ON BRAUER GROUPS OF REAL ENRIQUES SURFACES

V. V. Nikulin and R. Sujatha

Abstract. Let Y be a real Enriques surface, ${}_2Br(Y)$ the subgroup of elements of order 2 of $Br(Y)$, and s, s_{or}, and s_{nor} the number of all connected, connected orientable, and connected non-orientable components of $Y(\mathbf{R})$ respectively. Using universal covering $K3$-surface X of Y, we connect $\dim_2 Br(Y)$ with s, s_{or} and s_{nor}. As a geometric corollary of our considerations, we show that $s \le 6$ and $s_{nor} \le 4$.

§0. Introduction

Let X be a smooth projective irreducible real surface and

$$Br(X) = H^2_{et}(X; \mathbf{G}_m)$$

denote the Brauer group (cohomological) of X. This group is known to be a birational invariant of the surface. Let $X(\mathbf{R})$ denote the space of $\mathbf{R}$-rational points of X with the Euclidean topology and s be the number of real connected components of this space. Let ${}_2Br(X)$ be the group of 2-torsion elements in $Br(X)$. If $P \in X$ is a real point of X, we get a natural map ${}_2Br(X) \to {}_2Br(P) \cong \mathbf{Z}/2$. In [CT-P], it is proved that this map ${}_2Br(X) \to \mathbf{Z}/2$ does not depend on a choice of the point P in a connected component of $X(\mathbf{R})$. Thus, the canonical map

$$(0\text{–}1) \qquad {}_2Br(X) \to (\mathbf{Z}/2)^s$$

is defined where s is the number of connected components of $X(\mathbf{R})$.

A calculation of dim ${}_2Br(X)$ and the dimension of the kernel of the map (0–1) is a very interesting and difficult problem. It is important for a calculation of a such invariant of X as the Witt group $W(X)$, see [Su]. Probably, these calculations are now known only for real surfaces X rational over $\mathbf{C}$. In this case [Su], if $s > 0$, the dimension dim ${}_2Br(X) = 2s - 1$, the map (0–1) is epimorphic [CT-P, 3.1], and $W(X) \cong (\mathbf{Z})^s \oplus (\mathbf{Z}/2)^{s-1}$. We mention that in [CT-P, 3.1] it is proved that the map (0–1) is an epimorphism for surfaces X with $H^3(X(\mathbf{C}); \mathbf{Z}/2) = 0$. In particular, it is true for rational surfaces.

In this paper, we start studying similar problems for real Enriques surfaces. Here, for a real Enriques surface Y, we calculate dim ${}_2Br(Y)$ and connect it with the numbers s, s_{or} and s_{nor} of connected, orientable and non-orientable components of $Y(\mathbf{R})$ respectively.

By a real Enriques surface Y, we mean that $Y_{\mathbf{C}} = Y \otimes \mathbf{C}$ is an irreducible non-singular minimal projective algebraic surface over $\mathbf{C}$ with invariants $\varkappa(Y_{\mathbf{C}}) = p_g(Y_{\mathbf{C}}) = q(Y_{\mathbf{C}}) = 0$, [A]. For Enriques surfaces, the invariants $p_g(Y_{\mathbf{C}})$ and $q(Y_{\mathbf{C}})$ are the same as for rational surfaces. But $H^3(Y(\mathbf{C}); \mathbf{Z}/2) = \mathbf{Z}/2$. Thus, it is very interesting to calculate $\dim\ {}_2Br(Y)$, study the map (0–1) and $W(Y)$ for real Enriques surfaces Y, and compare these calculations with calculations for rational surfaces we mentioned above. We mention that topologically Enriques surfaces are much more complicated than rational ones because they are not simply-connected: the fundamental group $\pi_1(Y(\mathbf{C}))\cong \mathbf{Z}/2$. It makes these calculations very delicate.

Now, we formulate basic results of the paper. Let Y be a real Enriques surface. Let $Y(\mathbf{C})$ denote the underlying complex manifold of $Y_\mathbf{C}$ and G be the Galois group $G(\mathbf{C}/\mathbf{R})$. We identify the generator of this group with the corresponding antiholomorphic involution θ on $Y(\mathbf{C})$. Everywhere below for a 2-elementary group $\dim(\mathbf{Z}/2)^a=a$. For any G-module A, $A^G=A^\theta$ denotes the set of elements of A fixed by G.

We introduce the following basic invariants of a real Enriques surface Y:

Definition 0.1. The invariant $\epsilon(Y)=1$ if the differential

$$d_2^{0,2}: E_2^{0,2}=H^0(Y(\mathbf{C});\mathbf{Z}/2)^G\to E_2^{2,1}=H^2(G;H^1(Y(\mathbf{C});\mathbf{Z}/2))\cong\mathbf{Z}/2$$

of the Hochschild–Serre spectral sequence (see [Mi]) vanishes, and $\epsilon(Y)=0$ otherwise. For Enriques surfaces, $H^1(Y(\mathbf{C});\mathbf{Z}/2)\cong\mathbf{Z}/2$.

The invariant

$$b(Y)=\dim H^2(Y(\mathbf{C});\mathbf{Z}/2)^\theta-\dim(\mathrm{Pic}\ Y_\mathbf{C})^\theta/2(\mathrm{Pic}\ Y_\mathbf{C})^\theta+1.$$

First of all we prove

Theorem 0.1. *Let Y be a real Enriques surface and $Y(\mathbf{R})\neq\emptyset$.*

Then

$$\dim\ {}_2Br(Y)=b(Y)+\epsilon(Y).$$

Proof. See Theorem 1.2 (in §1).

Theorem 0.1 shows that to estimate $\dim\ {}_2Br(Y)$, it is important to estimate the invariant $b(Y)$, because $0\le\epsilon(Y)\le 1$.

We prove the following general estimates from below for the $b(Y)$ and $\dim\ {}_2Br(Y)$.

Theorem 0.2. *Let Y be an arbitrary real Enriques surface and s the number of connected components of $Y(\mathbf{R})$. Then*

$$b(Y)\ge 2s-2.$$

It follows (by Theorem 0.1)

$$\dim\ {}_2Br(Y)\ge 2s-2+\epsilon(Y)\ge 2s-2.$$

Proof. See Theorem 2.1 (in §2).

Considering the image of the ${}_2Br(\mathrm{Spec}\ \mathbf{R})$ of the basic field $\mathbf{R}$ in ${}_2Br(X)$, we evidently get

Proposition 0.3. *For any real surface X, the element*

$$\underbrace{(1, \cdots, 1)}_{s \text{ times}}$$

belongs to the image of the map (0–1).

Thus, from these three statements we get the

Corollary 0.4. *For a real Enriques surface Y*

$$\dim_2 Br(Y) \geq s.$$

This corollary is interesting because it is not now known that the map (0–1) is an epimorphism for real Enriques surfaces.

In §3 of the paper, we give some precise formula for the invariant $b(Y)$. This is the most non-trivial result of the paper. We recall [C-D] that the universal covering surface of an Enriques surface Y is a $K3$-surface X, and the universal covering $\pi : X(\mathbf{C}) \to Y(\mathbf{C})$ is 2-sheeted. We denote by τ the holomorphic involution of X corresponding to this covering. It is not difficult to see that if $Y(\mathbf{R}) \neq \emptyset$, then there are two liftings σ and $\tau\sigma$ of the antiholomorphic involution θ of $Y(\mathbf{C})$ to antiholomorphic involutions of $X(\mathbf{C})$.

Theorem 0.5. *Let Y be a real Enriques surface and suppose there are two liftings σ and $\tau\sigma$ of antiholomorphic involution θ of $Y(\mathbf{C})$ to antiholomorphic involutions of the universal covering $K3$-surface $X(\mathbf{C})$ (for example this is true if $Y(\mathbf{R}) \neq \emptyset$). Let s_{or} and s_{nor} be the number of orientable and non-orientable connected components of $Y(\mathbf{R})$ respectively.*

Then

$$b(Y) = s_{nor} + 2s_{or} - z(\sigma) + \dim H(\sigma)_- - \dim(H(\sigma)_+)^\perp \cap H(\sigma)_- + \beta(Y).$$

Proof. See Theorems 3.5.1, 3.5.2 and 3.5.3 and the formula (3–5–1) (in §3).

Here we cannot give precise definitions of the summands of this formula. We only mention the following inequalities for these integers: $0 \leq z(\sigma) \leq 2$, $0 \leq \dim H(\sigma)_- - \dim(H(\sigma)_+)^\perp \cap H(\sigma)_- \leq 2$, $0 \leq \beta(Y) \leq 1$. See §3 for precise definitions of these integers. It is very important that integers $s_{nor} + 2s_{or}$, $z(\sigma)$, $\dim H(\sigma)_-$ and $\dim(H(\sigma)_+)^\perp \cap H(\sigma)_-$ are defined using only the action of the involutions σ and $\tau\sigma$ on $H^2(X(\mathbf{C}); \mathbf{Z})$. In some cases we can prove that the invariant $\beta(Y) = 0$. Thus, using global Torelli Theorem [PŠ-Š] and epimorphicity of Torelli map [Ku] for $K3$-surfaces, and results of [N4], we can construct Enriques surfaces with these prescribed invariants.

By the inequalities above, Theorem 0.5 gives the estimate for $b(Y)$ from above. Since $2s = 2s_{nor} + 2s_{or}$ and Theorem 0.2 gives the inequality for $b(Y)$ from below, the formula of Theorem 0.5 and the inequality of Theorem 0.2 give very strong estimates on the numbers s_{nor}, s_{or} and show that the inequality of the Theorem 0.2 is not far from being an equality. For a classification of real Enriques surfaces using these considerations, see [N6].

We mention that in Theorem 3.4.7 we give another formula for the number $b(Y)$ which is more useful in some cases.

From Theorem 0.5 and statements 0.1 —0.3 above, we get the following result when we calculate dim ${}_2Br(Y)$ precisely. To formulate this result, we should introduce some invariants of Y.

Let $H^2 = H^2(Y(\mathbf{C}); \mathbf{Z})/\mathrm{Tor}$. The H^2 is an unimodular lattice with respect to the intersection pairing. The involution θ acts on H^2. The invariant $r(\theta)$ is defined by $r(\theta) = \mathrm{rk}\ (H^2)^\theta$. The intersection pairing defines a group $A_{(H^2)^\theta} = ((H^2)^\theta)^*/(H^2)^\theta$. Since the lattice H^2 is unimodular, this group is 2-elementary: $A_{(H^2)^\theta} \cong (\mathbf{Z}/2)^{a(\theta)}$. This defines the invariant $a(\theta)$. Another definition of this invariant gives the formula dim $(H^2/2H^2)^\theta = \mathrm{rk}\ H^2 - a(\theta)$.

Theorem 0.6. *Let Y be a real Enriques surface and $Y(\mathbf{R}) \neq \emptyset$.*

Then $b(Y) = 0$ iff the surface $Y(\mathbf{R})$ is connected non-orientable and for the invariants $r(\theta)$ and $a(\theta)$ we have the equality $r(\theta) = a(\theta)$.

By Theorem 0.1 and Proposition 0.3, for this surface Y the invariant $\epsilon(Y) = 1$ and dim ${}_2Br(Y) = 1$.

Proof. See Theorem 3.6.1.

The idea of the proof of Theorem 0.1 is to use the Kummer sequence and estimate the dimension of ${}_2Br(Y)$ using the Hochschild–Serre spectral sequence. The proof of Theorem 0.2 is not difficult and uses Lefschetz formula for fixed points and Smith exact sequence. The proof of the Theorems 0.5 and 0.6 is hard and uses results of [Ha1], [N3], [N4]. The most important ingredient of our proof is the use of the theory of involutions of lattices (integral quadratic forms) with condition on sublattice which was developed in [N4] (and also [N3]). We apply this theory to the action of antiholomorphic involutions on the 2-cohomology lattice of $Y(\mathbf{C})$ and its universal covering $K3$-surface $X(\mathbf{C})$. The proof of Theorems 0.2, 0.5 and 0.6 is a part of the general problem of the classification of real Enriques surfaces which is studied in [N6] (as an example, see Sect. 3.7 at the end of the paper).

