arXiv:hep-ph/0608030v1 3 Aug 2006

CP violation in neutrino mass matrix

Utpal Sarkar and Santosh K. Singh

[2] *Physical Research Laboratory, Ahmedabad 380009, India*

Abstract

We constructed rephasing invariant measures of CP violation with elements of the neutrino mass matrix, in the basis in which the charged lepton mass matrix is diagonal. We discuss some examples of neutrino mass matrices with texture zeroes, where the present approach is applicable and demonstrate how it simplifies an analysis of CP violation. We applied our approach to study CP violation in all the phenomenologically acceptable 3-generation two-zero texture neutrino mass matrices and shown that in any of these cases there is only one CP phase which contributes to the neutrino oscillation experiment and there are no Majorana phases.

1 Introduction

In the standard model there is only one source of CP violation, which is in the charged-current mixing matrix in the quark sector. The charged-current mixing matrix in the quark sector contains one CP phase, which has been observed. It is not possible to identify the position of the CP phase, since it is possible to make any phase transformations to the quarks. However, it is possible to define a rephasing invariant quantity as product of elements of the mixing matrix that remains invariant under any rephasing of the quarks [1, 2]. This is known as Jarlskog invariant.

In the leptonic sector, standard model does not allow any CP violation. If one considers extensions of the standard model to accommodate the observed neutrino masses, then there can be several CP phases [3, 4, 5, 6]. In the simplest scenario of three generations, there could be one CP phase in the mixing matrix in the leptonic sector, similar to the quark sector. In addition, if neutrinos are Majorana particles they can have two more Majorana CP phases [4]. In this case it is possible to work in a parametrization, in which all the three CP phases could be in the charged-current mixing matrix in the leptonic sector. One of these CP phase will contribute to the neutrino oscillation experiments, while the other two will contribute to lepton number violating process like neutrinoless double beta decay. A natural explanation for the smallness of the neutrino masses comes from the see-saw mechanism [7]. The origin of small neutrino mass then relates to a large lepton number violating scale. It is quite natural that this lepton number violation at the high scale would also explain the baryon asymmetry of the universe through leptogenesis [8, 9]. This connection between the neutrino mass and leptogenesis makes the question of CP violation in the leptonic sector more interesting [5, 6].

The CP phases in the leptonic sector has been studied and rephasing invariants for both lepton number conserving as well as lepton number violating CP violation have been constructed [3]. In this article we try to study this question only in terms of neutrino masses. Since neutrinos are produced only through weak interactions, it is possible to work in the weak interaction basis, in which the charged lepton mass matrix is diagonal. The neutrino mass matrix in this basis will then contain all the information about CP violation. We try to find rephasing invariant combinations of the neutrino mass elements, so that with those invariants some general comments can be made about CP violation in the model without deriving the structure of the charged-current mixing matrix.

2 CP Violation in the Quark Sector

We briefly review the rephasing invariants in terms of the mixing matrices and then show how the same results can be obtained from the mass matrix without taking the trouble of diagonalizing them in the leptonic sector. Consider first the quark sector, where the up and the down quark mass matrices are diagonalized by the bi-unitary transformations. We write the corresponding unitary matrices that relates the left-handed and right-handed physical (with definite masses) up and down quarks fields to their weak (diagonal charged current) basis as: U_L, D_L, U_R and D_R. Then the charged current interactions in terms of the physical fields will contain the Kobayashi-Cabibbo-Maskawa mixing matrix

$$V = D_L^\dagger U_L.$$

Since the right-handed fields are singlets under the standard model interactions, they do not enter in the charged current interactions. In any physical processes, only this CKM mixing matrix would appear and hence the matrices U_R and D_R becomes redundant. So, the up and down quark masses have much more freedom and the physical observables that can determine the $V_{\alpha i}$ cannot infer about the up and down quark masses uniquely.

For the CP violation, one needs to further consider the rephasing of the left-handed fields. Any phase transformation to the up and down quarks will also transform the CKM matrix

$$V_{\alpha i} \to e^{-i(d_\alpha - u_i)}.$$

However, if there is any CP phase in the CKM matrix, which cannot be removed by any phase transformations of the up and the down quarks, should be present in the Jarlskog invariant [1, 2]

$$J_{\alpha i \beta j} = \mathrm{Im}[V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i}]. \tag{1}$$

Thus if the Jarlskog invariant is non-vanishing, that would imply CP violation in the quark mixing. It is apparent from the definition that any phase transformations to the up and down quarks cannot change $J_{\alpha i\beta j}$. In a three generation scenario there can be only one such invariant and hence the CKM matrix can have only one CP phase, which is invariant under rephasing of the up and the down quarks.

