

Correlators and fractional statistics in the quantum Hall bulk

Smitha Vishveshwara¹, Michael Stone¹, and Diptiman Sen²

¹ *Department of Physics, University of Illinois at Urbana-Champaign, 1110 W. Green St, Urbana, IL 61801, USA*

² *Centre for High Energy Physics, Indian Institute of Science, Bangalore 560012, India*

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We derive single-particle and two-particle correlators of anyons in the presence of a magnetic field in the lowest Landau level. We show that the two-particle correlator exhibits signatures of fractional statistics which can distinguish anyons from their fermionic and bosonic counterparts. These signatures include the zeroes of the two-particle correlator and its exclusion behavior. We find that the single-particle correlator in finite geometries carries valuable information relevant to experiments in which quasiparticles on the edge of a quantum Hall system tunnel through its bulk.

Given the increasing interest in topological quantum computation and the rapid experimental progress in quantum Hall physics, the study of “anyons”—quasiparticles that obey fractional statistics interpolating between the statistics of fermions and bosons—has gained attention not only as a fascinating academic topic but also as one of applied value. In certain systems[1, 2] the detection of anyons would establish the existence of topological order [3]. The quantum Hall (QH) effect, where electrons are trapped in two dimensions in the presence of a magnetic field, provides the paradigm for anyon-hosting many-body systems [4]. A variety of theoretical proposals [5, 6, 7, 8] and experimental attempts [9] have pursued the detection of fractional statistics quasiparticles. While much of the understanding of anyons in these geometries has stemmed from the edge-state description [10] of QH quasiparticles, an involved investigation of anyon bulk correlations [5] has received less attention, as has the problem of mapping the bulk correlations to the QH edge. In that anyons are intrinsically two-dimensional, a study of their bulk properties is much called for. In this Letter we formulate and analyze the anyon correlators in the presence of a magnetic field in the lowest Landau level (LLL). We show that these correlators contain valuable information on statistics. In the presence of boundaries we demonstrate that these bulk correlations become manifest in edge-state properties.

The objects of our attention are the single-particle kernel $K_1(\vec{r}_f, \vec{r}_i)$, which is the amplitude for a quasiparticle to propagate from an initial point \vec{r}_i to a final point \vec{r}_f , and the two-particle kernel $K_2(\vec{r}_{1f}, \vec{r}_{2f}, \vec{r}_{1i}, \vec{r}_{2i})$, which is the amplitude for two quasiparticles starting at points \vec{r}_{1i} and \vec{r}_{2i} to end at points \vec{r}_{1f} and \vec{r}_{2f} (see Fig. 1). As elucidated in what follows, the single-particle kernel K_1 lies at the heart of observable single-particle quantities such as two-point quasiparticle correlations along a QH edge and inter-edge tunneling matrix elements. As for K_2 , historically, two-particle kernels have played an ubiquitous role in a wide range of settings from particle scattering in nuclear physics to quantum optics and astrophysics phenomena whose study was instigated by the studies of Hanbury Brown and Twiss [11, 12]. It is well known that the amplitude for two incoming fermions in vacuum

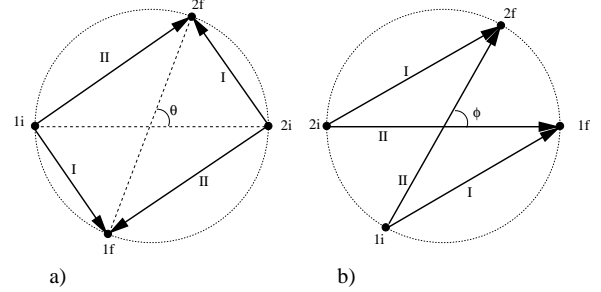


FIG. 1: Two representative configurations for anyons starting at points \vec{r}_{1i} and \vec{r}_{2i} to end at points \vec{r}_{1f} and \vec{r}_{2f} . As the particles are indistinguishable, it is not possible to determine which of two possible paths I and II each particle took.

to scatter at an angle of $\pi/2$ is zero in the absence of a magnetic field [11, 13] and that fermions tend to “anti-bunch” while bosons tend to “bunch.” Our analysis of the kernel K_2 will show that the situation is dramatically altered by the presence of a magnetic field and that statistical effects in anyons are distinctly different from those in fermions and bosons.