All the results of this paper may be generalized to the following more general situation: Instead of $K3$-surfaces one should consider complex smooth projective algebraic surfaces X such that 2-torsion ${}_2\mathrm{Pic}\ X = 0$. Instead of Enriques surfaces one should consider real surfaces $Y = X/\{id, \tau\}$ where τ is a holomorphic involution of X without fixed points. For example, statements 0.1 – 0.4 are true in this case. The situation is more complicated with Theorems 0.5 and 0.6 (see §3.4 and Lemma 3.4.3) but it is similar. We hope to generalize the results here for this more general case in subsequent publications.

In Sect. 3.8, we cite further results on real Enriques surfaces from [N5] and [N6], which were obtained by the first author during the time this paper was considered for publication.

We are grateful to Prof. J–L. Colliot-Thélène for useful comments.

The first author worked on this article during his stay at the first half of 1991 in Tata Institute of Fundamental Research, Bombay; Steklov Mathematical Institute, Moscow and Institut des Hautes Études Scientifiques, Paris. He is grateful to these Institutes for hospitality. Preliminary variant of this paper was published as a preprint [N-S].

§1. Estimation of the 2-torsion of Brauer group of real Enriques surfaces

At first, we give some basic facts which we will use about complex Enriques surfaces.

We recall that a complex $K3$-surface is a smooth projective algebraic surface X over $\mathbf{C}$ such that $X(\mathbf{C})$ is a simply-connected complex surface and the canonical class $K_X = 0$. It follows that there exists a 2-dimensional regular differential form $\omega_X \in \Omega[X]$ such that the divisor $(\omega_X) = 0$. Thus, for an arbitrary point $z \in X(\mathbf{C})$ and local coordinates z_1, z_2 at a neighbourhood of z, we can write down this form as $f(z_1, z_2)dz_1 \wedge dz_2$ where $f(z_1, z_2)$ is a holomorphic function and $f(z) \neq 0$. This form ω_X is unique up to multiplication on elements of $\mathbf{C}$.

We recall basic facts about $K3$-surfaces X we will use (see G. N. Tjurina, [A, Ch. 9]). We have:

$$\begin{aligned} &H^0(X(\mathbf{C});\mathbf{Z}) \cong H^4(X(\mathbf{C});\mathbf{Z}) \cong \mathbf{Z}, \\ &H^1(X(\mathbf{C});\mathbf{Z}) \cong H^3(X(\mathbf{C});\mathbf{Z}) \cong 0, \\ &H^2(X(\mathbf{C});\mathbf{Z}) \cong \mathbf{Z}^{22}. \end{aligned} \tag{1–0}$$

Besides, the group $H^2(X(\mathbf{C});\mathbf{Z})$ with the intersection pairing is an even unimodular lattice of signature $(3, 19)$.

By definition, an Enriques surface Y over $\mathbf{C}$ is a minimal smooth projective algebraic surface over $\mathbf{C}$ with invariants $\varkappa(Y) = p_g(Y) = q(Y) = 0$ (see B. G. Averbuh, [A, Ch. 10]). An equivalent definition of Enriques surfaces is that an Enriques surface Y is the quotient $Y = X/\{\mathrm{id}, \tau\}$ of a $K3$-surface X by an algebraic involution τ without fixed points. In the paper, we will use the last definition of Enriques surfaces. Thus, $X(\mathbf{C})$ is the universal covering of $Y(\mathbf{C})$, and

$$\pi_1(Y(\mathbf{C})) = H_1(Y(\mathbf{C});\mathbf{Z}) \cong \mathbf{Z}/2. \tag{1–1}$$

By [N1, §5],

$$\tau^*(\omega_X) = -\omega_X. \tag{1–2}$$

for an involution τ without fixed points on a $K3$-surface. It follows that the canonical class of Y

$$K_Y \neq 0, \ \mathit{but}\ 2K_Y = 0. \tag{1–3}$$

Using (1–0), (1–1) and standard topological facts: Lefschetz fixed point formula, universal coefficient formula, Poincaré duality [Sp] and elementary facts about actions of finite groups [Br, Ch. II, Sects. 2, 3], one can prove easily that:

$$\begin{aligned} &H^0(Y(\mathbf{C});\mathbf{Z}) \cong \mathbf{Z},\ H^1(Y(\mathbf{C});\mathbf{Z}) = 0, \\ &H^2(Y(\mathbf{C});\mathbf{Z}) \cong \mathbf{Z}^{10} \oplus \mathbf{Z}/2, \\ &H^3(Y(\mathbf{C});\mathbf{Z}) \cong \mathbf{Z}/2,\ H^4(Y(\mathbf{C});\mathbf{Z}) \cong \mathbf{Z}; \end{aligned} \tag{1–4}$$

$$\begin{aligned} &H^0(Y(\mathbf{C});\mathbf{Z}/2) \cong \mathbf{Z}/2,\ H^1(Y(\mathbf{C});\mathbf{Z}/2) \cong \mathbf{Z}/2, \\ &H^2(Y(\mathbf{C});\mathbf{Z}/2) \cong (\mathbf{Z}/2)^{12}, \\ &H^3(Y(\mathbf{C});\mathbf{Z}/2) \cong \mathbf{Z}/2,\ H^4(Y(\mathbf{C});\mathbf{Z}/2) \cong \mathbf{Z}/2; \end{aligned} \tag{1–5}$$

and we have an isomorphism:

$$\pi^* : H^2(Y(\mathbf{C}); \mathbf{Z})/\mathrm{Tor} \to H^2(X(\mathbf{C}); \mathbf{Z})^\theta \tag{1–6}$$

where $\pi : X \to Y$ is the quotient morphism. Besides, since $H^1(Y(\mathbf{C}); \mathbf{Z}) = 0$, the characteristic class map gives an isomorphism $\mathrm{Pic}\ Y \cong H^2(Y(\mathbf{C}); \mathbf{Z})$. By Poincaré duality and Hodge index theorem, the $H^2(Y(\mathbf{C}); \mathbf{Z})/\mathrm{Tor}$ with intersection pairing is an unimodular lattice of signature $(1, 9)$. By the formula for the genus of a curve on an algebraic surface and (1–2), this lattice is even.

Now let us consider real Enriques surfaces. By definition, a real Enriques surface is a smooth projective algebraic surface Y over $\mathbf{R}$ such that $Y_\mathbf{C} = Y \otimes_\mathbf{R} \mathbf{C}$ is a complex Enriques surface.

Let $G = Gal(\mathbf{C}/\mathbf{R}) = \{\mathrm{id}, \theta\}$. The θ acts on $Y(\mathbf{C})$ as an antiholomorphic involution and $Y(\mathbf{R}) = Y(\mathbf{C})^G$. Here and as follows, $U^G = U^\theta$ denote the set of fixed elements for an action of a group G on a set U.

We recall [Mi] that the Kummer exact sequence

$$0 \to \mu_2 \to \mathbf{G}_m \to \mathbf{G}_m \to 0$$

yields the exact sequence

$$0 \to \mathrm{Pic}\ Y/2\mathrm{Pic}\ Y \to H^2_{et}(Y; \mu_2) \to {}_2Br(Y) \to 0. \tag{1–7}$$

Since $Y(\mathbf{R})$ is non-empty, $\mathrm{Pic}\ Y = (\mathrm{Pic}\ Y_\mathbf{C})^G$, (see [Ma]). (One can deduce this from Hochschild–Serre spectral sequence and the canonical isomorphisms $\mathrm{Pic}\ Y \simeq H^1_{et}(Y; \mathbf{G}_m)$, $\mathrm{Pic}\ Y_\mathbf{C} \simeq H^1_{et}(Y_\mathbf{C}; \mathbf{G}_m)$, see [Mi].) By (1–7), we then get

$$\dim\ {}_2Br(Y) = \dim\ H^2_{et}(Y; \mu_2) - \dim(\mathrm{Pic}\ Y_\mathbf{C})^G/2(\mathrm{Pic}\ Y_\mathbf{C})^G. \tag{1–8}$$

The dimension of the étale cohomology group $H^2_{et}(Y; \mu_2)$ is estimated using the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(G; H^q_{et}(Y_\mathbf{C}; \mu_2)) \Longrightarrow H^{p+q}_{et}(Y; \mu_2) = E^{p+q}.$$

where for a complex algebraic manifold $Y_\mathbf{C}$ we have

$$H^q_{et}(Y_\mathbf{C}; \mu_2) = H^q(Y(\mathbf{C}); \mathbf{Z}/2),$$

see [Mi].

The following invariant of a real Enriques surface is very important. We recall $H^1(Y(\mathbf{C}); \mathbf{Z}/2) \simeq \mathbf{Z}/2$, by (1–5).

Definition 1.1. For a real Enriques surface Y an invariant $\epsilon(Y) = 1$ if the differential $d_2^{0,2} : E_2^{0,2} = H^0(Y(\mathbf{C}); \mathbf{Z}/2)^G \to E_2^{2,1} = H^2(G; H^1(Y(\mathbf{C}); \mathbf{Z}/2)) \simeq \mathbf{Z}/2$ of the Hochschild–Serre spectral sequence vanishes, and $\epsilon(Y) = 0$ otherwise.

We then have the following Lemma.

Lemma 1.1. *Let Y be a real Enriques surface and $Y(\mathbf{R}) \neq \emptyset$. Then*

$$\dim H^2_{et}(Y;\mu_2) = 1 + \dim H^2(Y(\mathbf{C});\mathbf{Z}/2)^G + \epsilon(Y).$$

Proof. Denoting $H^2_{et}(Y;\mu_2)$ by E^2 and using the usual spectral sequence notation, we see that

(1–9) $$\dim\ E^2 = \dim\ E^{2,0}_\infty + \dim\ E^{1,1}_\infty + \dim\ E^{0,2}_\infty.$$

We calculate $\dim\ E^{2,0}_\infty$, $\dim\ E^{1,1}_\infty$ and $\dim\ E^{0,2}_\infty$ below.

Since $Y(\mathbf{R})$ is not empty, consider a real point $P \in Y(\mathbf{R})$. By functoriality, this induces a spectral sequence homomorphism,

$$H^p(G;H^q(Y(\mathbf{C});\mathbf{Z}/2)) \to H^p(G;H^q(P;\mathbf{Z}/2)).$$

But the homomorphism

$$H^p(G;H^0(Y(\mathbf{C});\mathbf{Z}/2)) \to H^p(G;H^0(P;\mathbf{Z}/2))$$

is an isomorphism and the spectral sequence for the point is trivial. Therefore in the spectral sequence for Y, the differentials $d_r^{p-r,r-1}$ are identically zero. This implies that $E^{2,0}_\infty \simeq E^{2,0}_2 \simeq \mathbf{Z}/2$, and by (1–5), $E^{1,1}_\infty \simeq E^{1,1}_2 \simeq \mathbf{Z}/2$. Thus, $\dim\ E^{2,0}_\infty = \dim\ E^{1,1}_\infty = 1$. Further, by the above remark, the differential $d^{0,2}_3 : E^{0,2}_3 \to E^{3,0}_3$ is zero, and we have $E^{0,2}_3 \simeq E^{0,2}_\infty$ and an exact sequence

$$0 \to E^{0,2}_\infty \to E^{0,2}_2 \xrightarrow{d^{0,2}_2} E^{2,1}_2$$

where by (1–5), $E^{2,1}_2 = H^2(G;\mathbf{Z}/2) \simeq \mathbf{Z}/2$. Thus, by the definition 1.1,

(1–10) $$\dim\ E^{0,2}_2 = \dim\ E^{0,2}_\infty + 1 - \epsilon(Y).$$

By definition, $E^{0,2}_2 = H^0(G;H^2(Y(\mathbf{C});\mathbf{Z}/2)) = H^2(Y(\mathbf{C});\mathbf{Z}/2)^G$, and by (1–10),

$$\dim\ E^{0,2}_\infty = \dim\ H^2(Y(\mathbf{C});\mathbf{Z}/2)^G - 1 + \epsilon(Y).$$

This proves the Lemma.

Definition 1.2. We denote by $b(Y)$ the following invariant of a real Enriques surface Y with an antiholomorphic involution θ:

$$b(Y) = \dim H^2(Y(\mathbf{C});\mathbf{Z}/2)^\theta - \dim(\text{Pic } Y_\mathbf{C})^\theta/2(\text{Pic } Y_\mathbf{C})^\theta + 1.$$

By Lemma 1.1 and (1–8), we get the following

Theorem 1.2. *Let Y be a real Enriques surface with an antiholomorphic involution θ and $Y(\mathbf{R}) \neq \emptyset$.*

Then

$$\dim\ {}_2Br(Y) = b(Y) + \epsilon(Y)$$

where $\epsilon(Y) = 0$ or 1. See Definitions 1.1 and 1.2 above.

