Power dissipation for systems with junctions of multiple quantum wires

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We study power dissipation for systems of multiple quantum wires meeting at a junction, in terms of a current splitting matrix ($\mathbb{M}$) describing the junction. We present a unified framework for studying dissipation for wires with either interacting electrons (i.e., Tomonaga-Luttinger liquid wires with Fermi liquid leads) or non-interacting electrons. We show that for a given matrix $\mathbb{M}$, the eigenvalues of $\mathbb{M}^T\mathbb{M}$ characterize the dissipation, and the eigenvectors identify the combinations of bias voltages which need to be applied to the different wires in order to maximize the dissipation associated with the junction. We use our analysis to propose and study some microscopic models of a dissipative junction which employ the edge states of a quantum Hall liquid. These models realize some specific forms of the $\mathbb{M}$-matrix whose entries depends on the tunneling amplitudes between the different edges.

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I. INTRODUCTION

One-dimensional systems of strongly correlated electrons have been studied extensively for several years both experimentally, in the form of quantum wires and carbon nanotubes, and theoretically [1–6]. Junctions of several quantum wires have also been studied in recent years since they can now be experimentally realized in carbon nanotubes [7–11]. The existing studies of junctions of quantum wires, which are usually modeled as Tomonaga-Luttinger liquids (TLL), have mainly looked at their low-temperature fixed points and the corresponding conductance matrices [12–24]. Many of these studies have focused on situations in which there is no power dissipation in the system. The aim of our work will be to include dissipation in the discussion. For simplicity, we will consider only spinless electrons and will restrict ourselves to the zero frequency limit (DC) in our work.

A motivation for studying dissipation is as follows. In Ref. 16, a two-parameter description of a junction of three TLLs has been discussed. The dissipationless fixed points were shown to lie on the circumference of a circle, while the interior of the circle corresponds to dissipative junction. In this context, the center of the circle which corresponds to the current splitting matrix with all its elements equal to 1/3 is of particular interest as it corresponds to the maximum possible dissipation allowed by constraint of current conservation. It therefore seems useful to understand dissipative junctions in a general way and to study whether any of the points inside the circle correspond to fixed points of some renormalization group (RG) equations.

The plan of this paper is as follows. In Sec. II, we introduce the idea of a current splitting matrix $\mathbb{M}$ at a junction of N wires and discuss the cases of both non-interacting and interacting electrons. In Sec. III, we obtain an expression for the power dissipated in terms of this matrix. The measure of the degree of dissipation is then defined in terms of the eigenvalues of $\mathbb{M}^T\mathbb{M}$. For two-wire and three-wire junctions, we write down the most general form of the $\mathbb{M}$-matrix allowed by current conservation, thus providing a complete parametrization of the dissipation at the junction. In general, an $\mathbb{M}$-matrix which respects current conservation can have both positive and negative matrix elements. As we will show, an $\mathbb{S}$-matrix describing non-interacting electrons scattering at the junction can be related to a matrix $\mathbb{M}$ all of whose elements are positive; this relation will follow from the assumption that there are no phase correlations between electrons coming from different reservoirs which lie far away from the junction. But when $\mathbb{M}$ has negative elements, such a relation does not exist and the matrix then necessarily corresponds to a system of interacting electrons.

In Sec. IV, we introduce a simple model involving three patches of the edge states of a quantum Hall liquid with filling fraction ν. The patches are taken to be mutually coupled to each other by local electron tunnelings between three distinct points lying on the three patches with amplitudes σ_{ij}, where i, j denote the patch index. Then a parametrization of the $\mathbb{M}$-matrix is obtained in terms of the conductance amplitudes σ_{ij}. In this way we obtain interesting dissipationless matrices in the limits $\sigma_{ij} \to 0$ and ∞ respectively. These matrices were shown to represent dual fixed points in the theory of a junction of TLL wires in Ref. 16 using more involved calculations. In Sec. V, we introduce a more complex model of a junction of three quantum wires in which the junction consists of a ring-shaped region with edges of its own. Once again, the matrix $\mathbb{M}$ of the entire system can be found in terms of the coupling of each wire to the ring and the tunneling amplitudes across the two edges of the ring. Even though such a geometry is complicated, from an experimental point of view it allows for easier tunability as far as realizing various types of $\mathbb{M}$ matrices is concerned. In Sec. VI, we make some concluding remarks.

II. THE CURRENT SPLITTING MATRIX

A junction is a meeting point of N wires each of which has an incoming and an outgoing mode. Physically, if the junction is made of a material like a carbon nanotube, then the incoming and outgoing modes (which carry currents) belonging to a single wire are not separated in space. But if these are quantum wires made out of the edge states of a quantum Hall liquid [25], then the incoming and outgoing modes are spatially separated. In the following discussion, we will consider a junction of several quantum wires, each with two spatially separated chiral current carrying edges, one incoming and one outgoing. Each chiral mode (incoming or outgoing) is labeled by an index i which runs from 1 to N and is parametrized by a coordinate x. We will take x to run from 0 to ∞; the point $x = 0$ will be common to all the wires and will denote the junction. The outgoing currents in the system are related to the incoming currents by a current splitting matrix $\mathbb{M}$ given by

$$J_{Oi} \ = \ \sum_j \mathbb{M}_{ij} \ J_{Ij}. \tag{1}$$

Current conservation at the junction therefore implies that each column of $\mathbb{M}$ must add up to 1. Let us also assume that the incoming current on wire i is proportional to the applied bias voltage V_{Ii}, with the constant of proportionality being the same for all wires. Then if all the wires have the same bias voltage, the net outgoing current $J_{Oi} - J_{Ii}$ on each wire i must vanish which implies that each row of $\mathbb{M}$ must also add up to 1.