Our starting point is a two-dimensional system of two anyons in a perpendicular magnetic field, whose common wavefunction by definition picks up a phase of $e^{i\pi\nu}$ ($e^{-i\pi\nu}$) upon anticlockwise (clockwise) exchange of particles. The real parameter ν lies in the range $-1 < \nu \leq 1$. The limiting cases of $\nu = 0$ and $\nu = 1$ correspond to bosons and fermions, respectively. Such LLL anyon models provide an effective description of quasihole excitations associated with the addition of vortices to the QH bulk [14, 15, 16]. In particular, for Laughlin states [15] quasiholes have fractional charge $q = e/m$ and anyon phase $\nu = 1/m$, where m is an odd integer [3, 17, 18]. While it is known that interactions can alter anyon correlations in vacuum in the absence of a magnetic field [19], as a simple and realistic case, we treat the anyons as non-interacting. Coulomb interactions do exist between QH quasiholes but they are expected to be screened by the background charge and can be treated perturbatively [5]. Thus, our treatment of the two-anyon model ought to be applicable to QH excitations on lengthscales much larger than the magnetic length.

The Hamiltonian for two anyons in a perpendicular magnetic field $\vec{B} = B\hat{z}$, in terms of center of mass and relative variables, has the decoupled form

$$H = \frac{1}{4\mu} \left(P_x + \frac{qB}{c} Y \right)^2 + \frac{1}{4\mu} \left(P_y - \frac{qB}{c} X \right)^2 + \frac{1}{\mu} \left(p_x + \frac{qB}{4c} y \right)^2 + \frac{1}{\mu} \left(p_y - \frac{qB}{4c} x \right)^2. \quad (1)$$

We make the particles into anyons by requiring that when the two particles are exchanged in a clockwise fashion their wavefunction gains a phase factor $e^{i\pi\nu}$. Note that in an alternate formalism, the phase factor can be gauged out of the wavefunction by treating the anyons as bosons carrying flux lines which can be incorporated into the Hamiltonian [16]. Here the anyons are assumed to each have mass μ (which is immaterial when states are projected to the LLL) and charge q . The symmetric gauge is assumed for the vector potential $\vec{A} = B(-y\hat{x} + x\hat{y})/2$. The center of mass co-ordinate and momentum are given by $\vec{R} = (\vec{r}_1 + \vec{r}_2)/2$ and $\vec{P} = \vec{p}_1 + \vec{p}_2$ while the relative co-ordinate and momentum are given by $\vec{r} = \vec{r}_1 - \vec{r}_2$ and $\vec{p} = (\vec{p}_1 - \vec{p}_2)/2$, respectively. The Hamiltonian can also be employed to study the properties of a single particle by restricting it to the (\vec{R}, \vec{P}) sector where the variables now describe the co-ordinate and momentum of the single particle.

The eigenstates of Eq. (1) are products of eigenstates for the center of mass and relative coordinate systems. In the LLL, the center of mass eigenstates are given by

$$\psi_n(\vec{R}) = \frac{1}{l\sqrt{m\pi n!}} \left(\frac{Z}{\sqrt{m}} \right)^n \exp \left[-\frac{|Z|^2}{2m} \right], \quad (2)$$

where $n = 0, 1, 2, \dots$. The complex parameter $Z = (X + iY)/l$ represent the components of \vec{R} , rescaled by the magnetic length $l = \sqrt{\hbar c/eB}$. The relative-coordinate eigenstates are given by

$$\psi_p(\vec{r}) = \frac{(4\pi m)^{-1/2}}{\sqrt{\Gamma(2p + \nu + 1)l}} \left(\frac{z}{2\sqrt{m}} \right)^{2p+\nu} \exp \left[-\frac{|z|^2}{8m} \right], \quad (3)$$

where $p = 0, 1, 2, \dots$, and $z = (x + iy)/l$ represents rescaled components of \vec{r} . These relative-coordinate eigenstates respect the anyon property.

