu(0) = \Big(2\alpha\eta N\Big)^{1/2}$$
$$\text{and} \quad E(0) = \frac{2}{3}\, N\mu(0). \tag{23}$$

The thermodynamic limit is again defined by taking $N \to \infty$ keeping $\eta N$ fixed. To study finite temperature, we define an auxiliary variable

$$\nu = \int_0^\infty d\epsilon\ n(\epsilon). \tag{24}$$

Eq. (6) implies that

$$\eta \left(\frac{\partial N}{\partial \mu}\right)_T = \nu$$
$$\text{and} \quad \left(\frac{\partial \nu}{\partial \mu}\right)_T = n(0). \tag{25}$$

As in Eqs. (15, 17), we deduce that

$$\mu(\nu, T) = \alpha\nu + T\ \log[1 - \exp(-\nu/T)]$$
$$\text{and} \quad \eta N(\nu, T) = \frac{1}{2}\,\alpha\nu^2 + T^2 \int_o^{\nu/T} dy\ \frac{y}{e^y - 1}. \tag{26}$$

Using (7), we can compute the energy

$$E(\nu, T) = 2 \int_0^\nu dy\ N(y, T)\ \frac{\partial}{\partial y}\ \mu(y, T). \tag{27}$$

Thus $\mu$, $N$ and $E$ are all known in terms of $\nu$ and $T$.

Since a particle moving in a two-dimensional harmonic oscillator potential or moving freely in four dimensions also has a linear density of states, we may ask whether our system exhibits BE condensation at low temperature for a *finite* range of $\alpha$. If $\alpha = 0$, we see that (26) has no solution once $T$ falls below

$$T_c = \sqrt{6\eta N}/\pi. \tag{28}$$

For $T \leq T_c$ , $\nu$ and $\mu$ stay at $\infty$ and 0 respectively. Therefore the second equation in (26) can only be satisfied if there is a macroscopic occupation $N(0)$ of the $\epsilon = 0$ state, that is, if $\eta N = \eta N(0) + \pi^2 T^2/6$ .

However, for any $\alpha > 0$, Eq. (26) has a solution right down to $T = 0$ and there is no condensation at finite temperature. In Fig. 2, we present $\mu$ vs $T$ with $\eta N = 1$ and $\alpha = 0,\ 0.02,\ 0.1$ and $0.25$. It will be seen that $\mu(T)$ is nonanalytic only for $\alpha = 0$ at $T = \sqrt{6}/\pi = 0.7797...$ .

To summarize, we have developed a TF method to study the thermodynamics of particles obeying the generalized statistics of Eq. (1) in any given external potential . We can numerically compute the one-particle density for any $\alpha$ and $T$. If $g(\epsilon)$ varies as a (nonnegative) integer power of $\epsilon$, we can solve the thermodynamics exactly. For a linear density of states, there is no BE condensation if $\alpha > 0$. This also implies that a $1/x^2$ interaction between bosons, no matter how weak, destroys BE condensation.

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**Figure Captions**

1. The one-particle density $\rho(x, T)$ vs $x$ in a harmonic oscillator potential with $T = \omega N/2$ and $\alpha = 1/2,\ 1$ and $2$. The $T = 0$ (or $\alpha \to \infty$ ) semicircle is shown as a solid line.
2. The chemical potential $\mu$ vs $T$ for a linear density of states with $\eta N = 1$ and $\alpha = 0,\ 0.02,\ 0.1$ and $0.25$. The solid line ($\alpha = 0$) is nonanalytic at $T = \sqrt{6}/\pi$.