For non-interacting electrons, the junction can be described in terms of a scattering matrix $\mathbb{S}$ which provides a linear relation between the incoming and outgoing electron fields at the junction. Namely, the incoming and outgoing electron fields, $\psi_{Ii}(x,t)$ and $\psi_{Oi}(x,t)$, are related at all times t as

$$\psi_{Oi}(0,t) \ = \ \sum_j \mathbb{S}_{ij} \ \psi_{Ij}(0,t). \tag{2}$$

Current conservation implies that $\mathbb{S}$ must be an $N \times N$ unitary matrix. Any deviation from the linear boundary condition for the electron fields at the junction will imply the existence of local inter-electron interactions at the junction even if the electrons in the bulk of the wires are left non-interacting. The scattering matrix description can also be used to describe electrons which are weakly interacting in the bulk of the wire, by treating the effects due to interactions perturbatively [14, 15, 17].

Given a scattering matrix $\mathbb{S}$ for non-interacting electrons, we will now see how the elements of the current splitting matrix $\mathbb{M}$ can be found. The incoming and outgoing currents J_{Ii} and J_{Oi} in wire i are proportional to $|\psi_{Ii}|^2$ and $|\psi_{Oi}|^2$ respectively. Eq. (2) implies that $|\psi_{Oi}|^2 = \sum_{jk} \mathbb{S}^*_{ij} \mathbb{S}_{ik} \psi^*_{Ij} \psi_{Ik}$. We now assume that there are no phase correlations between the incoming electrons on different wires j and k since they are coming from different reservoirs whose distances from the junction are taken to be much larger than the phase coherence length; the absence of such phase correlations is crucial for the validity of the Landauer-Büttiker theory of electronic transport in mesoscopic systems [26]. Hence terms like $\psi^*_{Ij} \psi_{Ik}$ can be set equal to zero if $j \neq k$. We thus obtain $|\psi_{Oi}|^2 = \sum_j |\mathbb{S}_{ij}|^2 |\psi_{Ij}|^2$. This is of the same form as in Eq. (1) if we identify $M_{ij} = |S_{ij}|^2$.

On the other hand, if we have strongly interacting electrons in one dimension, then it is natural to use bosonization. The electrons in the wire are then expressed in terms of free bosonic excitations described by TLL theory, and the fixed point theory of the junction can be described in terms of a current splitting matrix $\mathbb{M}$ which is obtained by imposing a linear boundary condition on the incoming and outgoing bosonic fields at the junction [12, 13, 16, 18–24]. One can use free bosonic fields to describe either non-interacting or interacting electrons in the bulk of the one-dimensional wires depending on whether the Luttinger parameter g is equal to or not equal to 1. Note that even when we have $g = 1$ (non-interacting electrons) in the bulk of the wire, within the bosonization approach the current splitting matrix $\mathbb{M}$ representing a linear relation between the incoming and outgoing boson fields at the junction corresponds to the presence of non-zero inter-electron interaction at the junction. This is because the boson fields are related to the corresponding electron fields by a non-linear bosonization identity, $\psi_{I/O}(x) = (1/\sqrt{2\pi\alpha}) \ F_{I/O} e^{i\phi_{I/O}(x)}$ [1–6], where $\psi_{I/O}(x)$ are the incoming and outgoing electron fields, $\phi_{I/O}$ are the incoming and outgoing chiral bosonic fields, and $F_{I/O}$ are the corresponding Klein factors. Thus there is a subtle difference between using a scattering matrix $\mathbb{S}$ for non-interacting electrons and a current splitting matrix $\mathbb{M}$ in bosonized TLL theory for electrons which are non-interacting ($g = 1$) in the bulk of the wire. In the latter case the $\mathbb{M}$-matrix description of the junction corresponds to an interacting theory of electrons where the interaction is localized at the junction. Now, if the incoming and outgoing boson fields are linearly related to each other at the junction, i.e., if $\phi_{Oi}(x = 0,t) = \sum_j \mathbb{M}_{ij} \phi_{Ij}(x = 0,t)$, then $\mathbb{M}$ must be a real and orthogonal field splitting matrix in order that both the incoming and outgoing bosonic fields satisfy the canonical commutation relations [18, 27, 28]. Such a description of the junction given by an orthogonal $\mathbb{M}$ represents fixed points of the junction as was shown in Ref. 18. The current at any point of wire i is given by $-(1/2\pi)\partial\phi/\partial t$. Hence we note that the above condition at the junction implies that the outgoing and incoming currents also satisfy Eq. (1), i.e., the field splitting $\mathbb{M}$ matrix can be taken to be the same as the current splitting matrix. Hence in the bosonic formalism, $\mathbb{M}$ must be an $N \times N$ real and orthogonal matrix each of whose rows and columns add up to 1 [22].