Theorem 1.2 shows that to estimate $\dim\ {}_2Br(Y)$ it is very important to estimate the invariant $b(Y)$. The following §2 and §3 will be devoted to this problem. We want to connect $b(Y)$ with the numbers s, s_{nor} and s_{or} of connected, non-orientable and orientable components of $Y(\mathbf{R})$.

§2. Inequalities: $b(Y) \geq 2s - 2$, $\dim\ {}_2Br(Y) \geq 2s - 2 + \epsilon(Y)$

For a real rational surface X with $X(\mathbf{R}) \neq \emptyset$ we have [Su]: $\dim\ {}_2Br(X) = 2s - 1$. In this paper, for a real Enriques surface Y, we can only prove similar inequalities.

Theorem 2.1. *Let Y be an arbitrary real Enriques surface.*

Then $b(Y) \geq 2s - 2$.

It follows (by Theorem 1.2) that $\dim\ {}_2Br(Y) \geq 2s - 2 + \epsilon(Y)$.

By Theorem 2.1 and Proposition 0.3, we also get the following result which is interesting because for a real Enriques surface we don't know that the homomorphism (0–1) is an epimorphism.

Corollary 2.2. *Let Y be an arbitrary real Enriques surface and s the number of connected components of $Y(\mathbf{R})$.*

Then $\dim\ {}_2Br(Y) \geq s$.

Proof of Theorem 2.1. We introduce an important invariant of a real Enriques surface Y with an antiholomorphic involution θ:

$$r(\theta) = \mathrm{rk}\ (H^2(Y(\mathbf{C}); \mathbf{Z})/\mathrm{Tor})^{\theta}. \tag{2–1}$$

We prove the following formula for an arbitrary real Enriques surface:

$$\dim(\mathrm{Pic}\ Y_{\mathbf{C}})^{\theta}/2(\mathrm{Pic}\ Y_{\mathbf{C}})^{\theta} = 11 - r(\theta). \tag{2–2}$$

By (1–4), $H^1(Y(\mathbf{C}); \mathbf{R}) = 0$. Thus, we have the canonical isomorphism of characteristic class

$$\mathrm{Pic}\ Y_{\mathbf{C}} \cong H^2(Y(\mathbf{C}); \mathbf{Z}).$$

For this isomorphism, the action of the involution θ on the Picard group goes to the action of the involution $-\theta$ on the $H^2(Y(\mathbf{C}); \mathbf{Z})$ (this is well-known, for example, see [Si, Sect. I, 4]). It follows therefore, that we have an isomorphism,

$$(\mathrm{Pic}\ Y_{\mathbf{C}})^{\theta} \cong H^2(Y(\mathbf{C}); \mathbf{Z})_{\theta}.$$

Here for a module M and an involution θ of this module we denote $M_{\theta} = \{x \in M \mid \theta(x) = -x\}$.

By (1–4), $H^2(Y(\mathbf{C}); \mathbf{Z}) \cong \mathbf{Z}/2 \oplus \mathbf{Z}^{10}$ where

$$\mathbf{Z}^{10} \cong H^2(Y(\mathbf{C}); \mathbf{Z})/\mathrm{Tor}\ H^2(Y(\mathbf{C}); \mathbf{Z}).$$

We use the following obvious statement: Let $M = \mathbf{Z}/2 \oplus \mathbf{Z}^m$ and a group G of order two acts on M. Then $M^G \cong \mathbf{Z}/2 \oplus \mathbf{Z}^n$ if $(M/\mathrm{Tor}\ M)^G \cong \mathbf{Z}^n$.

By definition of the invariant $r(\theta)$, it follows that

$$\mathrm{rk}\ (H^2(Y(\mathbf{C}); \mathbf{Z})/Tor)_{\theta} = 10 - r(\theta).$$

By the remark above, we get the statement (2–2).

By (1–4), Lefschetz's fixed point formula (see [Sp, Ch. 5, Sect. 7, Theorem 6]) gives for the involution θ

$$\chi(Y(\mathbf{R})) = 2r(\theta) - 8$$

It follows that

$$r(\theta) = (1/2)\chi(Y(\mathbf{R})) + 4. \tag{2–3}$$

From [Kr, Theorem 2.3] or from Smith exact sequence (see [Br, Ch.3]), we have

$$\dim\ H^*(Y(\mathbf{R});\mathbf{Z}/2) \leq \dim\ H^1(G; H^*(Y(\mathbf{C});\mathbf{Z}/2)).$$

Here $H^1(G; H^*(Y(\mathbf{C});\mathbf{Z}/2)) = H^*(Y(\mathbf{C});\mathbf{Z}/2)^\theta/(1+\theta)H^*(Y(\mathbf{C});\mathbf{Z}/2)$. From the exact sequence

$$0 \to H^*(Y(\mathbf{C});\mathbf{Z}/2)^\theta \to H^*(Y(\mathbf{C});\mathbf{Z}/2) \to (1+\theta)H^*(Y(\mathbf{C});\mathbf{Z}/2) \to 0$$

we get $\dim\ H^1(G; H^*(Y(\mathbf{C});\mathbf{Z}/2)) = 2\dim\ H^*(Y(\mathbf{C});\mathbf{Z}/2)^\theta - \dim H^*(Y(\mathbf{C});\mathbf{Z}/2)$. Thus, from the inequality above, we have

$$\dim\ H^*(Y(\mathbf{R});\mathbf{Z}/2) \leq 2\dim\ H^*(Y(\mathbf{C});\mathbf{Z}/2))^\theta - \dim\ H^*(Y(\mathbf{C});\mathbf{Z}/2).$$

By (1–5), it follows that

$$\dim H^2(Y(\mathbf{C});\mathbf{Z}/2)^\theta \geq 4 + (1/2)\dim H^*(Y(\mathbf{R});\mathbf{Z}/2). \tag{2–4}$$

From (2–2),

$$\dim H^2(Y(\mathbf{C});\mathbf{Z}/2)^\theta - \dim(\mathrm{Pic}\ Y_{\mathbf{C}})^\theta/2(\mathrm{Pic}\ Y_{\mathbf{C}})^\theta + 1 =$$

$$= \dim H^2(Y(\mathbf{C});\mathbf{Z}/2)^\theta + r(\theta) - 10. \tag{2–5}$$

Thus, from (2–3), (2–4) and (2–5), we get that

(2–6)

$$\begin{aligned} b(Y) = \dim H^2(Y(\mathbf{C});\mathbf{Z}/2)^\theta - \dim(\mathrm{Pic}\ Y_{\mathbf{C}})^\theta/2(\mathrm{Pic}\ Y_{\mathbf{C}})^\theta + 1 \\ \geq (1/2)\dim H^*(Y(\mathbf{R});\mathbf{Z}/2) + (1/2)\chi(Y(\mathbf{R})) - 2. \end{aligned}$$

For a closed connected surface F, we have $\dim H^*(F;\mathbf{Z}/2) + \chi(F) = 4$. Thus, the right side of the inequality (2–6) is equal to $2s - 2$.

This proves the Theorem 2.1.

§3. Calculation of the invariant $b(Y)$ using universal covering $K3$-surface X

3.1. Notation.

We use the notations set up in the previous §1 and §2. Thus, Y is a real Enriques surface, $Y(\mathbf{R})$ is the set of real points of Y, and $Y(\mathbf{C})$ the set of complex points with the corresponding antiholomorphic involution θ. We want to obtain in this section a precise formula for the invariant $b(Y)$ (see Definition 1.2) using the universal covering $K3$-surface X.

3.2. Real part of Enriques surface Y and the universal covering $K3$-surface. By definition (see the beginning of §1), an Enriques surface is a quotient surface $Y_{\mathbf{C}} = X/\{\mathrm{id}, \tau\}$ where X over $\mathbf{C}$ is a complex $K3$-surface and τ an algebraic involution of X without fixed points. Let $\pi : X \to Y_{\mathbf{C}}$ be the quotient morphism. Since $X(\mathbf{C})$ is simply-connected, $\pi : X(\mathbf{C}) \to Y(\mathbf{C})$ is the 2-sheeted universal covering with the holomorphic involution τ of the covering. Thus, $\pi\tau = \pi$.

Let θ be the antiholomorphic involution of the complex surface $Y(\mathbf{C})$ corresponding to the real surface Y. Let us suppose that

$$Y(\mathbf{R}) \neq \emptyset$$

and $\bar{y} \in Y(\mathbf{R})$. Let $y \in X(\mathbf{C})$ such that $\pi(y) = \bar{y}$. Since π is the universal covering, there exists an antiholomorphic automorphism σ of $X(\mathbf{C})$ such that $\pi\sigma = \theta\pi$ and $\sigma(y) = y$. It follows therefore that σ^2 is holomorphic, $\sigma^2(y) = y$, and $\pi\sigma^2 = \pi$. Thus, σ^2 is an element of the group of the unramified covering π. Since σ^2 has the fixed point y, $\sigma^2 = id$. Hence it follows that σ is an antiholomorphic involution of $X(\mathbf{C})$ with the fixed point $y \in X_\sigma(\mathbf{R}) = X(\mathbf{C})^\sigma$. Evidently, the point $\tau(y) \in X_\sigma(\mathbf{R})$ too. Here we denote by X_σ the real $K3$-surface defined by the antiholomorphic involution σ. Thus, $X_\sigma(\mathbf{R})$ is the set of real points of X_σ.

The composition $\tilde{\sigma} = \tau\sigma$ is also an antiholomorphic automorphism of $X(\mathbf{C})$, $\pi\tilde{\sigma} = \theta\pi$, and $\tilde{\sigma}(y) = \tau(y)$. It follows that $\tilde{\sigma}^2 = id$. Thus, $\tilde{\sigma} = \tau\sigma$ is another lifting on $X(\mathbf{C})$ of the antiholomorphic involution θ of $Y(\mathbf{C})$. Thus, $\tau\sigma$ defines another real $K3$-surface $X_{\tau\sigma}$ with the set of real points $X_{\tau\sigma}(\mathbf{R}) = X(\mathbf{C})^{\tau\sigma}$. As in the case of $X_\sigma(\mathbf{R})$, the real part $X_{\tau\sigma}(\mathbf{R})$ is invariant by the action of τ. It is obvious that there do not exist other liftings of θ to X except σ and $\tau\sigma$.

Other speaking, we lift the group $G = \{\mathrm{id}, \theta\}$ of order 2 on Y to the group $\Gamma = \{\mathrm{id}, \tau, \sigma, \tau\sigma\}$ of order 4 on X. The group $\Gamma \cong (\mathbf{Z}/2)^2$. It is difficult to connect the action of G on cohomology of Y with the real part $Y(\mathbf{R})$. Therefore, we want to connect the action of Γ on cohomology of X with the $X_\sigma(\mathbf{R})$, $X_{\tau\sigma}(\mathbf{R})$ and $Y(\mathbf{R})$.

The real parts $X_\sigma(\mathbf{R})$ and $X_{\tau\sigma}(\mathbf{R})$ have no common point, since such a point would be the fixed point of the involution τ, but τ has no fixed point in $X(\mathbf{C})$.