We are now equipped to evaluate the single- and two-particle kernels, defined in imaginary time τ , by

$$K_1(\vec{R}_f; \vec{R}_i) = \sum_n \psi_n(\vec{R}_f) \psi_n^*(\vec{R}_i) e^{-E_n \tau / \hbar}, \\ K_2(\vec{r}_{1f}, \vec{r}_{2f}; \vec{r}_{1i}, \vec{r}_{2i}) = \sum_n \psi_n(\vec{R}_f) \psi_n^*(\vec{R}_i) e^{-E_n \tau / \hbar} \\ \times \sum_p \psi_p(\vec{r}_f) \psi_p^*(\vec{r}_i) e^{-E_p \tau / \hbar}. \quad (4)$$

In the LLL, all energies E_n are degenerate, and so we set them to zero. Thus the kernels have no explicit time dependence. In terms of Eqs. (2,3), the single-particle

kernel takes the explicit form

$$K_1(z_f; z_i) = \frac{1}{2\pi} \exp \left[-\frac{1}{4} (|z_f|^2 + |z_i|^2) + \frac{1}{2} z_f z_i^* \right], \quad (5)$$

and the two-particle kernel takes the form

$$K_2(\vec{r}_{1f}, \vec{r}_{2f}; \vec{r}_{1i}, \vec{r}_{2i}; \tau) \\ = \frac{1}{(2\pi m)^2} \exp \left[-\frac{1}{4m} (|z_{1f}|^2 + |z_{2f}|^2 + |z_{1i}|^2 + |z_{2i}|^2) \right. \\ \left. + \frac{1}{4m} (z_{1f} + z_{2f})(z_{1i}^* + z_{2i}^*) \right] \\ \times \sum_{p=0}^{\infty} \frac{[(z_{1f} - z_{2f})(z_{1i}^* - z_{2i}^*)/4m]^{2p+1/m}}{\Gamma(2p + 1/m + 1)}. \quad (6)$$

It must be remarked that, through different reasoning, a similar form for K_2 was presented by R. Laughlin in Ref. [15].

In the limiting case of fermions/bosons, it can be shown that the two-particle kernel can be separated into products of individual paths, i.e., $K_2(\vec{r}_{1f}, \vec{r}_{2f}; \vec{r}_{1i}, \vec{r}_{2i}) = K_1(\vec{r}_{1f}; \vec{r}_{1i}) K_1(\vec{r}_{2f}; \vec{r}_{2i}) \mp K_1(\vec{r}_{2f}; \vec{r}_{1i}) K_1(\vec{r}_{1f}; \vec{r}_{2i})$. From this property, several consequences follow. In particular, in the case of Fig. 1a, K_2 can only vanish if the magnitudes of the kernels along paths of type *I* and *II* equal each other. This condition implies that $(z_{1f} - z_{2f})(z_{1i}^* - z_{2i}^*)$ is imaginary, or, in other words, that $\vec{r}_{1f} - \vec{r}_{2f}$ is perpendicular to $\vec{r}_{1i} - \vec{r}_{2i}$ ($\theta = \pi/2$). Furthermore, for the two separable parts to cancel one another, their phase difference is required to be $0/\pi$ for fermions/bosons. This second condition translates to the requirement that the quantity $eB\hat{z} \cdot (\vec{r}_{1i} - \vec{r}_{2i}) \times (\vec{r}_{1f} - \vec{r}_{2f}) / (\hbar c)$ be an even/odd integer for fermions/bosons. A geometric interpretation of these arguments is that K_2 vanishes when the phase difference between paths of type *I* and *II* is π and that the phase is given by the sum of the Aharonov-Bohm phase picked up by the loop in Fig. 1. a and the phase $\pi/0$ due to anti-clockwise exchange of the two fermions/bosons. Upon setting $\vec{B} = 0$, one retrieves the result that in the absence of a magnetic field, the two-particle kernel vanishes at an angle $\theta = \pi/2$ for fermions.

For the case of anyons, neither the two-particle wavefunction nor the two-particle kernel is of a separable form. However, a detailed analysis [20] of the two-particle kernel shows surprisingly that the geometric arguments presented above still hold. Thus, in the configuration of Fig. 1. a, the two-particle kernel vanishes for the same conditions stated for fermions and bosons except that the statistical phase picked up by the anyons for a closed loop along the $\vec{r}_{1i} \rightarrow \vec{r}_{1f}$ direction is $\pm\pi/m$ for a clockwise/anticlockwise loop. Hence, as shown in Fig. 2a, the kernel vanishes along the direction $\theta = \pi/2$ for a discrete set of radii satisfying the constraint $r^2/m = (n - 1/2 + 1/(2m))\pi$.