As we will see later, the orthogonality condition on the $\mathbb{M}$ matrix also renders it dissipationless irrespective of its origin, i.e., this is true for both a junction of non-interacting electrons described by an $\mathbb{S}$ matrix or a junction of TLL wires described by a field splitting matrix $\mathbb{M}$. Hence, to include dissipation in the analysis we have to relax the condition of orthogonality on $\mathbb{M}$. This also implies that the formalism of bosonization cannot be used directly since a non-orthogonal $\mathbb{M}$ does not allow the bosonic commutation relations to be satisfied. However, $\mathbb{M}$ must continue to be real since it relates incoming and outgoing currents which are all real, and each of its rows and columns must add up to 1 as argued before. Hence the main emphasis of this section is on the fact that various situations comprising of either non-interacting electrons or interacting electrons, where the interaction is either localized at the junction or extended all over the wire, can be described just in terms of a current splitting matrix (Eq. (1)). We will see in the following section that this information is enough to characterize the dissipation associated with the junction.

III. DISSIPATION

We will now consider the specific case involving edge states in a quantum Hall liquid for discussing dissipation in a junction. In such systems, currents flow only along the edges as all states in the bulk are localized; such edge modes are chiral in nature and can be described by theories of chiral bosons [29]. In the linear response regime, if a voltage V is applied to an Ohmic contact which is assumed to be perfectly coupled to the edge, then for filling fraction ν, the current J injected into the edge from that contact is given by $J = GV$, where $G = \nu e^2/h$ is the conductance.

Now let us derive an expression for the power dissipated by such a system which is governed by a current splitting matrix $\mathbb{M}$. The power P_d dissipated near the junction is given by the difference of the total incoming and outgoing powers [30, 31],

$$\begin{aligned} P_d &= \frac{1}{2} \sum_i \left(J_{Ii} V_{Ii} - J_{Oi} V_{Oi} \right) \\ &= \frac{1}{2G} J_I^T \left(I - \mathbb{M}^T \mathbb{M} \right) J_I, \end{aligned} \tag{3}$$

where we have introduced a matrix notation in the second line of Eq. (3), with J_I being a column made up of the incoming currents J_{Ii}, and I being the $N \times N$ identity matrix. On physical grounds, the power dissipated near the junction cannot be negative. This implies that the eigenvalues λ_i of $\mathbb{M}^T\mathbb{M}$ must necessarily lie in the range $[0, 1]$. If the incoming current J_I is proportional to an eigenvector of $\mathbb{M}^T\mathbb{M}$ with eigenvalue λ_i, the power dissipated will be proportional to $1 - \lambda_i$. The set of values of $1 - \lambda_i$ therefore provides a measure of the amount of dissipation associated with a system characterized by the current splitting matrix $\mathbb{M}$.

We emphasize here that Eq. (3) describes the power dissipated in the region close to the junction, and not in the leads which are assumed to be far away from the junction. For instance, for a two-wire junction with perfect transmission of the currents, i.e., for a matrix $\mathbb{M}$ given by $M_{11} = M_{22} = 0$ and $M_{12} = M_{21} = 1$, the expression in Eq. (3) vanishes; however, we know that dissipation occurs in the leads because the outgoing electrons eventually equilibrate to the chemical potential there, leading to a contact resistance of e^2/h (for spinless electrons). Thus, $M^T M = I$ only means that there is no dissipation associated with the junction, although dissipation can still occur in the leads.

Since each row and column of $\mathbb{M}$ adds up to unity, both $\mathbb{M}$ and $\mathbb{M}^T$ must have one eigenvalue equal to 1, the corresponding eigenvector being given by a column all of whose entries are equal to each other. This column is therefore an eigenvector of $\mathbb{M}^T\mathbb{M}$ with eigenvalue equal to 1 which corresponds to a situation where the bias voltages V_{Ii} (or incoming currents J_{Ii}) on all the wires are equal, and no power is dissipated. Also note that the power dissipated vanishes for *all* possible values of the incoming currents if $\mathbb{M}$ is orthogonal. On the other hand, the dissipated power is maximized if all the eigenvalues of $\mathbb{M}^T\mathbb{M}$ are equal to 0 except for one eigenvalue which is necessarily equal to 1. This occurs when all the entries of $\mathbb{M}$ are equal to $1/N$. Hence if we think of a situation where the dissipation happens at the junction and not in the leads, the entire incoming power will be converted to heat at the junction and the outgoing power will vanish.

In general, a current splitting matrix $\mathbb{M}$ corresponding to an N-wire junction has $(N-1)^2$ independent parameters. This is because the first $(N-1) \times (N-1)$ block of $\mathbb{M}$ can have arbitrary entries while the entries of the last row and column of $\mathbb{M}$ are then fixed by the conditions that each row and column must add up to 1. For $N = 2$, we need only one parameter and the matrix is given by

$$\mathbb{M} = \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}, \tag{4}$$

where a must lie in the range $[0, 1]$ to ensure that the dissipated power is always non-negative. No power will be dissipated if $a = 0$ or 1 (i.e., $\mathbb{M}$ is orthogonal), while the maximum power can be dissipated if $a = 1/2$. It is interesting to note that this one parameter family of $\mathbb{M}$-matrices can be obtained from the following electronic $\mathbb{S}$-matrix describing scattering of non-interacting electrons,

$$\mathbb{S} = \begin{pmatrix} \pm\sqrt{a} & \sqrt{1-a} \\ \sqrt{1-a} & \mp\sqrt{a} \end{pmatrix}. \tag{5}$$

The case of maximum dissipation, ı.e, $a = 1/2$ for the non-interacting electrons case also corresponds to extremal shot noise [32] as is expected.