3 CP Violation in the Leptonic Sector

In the leptonic sector, the charged lepton mass matrix can in general be diagonalized by a bi-unitary transformation. In addition, the neutrinos are produced in weak interactions, so the flavour of the neutrinos at the time of production is always same as the flavour of the charged lepton. The charged-current interaction is given by

$$\mathcal{L}_{cc} = \frac{g}{\sqrt{2}}\ \bar{\ell}_{iL}\gamma^{\mu}\nu_L\ W^-_{\mu} \tag{2}$$

in the basis $(\ell_{iL}, i = e, \mu, \tau)$ in which charged lepton mass matrix is diagonal, *i.e.*, the states e, μ, τ correspond to physical states. Without loss of generality we further assume that elements of the diagonal charged lepton mass matrix are real and positive.

The neutrino mass term can be written as

$$\mathcal{L}_M = M_{\nu ij}\ \overline{\nu^c}_{iR}\ \nu_{jL} = M_{\nu ij}\ \nu^T_{iL}\ C^{-1}\ \nu_{jL} \tag{3}$$

so that the neutrino mass matrix can be diagonalized by a single unitary matrix U through

$$U^T\ M_\nu\ U = K^2\ \hat{M}_\nu \tag{4}$$

where $\hat{M}_\nu = \mathrm{diag}[m_1,\ m_2,\ m_3]$ is a real diagonal matrix and K is a diagonal phase matrix. The unitary matrix U_{ia} (with $i, j = e, \mu, \tau$ and $a, b = 1, 2, 3$) relates the physical neutrino states ν_a (with masses m_a) to the weak states

$$\nu_a = U^*_{ai}\ \nu_i + K^{*\ 2}_{aa}\ U_{ai}\ \nu^c_i, \tag{5}$$

so that the physical neutrinos satisfy the Majorana condition

$$\nu = K^2\ \nu^c. \tag{6}$$

The Unitary matrix U thus gives the mixing of the neutrinos and hence neutrino oscillations, which is known as the PMNS mixing matrix [10] and K is the Majorana phase matrix containing the Majorana phases, which are the new sources of CP violation entering due to the Majorana nature of the neutrinos. The PMNS matrix U also can contain CP violating phases, which should be observed in the neutrino oscillation experiments. We call these phases in the PMNS mixing matrix U as *Dirac phases* to distinguish them from the *Majorana phases*. The main difference betwen a Majorana phase and a Dirac phase is that the Majorana phases do not affect any lepton number conserving process like neutrino oscillations. On the other hand, the Dirac phases may contribute to both lepton number conserving as well as lepton number violating processes.

From the above discussions it is apparent that the information about the CP phases can be obtained from either U and K or only from the mass matrix M_ν. In the literature the question of CP violation is usually discussed by studying U and K. In this article we point out that it is possible to study the question of CP phases only by studying the neutrino mass matrix M_ν. In particular, the information about CP violation is conveniently obtained from the rephasing invariant combinations of neutrino mass elements. When the neutrino masses originate from see-saw mechanism, the question of CP violation has been studied in details and similar invariants have been constructed [6]. Our approach is different and we do not restrict our analysis to any specific origin of the neutrino masses. Our results are general and applicable to any models of neutrino masses.

Consider the transformation of different quantities under the rephasing of the neutrinos

$$\begin{aligned}
\nu_i &\rightarrow e^{i\delta_i}\nu_i \\
\ell_a &\rightarrow e^{i\eta_a}\ell_a \\
U_{ai} &\rightarrow e^{-i(\eta_a-\delta_i)}U_{ai} \\
K_i &\rightarrow e^{i\delta_i}K_i.
\end{aligned} \tag{7}$$

From these transformations it is possible to construct the rephasing invariants [3]

$$s_{aij} = U_{ai}U_{aj}^*K_i^*K_j. \tag{8}$$

In the three generation case there will be three independent rephasing invariant measures. There is another rephasing invariant which is similar to the Jarlskog invariant in the quark sector,

$$t_{aibj} = U_{ai}U_{bj}U_{aj}^*U_{bi}^*, \tag{9}$$

so that $T_{aibj} = \text{Im } t_{aibj}$ and $S_{aij} = \text{Im } s_{aij}$ becomes the measure of CP violation. T_{aibj} contains the information about the Dirac phase, while s_{aji} contains information about both Dirac as well as Majorana phases. One can then use the relation

$$t_{aibj} = s_{aij} \cdot s_{aji}$$

to eliminate the invariants T's or else keep the T's as independent measures and reduce the number of independent S's. One convenient choice for the independent measures is the independent t_{aibj}s and s_{1ij}s. In the three generation case there is only one t_{aibj} and two s_{1ij}s. The advantage of this parametrization is that the measure T_{aibj} gives the CP violation in any neutrino oscillation experiment, while the measures S_{1ij} corresponds to CP violation in lepton number violating interactions like the neutrinoless double beta decay or scattering processes like $W^- + W^- \rightarrow \ell_i^- + \ell_j^-$ also.