Let ω_X be a non-zero 2-dimensional holomorphic differential form on X (see the beginning of §1). This form is unique up to multiplication by non-zero elements of $\mathbf{C}$ and it has no zeros on $X(\mathbf{C})$. (Roughly speaking, it is a complex volume form.) By (1–2), $\tau^*(\omega_X) = -\omega_X$. We can choose the form ω_X by the condition that $\sigma^*(\omega_X) = \overline{\omega_X}$. This defines the form ω_X up to multiplication by real numbers. We denote this form as ω_X^σ. The form ω_X^σ gives the canonical (up to multiplication by non-zero reals) volume form on the real part $X_\sigma(\mathbf{R})$ and defines the canonical orientation on it (up to a change of this orientation on all connected components of $X_\sigma(\mathbf{R})$ simultaneously). See [N2, Theorem 3.10.6] for details. Since $\tau^*(\omega_X^\sigma)$

$-\omega_X^\sigma$, the involution τ restricted on $X_\sigma(\mathbf{R})$ changes this orientation. The same statement holds for the real part $X_{\tau\sigma}(\mathbf{R})$. Thus, we have proved the following very important

Lemma 3.2.1. *Let Y be a real Enriques surface, θ the corresponding antiholomorphic involution on $Y(\mathbf{C})$ and the real part $Y(\mathbf{R}) \neq \emptyset$. Let $\pi : X(\mathbf{C}) \to Y(\mathbf{C})$ be the universal two sheeted covering where $X(\mathbf{C})$ is $K3$-surface, and τ the involution of the covering π.*

There are precisely two liftings σ and $\tilde{\sigma} = \tau\sigma$ of θ to antiholomorphic involutions of $X(\mathbf{C})$. Let $X_\sigma(\mathbf{R}) = X(\mathbf{C})^\sigma$ and $X_{\tau\sigma}(\mathbf{R}) = X(\mathbf{C})^{\tau\sigma}$ be the real parts corresponding to these antiholomorphic involutions equipped with canonical orientations defined by the canonical volume forms ω_X^σ and $\omega_X^{\tau\sigma}$ respectively. Then $X_\sigma(\mathbf{R}) \cap X_{\tau\sigma}(\mathbf{R}) = \emptyset$, $\tau(X_\sigma(\mathbf{R})) = X_\sigma(\mathbf{R})$ and $\tau(X_{\tau\sigma}(\mathbf{R})) = X_{\tau\sigma}(\mathbf{R})$, and the involution τ changes the canonical orientation of this sets (defined by volume forms ω_X^σ and $\omega_X^{\tau\sigma}$ respectively) to the opposite one. Moreover, with respect to this action of τ the morphism π is the factorisation morphism which defines the canonical identification

$$Y(\mathbf{R}) = X_\sigma(\mathbf{R})/\{id, \tau\} \coprod X_{\tau\sigma}(\mathbf{R})/\{id, \tau\}.$$

In particular, for the numbers $s(\sigma)$ and $s(\tau\sigma)$ of connected components of $X_\sigma(\mathbf{R})$ and $X_{\tau\sigma}(\mathbf{R})$ respectively, and the numbers s_{or} and s_{nor} of orientable and non-orientable connected components of $Y(\mathbf{R})$ respectively, we have the equality

$$s(\sigma) + s(\tau\sigma) = s_{nor} + 2s_{or}.$$

In the §2, we connected the invariant $b(Y)$ with the number $s = s_{or} + s_{nor}$. In the following sections, we want to connect the $b(Y)$ with the number $s_{nor} + 2s_{or}$. Lemma 3.2.1 is important for us, because, by this Lemma, the number $s_{nor} + 2s_{or}$ is the same as $s(\sigma) + s(\tau\sigma)$.

3.3. Invariants of the action of $\Gamma = \{\mathbf{id}, \tau, \sigma, \tau\sigma\}$ on $H^2(X(\mathbf{C}); \mathbf{Z})$ and the sets $X_\sigma(\mathbf{R})$ and $X_{\tau\sigma}(\mathbf{R})$.

Henceforth, let Y be a real Enriques surface such that the antiholomorphic involution θ of $Y(\mathbf{C})$ has two liftings σ and $\tau\sigma$ to antiholomorphic involutions of the $K3$-surface $X(\mathbf{C})$. By Lemma 3.2.1, this is true if $Y(\mathbf{R}) \neq \emptyset$.

Let L be a lattice $H^2(X(\mathbf{C}); \mathbf{Z})$ with the intersection pairing. The L is an even unimodular lattice of signature (3,19) (see the beginning of §1).

For the module M and an involution ϕ of this module we denote

$$M^\phi = \{x \in M \mid \phi(x) = x\}, \qquad M_\phi = \{x \in M \mid \phi(x) = -x\}.$$

Let ϕ be an involution on a complex $K3$-surface. We suppose that ϕ is either holomorphic with the condition $\phi^*\omega_X = -\omega_X$ or antiholomorphic. This involution ϕ acts on L. As for any involution of an even unimodular lattice, invariants

$$(r(\phi), a(\phi), \delta(\phi))$$

of this action are defined (see [N3, §3]). Here

(3.3.1) $$r(\phi) = \operatorname{rk} L^\phi$$

It is known (see [N2, §4] for the holomorphic case and [H, appendix], [N3, §3] for the antiholomorphic one) that the lattice L^ϕ has the signature $(1, r(\phi)-1)$, i.e. it is hyperbolic.

Since the lattice L is unimodular, the lattice L^ϕ is 2-elementary. This means that the discriminant group $A_{L^\phi} = (L^\phi)^*/L^\phi$ is isomorphic to a 2 -elementary group:

$$A_{L^\phi} = (L^\phi)^*/L^\phi \simeq (\mathbf{Z}/2)^{a(\phi)}. \tag{3–3–2}$$

(3–3–3) The invariant $\delta(\phi)$ is equal to 0 if the discriminant quadratic form q_{L^ϕ} of the lattice L^ϕ is even, i.e. $q_{L^\phi}(u) \in \mathbf{Z}/2\mathbf{Z}$ for any $u \in A_{L^\phi}$, and $\delta(\phi) = 1$ otherwise.

Here and as follows we denote by A_S the discriminant group $A_S = S^*/S$ of a lattice S and by q_S and b_S the discriminant quadratic and bilinear form of the lattice S (the discriminant quadratic form is defined for an even lattice S only). See [N3, §1]. We recall that these forms are defined by extension of the form of the lattice S to S^*. The discriminant quadratic form q_S takes values in $\mathbf{Q}/2\mathbf{Z}$. And b_S in $\mathbf{Q}/\mathbf{Z}$.

These invariants $(r(\phi), a(\phi), \delta(\phi))$ define the involution ϕ uniquely up to automorphisms of the unimodular lattice L (see [N3, §3]). Also, they define the topology of the fixed part of the action of ϕ on $X(\mathbf{C})$ (see [N2, §4] for the holomorphic case and [H, appendix] and [N3, §3] for the antiholomorphic one):

$$X(\mathbf{C})^\phi = \begin{cases} \emptyset, & \text{if } (r(\phi), a(\phi), \delta(\phi)) = (10, 10, 0); \\ 2T_1, & \text{if } (r(\phi), a(\phi), \delta(\phi)) = (10, 8, 0); \\ T_{g(\phi)} \amalg k(\phi) T_0, & \text{where } g(\phi) = (22 - r(\phi) - a(\phi))/2, \\ & k(\phi) = (r(\phi) - a(\phi))/2 \text{ otherwise.} \end{cases} \tag{3–3–4}$$

Here T_g is a real orientable compact surface of genus g.

We mention the basic formulae necessary to get (3–3–4).

The invariant $\delta(\phi)$ has the following geometrical sense:

$$X(\mathbf{C})^\phi \sim 0 \mod 2 \ in\ H_2(X(\mathbf{C}); \mathbf{Z})\ iff\ \delta(\phi) = 0. \tag{3–3–5}$$

In particular, it follows that the invariant $\delta(\phi) = 0$ if $X(\mathbf{C})^\phi = \emptyset$.

Using (1–0), from the Lefschetz fixed point formula (see [Sp, Ch. 5, Sect. 7, Theorem 6]),

$$r(\phi) = \chi(X(\mathbf{C})^\phi)/2 + 10. \tag{3–3–6}$$

From [H, Lemma 3.7] (it is the consequence of Smith exact sequence applied to the involution ϕ, see [Br, Ch.III]),

$$\dim H^*(X(\mathbf{C})^\phi; \mathbf{Z}/2) = \dim H^*(X(\mathbf{C}); \mathbf{Z}/2) - 2a(\phi),$$

if $X(\mathbf{C})^\phi \neq \emptyset$. It follows therefore that

$$a(\phi) = \begin{cases} 12 - \dim H^*(X(\mathbf{C})^\phi; \mathbf{Z}/2)/2, & \text{if } X(\mathbf{C})^\phi \neq \emptyset, \\ 10, & \text{if } X(\mathbf{C})^\phi = \emptyset. \end{cases} \tag{3–3–7}$$

For any connected closed surface F (orientable or not) we have the formula

$$\chi(F)/4 + \dim H^*(F; \mathbf{Z}/2)/4 = 1$$

Then, from (3–3–6) and (3–3–7), for the number $s(\phi)$ of connected components of $X(\mathbf{C})^{\phi}$, we get the following formula:

$$s(\phi)=\begin{cases}(r(\phi)-a(\phi))/2+1 & \text{if } s(\phi)>0,\\ (r(\phi)-a(\phi))/2 & \text{if } s(\phi)=0.\end{cases} \tag{3–3–8}$$

Since the involution τ has no fixed points, we get

$$(r(\tau),a(\tau),\delta(\tau))=(10,10,0).$$

Further, we want to connect the invariants

$$(r(\sigma),a(\sigma),\delta(\sigma)),\ and\ (r(\tau\sigma),a(\tau\sigma),\delta(\tau\sigma))$$

of the involutions σ and $\tau\sigma$.

Let L be an even unimodular lattice, $S\subset L$ a primitive sublattice of L and $\theta\mid S$ an involution of S. In [N4] all genus invariants of an extension of θ to an involution σ of the unimodular lattice L were found and studied. In [N4] it was formulated as a studying of triplets

$$(L,S,\sigma).$$

Here L is an even unimodular lattice, S is a primitive sublattice of L (i.e. L/S is free) and σ an involution of L such that $\sigma(S)=S$ and $\sigma\mid S=\theta$ where θ is a fixed involution of S. We apply this theory to the triplet (L,L^{τ},σ).

We give some invariants from [N4] of the triplet (L,L^{τ},σ) which are necessary for us to prove our results here. The full investigation of the invariants from [N4] for real Enriques surfaces is in [N6].

At first, we have to study invariants of the action $\theta=\sigma\mid L^{\tau}$.

The lattice L^{τ} is an even 2-elementary lattice with invariants $r(\tau)=\text{rk } L^{\tau}=10$, $a(\tau)=\dim(L^{\tau})^*/L^{\tau}=10$ and an even discriminant 2-elementary form $q_{L^{\tau}}$ (see (3–3–3)). It follows that the lattice $L^{\tau}(1/2)$ is an even unimodular lattice of signature (1,9). Here we use the following notation:

Notation 3.3.1. For a lattice M, the lattice $M(a)$ is defined by multiplying the form of the lattice M by $a\in\mathbf{Q}$.

Thus, the restriction of the involution σ on the lattice L^{τ} is defined by the action of σ on the unimodular lattice $L^{\tau}(1/2)$. By (1–6), we have the canonical isomorphism of lattices:

$$\pi^*:\ H^2(Y(\mathbf{C});\mathbf{Z})/\text{Tor}\ \cong L^{\tau}(1/2).$$

because $\pi^*(x)\cdot\pi^*(y)=2x\cdot y$ for $x,y\in H^2(Y(\mathbf{C});\mathbf{Z})$. The map π^* is equivariant with respect to the action of θ and its lifting σ. Hence it follows that the action of σ on L^{τ} is the same as the action of θ on $H^2(Y(\mathbf{C});\mathbf{Z})/\text{Tor}$. Thus, our notation $\theta=\sigma\mid L^{\tau}$ agrees with our previous notation θ for an antiholomorphic involution of Y. For the involution $\theta=\sigma\mid L^{\tau}$ on the even unimodular lattice $L^{\tau}(1/2)$, we have a similar triplet of invariants as above (for σ and $\tau\sigma$ on L):

$$(r(\theta),a(\theta),\delta(\theta)) \tag{3–3–9}$$

where

$$r(\theta) = \text{rk } L^{\tau,\sigma}$$

(we denote $L^{\tau,\sigma} = (L^\tau)^\sigma$ and $L_{\tau,\sigma} = (L_\tau)_\sigma$);

$$(L^{\tau,\sigma}(1/2))^*/L^{\tau,\sigma}(1/2) \cong (\mathbf{Z}/2)^{a(\theta)};$$

$\delta(\theta)$ is zero if the discriminant quadratic form $q_{L^{\tau,\sigma}(1/2)}$ takes values in $\mathbf{Z}/2\mathbf{Z}$. Otherwise, $\delta(\theta) = 1$.