For $N = 3$, we require four parameters to specify $\mathbb{M}$ in general as we can see below

$$\mathbb{M} = \begin{pmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ 1-a-c & 1-b-d & a+b-c-d-1 \end{pmatrix}. \quad (6)$$

(The ranges of the parameters $a-d$ are fixed by the condition that the power dissipated must be non-negative; hence we will not specify these ranges here). If we demand that no power be dissipated, i.e., that $\mathbb{M}$ be orthogonal, then we only need to specify one parameter as will be discussed below. Note that the three-wire case is quite different from the two-wire case discussed earlier. In the three-wire case, it was possible for $\mathbb{M}$ to have some negative elements without violating current conservation and non-negativity of the dissipated power, in sharp contrast to the two-wire case. To get a better feel for this, let us consider a one-parameter family of $\mathbb{M}$-matrices which corresponds to a highly symmetric junction given by

$$\mathbb{M} = \begin{pmatrix} a & (1-a)/2 & (1-a)/2 \\ (1-a)/2 & a & (1-a)/2 \\ (1-a)/2 & (1-a)/2 & a \end{pmatrix}. \quad (7)$$

This matrix corresponds to a situation in which the reflected current in each wire and the transmitted currents from one wire to the other two are the same for all the wires. Using the condition of non-negativity of the net dissipated power, we can show that the parameter a must lie between $-1/3$ and 1. For $a = -1/3$ and 1, $\mathbb{M}$ is orthogonal and is therefore dissipationless. For all values of a lying between $-1/3$ and 0 the diagonal elements of $\mathbb{M}$ are negative. It is easy to see that $\mathbb{M}$-matrices with negative entries cannot be obtained from any unitary $\mathbb{S}$-matrix, i.e., cannot be obtained from any non-interacting electron theory. Hence such $\mathbb{M}$-matrices necessarily correspond to situations in which the inter-electron interaction strength is non-zero. We emphasize that such current splitting matrices only exist for a junction of three or more wires and are absent for the two-wire case.

Next, let us consider a situation where the power associated with the incoming current is set to unity in units of $1/(2G)$. Then the three-element column given by J_I can be identified with a unit vector in three dimensions which can be parametrized as $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The maximum dissipation occurs for the $\mathbb{M}$-matrix which has all its elements equal to $1/N$. This corresponds to $a = 1/3$ in Eq. (7). Using Eq. (3) for the case of $a = 1/3$, we find that the power associated with the outgoing currents is given by

$$P_o = \frac{1}{3}\left[cos\theta + sin\theta(cos\phi + sin\phi)\right]^2. \quad (8)$$

Note that P_o is bounded by $[0, 1]$ and is symmetric under $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$. To visualize the expression in Eq. (8), we present a contour plot of P_o as a function of ϕ and θ in Fig. 1. An interesting point to note in the figure is the existence of a line of points on which

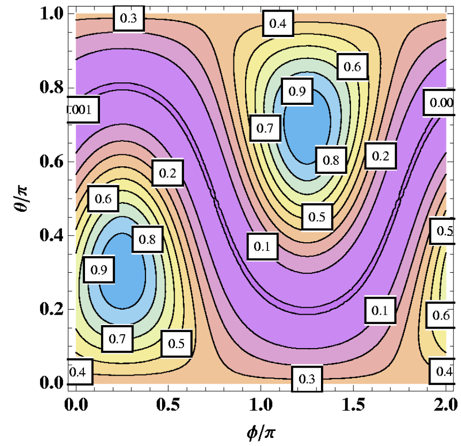

FIG. 1: (Color online) Contour plot of P_o given in Eq. (8) as a function of ϕ and θ.

the outgoing power is zero and hence the power dissipation is maximum. This implies that there is a family of bias voltage or incoming current configurations for which the power dissipated is maximum. On the other hand, there are two points at which the outgoing power is unity which correspond to zero power dissipation. To understand these patterns, we recall that the direction of maximum power dissipation in the space of incoming current vectors corresponds to the two distinct eigenvectors of the $\mathbb{M}^T\mathbb{M}$ with zero eigenvalue. For any value of a in Eq. (7), an orthonormal set of eigenvectors of $\mathbb{M}^T\mathbb{M}$ is given by $\mathbf{V}_1 = (1,1,1)/\sqrt{3}, \mathbf{V}_2 = (1,-1,1)/\sqrt{2}, \mathbf{V}_3 = (1,1,-2)/\sqrt{6}$ and the corresponding eigenvalues are 1, $(1-3a)^2/4$ and $(1-3a)^2/4$. Note that these eigenvectors are independent of the parameter a. This is so because of the symmetric form of $\mathbb{M}$ matrix. Hence, for this entire family of $\mathbb{M}$-matrices (Eq. (7)), the eigenvectors (i.e., the combinations of incoming currents) which give the maximum power dissipation are independent of a. Second, the eigenvalue which is different from unity is a quadratic function of a which is zero for $a = 1/3$ (maximum dissipation) and unity for $a = -1/3$ and $a = 1$ (both corresponding to zero dissipation). The line of maximum dissipation appearing in Fig. 1 corresponds to an incoming current column J_I which is a linear combination of the two degenerate eigenvectors corresponding to the zero eigenvalue given by $J_I = cos\delta\ \mathbf{V}_2 + sin\delta\ \mathbf{V}_3$, where δ lies in the interval $[0, 2\pi]$. The existence of such a line of maximum dissipation is encouraging from an experimental point of view since this implies that we only need to vary a single parameter in an experiment to encounter the point of maximum dissipation. The two points in Fig. 1 which have $P_o = 1$ (zero dissipation) correspond to the eigenvector $\mathbf{V}_1$. There are two such points because we get zero dissipation if the incoming current is prepared either in the direction of this eigenvector or opposite to it; in Fig. 1, these points lie at $(\phi, \theta) = (\pi/4, \cos^{-1}(1/\sqrt{3}))$

and $(5\pi/4, \pi - \cos^{-1}(1/\sqrt{3}))$. To conclude, we see that a study of the eigenvectors of $\mathbb{M}^T\mathbb{M}$ can lead to a complete understanding of dissipation in a junction as a function of the bias voltages applied in the various wires.