4 Rephasing Invariants with Neutrino Masses

We shall now proceed to construct such measures of CP violation in terms of the mass matrix itself. The rephasing invariant measures with the mixing matrix can allow all the rephasing invariants non-vanishing even when there is only one Dirac phase. However, in the present formalism, the number of rephasing invariants is same as the number of CP phases. So, we can find out if there is any Majorana phase or not. Since the neutrino mass matrix is diagonalized by a single unitary matrix, the mass matrix contains all the information about the PMNS mixing matrix and also the mass eigenstates. However, this is not obvious with the CP phase. When the neutrinos are given a phase transformation, the mass matrix will be transformed the same way. Since the we are working in the weak basis, any transformation to the charged leptons can be transformed to the mixing matrix and in turn to the neutrino masses. Thus the phase transformation to the mass matrix will become

$$\begin{aligned} \nu_i &\rightarrow e^{i\delta_i}\nu_i \\ \ell_i &\rightarrow e^{i\eta_i}\ell_i \\ M_{\nu ij} &\rightarrow e^{i(\delta_i+\delta_j-\eta_i-\eta_j)}M_{\nu ij}. \end{aligned} \tag{10}$$

Consider the transformation $E \to XE$, where X is the phase transformation to the charged leptons. The mixing matrix will transform as $U \to X^*U$. However, in equation 4 this transformation can be interpreted as a transformation to the mass matrix, $M \to X^*MX^*$. Thus any rephasing invariant measure constructed with only the mass matrix will contain the information about CP violation.

Unlike the mixing matrices, the mass matrix is not unitary and instead it is symmetric. We write the elements of the mass matrix M_ν as m_{ij} and try to construct the rephasing invariants in terms of m_{ij}. This analysis do not depend on the origin of neutrino masses. We work with the neutrino mass matrix after integrating out any heavier degrees of freedom and in the weak basis. Any quadratic terms that can be constructed from the elements of the neutrino mass matrix are all real, $m_{ij}^* m_{ij} = |m_{ij}|^2$, as expected. Let us next consider the quartic terms

$$\mathcal{I}_{ijkl} = m_{ij} m_{kl} m_{il}^* m_{kj}^*. \tag{11}$$

It is easy to check that any three factors of the above quartic invariant can be made real by appropriate rephasing, but fourth one will remain complex. Since there are n re-phasing phases (δ_i), one can get n number of linear equations to make mass elements of the mass matrix to be real. So n number of entries (excluding symmetric elements) of the mass matrix can be made real, but positions of the mass entries can not be chosen randomly. That is the reason why all the above rephasing quartic invariants can not be made real in general. An $n \times n$ symmetric matrix has $n(n+1)/2$ independent entries and so it has the same number of phases. By appropriate rephasing, as argued above, n independent phases can be removed. Then, one is left with $n(n-1)/2$ number of independent phases.

To find out the minimal set of rephasing invariants we list some of the transitive and conjugation properties of the invariants:

$$\begin{aligned}
\mathcal{I}_{ijpl} \mathcal{I}_{pjkl} &= |m_{pj} m_{pl}|^2 \mathcal{I}_{ijkl} \\
\mathcal{I}_{ijkp} \mathcal{I}_{ipkl} &= |m_{ip} m_{kp}|^2 \mathcal{I}_{ijkl}
\end{aligned}$$

and

$$\mathcal{I}_{ijkl} = \mathcal{I}_{ilkj}^* = \mathcal{I}_{klij} = \mathcal{I}_{kjil}^* \tag{12}$$

Using these relations it can be shown that all the $\mathcal{I}_{ijkl}$ are not independent and they can be expressed in terms of a subset of these invariants $\mathcal{I}_{ij\alpha\alpha}$ and the quadratic invariants as

$$\mathcal{I}_{ijkl} = \frac{\mathcal{I}_{ij\alpha\alpha}\mathcal{I}_{kl\alpha\alpha}\mathcal{I}^*_{li\alpha\alpha}\mathcal{I}^*_{kj\alpha\alpha}}{|m_{\alpha\alpha}|^4|m_{i\alpha}m_{j\alpha}m_{k\alpha}m_{l\alpha}|^2} \tag{13}$$

Where $i, j \neq \alpha$ and $\alpha = 1, 2, ..., n$, where n is the number of generations. On the other hand, any quartics of the form $\mathcal{I}_{ij\alpha\alpha}$ can be expressed in terms of $\mathcal{I}_{\beta\beta\alpha\alpha}$ as

$$\text{Im}\ [\mathcal{I}_{ij\alpha\alpha}] = -\text{Im}\ [\mathcal{I}_{i\alpha\alpha j}] = -\frac{\text{Im}[\mathcal{I}_{ii\alpha\alpha} \cdot \mathcal{I}_{\alpha\alpha jj} \cdot \mathcal{I}_{iijj}]}{\text{Re}\ [\mathcal{I}_{i\alpha\alpha j}]\ (|m_{ii}|^2\ |m_{jj}|^2)}. \tag{14}$$

Thus we can express all other invariants in terms of $\mathcal{I}_{iijj}$ and hence consider them to be of fundamental importance. However, when there are texture zeroes in the neutrino mass matrix, some or all of these invariants $\mathcal{I}_{iijj}$ could be vanishing. In that case, it is convenient to use the $\mathcal{I}_{ij\alpha\alpha}$ as the measure of CP violation. For the present we shall concentrate on the more general case with neutrino mass matrices without any texture zeroes, when the simplest rephasing invariants are $\mathcal{I}_{iijj}$.