Since the lattices L^τ, L^σ and $L^{\tau\sigma}$ are hyperbolic, the lattice $L^{\tau,\sigma}$ is negative and the lattice L^τ_σ is hyperbolic of rk $L^\tau_\sigma = 10 - r(\theta)$. Using results of [N3] (or [N4]) one can describe very easily all possibilities for triplets of invariants $(r(\theta), a(\theta), \delta(\theta))$. They are:

(3–3–10)
$$\begin{aligned}(r(\theta), a(\theta), \delta(\theta)) =&(0,0,0),\ (1,1,1),\ (2,2,1),\ (3,3,1),\ (4,2,0),\ (4,4,1),\\ &(5,3,1),\ (5,5,1),\ (6,2,1),\ (6,4,1),\ (7,1,1),\ (7,3,1),\\ &(8,0,0),\ (8,2,0),\ (8,2,1),\ (9,1,1).\end{aligned}$$

We need another interpretation of the invariants $(r(\theta), a(\theta), \delta(\theta))$ using the lattices $L^{\tau,\sigma}$ and L^τ_σ. We give it below.

For the invariant $r(\theta)$ we evidently have $r(\theta) = \text{rk } L^{\tau,\sigma}$, rk $L^\tau_\sigma = 10 - r(\theta)$.

We have the canonical subgroup

$$\Gamma(L^\tau(1/2)) \subset A_{L^{\tau,\sigma}(1/2)} \oplus A_{L^\tau_\sigma(1/2)},$$

where $\Gamma(L^\tau(1/2)) = L^\tau(1/2)/(L^{\tau,\sigma}(1/2) \oplus L^\tau_\sigma(1/2))$. For the discriminant form $q_{L^{\tau,\sigma}(1/2)} \oplus q_{L^\tau_\sigma(1/2)}$ on the group $A_{L^{\tau,\sigma}(1/2)} \oplus A_{L^\tau_\sigma(1/2)}$ the subgroup $\Gamma(L^\tau(1/2))$ is isotropic and

$$(\Gamma(L^\tau(1/2)))^\perp = \Gamma(L^\tau(1/2))$$

since the lattice $L^\tau(1/2)$ is unimodular. Moreover,

$$\Gamma(L^\tau(1/2)) \cap A_{L^{\tau,\sigma}(1/2)} = \Gamma(L^\tau(1/2)) \cap A_{L^\tau_\sigma(1/2)} = \{0\}.$$

Let p_1 and p_2 be the projections on $A_{L^{\tau,\sigma}(1/2)}$ and $A_{L^\tau_\sigma(1/2)}$ respectively. Then the map $(p_2 \mid \Gamma(L^\tau(1/2)))(p_1)^{-1}$ is an isomorphism of the discriminant quadratic forms $q_{L^{\tau,\sigma}(1/2)}$ and $-q_{L^\tau_\sigma(1/2)}$ on the groups $A_{L^{\tau,\sigma}(1/2)}$ and $A_{L^\tau_\sigma(1/2)}$ respectively. One sees very easily by considering an orthogonal decomposition of 2-adic lattices $L^{\tau,\sigma} \otimes \mathbf{Z}_2$, $L^\tau_\sigma \otimes \mathbf{Z}_2$ as a sum of elementary lattices of rank one or two, that we have the following identifications where for a group A we denote $A^{(1)} = Ann(2\ in\ A)$:

$$A_{L^{\tau,\sigma}(1/2)} = (L^{\tau,\sigma}(1/2))^*/L^{\tau,\sigma}(1/2) = 2(L^{\tau,\sigma})^*/L^{\tau,\sigma} =$$

$$= 2A_{L^{\tau,\sigma}} = \text{Ker } (b_{L^{\tau,\sigma}} \mid A^{(1)}_{L^{\tau,\sigma}}).$$

The same is true for the lattice L^τ_σ. We have

$$A_{L^\tau_\sigma(1/2)} = (L^\tau_\sigma(1/2))^*/L^\tau_\sigma(1/2) = 2(L^\tau_\sigma)^*/L^\tau_\sigma$$

$$= 2A_{L^\tau_\sigma} = \mathrm{Ker}\ (b_{L^\tau_\sigma} \mid A^{(1)}_{L^\tau_\sigma}).$$

Let us denote

$$\Gamma(\sigma)_+ = 2A_{L^{\tau,\sigma}} = \mathrm{Ker}\ (b_{L^{\tau,\sigma}} \mid A^{(1)}_{L^{\tau,\sigma}})$$

and

$$\Gamma(\sigma)_- = 2A_{L^\tau_\sigma} = \mathrm{Ker}\ (b_{L^\tau_\sigma} \mid A^{(1)}_{L^\tau_\sigma}).$$

Moreover, arguing similarly, we see that

$$q_{L^{\tau,\sigma}} \mid \Gamma(\sigma)_+ = 0 \ \ iff \ \ \delta(\theta) = 0.$$

And

$$q_{L^\tau_\sigma} \mid \Gamma(\sigma)_- = 0 \ \ iff \ \ \delta(\theta) = 0.$$

Thus, we get the following interpretation:

Lemma 3.3.1. *Let Y be an arbitrary real Enriques surface with an antiholomorphic involution θ. Let $(r(\theta), a(\theta), \delta(\theta))$ are invariants described above of the action of θ on the lattice $H^2(Y(\mathbf{C}); \mathbf{Z})/Tor$ (equivalently, invariants of the action of the lifting σ of θ on the lattice $L^\tau(1/2)$).*

Then: $r(\theta) = rk\ L^{\tau,\sigma}$ and $rk\ L^\tau_\sigma = 10 - r(\theta)$.

For the subgroup

$$\Gamma_{12} = L^\tau/(L^{\tau,\sigma} \oplus L^\tau_\sigma) \subset A_{L^{\tau,\sigma}} \oplus A_{L^\tau_\sigma}$$

and the projections p_1 and p_2 on the groups $A_{L^{\tau,\sigma}}$ and $A_{L^\tau_\sigma}$ respectively we have:

$$p_1 : (\Gamma_{12}) \cong \Gamma(\sigma)_+ = Ker\ b_{L^{\tau,\sigma}} \mid A^{(1)}_{L^{\tau,\sigma}} \cong (\mathbf{Z}/2)^{a(\theta)}$$

and

$$p_2 : (\Gamma_{12}) \cong \Gamma(\sigma)_- = Ker\ b_{L^\tau_\sigma} \mid A^{(1)}_{L^\tau_\sigma} \cong (\mathbf{Z}/2)^{a(\theta)}$$

(where $A^{(1)} = Ann(2)$ for an abelian group A).

The invariant $\delta(\theta) = 0$ iff $q_{L^{\tau,\sigma}} \mid \Gamma(\sigma)_+ = 0$, equivalently, $q_{L^\tau_\sigma} \mid \Gamma(\sigma)_- = 0$).

Besides, we have the following property:

$$b_{L^{\tau,\sigma}}(x,x) = 0 \ for\ any\ x \in A^{(1)}_{L^{\tau,\sigma}};$$

and

$$b_{L^\tau_\sigma}(x,x) = 0 \ for\ any\ x \in A^{(1)}_{L^\tau_\sigma}.$$

Proof. We should only explain the last statement. This is true because the lattice $L^\tau(1/2)$ is even: $z^2 \equiv 0 \mod 2$ for any $z \in L^\tau(1/2)$.

Thus, we studied the action $\theta = \sigma \mid L^\tau = \tau\sigma \mid L^\tau$ introducing the invariants $(r(\theta), a(\theta), \delta(\theta))$ above. Now we fix these invariants (one of triplets (3–3–10)) and want to give some invariants of extension σ of θ on the unimodular lattice L. Thus, we choose an involution between σ and $\tau\sigma$.

We consider the decomposition $L \supset L^\sigma \oplus L_\sigma$ of finite index. Up to automorphisms of L this is defined by the invariants

$$(r(\sigma), a(\sigma), \delta(\sigma)) \tag{3–3–11}$$

we had introduced above (see (3–3–1), (3–3–2), (3–3–3)). These invariants are the most important invariants of the extension σ of θ. But there are some other invariants we want to describe.

Let us consider the corresponding decomposition of the discriminant groups $A_{L^\sigma} \oplus A_{L_\sigma}$ with the discriminant form $q_{L^\sigma} \oplus q_{L_\sigma}$. The subgroup $\Gamma(L^\sigma, L_\sigma) = L/(L^\sigma \oplus L_\sigma) \subset A_{L^\sigma} \oplus A_{L_\sigma}$ is isotropic, $\Gamma(L^\sigma, L_\sigma)^\perp = \Gamma(L^\sigma, L_\sigma)$ and $\Gamma(L^\sigma, L_\sigma) \cap A_{L^\sigma} = \Gamma(L^\sigma, L_\sigma) \cap A_{L_\sigma} = \{0\}$. It follows that $\Gamma(L^\sigma, L_\sigma)$ is the graph of the isomorphism $\gamma(L^\sigma, L_\sigma) : q_{L^\sigma} \cong -q_{L_\sigma}$ of the discriminant quadratic forms. Using this isomorphism, we identify discriminant groups and quadratic forms:

$$q(\sigma) = q_{L^\sigma} = -q_{L_\sigma},\ A_{q(\sigma)} = A_{L^\sigma} = A_{L_\sigma}. \tag{3–3–12}$$

We now consider the following decomposition of lattices, $L^{\tau,\sigma} \oplus L_\sigma$, and the corresponding decomposition of the discriminant groups $A_{L^{\tau,\sigma}} \oplus A_{L_\sigma}$ with the discriminant form $q_{L^{\tau,\sigma}} \oplus q_{L_\sigma}$. The subgroup $\Gamma(L^{\tau,\sigma}, L_\sigma) = (L^\sigma_\tau)^\perp/(L^{\tau,\sigma} \oplus L_\sigma) \subset A_{L^{\tau,\sigma}} \oplus A_{L_\sigma}$ is isotropic, $\Gamma(L^{\tau,\sigma}, L_\sigma) \cap A_{L^{\tau,\sigma}} = \Gamma(L^{\tau,\sigma}, L_\sigma) \cap A_{L_\sigma} = \{0\}$. Thus, the subgroup $\Gamma(L^{\tau,\sigma}, L_\sigma)$ is the graph of the embedding $\gamma(L^{\tau,\sigma}, L_\sigma)$ of a subgroup $H(\sigma)_+ \subset A^{(1)}_{L^{\tau,\sigma}}$ into the group $A_{q(\sigma)} = A_{L_\sigma}$ with the discriminant form $q(\sigma)$:

$$H(\sigma)_+ \subset A^{(1)}_{L^{\tau,\sigma}},\ \ \gamma(L^{\tau,\sigma}, L_\sigma) : H(\sigma)_+ \hookrightarrow A_{q(\sigma)}. \tag{3–3–13}$$

Evidently, $\gamma(L^{\tau,\sigma}, L_\sigma)$ gives the embedding of quadratic forms

$$\gamma(L^{\tau,\sigma}, L_\sigma) : q_{L^{\tau,\sigma}} \mid H(\sigma)_+ \to q(\sigma) \mid A_{q(\sigma)}. \tag{3–3–14}$$

We repeat the same construction for $L^\tau_\sigma \oplus L^\sigma$. We have the decomposition $A_{L^\tau_\sigma} \oplus A_{L^\sigma}$ with the discriminant form $q_{L^\tau_\sigma} \oplus q_{L^\sigma}$. The subgroup $\Gamma(L^\tau_\sigma, L^\sigma) = (L_{\tau,\sigma})^\perp/(L^\tau_\sigma \oplus L^\sigma) \subset A_{L^\tau_\sigma} \oplus A_{L^\sigma}$ is isotropic, $\Gamma(L^\tau_\sigma, L^\sigma) \cap A_{L^\tau_\sigma} = \Gamma(L^\tau_\sigma, L^\sigma) \cap A_{L^\sigma} = \{0\}$. The subgroup $\Gamma(L^\tau_\sigma, L^\sigma) \subset A_{L^\tau_\sigma} \oplus A_{L^\sigma}$ is the graph of the embedding $\gamma(L^\tau_\sigma, L^\sigma)$ of a subgroup $H(\sigma)_- \subset A^{(1)}_{L^\tau_\sigma}$ into the group $A_{q(\sigma)} = A_{L^\sigma}$ with the discriminant form $q(\sigma)$:

$$H(\sigma)_- \subset A^{(1)}_{L^\tau_\sigma},\ \ \gamma(L^\tau_\sigma, L^\sigma) : H(\sigma)_- \hookrightarrow A_{q(\sigma)}. \tag{3–3–15}$$

Evidently, $\gamma(L^\tau_\sigma, L^\sigma)$ gives the embedding of quadratic forms

$$\gamma(L^\tau_\sigma, L^\sigma) : -q_{L^\tau_\sigma} \mid H(\sigma)_- \to q(\sigma) \mid A_{q(\sigma)}. \tag{3–3–16}$$

Above, we had defined subgroups $\Gamma(\sigma)_+ \subset A^{(1)}_{L^{\tau,\sigma}}$ and $\Gamma(\sigma)_- \subset A^{(1)}_{L^\tau_\sigma}$. One can see very easily (see [N4, §1]) that $\Gamma(\sigma)_\pm \subset H(\sigma)_\pm$ and

$$\begin{aligned} &\gamma(L^{\tau,\sigma}, L_\sigma)(H(\sigma)_+) \cap \gamma(L^\tau_\sigma, L^\sigma)(H(\sigma)_-) = \\ &= \gamma(L^{\tau,\sigma}, L_\sigma)(\Gamma(\sigma)_+) = \gamma(L^\tau_\sigma, L^\sigma)(\Gamma(\sigma)_-). \end{aligned} \tag{3–3–17}$$

The triplet of the basic invariants $(r(\sigma), a(\sigma), \delta(\sigma))$ and subgroups $H(\sigma)_+ \subset A^{(1)}_{L^{\tau,\sigma}}$ and $H(\sigma)_- \subset A^{(1)}_{L^\tau_\sigma}$ with the pairing between them defined by embeddings $\gamma(L^{\tau,\sigma}, L_\sigma)$ and $\gamma(L^\tau_\sigma, L^\sigma)$ respectively and the discriminant form $q(\sigma)$ are the most important invariants from [N4] of the triplet (L, L^τ, σ).