IV. A THREE-WIRE MODEL WITH DISSIPATION

In this section, we develop a microscopic model for a three-wire system with a dissipative junction. A schematic picture of the system is presented in Fig. 2. The currents and voltages on each wire will be assumed to be governed by $J = GV$ (where $G = \nu e^2/h$) on all the incoming and outgoing chiral wires. (The symbols V_i in the figure denote the incoming voltages which drive the incoming currents; the outgoing currents and voltages are then determined by the V_i and the matrix $\mathbb{M}$ which will be derived below). The junction region consists of three points a, b, c, one point lying on each of the three wires as shown in Fig. 2. Electrons can tunnel between any two of these points, say, i and j. If the tunneling amplitude is denoted by ξ_{ij}, the corresponding tunneling conductance $\sigma_{ij}G$ will be proportional to $|\xi_{ij}|^2$. Here we have introduced the quantity G so that σ_{ij} is dimensionless. The conductances satisfy $\sigma_{ij} = \sigma_{ji} \geq 0$. If the voltages at the two points are given by V_i and V_j, the current flowing from i to j will be given by $\sigma_{ij}G(V_i - V_j)$. We will now see that this model gives rise to a current splitting matrix $\mathbb{M}$ which is generally dissipative.

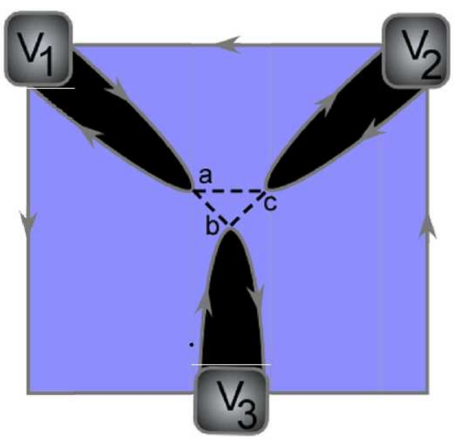

FIG. 2: (Color online) Picture of a three-wire model with tunneling conductances σ_{ij} between points i, j which can take values a, b, c. V_i denote the incoming voltages.

In our analysis, we will work directly with the currents without introducing any fermionic or bosonic fields. To derive the matrix $\mathbb{M}$, we have to determine the outgoing currents (J_{O1}, J_{O2}, J_{O3}) in terms of the incoming currents (J_{I1}, J_{I2}, J_{I3}). The incoming and outgoing currents (and therefore voltages) will generally change discontinuously at the three junction points. The corresponding incoming and outgoing voltages are obtained by dividing the currents by G. We assume that the voltages at each of the three points of the junction are given by the mean values of the corresponding incoming and outgoing voltages. Namely,

$$V_i \ = \ \frac{1}{2} \ (\ V_{Ii} \ + \ V_{Oi} \ ) \ = \ \frac{1}{2G} \ (\ J_{Ii} \ + \ J_{Oi} \ ) \qquad (9)$$

for $i = 1, 2, 3$.

The mean value assumption made in Eq. (9) can be justified as follows. We can begin with a model in which the tunneling region is not a point but has a finite length l, and there is a tunneling conductance per unit length given by $\tilde{\sigma}_{ij}$ [27, 33]. Tunneling will then occur from every point lying in the tunneling region in wire i to the corresponding point lying in wire j. We then find that the current J_i on wire i changes smoothly from J_{Ii} to J_{Oi} as we go from one end of the tunneling region to the other. Hence the voltage $V_i(x_i) = J_i(x_i)/G$ will also change smoothly, where i runs over $1, 2, 3$, and x_i runs over the tunneling region from 0 to l. The current $J_i(x_i)$ can be obtained by solving equations of continuity given by [27, 33]

$$\frac{\partial J_i}{\partial x_i} \ = \ - \ \sum_{j \neq i} \ \tilde{\sigma}_{ij} \ [J_i(x_i) - J_j(x_j)], \qquad (10)$$

We can solve these equations to obtain the dependence of the current $J_i(x_i)$ and voltage $V_i(x_i) = J_i(x_i)/G$ on the coordinates x_i. If we now take the limit $l \to 0$ with $l\tilde{\sigma}_{ij} = \sigma_{ij}$ being held fixed, we recover the earlier model of tunneling between three points, with the voltages at the three points being given by Eq. (9).