We can thus define the independent CP violating measures as

$$I_{ij} = \text{Im}\ [\mathcal{I}_{iijj}] = \text{Im}\ [\mathcal{I}_{iijj}] = \text{Im}\ [m_{ii}m_{jj}m^*_{ij}m^*_{ji}], \qquad (i < j) \tag{15}$$

These are the minimal set of CP violating measures one can construct and this gives the independent CP violating quantities. Since I_{ij} satisfies

$$I_{ij} = I_{ji} \qquad \text{and} \qquad I_{ii} = 0,$$

there are $n(n-1)/2$ independent measures for n generations.

We ellaborate with some examples starting with a 2-generation scenario. There are three $\mathcal{I}_{ijkl}$, two of which are real: $\mathcal{I}_{1211} = |m_{11}m_{12}|^2$; and $\mathcal{I}_{1222} = |m_{12}m_{22}|^2$. The third one can have imaginary phase, which is $I_{12} = \text{Im}[\mathcal{I}_{1122}] = \text{Im}\ [m_{11}m_{22}m^*_{12}m^*_{21}]$. In the 3-generation case there are thus three independent measures I_{12}, I_{13}, I_{23}. Imaginary phases in all other quartics $\mathcal{I}_{ijkl}$ are related to only these three independent measures. For example,

$$\mathcal{I}^2_{1223} = \frac{I^*_{12} \cdot I^*_{23} \cdot I_{13}}{|m_{11}|^2\ |m_{33}|^2}.$$

Similarly, for 4-generations there will be six rephasing invariant independent phases, which are $I_{12}, I_{23}, I_{31}, I_{14}, I_{24}, I_{34}$.

The above arguments have been stated without considering any texture zeroes in the mass matrix. If any element of the mass matrix is zero, then these discussions have to be generalized. It is because some quartic invariants can become undefined because of vanishing denominator of the right hand side of the expression 13 and 14. In that case one needs to consider all possible invariants $\mathcal{I}_{ijkl}$, which could be non-vanishing. In addition, even if all the quartic invariants vanish, the product of six mass matrix elements of the form

$$\mathcal{I}_{ijklpq} = m_{ij}\ m_{kl}\ m_{pq}\ m_{il}^*\ m_{kq}^*\ m_{pj}^*$$

could be non-vanishing and can contribute to CP violation. When there are no texture zeroes, the product of six mass elements do not contain any new information about CP phases, they are related to the quartic invariants

$$\mathcal{I}_{ijklpq} = \frac{\mathcal{I}_{ijkl}\ \mathcal{I}_{pqkj}}{|m_{kj}|^2}. \tag{16}$$

Other products of six mass elements are of the form, $m_{ij}\ m_{kl}\ m_{pq}\ m_{il}^*\ m_{kj}^*\ m_{pq}^* = |m_{pq}|^2\ \mathcal{I}_{ijkl}$ or $|m_{ij}\ m_{kl}\ m_{pq}|^2$.

We summarize this section by restating that when all elements of the neutrino mass matrix are non-vanishing, $I_{ij},\ (i<j)$ gives the total number of Dirac and Majorana phases. If some of the elements of the mass matrix vanishes, then either $\mathcal{I}_{ijkl}$ or $\mathcal{I}_{ijklpq}$ could also represent some of the independent phases.

5 CP Violation in Lepton Number Conserving Processes

The rephasing invariant independent phases contained in $I_{ij}, i<j$, are inclusive of the Dirac phases as well as the Majorana phases. We shall now identify the rephasing invariant measures, which is independent of the Majorana phases, which would enter in the neutrino oscillation experiments. The mass matrix (M_ν) in terms of the diagonal mass matrix ($\hat{M}_\nu$) can be expressed following equation 4 as

$$M_\nu = U^*\ K^2\ \hat{M}_\nu\ U^\dagger.$$

Thus the products

$$\tilde{M} = (M_\nu^\dagger\ M_\nu) = (M_\nu\ M_\nu^\dagger)^* = U\ \hat{M}_\nu^2\ U^\dagger \tag{17}$$

are independent of the Majorana phases K and any rephasing invariant measure constructed with elements $\tilde{m}_{ij}$ of $\tilde{M}$ will contain only the Dirac phases and hence should contribute to any lepton number conserving processes.