Other more delicate invariants of the triplet (L, L^τ, σ) can be found in [N4]. We will use some of them in Sect. 3.4. A complete description of invariants from [N4] for real Enriques surfaces can be found in [N6].

Notation 3.3.2. To simplify our notations, we will identify groups $H(\sigma)_+$ and $H(\sigma)_-$ with their images $\gamma(L^{\tau,\sigma}, L_\sigma)(H(\sigma)_+)$ and $\gamma(L_\sigma^\tau, L^\sigma)(H(\sigma)_-)$ in $A_{q(\sigma)}$. See (3–3–13) and (3–3–15). Thus, by (3–3–17), we have

$$H(\sigma)_+ \cap H(\sigma)_- = \Gamma(\sigma)_\pm$$

(we use similar identification for groups $\Gamma(\sigma)_\pm$). Using the discriminant quadratic form $q(\sigma)$ on $A_{q(\sigma)}$, we then have a bilinear pairing between $H(\sigma)_+$ and $H(\sigma)_-$ and can consider the orthogonal complements $H(\sigma)_\pm^\perp$ to $H(\sigma)_\pm$ in $A_{q(\sigma)}$. We remark that by (3–3–14) and (3–3–16), we have

$$q_{L^{\tau,\sigma}} \mid H(\sigma)_+ = q(\sigma) \mid H(\sigma)_+,\ -q_{L_\sigma^\tau} \mid H(\sigma)_- = q(\sigma) \mid H(\sigma)_-.$$

The following very important and non-trivial Lemma connects the invariants $(r(\sigma), a(\sigma), \delta(\sigma))$ and $(r(\tau\sigma), a(\tau\sigma), \delta(\tau\sigma))$ of the involutions σ and $\tau\sigma$. We will not use the third statement of the Lemma here, and include it for the sake of completeness.

Lemma 3.3.2. *Let Y be a real Enriques surface such that there are two liftings σ and $\tau\sigma$ of the antiholomorphic involution θ of Y to the antiholomorphic involutions of the $K3$-surface $X(\mathbf{C})$ (it is true for example if $Y(\mathbf{R}) \neq \emptyset$).*

Then

$$r(\sigma) + r(\tau\sigma) = 12 + 2r(\theta);$$

$$\begin{aligned} &a(\tau\sigma) - a(\sigma) = \\ &= 10 + 2a(\theta) - 2\dim H(\sigma)_+ - 2\dim (H(\sigma)_+)^\perp \cap H(\sigma)_-; \end{aligned}$$

$$\delta(\sigma) + \delta(\tau\sigma) \equiv \delta(\theta) \mod 2.$$

Proof. We consider the decomposition

$$L^{\tau,\sigma} \oplus L_\sigma^\tau \oplus L_\tau^\sigma \oplus L_{\tau,\sigma} \subset L$$

of finite index and the corresponding direct sum of the discriminant groups

$$A_{L^{\tau,\sigma}} \oplus A_{L_\sigma^\tau} \oplus A_{L_\tau^\sigma} \oplus A_{L_{\tau,\sigma}}$$

with the discriminant quadratic form

$$q_{L^{\tau,\sigma}} \oplus q_{L_\sigma^\tau} \oplus q_{L_\tau^\sigma} \oplus q_{L_{\tau,\sigma}}$$

and isotropic subgroup

$$\Gamma = L/(L^{\tau,\sigma} \oplus L_\sigma^\tau \oplus L_\tau^\sigma \oplus L_{\tau,\sigma}) \subset A_{L^{\tau,\sigma}} \oplus A_{L_\sigma^\tau} \oplus A_{L_\tau^\sigma} \oplus A_{L_{\tau,\sigma}}$$

Let p_1, p_2, p_3, p_4 be the corresponding projections on the $A_1 = A_{L^{\tau,\sigma}}$, $A_2 = A_{L_\sigma^\tau}$, $A_3 = A_{L_\tau^\sigma}$, $A_4 = A_{L_{\tau,\sigma}}$ respectively. We denote by q_i respectively the corresponding discriminant form on the group A_i. Evidently, $\Gamma \cap A_i = \{0\}, 1 \leq i \leq 4$.

Let $\Gamma_{ij} = \Gamma \cap (A_i \oplus A_j)$ for $1 \leq i < j \leq 4$. We use a similar definition for Γ_{ijk}, where $1 \leq i < j < k \leq 4$. We remark that Lemma 3.3.1 gives the description of the subgroup Γ_{12}.

Evidently, $\text{rk } L^\sigma = \text{rk } L^{\tau,\sigma} + \text{rk } L_\tau^\sigma$ and $\text{rk } L^{\tau\sigma} = \text{rk } L^{\tau,\sigma} + \text{rk } L_{\tau,\sigma}$. Thus, it follows that $\text{rk } L^\sigma + \text{rk } L^{\tau\sigma} = 2\text{rk } L^{\tau,\sigma} + \text{rk } L_\tau^\sigma + \text{rk } L_{\tau,\sigma} = 2\text{rk } L^{\tau,\sigma} + \text{rk } L_\tau = 2r(\theta) + 12$. This proves the first statement.

We now prove the second part. The following statement will be useful to us.

Proposition 3.3.3. *Let $S_1 \oplus S_2 \oplus S_3 \subset L$ be an orthogonal decomposition up to finite index of an unimodular lattice L with all sublattices $S_i, 1 \le i \le 3$ are primitive in L. Let $\Gamma(L) = L/(S_1 \oplus S_2 \oplus S_3) \subset A_{S_1} \oplus A_{S_2} \oplus A_{S_3}$ be the corresponding isotropic subgroup with respect to the discriminant form $q_{S_1} \oplus q_{S_2} \oplus q_{S_3}$. Let $\Gamma(S_i, S_j) = \Gamma(L) \cap (A_{S_i} \oplus A_{S_j})$ where $1 \le i < j \le 3$.*

Then the orthogonal complement to $\Gamma(S_1, S_2)$ in $\Gamma(L)$ with respect to the form q_{S_2} is equal to $\Gamma(S_2, S_3) + \Gamma(S_1, S_3)$.

Proof. Evidently, these subgroups are orthogonal to one another. Let p_i be the projection on A_{S_i}.

We prove that $p_2(\Gamma(S_1, S_2))$ and $p_2(\Gamma(S_2, S_3))$ are the orthogonal complements to one another with respect to the form q_{S_2}. The projection p_2 is an embedding of both these subgroups since the sublattices S_i are primitive. It follows that

$$\sharp A_{S_1} = \sharp A_{(S_1)^\perp_L} = (\sharp A_{S_2} \sharp A_{S_3})/(\sharp\Gamma(S_2, S_3))^2$$

and

$$\sharp A_{S_3} = \sharp A_{(S_3)^\perp_L} = (\sharp A_{S_1} \sharp A_{S_2})/(\sharp\Gamma(S_1, S_2))^2.$$

It follows therefore that

$$\sharp A_{S_2} = \sharp\Gamma(S_1, S_2)\sharp\Gamma(S_2, S_3).$$

This equality is equivalent to the statement we had to prove since the bilinear form of the form q_{S_2} is non-degenerate.

We now consider the projection $p_2 : \Gamma(L) \to A_{S_2}$. Evidently, $ker\ p_2 = \Gamma(S_1, S_3)$. From the result proved above, $(p_2)^{-1}(p_2(\Gamma(S_1, S_2))^\perp) = \Gamma(S_2, S_3) + \Gamma(S_1, S_3)$, where we take the orthogonal complement in A_{S_2} using the bilinear form of q_{S_2}. Hence the Proposition follows.

From the definition of $r(\theta)$ and $a(\theta)$,

(3–3–18) $$\sharp A_{L^{\tau,\sigma}} = 4^{a(\theta)} 2^{r(\theta)-a(\theta)} = 2^{r(\theta)+a(\theta)};$$

(3–3–19) $$\sharp A_{L^\tau_\sigma} = 4^{a(\theta)} 2^{10-r(\theta)-a(\theta)} = 2^{10-r(\theta)+a(\theta)};$$

(3–3–20) $$\begin{aligned}\sharp A_{L^\sigma_\tau} = \sharp A_{(L^\sigma_\tau)^\perp_L} &= (\sharp A_{L^{\tau,\sigma}} \sharp A_{L_\sigma})/(\sharp H(\sigma)_+)^2 \\ &= 2^{r(\theta)+a(\theta)+a(\sigma)-2\dim H(\sigma)_+};\end{aligned}$$

(3–3–21)
$$\begin{aligned}\sharp A_{L_{\tau,\sigma}} = \sharp A_{(L_{\tau,\sigma})^\perp_L} &= (\sharp A_{L^\tau_\sigma} \sharp A_{L^\sigma})/(\sharp H(\sigma)_-)^2 \\ &= 2^{10-r(\theta)+a(\theta)+a(\sigma)-2\dim H(\sigma)_-}.\end{aligned}$$

By the projection $p_1 \oplus p_2$, we have the identifications

$$q(\sigma) \mid A \ldots \cong q \oplus q \mid \Gamma/(\Gamma + \Gamma);$$

$$\gamma(L^{\tau,\sigma}, L_\sigma)H(\sigma)_+ = \Gamma_{124}/\Gamma_{24} \subset \Gamma/(\Gamma_{13}+\Gamma_{24});$$

$$\gamma(L_\sigma^\tau, L^\sigma)H(\sigma)_- = \Gamma_{123}/\Gamma_{13} \subset \Gamma/(\Gamma_{13}+\Gamma_{24}).$$

By Proposition 3.3.3 and this identification,

$$(\gamma(L^{\tau,\sigma}, L_\sigma)H(\sigma)_+)^\perp_{q(\sigma)} = (\Gamma_{13}+\Gamma_{234})/(\Gamma_{13}+\Gamma_{24}).$$

We hence have

$$(\gamma(L^{\tau,\sigma}, L_\sigma)H(\sigma)_+)^\perp_{q(\sigma)} \cap \gamma(L_\sigma^\tau, L^\sigma)H(\sigma)_- =$$

$$= (\Gamma_{13}+\Gamma_{234}) \cap (\Gamma_{24}+\Gamma_{123})/(\Gamma_{13}+\Gamma_{24}) =$$

$$= (\Gamma_{13}+\Gamma_{24}+\Gamma_{234}\cap\Gamma_{123})/(\Gamma_{13}+\Gamma_{24}) \cong \Gamma_{23}.$$

Thus,

$$\sharp(\gamma(L^{\tau,\sigma}, L_\sigma)H(\sigma)_+)^\perp_{q(\sigma)} \cap \gamma(L_\sigma^\tau, L^\sigma)H(\sigma)_- = \sharp\Gamma_{23}.$$

On the other hand, by definition, and calculations above

$$2^{a(\tau\sigma)} = \sharp A_{L^{\tau\sigma}} = \sharp A_{L_{\tau\sigma}} = \sharp A_{L_\sigma^\tau}\sharp A_{L_\tau^\sigma}/(\sharp\Gamma_{23})^2 =$$

$$= 2^{10-r(\theta)+a(\theta)}2^{r(\theta)+a(\theta)+a(\sigma)-2\dim H(\sigma)_+}2^{-2\dim\Gamma_{23}}.$$

This gives the proof of the second statement of the Lemma.