We now return to our original model and write down equations of continuity for the currents at the three tunneling points,

$$J_{Oi} \ - \ J_{Ii} \ = \ - \ G \sum_{j \neq i} \sigma_{ij} \ (V_i \ - \ V_j), \qquad (11)$$

for $i = 1, 2, 3$. Using Eq. (9), we can solve for the J_{Oi} in terms of the J_{Ii}. This enables us to obtain the matrix $\mathbb{M}$ which relates the two sets of currents. We find that

$$\mathbb{M}_{11} \ = \ \frac{1 \ + \ \sigma_{23} - (1/4)S_2}{1 \ + \ S_1 \ + \ (3/4)S_2},$$
$$\mathbb{M}_{12} \ = \ \frac{\sigma_{12} \ + \ (1/2)S_2}{1 \ + \ S_1 \ + \ (3/4)S_2}, \qquad (12)$$

where $S_1 \equiv \sigma_{12} + \sigma_{23} + \sigma_{31}$ and $S_2 \equiv \sigma_{12}\sigma_{23} + \sigma_{23}\sigma_{31} + \sigma_{31}\sigma_{12}$. All the other entries of $\mathbb{M}$ can be found by symmetry. Note that each row and column of $\mathbb{M}$ adds up to 1 as desired. In addition, $\mathbb{M}$ being a symmetric matrix is a special feature of this specific model.

We can show in general that the expression for power dissipation given in Eq. (3) agrees with the sum of the powers dissipated by the three tunneling processes. Namely, if we substitute the expression for $\mathbb{M}$ given in Eq.

(12) in Eq. (3), and compare that with the expression for the power dissipated ($= I \times V$) by the three tunnelings, namely,

$$G[\sigma_{12}\ (V_1 - V_2)^2 + \sigma_{23}\ (V_2 - V_3)^2 + \sigma_{31}(V_3 - V_1)^2], \quad (13)$$

(where $V_i - V_j$ appears in Eq. (11)), we find that the two agree for all values of the incoming currents J_{Ii}.

In the special case that $\sigma_{12} = \sigma_{23} = \sigma_{31} = \sigma$, the expression for $\mathbb{M}$ simplifies to

$$\mathbb{M} = \frac{1}{1 + 3\sigma/2} \begin{pmatrix} 1-\sigma/2 & \sigma & \sigma \\ \sigma & 1-\sigma/2 & \sigma \\ \sigma & \sigma & 1-\sigma/2 \end{pmatrix}, \quad (14)$$

which is of the form given in Eq. (7). Three particular values of this $\mathbb{M}$ are worth noting, namely,

$$\begin{aligned} \mathbb{M} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } \sigma = 0, \\ &= \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \quad \text{for } \sigma = 2/3, \\ &= \begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix} \quad \text{for } \sigma = \infty. \quad (15) \end{aligned}$$

For the $\mathbb{M}$-matrix given in Eq. (14), one of the eigenvalues of $\mathbb{M}^T\mathbb{M}$ is equal to 1, while the other two are equal to $[(2 - 3\sigma)/(2 + 3\sigma)]^2$. There is no power dissipation ($\mathbb{M}$ is orthogonal) if $\sigma = 0$ or ∞, while there is maximum power dissipation if $\sigma = 2/3$. On physical grounds it is natural to expect that there is no dissipation if $\sigma = 0$. But the dissipation also turns out to to be zero for $\sigma = \infty$ which is somewhat surprising. This can be traced back to the analysis done in Ref. 16 for the junction of three TLL wires. In that paper, the authors started with a situation where there is a perfectly reflecting (disconnected) junction of three TLL wires effectively described by a $\mathbb{M}$-matrix corresponding to $\sigma = 0$ in Eq. (14), and then switched on electron tunneling operators between each pair of wires such that the amplitudes of all the three tunneling operators are equal. Using the technique of bosonization, they then established that as the strengths of all the tunneling operators go to infinity under an RG flow, the system is described by the $\mathbb{M}$-matrix which is obtained by taking the $\sigma = \infty$ limit in Eq. (14). In their analysis, the $\mathbb{M}$-matrices corresponding to both $\sigma = 0$ and $\sigma = \infty$ are fixed points of the theory, and they are connected to each other by a duality transformation. These statements make our model seem quite attractive because even though it is rather simple, it manages to capture the essential non-trivial physics related to dissipation in a three-wire junction without getting into the technicalities of bosonization.

If the various edges shown in Fig. 2 are the edges of a quantum Hall system, the tunneling operators will satisfy some RG equations. Depending on the filling fraction ν and the location of the quantum Hall liquid with respect to the edges [18], the tunneling operators will be either irrelevant or relevant, and the corresponding tunneling conductances will then flow to 0 or ∞ respectively. This implies that the RG fixed point for $\mathbb{M}$ will be given by either the first matrix or the last matrix in Eq. (15). Thus, the second matrix in (15) which corresponds to maximum dissipation does not appear to be a fixed point of the model.

V. A MORE COMPLEX THREE-WIRE MODEL

We now consider another model for a dissipative junction of three wires. This model consists of a ring shaped region (with two chiral edges) and three external wires (each with two chiral edges: incoming and outgoing) which connect to the ring at three different points. All the edges carry currents and can be modeled by TLLs. Further, each of the external wires can have different bias voltages which determine the incoming currents impinging on the ring region. Along the ring, the co-propagating currents can tunnel between the two edges. For simplicity we will assume equal tunneling amplitudes at all points.

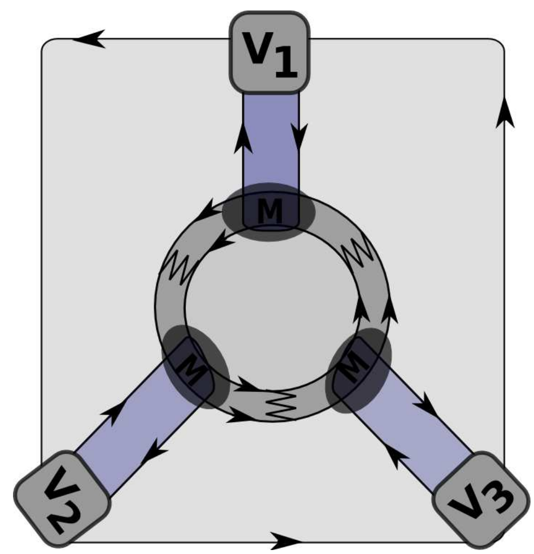

FIG. 3: (Color online) Picture of a three-wire model where the external wires connect to a ring which has two co-propagating edges with interedge tunneling. V_i denote the incoming voltages.