The mass-squared elements $\tilde{m}_{ij}$ transforms under rephasing of the neutrinos and charged leptons as

$$\tilde{m}_{ij} \to e^{i(\delta_i - \delta_j)} \tilde{m}_{ij}. \tag{18}$$

Neutrino rephasing does not appear because it cancels in $\tilde{M}$. Since the mass-squared matrix $\tilde{M}_\nu$ is Hermitian, $\tilde{M}_\nu^\dagger = \tilde{M}_\nu$, the mass elements satisfy

$$\tilde{m}_{ij} = \tilde{m}_{ji}^*. \tag{19}$$

Thus the simplest rephasing invariant that can be constructed from the mass-squared matrix $\tilde{M}_\nu$ is just $\tilde{m}_{11}$. However, from equation 19 it is obvious that this is a real quantity. The next possible rephasing invariant would be a quadratic term, but even that is also real

$$\tilde{m}_{ij} \tilde{m}_{ji} = \tilde{m}_{ij} \tilde{m}_{ij}^* = |\tilde{m}_{ij}|^2.$$

Thus the simplest rephasing invariant combination that can contain the complex CP phase is of the form

$$\mathcal{J}_{ijk} = \tilde{m}_{ij} \ \tilde{m}_{jk} \ \tilde{m}_{ki} \qquad (i \neq j \neq k). \tag{20}$$

$\mathrm{Im}[\mathcal{J}_{ijk}]$ are antisymmetric under interchange of any two indices and hence vanishes when any two of the indices are same. We can express $\mathcal{J}_{ijk}$ in terms of M matrix elements as,

$$\begin{aligned} \mathcal{J}_{ijk} &= \tilde{m}_{ij} \tilde{m}_{jk} \tilde{m}_{kl} \\ &= \left(\sum_\alpha m_{i\alpha}^* m_{j\alpha} \right) \left(\sum_\beta m_{j\beta}^* m_{k\beta} \right) \left(\sum_\gamma m_{k\gamma}^* m_{l\gamma} \right) \end{aligned} \tag{21}$$

Where $\sum_\alpha m_{i\alpha}^* m_{j\alpha}$ can be interpreted as scalar product of ith and jth row. A similar invariant was constructed in the case of see-saw model of neutrino masses in ref. [6], although the approach to the problem is completely different. In this expression, if any one scalar product vanishes then number of independent rephasing measure $\mathrm{Im}[\mathcal{J}_{ijk}]$ which are independent of the Majorana phases will be reduced by one.

It is possible to express all the rephasing invariants containing the Dirac phases $\mathcal{J}_{ijk}$ in terms of a minimal set of $\frac{(n-1)(n-2)}{2}$ invariants $\mathcal{J}_{ijn},\ (i<j<n)$ as

$$\mathcal{J}_{ijk} = \frac{\mathcal{J}_{ijn}\mathcal{J}_{jkn}\mathcal{J}_{kin}}{|m_{in}|\ |m_{jn}|\ |m_{kn}|} \tag{22}$$

where n is the index corresponding to the number of generations. Thus we define the measures of CP violation in lepton number conserving processes as

$$J_{ijn} = \mathrm{Im}[\mathcal{J}_{ijn}] \qquad\qquad (i<j<n). \tag{23}$$

These invariants $\mathrm{Im}[\mathcal{J}_{ijk}]$ are not independent of the invariants $\mathcal{I}_{ijkl}$ and can be exressed as

$$\mathcal{J}_{ijk} = \sum_{a,b,c} \frac{\mathcal{I}_{iajb}\cdot\mathcal{I}_{kaic}}{|m_{ia}|^2}. \tag{24}$$

So, the independent measures I_{ij} include these independent measures of Dirac CP phases $\mathrm{Im}[J_{ijn}],\ (i<j<n)$.

There are $n(n-1)/2$ phases present in $\tilde{M}$ for n generations, but all of them are not independent. $(n-1)$ of these phases can be removed by redefining the phases of the leptons. That leaves $\frac{n(n-1)}{2} - n = \frac{(n-1)(n-2)}{2} = ^{(n-1)}C_2$ independent phases in $\tilde{M}$. This is the number of Dirac phases and may be observed in neutrino oscillation experiments. Let us assume that some particular $n-1$ entries are made real with appropriate rephasing. We can take all possible pair-product of these real entries. To have non-real rephasing invariant $\mathcal{J}_{ijk}$, one will have to multiply pair-product with some complex entry. For each real pair-product there correspond only one complex entry so that there product is a complex rephasing invariant defined as in equation 20. So number of all possible pair of real entries will give the number of non vanishing rephasing measures independent of Majorana phases which is $^{(n-1)}C_2 = \frac{(n-1)(n-2)}{2}$. This number is same as the number of physical phases present in $\tilde{M}$ as it has been analyzed earlier.

In the 2-generation case there is only one CP phase which is a Majorana phase. Which implies there should not be any non-vanishing $\mathcal{J}_{ijk}$, which is trivial to check. In the 3-generation case there is only one Dirac CP phase, which is

$$\mathcal{J}_{123} = \tilde{m}_{12}\ \tilde{m}_{23}\ \tilde{m}_{31} = \sum_{a,b,c} \left[m^*_{a1}\ m_{a2}\ m^*_{b2}\ m_{b3}\ m^*_{c3} m_{c1}\right]. \tag{25}$$

Thus given a neutrino mass matrix one can readily say if this mass matrix will imply CP violation in the neutrino oscillation experiments.