We now prove the last statement. As before, $q(\sigma) \mid A_{q(\sigma)} = q_1 \oplus q_3 \mid \Gamma/(\Gamma_{13}+\Gamma_{24})$. It follows that

$$\delta(\sigma) = 0 \ iff \ q_1 + q_3 \equiv 0 \mod 1.$$

(This must be true on the group Γ.) The same holds for the involution $\tau\sigma$:

$$\delta(\tau\sigma) = 0 \ iff \ q_1 + q_4 \equiv 0 \mod 1.$$

We remark that

$$q_1 + q_2 + q_3 + q_4 = 0$$

since the subgroup Γ is isotropic. Moreover, we know that $\delta(\tau) = 0$. If follows that

$$q_1 + q_2 \equiv q_3 + q_4 \equiv 0 \mod 1.$$

It follows very easily that if $\delta(\sigma) = 0$ then $\delta(\tau\sigma) = 0$ iff $2q_1 \equiv 0 \mod 1$. On the other hand, one can see very easily that $2q_1 \equiv 0 \mod 1$ iff $\delta(\theta) = 0$. By symmetry, if $\delta(\tau\sigma) = 0$, then $\delta(\sigma) = 0$ iff $\delta(\theta) = 0$. This proves the last statement.

From Lemma 3.2.1, formula (3–3–18) and Lemma 3.3.2, we get the following very important

Theorem 3.3.4. *Let Y be a real Enriques surface such that there are two liftings σ and $\tau\sigma$ of the antiholomorphic involution θ of Y to antiholomorphic involutions of the $K3$-surface $X(\mathbf{C})$ (this is true for example if $Y(\mathbf{R}) \neq \emptyset$). Let $s(\sigma)$ and $s(\tau\sigma)$ be the number of connected components of the real parts $X_\sigma(\mathbf{R})$ and $X_{\tau\sigma}(\mathbf{R})$ respectively.*

Then

$$\begin{aligned} s(\sigma)+s(\tau\sigma) &= 2+(r(\sigma)+r(\tau\sigma))/2-(a(\sigma)+a(\tau\sigma))/2 \\ &= 3+r(\theta)-a(\theta)-a(\sigma)+\dim H(\sigma)_+ + \dim (H(\sigma)_+)^\perp \cap H(\sigma)_- \end{aligned}$$

if both $s(\sigma)>0$ and $s(\tau\sigma)>0$.

If either $s(\sigma)=0$ and $s(\tau\sigma)>0$ or $s(\sigma)>0$ and $s(\tau\sigma)=0$ then

$$\begin{aligned} s(\sigma)+s(\tau\sigma) &= 1+(r(\sigma)+r(\tau\sigma))/2-(a(\sigma)+a(\tau\sigma))/2 \\ &= 2+r(\theta)-a(\theta)-a(\sigma)+\dim H(\sigma)_+ + \dim (H(\sigma)_+)^\perp \cap H(\sigma)_-. \end{aligned}$$

If $s(\sigma)=s(\tau\sigma)=0$, then

$$\begin{aligned} 0=s(\sigma)+s(\tau\sigma) &= (r(\sigma)+r(\tau\sigma))/2-(a(\sigma)+a(\tau\sigma))/2 \\ &= 1+r(\theta)-a(\theta)-a(\sigma)+\dim H(\sigma)_+ + \dim (H(\sigma)_+)^\perp \cap H(\sigma)_-. \end{aligned}$$

Besides, $s(\sigma)+s(\tau\sigma)=s_{nor}+2s_{or}$ where s_{or} and s_{nor} is the number of orientable and non-orientable connected components of $Y(\mathbf{R})$ respectively.

3.4. Calculation of group cohomology invariants.

Here we calculate group cohomology using the invariants above.

First of all, we have the following simple

Proposition 3.4.1. *Let Y be an arbitrary real Enriques surface . Then*

$$\dim (Pic\ Y_{\mathbf{C}})^\theta / 2(Pic\ Y_{\mathbf{C}})^\theta = 11 - r(\theta).$$

Proof. See the proof of the formula (2–2) in §2.

We use the following

Proposition 3.4.2. *Let Y be a real Enriques surface and suppose there are two liftings σ and $\tau\sigma$ of the antiholomorphic involution θ of Y to antiholomorphic involutions of the $K3$-surface $X(\mathbf{C})$ (for example it is true if $Y(\mathbf{R}) \neq \emptyset$).*

Then there are the canonical isomorphisms:

$$\begin{aligned} H^2(X(\mathbf{C});\mathbf{Z}/2)^\tau &\cong (H^2(X(\mathbf{C});\mathbf{Z})^\tau \oplus H^2(X(\mathbf{C});\mathbf{Z})_\tau)/2H^2(X(\mathbf{C});\mathbf{Z}) \\ &\cong H^2(X(\mathbf{C});\mathbf{Z})_\tau / 2H^2(X(\mathbf{C});\mathbf{Z})_\tau; \end{aligned}$$

$$\begin{aligned} H^2(X(\mathbf{C});\mathbf{Z}/2)^{\tau,\sigma} &\cong (H^2(X(\mathbf{C});\mathbf{Z})_\tau / 2H^2(X(\mathbf{C});\mathbf{Z})_\tau)^\sigma \cong \\ &\cong (H^2(X(\mathbf{C});\mathbf{Z})_\tau^\sigma \oplus H^2(X(\mathbf{C});\mathbf{Z})_{\tau,\sigma})/2H^2(X(\mathbf{C});\mathbf{Z})_\tau; \end{aligned}$$

and

$$\begin{aligned} \dim H^2(X(\mathbf{C});\mathbf{Z}/2)^{\tau,\sigma} &= \dim (H^2(X(\mathbf{C});\mathbf{Z})_\tau / 2H^2(X(\mathbf{C});\mathbf{Z})_\tau)^\sigma \\ &= 12 - a(\theta) - a(\sigma) + \dim H(\sigma)_+ + \dim H(\sigma) \end{aligned}$$

$$(H^2(X(\mathbf{C});\mathbf{Z})^\tau/2H^2(X(\mathbf{C});\mathbf{Z})^\tau)^\sigma \cong$$
$$\cong (H^2(X(\mathbf{C});\mathbf{Z})^{\tau,\sigma} \oplus H^2(X(\mathbf{C});\mathbf{Z})^\tau_\sigma)/2H^2(X(\mathbf{C});\mathbf{Z})^\tau\ ,$$

and

$$\dim(H^2(X(\mathbf{C});\mathbf{Z})^\tau/2H^2(X(\mathbf{C});\mathbf{Z})^\tau)^\sigma = 10 - a(\theta).$$

Proof. As above, let $L = H^2(X(\mathbf{C});\mathbf{Z})$. Then, since a $K3$-surface has no torsion in cohomology (see the beginning of §1), $H^2(X(\mathbf{C});\mathbf{Z}/2) = L/2L$ and we should calculate $(L/2L)^\tau$. We claim (cl. [Ha1]) that

$$(L/2L)^\tau = (L^\tau \oplus L_\tau)/2L.$$

Let $x = (x_+ + x_-)/2 \mod 2L \in (L/2L)^\tau$ where $x_+ \in L^\tau, x_- \in L_\tau$. We have $x - \tau(x) = x_- \in 2L$. It then follows that $x = y_+ + y_-$ where $y_+ \in L^\tau, y_- \in L_\tau$. Thus, we have proved the statement.

By (1–0) and since $a(\tau) = 10$, we have: $\dim(L^\tau \oplus L_\tau)/2L = 22 - \dim L/(L^\tau \oplus L_\tau) = 12$. But the group $(L^\tau \oplus L_\tau)/2L$ contains the subgroup $L_\tau/2L_\tau$ which has dimension 12 too. Thus, these groups are isomorphic.

The same proof for L_τ and the involution σ of L_τ gives the second statement of the Proposition.

We now prove the third statement. We follow the notation in the proof of Lemma 3.3.2. We have the sequence of embeddings:

$$L_\tau \supset L_\tau^\sigma \oplus L_{\tau,\sigma} \supset 2L_\tau$$

where

$$\dim H^2(X(\mathbf{C});\mathbf{Z}/2)^{\tau,\sigma} = \dim(L_\tau^\sigma \oplus L_{\tau,\sigma})/2L_\tau.$$

It follows, therefore that

$$2^{\dim(L_\tau^\sigma \oplus L_{\tau,\sigma})/2L_\tau} = 2^{12}/\sharp(L_\tau/(L_\tau^\sigma \oplus L_{\tau,\sigma})).$$

On the other hand,

$$\sharp A_{L_\tau} = 2^{10} = (\sharp A_{L_\tau^\sigma} \sharp A_{L_{\tau,\sigma}})/\sharp(L_\tau/(L_\tau^\sigma \oplus L_{\tau,\sigma}))^2.$$

From the calculations above of $\sharp A_{L_\tau^\sigma}$ and $\sharp A_{L_{\tau,\sigma}}$ (see (3–3–20) and (3–3–21)), we get that

$$\dim L_\tau/(L_\tau^\sigma \oplus L_{\tau,\sigma}) = a(\theta) + a(\sigma) - \dim H(\sigma)_+ - \dim H(\sigma)_-.$$

Hence the third statement follows.

The proof of statements about the group $(H^2(X(\mathbf{C});\mathbf{Z})^\tau/2H^2(X(\mathbf{C});\mathbf{Z})^\tau)^\sigma$ is similar. We remark that from Lemma 3.3.1

$$\dim L^\tau/(L^{\tau,\sigma} \oplus L^\tau_\sigma) = a(\theta).$$

Besides, rk $L^\tau = 10$.

We need some information about complex Enriques surfaces.

From the universal coefficients formula (see [Sp]), we have a filtration

(3–4–1) $$\text{Tor } H^2(Y(\mathbf{C});\mathbf{Z}) \subset H^2(Y(\mathbf{C});\mathbf{Z})\otimes \mathbf{Z}/2 \subset H^2(Y(\mathbf{C});\mathbf{Z}/2),$$

where by (1–4) and (1–5),

$$\dim \text{Tor } H^2(Y(\mathbf{C});\mathbf{Z}) = 1,\ \dim H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2 = 11,\ \dim H^2(Y(\mathbf{C});\mathbf{Z}/2) = 12.$$

Using Smith exact sequence (see [Br, Ch. III, Sect. 3]) and (3–4–1) and (1–0), we have the exact sequence

(3–4–2)

$$0 \to \text{Tor } H^2(Y(\mathbf{C});\mathbf{Z}) \to H^2(Y(\mathbf{C});\mathbf{Z}/2) \xrightarrow{\pi^*} H^2(X(\mathbf{C});\mathbf{Z}/2)$$
$$\xrightarrow{\pi_*} H^2(Y(\mathbf{C});\mathbf{Z})\otimes \mathbf{Z}/2 \to 0.$$

This exact sequence is very important for us, and we clarify. Smith exact sequence for the involution τ of $X(\mathbf{C})$ gives an exact sequence (since τ has no a fixed point):

(3–4–2')

$$0 \to H^1(Y(\mathbf{C});\mathbf{Z}/2) \xrightarrow{\partial^1} H^2(Y(\mathbf{C});\mathbf{Z}/2) \xrightarrow{\pi^*} H^2(X(\mathbf{C});\mathbf{Z}/2)$$
$$\xrightarrow{\pi_*} H^2(Y(\mathbf{C});\mathbf{Z}/2) \xrightarrow{\partial^2} H^3(Y(\mathbf{C});\mathbf{Z}/2) \to 0.$$

(We used here that $H^1(X(\mathbf{C});\mathbf{Z}/2) = H^3(X(\mathbf{C});\mathbf{Z}/2) = 0$.) Since a $K3$-surface has no torsion in cohomology, by (3–4–1), we evidently get that Tor $H^2(Y(\mathbf{C});\mathbf{Z})$ lies in a kernel of the homomorphism $\pi^* : H^2(Y(\mathbf{C});\mathbf{Z}/2) \to H^2(X(\mathbf{C});\mathbf{Z}/2)$. By (1–4) and (1–5), we have $\dim \text{Tor } H^2(Y(\mathbf{C});\mathbf{Z}) = \dim H^1(Y(\mathbf{C});\mathbf{Z}/2) = 1$. Thus, in (3–4–2) and (3–4–2'), the kernels of π^* are identified: Tor $H^2(Y(\mathbf{C});\mathbf{Z}) = H^1(Y(\mathbf{C});\mathbf{Z}/2) \cong \mathbf{Z}/2$. Since a $K3$-surface has no torsion in cohomology, the homomorphism $\pi_* : H^2(X(\mathbf{C});\mathbf{Z}/2) \to H^2(Y(\mathbf{C});\mathbf{Z}/2)$ is the tensor product by $\mathbf{Z}/2$ of the corresponding homomorphism over $\mathbf{Z}$. It follows that $\pi_*(H^2(X(\mathbf{C});\mathbf{Z}/2)) \subset H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2 \subset H^2(Y(\mathbf{C});\mathbf{Z}/2)$. Since $\dim H^3(Y(\mathbf{C});\mathbf{Z}/2) = 1$ and $\dim H^2(Y(\mathbf{C});\mathbf{Z}/2) - \dim H^2(Y(\mathbf{C});\mathbf{Z}) \otimes \mathbf{Z}/2 = 1$, from the (3–4–2'), we get $\pi_*(H^2(X(\mathbf{C});\mathbf{Z}/2)) = H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2$. Thus, from the exact sequence (3–4–2'), the exact sequence (3–4–2) follows.