Fig. 3 gives a schematic picture of the model we have in mind. Each of the external wires is made of a TLL (or a single edge of a fractional quantum Hall system) and has an outgoing and an incoming chiral edge I_{Oi} and I_{Ii}, where i labels the wire. The ring is also made of a TLL (or a single mode fractional quantum Hall edge) and has two co-propagating modes, one on the outer edge (J_{Oi}) and the other on the inner edge (J_{Ii}).

At each of the 'point' junctions where an external wire meets the ring, we have three incoming modes and three outgoing modes (marked by arrows in Fig. 3). Such a three-wire junction can be described by a orthogonal 3×3 current splitting matrix ($\mathbb{M}$) whose rows and columns add up to 1. The orthogonality implies that (i) the junction relates the outgoing bosonic fields to the incoming bosonic fields in a way which preserves the chiral commutation relations of the fields, and (ii) the junction is dissipationless. For a three-wire charge-conserving and dissipationless junction, the matrix $\mathbb{M}$ can be parametrized by a single continuous parameter θ [16, 18, 28], and it can be classified into two classes for which (a) $\det \mathbb{M}_1 = 1$, and (b) $\det \mathbb{M}_2 = -1$. These two classes are expressed as

$$\mathbb{M}_1 = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}, \quad \mathbb{M}_2 = \begin{pmatrix} b & a & c \\ a & c & b \\ c & b & a \end{pmatrix}. \tag{16}$$

In Eq. (16), $a = (1 + 2\cos\theta)/3$, $b = (1 - \cos\theta + \sqrt{3}\sin\theta)/3$, and $c = (1 - \cos\theta - \sqrt{3}\sin\theta)/3$. In the $\mathbb{M}_1$ class, $\theta = 0$ corresponds to the disconnected N fixed point, $\theta = \pi$ to the D_P fixed point, and $\theta = \pm 2\pi/3$ to the chiral fixed points $\chi_\pm$ in the notation of Ref. 16.

Each of the three 'point' junctions in Fig. 3 is characterized by a dissipationless current splitting matrix $\mathbb{M}$. For simplicity we will now assume all the three junctions have the same $\mathbb{M}$ with the same orientation. We will also assume all the 'point' junction matrices to be identical and of the $\mathbb{M}_1$ type. Next, we will allow tunneling between the inner and outer edges of the ring, which can be thought of 'classically' as a resistor connecting the inner and outer current carrying wires. A more microscopic model of such a dissipative tunneling is given in Refs. 27 and 33. The main result is that at the ends of each tunneling region (of length L), the currents on the outgoing edges are a linear combination of the incoming currents and can be written as

$$\begin{pmatrix} J_{O1}(L) \\ J_{O2}(L) \end{pmatrix} = \begin{pmatrix} 1-t & t \\ t & 1-t \end{pmatrix} \begin{pmatrix} J_{I1}(0) \\ J_{I2}(0) \end{pmatrix}. \tag{17}$$

The parameter t can be expressed in terms of the microscopic tunneling conductance as

$$t = \frac{1}{2}\left(1 - e^{-2L\sigma h/(\nu e^2)}\right) \tag{18}$$

for the case of co-propagating edges, where σ is the tunneling conductance per unit length between the two edges of the ring [27].

We note again that the 'point' junction matrices connecting external wires to the ring are dissipationless. The only source of dissipation in our model is therefore the interedge tunneling between the co-propagating modes propagating on the ring. Now, starting from a given dissipationless current splitting matrix $\mathbb{M}_1$ at each 'point' junction and a given interedge tunneling parameter t, we can solve for the three outgoing and six interedge currents in terms of the three incoming currents. We then find that the $\mathbb{M}$-matrix of the system which relates the outgoing currents to the incoming currents is of the cyclic form

$$\begin{pmatrix} I_{O1} \\ I_{O2} \\ I_{O3} \end{pmatrix} = \begin{pmatrix} d & e & f \\ f & d & e \\ e & f & d \end{pmatrix} \begin{pmatrix} I_{I1} \\ I_{I2} \\ I_{I3} \end{pmatrix}, \tag{19}$$

where d, e and f are given by

$$\begin{aligned} d &= \frac{30t^2 - 48t + 27 + \left(60t^2 - 84t + 42\right)\cos\theta + \left(18t^2 - 30t + 12\right)\cos(2\theta)}{42t^2 - 48t + 33 + (28t^2 - 68t + 34)\cos\theta + (38t^2 - 46t + 14)\cos(2\theta)}, \\ e &= \frac{12t^2 + 6t - \left(24t^2 - 12t + 6\right)\cos\theta + \left(12t^2 - 18t + 6\right)\cos(2\theta)}{42t^2 - 48t + 33 + (28t^2 - 68t + 34)\cos\theta + (38t^2 - 46t + 14)\cos(2\theta)}, \\ f &= \frac{-6t + 6 - \left(8t^2 - 4t + 2\right)\cos\theta + \left(8t^2 + 2t - 4\right)\cos(2\theta)}{42t^2 - 48t + 33 + (28t^2 - 68t + 34)\cos\theta + (38t^2 - 46t + 14)\cos(2\theta)}. \end{aligned} \tag{20}$$