In the 4-generation case there are three CP phases in the PMNS mixing matrix and 3-Majorana phase. The independent rephasing invariants of Dirac phases will be given as $\mathcal{J}_{124}$, $\mathcal{J}_{134}$ and $\mathcal{J}_{234}$. One dependent rephasing invariant is $\mathcal{J}_{123}$ which can be expressed as

$$\mathcal{J}_{123} = \frac{\mathcal{J}_{124}\mathcal{J}_{234}\mathcal{J}^*_{134}}{|\tilde{m}_{14}\tilde{m}_{24}\tilde{m}_{34}|^2}.$$

In general, these invariants satisfy

$$\mathcal{J}_{ijk}\mathcal{J}^*_{ikl}\mathcal{J}_{ilj} = |\tilde{m}_{ij}\tilde{m}_{ik}\tilde{m}_{il}|^2\mathcal{J}_{jkl} \tag{26}$$

for n generations, where $i, j, k = 1, 2, ..., n$.

We summarize this section by restating for n-generation neutrino mass matrix without any texture zeroes, the rephasing invariants corresponding to the Dirac phase are J_{ijn}, $(i < j < n)$. If there are texture zeroes, then some of the J_{ijk}, $(i < j < k,\ k \neq n)$ could also be independent.

6 Texture Zeroes

In case neutrino mass matrix contains zero textures in all the columns, it is convenient to define independent rephasing invariants in slightly different form as,

$$\mathcal{R}_{ijnn} = \lim_{|m_{kn}|\to 0} \frac{m_{ij}m^*_{in}m^*_{jn}m_{nn}}{|m_{in}|\ |m_{jn}|\ |m_{nn}|} \qquad (\forall k \text{ for which } m_{kn} = 0 \text{ and } i, j \neq n) \tag{27}$$

Limit has to be taken for all zero textures present in nth column. Any other quartic rephasing invariant can be expressed in terms of these independent rephasing invariants $\mathcal{R}_{ijnn}$ as,

$$\mathcal{I}_{ijkl} = \mathcal{R}_{ijnn}\mathcal{R}_{klnn}\mathcal{R}^*_{linn}\mathcal{R}^*_{kjnn} \tag{28}$$

We can define independent rephasing invariant measures as,

$$I_{ijn} = \mathrm{Im}\ [\mathcal{R}_{ijnn}] \quad (i, j \neq n) \tag{29}$$

Advantage of defining the independent re-phasing invariants $\mathcal{R}_{ijnn}$ as the limiting case is that the expressions do not become undefined due to presence of vanishing denominators.

Let us write above expression in a different form as,

$$\mathcal{R}_{ijnn} = |m_{ij}|e^{i(\theta_{ij}+\theta_{nn}-\theta_{in}-\theta_{jn})} \quad (i, j \ \neq n \ and \ i \leq j)$$

Where θ_{kn} is the phase present at (k, n) entry of the mass matrix. If there are some zero textures in nth column, then the phase corresponding to this zero entry present in expression of $\mathcal{R}_{ijnn}$ must be unphysical. Let us assume that there is a zero texture at $(1, n)$ position in neutrino mass matrix. Then the unphysical phase $\theta_1 n$ will appear in some of the independent rephasing invariants as defined above in equation 25, which will allow us to make one of the independent rephasing invariants R_{1jnn} to be real eliminating corresponding CP measure. So one rephasing invariant measure vanishes corresponding to one zero texture in the nth column. One independent rephasing invariant (and so one CP measure) vanishes corresponding to the zero textures present in other than nth column (or nth row). Thus the number of independent CP measures N_{CP} for neutrino mass matrix having p zero textures and q zero rows for n generations is given by

$$N_{CP} = \frac{n(n-1)}{2} - p + q \tag{30}$$

In the same way we can study the mass-squared matrices $\tilde{M}$ and write down the number of rephasing invariant measures independent of Majorana phases $\widetilde{N}_{CP}$ is given as

$$\widetilde{N}_{CP} = \frac{(n-1)(n-2)}{2} - r + s$$

where r is the number of zero entries in $\tilde{M}$ and s is the number of those rows whose all the entries excluding diagonal one are zero. It should be noticed that above relation of $\tilde{N_{CP}}$ is only valid if $N_{CP}$is not zero.