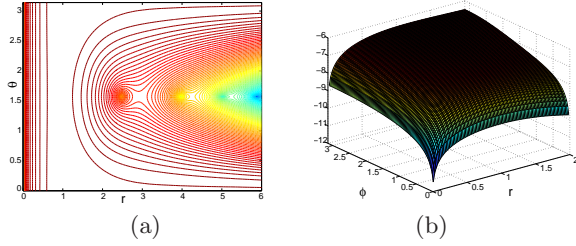


FIG. 2: Plots of the two-particle kernel corresponding to the configurations in Fig. 1 in which the particles lie on a circle. The configurations correspond to (a) $z_{1i} = -z_{2i} = r, z_{1f} = -z_{2f} = re^{i\theta}$, which shows zeros of the kernel at specific radii and angle $\theta = \pi/2$ and (b) $z_{1i} = -z_{1f} = r, z_{2i} = -z_{2f} = re^{i\phi}$, which shows power-law dependences of the kernel as a function of r and ϕ for small arguments.

The two-particle kernel clearly exhibits features that reflect the exclusion statistics of anyons [21]. As a specific instance, for the case shown in Fig. 1b, we find that as $\phi \rightarrow 0$, the kernel exhibits the power-law dependence $K_2 \sim |\phi|^{2/m}$. Physically, the amplitude for two-incoming anyons to start at nearby points and to have a small scattering angle vanishes as the angle becomes smaller. As another instance, the probability that two anyons are a distance ' r ' apart is related to the two-particle kernel whose arguments are $\vec{r}_{1i} = \vec{r}_{1f} = 0$ and $\vec{r}_{2i} = \vec{r}_{2f} = \vec{r}$. For this case, in the limit of small separation, the kernel has the limiting form $K_2 \rightarrow r^{2/m}$ as $r \rightarrow 0$. For the limit $m = 1$, we reproduce the result that the probability that a fermion is at a given distance ' r ' away from another fermion is proportional to r^2 . On the other hand, in the limit $m \rightarrow \infty$, for small enough separation, one particle does not experience the existence of another, which is indeed the situation for condensed bosons. For any intermediate value of m , the power-law behavior shows that the presence of one particle excludes that of another (thus rendering Laughlin quasiparticles to be fermion-like), and that this anti-bunching property becomes more pronounced for smaller values of m .

Having analyzed bulk features of the kernels, we turn to finite size geometries that are of relevance to the physical setting of the Hall bar. By studying the properties of the single-particle kernel in a geometry such as the one shown in Fig. 3, we provide a simple picture for deriving correlations along the edge and justifying assumptions made for single-particle tunneling events in previous treatments. The system is assumed to be confined along the y direction via a potential $V(y)$. The Landau gauge $\vec{A} = -By\hat{x}$ proves to be convenient for such a case. The corresponding single-particle eigenstates are of the form $\psi_{k,n}(x, y) = e^{ikx} f_{k,n}(y)$, where the function $f_{k,n}$ depends on the confining potential and the momentum $k = 2\pi p/L_x$ along the x direction, where p is an integer and L_x is the length of the strip.

We first consider the simple illustrative example of no external potential $V(y)$ except for hard boundaries confining the strip to a width L_y centered at $y = 0$. In the

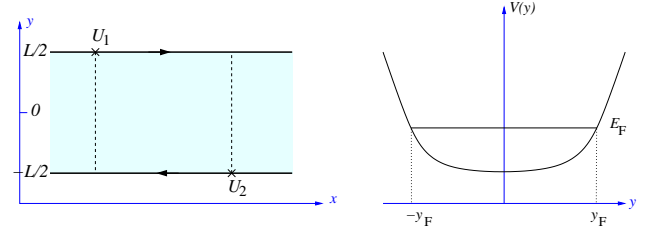


FIG. 3: Quantum Hall state in a strip geometry of length L_x and width L_y in the presence of a confining potential $V(y)$. States are filled to a Fermi energy E_F . Tunneling across the strip can take place via impurities denoted by U_i .

LLL ($n = 0$), the eigenfunctions $f_{0,k}$ are Gaussian packets proportional to $\exp[-\frac{eB}{2\hbar c}(y + \frac{\hbar ck}{eB})^2]$, where k ranges from $-eBL_y/(2\hbar c)$ to $eBL_y/(2\hbar c)$. The single-particle kernel can be evaluated using the relationship Eq. (4). We describe the salient features of the kernel (details are presented in Ref. [20]). For large width $L_y \gg l$, the kernel reproduces the behavior $K_1 \sim \exp[-(z_i - z_f)^2/4]$ when the points are well within the bulk or on opposite edges $y_i = -y_f = L_y/2$. In the large width limit, if the two points however lie along the same edge, i.e., $y_i = y_f = L_y/2$, in the limit $(x_i - x_f) \gg l$, the kernel obeys the power-law dependence $K_1 \sim 1/|x_i - x_f|$. This behavior is consistent with the power-law decay obtained from the edge-state picture for integer filling $\nu = 1$. In the interesting case of the width becoming comparable to the magnetic length, the kernel shows oscillations with a wavelength of about $4\pi l^2/L_y$ which become pronounced when the two points lie on opposite edges. This oscillatory behavior suggests that finite-size effects in realistic situations could cause significant deviations from predictions for extended systems.