If we take all the matrices at the three 'point' junctions to be identical and of the $\mathbb{M}_2$ type, we again find that the $\mathbb{M}$-matrix of the complete system is of the cyclic form given in Eq. (19), although the expressions for d, e, f are different from those given in Eq. (20). For the $\mathbb{M}$-matrix given in Eqs. (19-20), one of the eigenvalues of $\mathbb{M}^T\mathbb{M}$ is equal to 1 (non-dissipative), while the other two (degenerate and dissipative) are given by

$$\lambda = \frac{78t^2 - 84t + 33 + \left(28t^2 - 68t + 34\right)\cos\theta + \left(2t^2 - 10t + 14\right)\cos(2\theta)}{42t^2 - 48t + 33 + (28t^2 - 68t + 34)\cos\theta + (38t^2 - 46t + 14)\cos(2\theta)}. \tag{21}$$

A contour plot of λ in the $t-\theta$ plane is presented in Fig. 4. Since λ is symmetric under $\theta \to -\theta$, we have only plotted θ from 0 to π in the figure. We see that $\lambda = 1$ (no dissipation) if either $\theta = 0, \pi$ or $t = 0$. It was shown in Ref. 27 that an RG flow takes the variable $L\sigma$ to either 0 or ∞, depending on the value of the interaction parameter of the TLLs which constitute the two edges of the ring. Hence the fixed-point values of the parameter t are 0 and 1/2 according to Eq. (18). For $t \to 0$, the eigenvalue λ goes to 1 for any value of θ and we therefore get a dissipationless $\mathbb{M}$-matrix. But for $t \to 1/2$, we find that

$$\lambda = \frac{21 + 14\cos\theta + 19\cos(2\theta)}{39 + 14\cos\theta + \cos(2\theta)}. \qquad (22)$$

This is equal to 1 for $\theta = 0, \pi$, and is not equal to 0 for any value of θ. According to Fig. 4, the point of maximum dissipation ($\lambda = 0$) lies at $(t, \theta/\pi) \simeq (0.419, 0.583)$, and not at $t = 0$ or $1/2$, and it is therefore not a fixed point of this model.

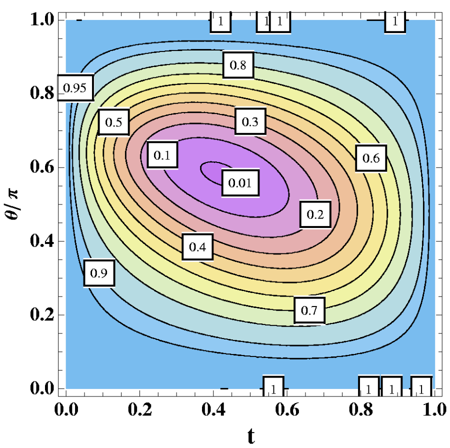

FIG. 4: (Color online) Contour plot of λ given in Eq. (21) as a function of t and θ.

VI. DISCUSSION AND SUMMARY

In this work we introduced a scheme to quantify dissipation for a N-wire junction for both non-interacting electrons and TLL wires. The quantification is achieved in terms of a real current splitting matrix $\mathbb{M}$. The dissipated power can be parametrized by the non-zero eigenvalues of $I - \mathbb{M}^T\mathbb{M}$ and hence there is no dissipation if $\mathbb{M}$ is orthogonal since $I - \mathbb{M}^T\mathbb{M}$ is then equal to 0. We have shown that if an eigenvalue of $I - \mathbb{M}^T\mathbb{M}$ is equal to 1, the corresponding eigenvector determines a combination of the applied bias voltages for which the input power is completely dissipated at the junction. For a three-wire junction, the matrix $\mathbb{M}$ with all entries equal to 1/3 has a doubly degenerate eigenvalue equal to 1. Hence any linear combination of the two eigenvectors corresponds to a combination of bias voltages which will lead to complete dissipation at the junction. This implies that the bias voltage combination which corresponds to maximum dissipation is not a unique point in the allowed parameter space but forms a one-parameter family of points as discussed in Sec. III. This fact makes it more likely to be accessible in an experimental situation.

We presented two microscopic models of dissipation for a three-wire system, one involving tunneling between three points (Sec. IV) and the other involving tunneling between three pairs of edges lying on a ring (Sec. V). The model in Sec. IV leads to a symmetric $\mathbb{M}$-matrix depending on three parameters σ_{ij}, while the model in Sec. V leads, for a particular choice of current splitting matrices at the 'point' junctions, to a cyclic $\mathbb{M}$-matrix depending on two parameters t, θ.

For both models, we have briefly discussed the RG flows of the various parameters. For the model in Sec. IV, the RG flow takes the system to one of two fixed points, both of which correspond to dissipationless $\mathbb{M}$-matrices. For the model in Sec. V, the RG flow again takes the system to one of two fixed points, one of which gives a dissipationless matrix while the other is generally dissipative (except for the special cases $\theta = 0, \pi$). In all cases, we find that the matrix corresponding to maximum dissipation (i.e., all elements of $\mathbb{M}$ being equal to 1/3) is not a fixed point of the RG equations. Hence within the models we have studied, it appears that there is nothing special about the maximally dissipative $\mathbb{M}$-matrix from an RG point of view.

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