7 Application to Two-zero Texture Mass Matrices

With our present formalism, we shall now study a class of 3-generation neutrino mass matrices with two-zero textures, which has been listed in ref. [11]. There are seven such mass matrices that are consistent with present information about neutrino masses:

$$A_1 : \begin{pmatrix} 0 & 0 & X \\ 0 & X & X \\ X & X & X \end{pmatrix}; \qquad (2 \leftrightarrow 3) \qquad A_2 : \begin{pmatrix} 0 & X & 0 \\ X & X & X \\ 0 & X & X \end{pmatrix};$$

$$B_1 : \begin{pmatrix} X & X & 0 \\ X & 0 & X \\ 0 & X & X \end{pmatrix}; \qquad (2 \leftrightarrow 3) \qquad B_2 : \begin{pmatrix} X & 0 & X \\ 0 & X & X \\ X & X & 0 \end{pmatrix};$$

$$B_3 : \begin{pmatrix} X & 0 & X \\ 0 & 0 & X \\ X & X & X \end{pmatrix}; \qquad (2 \leftrightarrow 3) \qquad B_4 : \begin{pmatrix} X & X & 0 \\ X & X & X \\ 0 & X & 0 \end{pmatrix};$$

$$C : \begin{pmatrix} X & X & X \\ X & 0 & X \\ X & X & 0 \end{pmatrix};$$

From our discussions in the previous section, there can be only one CP phase in all these cases. We shall now identify the rephasing invariants in all the cases. Although all these matrices differ in phenomenology, as far as CP violation is concerned, the interchange of the indices $(2 \leftrightarrow 3)$ will not change any discussion. So, we shall not explicitly discuss the models A_2, B_2, B_4, which can be obtained by changing the indices $(2 \leftrightarrow 3)$ from the matrices A_1, B_1, B_3 respectively.

Case A_1:

There is only one non-vanishing I_{ij}, which is I_{23}. The lepton number conserving rephasing invariant measure J_{123} is given by

$$\begin{aligned} [J_{123}] &= \mathrm{Im}\left[(m_{31}^* m_{32})(m_{22}^* m_{23} + m_{32}^* m_{33})(m_{33}^* m_{31})\right] \\ &= |m_{31}|^2 I_{23} \end{aligned} \tag{31}$$

Thus there is only one Dirac CP phase in this case, which will contribute to the lepton number conserving processes. The same result is valid for A_2.

Case B_1:

In this case all the measures I_{ij} are vanishing. Even the invariants of the form $\mathcal{I}_{ij}$ are all vanishing. However, there is one CP phase as discussed in the previous section. The invariant $\mathcal{I}_{122133}$ is non-vanishing, which cannot be related to to the lower invariants by $\mathcal{I}_{122133} = \mathcal{I}_{1221} \cdot \mathcal{I}_{3322}/|m_{22}^2$, since $m_{22} = 0$. The lepton number conserving invariant is related to this invariant by

$$\begin{aligned} [J_{123}] &= \mathrm{Im}\,[(m_{11}^* m_{12})(m_{32}^* m_{33})(m_{23}^* m_{21})] \\ &= \mathrm{Im}[\mathcal{I}_{122133}]. \end{aligned} \tag{32}$$

Again there are no Majorana CP phase. The analysis is same for the case B_2.

Case B_3:

There is only one non-vanishing CP violating measure I_{13}, which is related to the lepton number conserving measure by

$$\begin{aligned} [J_{123}] &= \mathrm{Im}\,[(m_{31}^* m_{32})(m_{32}^* m_{33})(m_{13}^* m_{11} + m_{33}^* m_{31})] \\ &= |m_{32}|^2 I_{13}. \end{aligned} \tag{33}$$

There are no more CP phase left in addition to the one entering in lepton number conserving processes. Replacing the indices $(2 \leftrightarrow 3)$ we get for the case B_4 a similar relation $[J_{123}] = |m_{32}|^2 I_{12}$.

Case C:

This is the most interesting case. There are no CP violating measures of the form I_{ij}, although the invariant $\mathcal{I}_{1123}$ is non-vanishing. So, there is one CP phase in this case, as expected. This is related to the CP violating measure that affects lepton number conserving processes by

$$\begin{aligned} J_{123} &= \mathrm{Im}[(m_{11}^* m_{12} + m_{31}^* m_{32})(m_{12}^* m_{13})(m_{13}^* m_{11} + m_{23}^* m_{21})] \\ &= |m_{12}|^2 \mathcal{I}_{1123} + |m_{13}|^2 \mathcal{I}_{1123}^*. \end{aligned}$$

Although this shows that the phase is a Dirac phase, in the special case of $m_{12} = m_{13}$, there will not be any CP violation in the neutrino oscillation experiments. This can be verified from the fact that for $m_{12} = m_{13}$ the third mixing angle and hence U_{13} vanishes. In this case the CP violation can originate from a Majorana phase, since J_{123} vanishes even when Im_{1123} is non-vanishing.

Another way to understand this is to write the mass matrix in a different basis. When $m_{12} = m_{13}$, we can write the mass matrix C as

$$\begin{pmatrix} X & X & 0 \\ X & X & 0 \\ 0 & 0 & X \end{pmatrix}.$$

In this case the third generation decouples from the rest and we know that for two generation there is only a Majorana phase, which corresponds to non-vanishing I_{12} and there is no Dirac phase, as we stated above. This is the only example of two-zero texture mass matrices where the CP violating phase could be a Majorana phase, but this mass matrix is not allowed phenomenologically.