As an application of the single-particle correlator relevant to several experiments and proposals, we now consider tunneling between integer quantum Hall edges caused by translational symmetry breaking along the x direction. As a simple model, we introduce localized impurities of the form

$$U(x, y) = \sum_n U_n \delta(x - x_n) \delta(y - y_n), \quad (7)$$

which act as scatterers. In the absence of a confining potential, all k states are degenerate and the scatterers cause mixing between all states. In reality, as shown in Fig. 3. b, the confining potential breaks the degeneracy, and electrons fill states up to a Fermi momentum k_F and an associated width $L_y = 2y_F$. The confining potential produces an effective electric field along the edge, $\mathcal{E} = -(dV/dy)_{y=y_F}$. Electrons experience a drift velocity given by $v_F = c|\mathcal{E}/B|$ and they move in opposite directions along the top and bottom edges. Treating the scatterers within the first-order Born approximation and assuming a linearized potential close to each edge (and

thus a linearized dispersion about the Fermi energy), we find that the scatterers couple each k state to corresponding $\pm k$ states [20]. The associated reflection co-efficient for a right-moving edge state $k \approx k_F$ to scatter to a left-mover $k \approx -k_F$ is given by

$$r = -\frac{i}{\hbar v_F} \sum_n U_n e^{i2k_F x_n} \exp\left[-\frac{eB}{\hbar c} y_n^2\right] \times \left(\frac{eB}{\pi \hbar c}\right)^{1/2} \exp\left[-\frac{eB}{\hbar c} y_F^2\right]. \quad (8)$$

The reflection co-efficient is directly related to the matrix element for particles to tunnel between edge states. Implicitly, it involves the single-particle propagation amplitude to traverse from one edge to another. Our method is simple enough that it can go beyond the strip geometry to any smooth confining potential and configuration of tunneling sites.

The form of Eq. (8) has several noteworthy features. As expected, the tunneling matrix element for each impurity decays exponentially over a magnetic length. For an impurity localized on an edge at a point x_i , tunneling to the other edge occurs along the shortest path. The treatment here was for fermions of charge e . In principle, we expect an identical form for any particle having charge e^* with this charge replacing e , in which case the decay of the *bare* tunneling matrix element is enhanced/suppressed by a factor e^*/e in the exponent. This reasoning is consistent with derivations of tunneling matrix elements that explicitly use the Laughlin wavefunction [22]. For the situation of more than one impurity, the reflection coefficient is sensitive to interference effects coming from multiple paths. In the case of two impurities of equal strength lying on either edge at points x_1 and x_2 , reflection processes off the two impurities have a phase difference of $2k_F(x_2 - x_1) = 2eBy_F(x_2 - x_1)/(\hbar c)$. Thus, as phenomenologically described in Ref. [8], we explicitly see Aharonov-Bohm interference coming from the particle traversing two different paths enclosing a rectangular area of length $x_1 - x_2$ and width $2y_F$.

In conclusion, we have derived and analyzed the form of two ubiquitous entities - the single-particle and two-particle anyon kernels - in the physically motivated situation of charged particles in a magnetic field in the LLL. We have shown that the two-particle kernel in the quantum Hall bulk contains information on statistics which is strikingly manifest in the zeros of the kernel. We have shown that the single-particle kernel in a finite geometry provides a faithful means of understanding features of bulk mediated tunneling between edge state quasiparticles, such as the tunneling amplitude and Aharonov-Bohm physics in a system with two tunneling centers. In principle, some of our predictions for the two-particle kernel ought to translate to realistic gate-defined Hall geometries. At the very least, our studies show that a complete explanation of experiments that measure two-

particle properties, whether of bulk or edge-state quasiparticles will need to take into account correlations and exclusion effects in the bulk. More spectacularly, our studies indicate that in the future, it may be possible to perform experiments in quantum Hall geometries, perhaps involving multi-edge tunneling, wherein correlations show signatures of fractional statistics in angular dependences such as those observed for fermions and bosons in scattering experiments.

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