Thus there are no phenomenologically acceptable two-zero texture neutrino mass matrices, which has any Majorana phase. The only CP phase possible in any two-zero texture 3-generation mass matrix is of Dirac type and should allow CP violation in neutrino oscillation experiments.

8 Summary

In summary, we constructed rephasing invariant measures of CP violation with elements of the neutrino masses in the weak basis. For an n-generation scenario, in the absence of any texture zeroes there are $n(n-1)/2$ independent measures of CP violation, given by

$$I_{ij} = \text{Im}\left[m_{ii} m_{jj} m_{ij}^* m_{ji}^*\right] \qquad (i < j)$$

which corresponds to $n(n-1)/2$ independent CP violating phases. Only $(n-1)(n-2)/2$ of these phases of CP violation can contribute to the neutrino oscillation experiments and are independent of the Majorana phases for which the rephasing invariant measures of CP violation can be defined as

$$J_{ijn} = \sum_{a,b,c} \mathrm{Im}\left[(m^*_{ia}\ m_{ja})\ (m^*_{jb}\ m_{nb})\ (m^*_{nc}\ m_{ic})\right] \qquad (i < j < n).$$

We then defined invariants for mass matrices with texture zeroes and ellaborated with some examples. We studied all the phenomenologically acceptable 3-generation two-zero texture neutrino mass matrices. We show that there are no Majorana phase in any of the allowed cases.

References

[1] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985); Z. Phys. **C 29**, 491 (1985);

[2] O.W. Greenberg, Phys. Rev. **D 32** (1985) 1841;
D. Wu, Phys. Rev. **D 33** (1986) 860;
I. Dunietz, O.W. Greenberg and D. Wu, Phys. Rev. Lett. **55** (1985) 2935.

[3] J.F. Nieves and P.B. Pal, Phys. Rev. **D 36** (1987) 315.

[4] B. Kayser, Phys. Rev. **D 30** (1984) 1023.

[5] J. R. Ellis and M. Raidal, Nucl. Phys. B **643** (2002) 229.

[6] G. C. Branco, T. Morozumi, B. M. Nobre and M. N. Rebelo, Nucl. Phys. B **617** (2001) 475;
G. C. Branco, R. Gonzalez Felipe, F. R. Joaquim, I. Masina, M. N. Rebelo and C. A. Savoy, Phys. Rev. D **67**, 073025 (2003).

[7] P. Minkowski, Phys. Lett. **B 67**, 421 (1977);
M. Gell-Mann, P. Ramond and R. Slansky in *Supergravity* (P. van Niewenhuizen and D. Freedman, eds), (Amsterdam), North Holland, 1979; T. Yanagida in *Workshop on Unified Theory and Baryon number in the Universe* (O. Sawada and A. Sugamoto, eds), (Japan), KEK 1979;
R.N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. **44**, 912 (1980);
J. Schechter and J.W.F. Valle, Phys. Rev. **D 22**, 2227 (1980).

[8] M. Fukugita and T. Yanagida, Phys. Lett. B**174**, 45 (1986).

[9] M.A. Luty, Phys. Rev. D**45**, 455 (1992);
R.N. Mohapatra and X. Zhang, Phys. Rev. D**46**, 5331 (1992);
A. Acker, H. Kikuchi, E. Ma and U. Sarkar, Phys. Rev. **D 48**, 5006 (1993);
M. Flanz, E.A. Paschos and U. Sarkar, Phys. Lett. **B 345**, 248 (1995);
M. Flanz, E.A. Paschos, U. Sarkar and J. Weiss, Phys. Lett. **B 389**, 693 (1996);
E. Ma and U. Sarkar, Phys. Rev. Lett. **80**, 5716 (1998);

T. Hambye, E. Ma and U. Sarkar, Nucl. Phys. B **602**, 23 (2001);

G. F. Giudice, A. Notari, M. Raidal, A. Riotto and A. Strumia, Nucl. Phys. B **685**, 89 (2004);

W. Buchmuller, P. Di Bari and M. Plumacher, Nucl. Phys. **B 665**, 445 (2003); Annals Phys. **315**, 305 (2005);

W. Buchmuller, R. D. Peccei and T. Yanagida, Ann. Rev. Nucl. Part. Sci. **55**, 311 (2005).

[10] Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962);
B. Pontecorvo, Sov. Phys. JETP **7**, 172 (1958) [Zh. Eksp. Teor. Fiz. **34**, 247 (1957)]; Sov. Phys. JETP **6**, 429 (1957) [Zh. Eksp. Teor. Fiz. **33**, 549 (1957)].

[11] P.H. Frampton, S.L. Glashow and D. Marfatia, Phys. Lett. **B 536**, 79 (2002);
W. Grimus, A.S. Joshipura, L. Lavoura and M. Tanimoto, Eur. Phys. J **C 36**, 227 (2004);
W. Grimus, PoS (HEP 2005) 186 [hep-ph/0511078].