For $\mathcal{L} = L/2L$ where (we recall) $L = H^2(X(\mathbf{C});\mathbf{Z})$ and $\mathcal{F} = Im\ \pi^*$ we have a filtration of subgroups (we use (1–6)):

(3–4–3) $$\mathcal{L}^{(\tau)} = L^\tau/2L^\tau \subset \mathcal{F} \subset \mathcal{L}^\tau = \mathcal{L}_{(\tau)} = L_\tau/2L_\tau \subset \mathcal{L} = L/2L,$$

where $\dim \mathcal{L}^{(\tau)} = 10, \dim \mathcal{F} = 11, \dim \mathcal{L}_{(\tau)} = 12, \dim \mathcal{L} = 22$. Then (3–4–2) and (3–4–3) are connected by the canonical isomorphisms

(3–4–4) $$(H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)/\text{Tor } H^2(Y(\mathbf{C});\mathbf{Z}) \cong \pi^*(H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2) = \mathcal{L}^{(\tau)};$$

(3–4–5) $$H^2(Y(\mathbf{C});\mathbf{Z}/2)/\text{Tor } H^2(Y(\mathbf{C});\mathbf{Z}) \cong \pi^*(H^2(Y(\mathbf{C});\mathbf{Z}/2)) = \mathcal{F};$$

(3–4–6) $$H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2 = \pi_*(\mathcal{L}) \cong \mathcal{L}/\mathcal{F}.$$

We want to find $\mathcal{F}$. Let $\mathbf{Z}_2$ be the ring of 2-adic integers. The lattice $L_\tau \otimes \mathbf{Z}_2 \cong U(1) \oplus 4U(2)$ where $U = \langle\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\rangle$. It follows that the quadratic form $x^2/2 \mod 2$ with symmetric bilinear form $x\cdot y \mod 2$ ($x, y \in L_\tau$) on $\mathcal{L}_{(\tau)}$ has 10-dimensional kernel $\mathcal{L}^{(\tau)}$ and the induced quadratic form ($x^2/2 \mod 2$) on $H = \mathcal{L}_{(\tau)}/\mathcal{L}^{(\tau)} = \{0, g_1, g_2, f\}$ has values $(g_1)^2/2 = (g_2)^2/2 = 0 \mod 2$ and $f^2/2 = 1 \mod 2$. The subgroup $F = \mathcal{F}/\mathcal{L}^\tau \subset \mathcal{L}_{(\tau)}/\mathcal{L}^{(\tau)} = H$ has order two. The following statement characterizes this subgroup F since the element f with $f^2/2 = 1 \mod 2$ is unique.

Lemma 3.4.3. *$F = \mathcal{F}/\mathcal{L}^\tau = \{0, f\}$, where $f^2/2 = 1 \mod 2$. In other words, for $x \in L$ such that $x + 2L \in \mathcal{F}$, but $x + 2L \notin L^\tau + 2L$ we have: $x^2/2 = 1 \mod 2$.*

Proof. It is known that the moduli space of complex Enriques surfaces is connected (at first, it was remarked at the end of [N0]; also see [Ho]). Thus, it is sufficient to prove this statement for one Enriques surface.

Let τ be an involution on the lattice L which acts on L like the involution τ of Enriques surfaces. Then the lattice $L_\tau = U \oplus U(2) \oplus E_8(2)$, where E_8 is an even unimodular negative lattice of the rank 8. Let c_1, c_2 be a basis of U such that $(c_1)^2 = (c_2)^2 = 0$ and $c_1 \cdot c_2 = 1$. Let us consider an involution σ on L such that $\sigma(c_1) = c_2, \sigma(c_2) = c_1$ and $\sigma \mid (U)^\perp_L = -id$. The involution σ evidently has $L^\sigma = [c_1 + c_2]$, where $(c_1 + c_2)^2 = 2$. Thus, L^σ is a hyperbolic lattice. We consider a general element $\omega_+ \in L^\sigma \otimes \mathbf{R}$ and $\omega_- \in L_{\tau,\sigma} \otimes \mathbf{R}$, where $(\omega_+)^2 = (\omega_-)^2 > 0$. By global Torelli theorem [PŠ-Š] and epimorphicity of Torelli map [Ku] for $K3$-surfaces, there exists a real $K3$-surface X with periods $\omega_+ + i\omega_-$ such that $L = H^2(X(\mathbf{C}); \mathbf{Z})$, Pic $X_\mathbf{C} = L^\tau$,the action of the antiholomorphic involution of X is σ. See [N3] and [N4] for details. Moreover, since Pic $X_\mathbf{C} = L^\tau$, $X(\mathbf{C})$ has a holomorphic involution which acts on L as τ. We denote this automorphisms of X by the same letters σ and τ respectively. By construction, the involution τ has no fixed points on $X(\mathbf{C})$ (see (3–3–4)) and by global Torelli Theorem $\tau\sigma = \sigma\tau$ (since this is true for their action on cohomology). It follows that $X(\mathbf{C})/\{id, \tau\}$ is an Enriques surface with antiholomorphic involution θ such that σ is a lifting of θ.

For $X(\mathbf{C})$, the group $\mathcal{L}_{(\tau)}/\mathcal{L}^{(\tau)} = \{c_1 \mod 2, c_2 \mod 2, c_1 + c_2 \mod 2\}$. The subgroup $\mathcal{F}/\mathcal{L}^{(\tau)}$ should be σ-invariant, because it is defined uniquely by the topology of $X(\mathbf{C})$ and $Y(\mathbf{C})$.

By construction of σ, we have $\sigma(c_1 + L^\tau) = (c_2 + L^\tau)$, $\sigma(c_2 + L^\tau) = (c_1 + L^\tau)$ and $\sigma(c_1 + c_2 + L^\tau) = (c_1 + c_2 + L^\tau)$. Thus, $\mathcal{F}/\mathcal{L}^{(\tau)} = (c_1 + c_2 + L^\tau)$. We have $(c_1 + c_2)^2/2 = 1 \mod 2$. Hence the Proposition follows.

Remark 3.4.1. It would be interesting to find a topological proof of Lemma 3.4.3 if it does exist. The Lemma is very important for complex Enriques surfaces. For example, using this statement, one can get more easily results of [M-N].

We need the following statement too.

Lemma 3.4.4. *Let Y be a real Enriques surface and suppose there are two liftings σ and $\tau\sigma$ of the antiholomorphic involution θ of Y to antiholomorphic involutions of the surface $X(\mathbf{C})$ (for example this is true if $Y(\mathbf{R}) \neq \emptyset$). Then*

$$\text{(3–4–7)} \qquad \dim H(\sigma)_+ + \dim H(\sigma)_- \leq a(\sigma) \leq \dim H(\sigma)_+ + \dim H(\sigma)_- + 1,$$

where the right inequality is a consequence of the following two inequalities:

$$\text{(3–4–8)} \qquad r(\sigma) - a(\sigma) \geq -2 \dim H(\sigma)_+ + 2r(\theta)$$

and

$$\text{(3–4–9)} \qquad r(\sigma) + a(\sigma) \leq 2 \dim H(\sigma)_- + 2r(\theta) + 2.$$

Proof. The inequality $\dim H(\sigma)_+ + \dim H(\sigma)_- \leq a(\sigma)$ follows from the general inequality of the condition 1.8.1 from [N4]. But we give a proof

We use Notation 3.3.2. By Lemma 3.3.1, $\Gamma_\pm = H(\sigma)_+ \cap H(\sigma)_-$ is an isotropic subgroup of $A_{q(\sigma)}$ and is orthogonal to $H(\sigma)_+ + H(\sigma)_-$ with respect to the bilinear form of $q(\sigma)$. Thus, with respect to the bilinear form of $q(\sigma)$ on $A_{q(\sigma)}$ we have $H(\sigma)_+ + H(\sigma)_- \subset \Gamma_\pm^\perp \subset A_{q(\sigma)}$, where $\dim \Gamma_\pm^\perp = \dim A_{q(\sigma)} - \dim \Gamma_\pm = a(\sigma) - \dim \Gamma_\pm$, since the bilinear form of $q(\sigma)$ is non-degenerate. It follows that $\dim H(\sigma)_+ + \dim H(\sigma)_- - \dim \Gamma_\pm = \dim(H(\sigma)_+ + H(\sigma)_-) \le \dim \Gamma_\pm^\perp = a(\sigma) - \dim \Gamma_\pm$. Thus, $\dim H(\sigma)_+ + \dim H(\sigma)_- \le a(\sigma)$.

The inequalities 1.8.2, 2) and 4) from the paper [N4], for our case give the inequalities (3–4–8) and (3–4–9). We mention that these inequalities are equivalent to the inequalities $\dim A_{L_\tau^\sigma} \le \text{rk } L_\tau^\sigma$ and $\dim A_{L_{\tau,\sigma}} \le \text{rk } L_{\tau,\sigma}$ and one can prove them for this case independently. Notation in [N4] and here are connected as follows: $t_{(+)} + t_{(-)} = r(\sigma), a = a(\sigma), a_{H_+} = \dim H(\sigma)_+, a_{H_-} = \dim H(\sigma)_-, p_{(+)} + p_{(-)} = r(\theta), p_0 = s_0 = 0, l(A_{(S_+/C_+)^2}) = r(\theta), s_{(+)} + s_{(-)} = 10, l(A_{(S_-/C_-)^2}) = 10 - r(\theta), l_{(+)} + l_{(-)} = 22$.

Multiplying the inequality (3–4–8) on -1, we get the inequality

$$a(\sigma) - r(\sigma) \le 2 \dim H(\sigma)_+ - 2r(\theta). \tag{3–4–10}$$

Adding the inequalities (3–4–9) and (3–4–10), we get the right inequality (3–4–7).

Using Lemma 3.4.4, we can introduce a new invariant which is very important:

Definition 3.4.1. Let Y be a real Enriques surface and σ is a lifting of the antiholomorphic involution θ of Y to an antiholomorphic involution of X. The invariant $\alpha(\sigma)$ is equal to

$$\alpha(\sigma) = a(\sigma) - \dim H(\sigma)_+ - \dim H(\sigma)_-.$$

By Lemma 3.4.4, $\alpha(\sigma) = 0$ or 1.

By (3–4–1), the subspace

$$H^2(Y(\mathbf{C}); \mathbf{Z}) \otimes \mathbf{Z}/2 \subset H^2(Y(\mathbf{C}); \mathbf{Z}/2)$$

is a subspace of codimension 1. Thus, we can introduce the invariant:

Definition 3.4.2. Let Y be a real Enriques surface with an antiholomoprhic involution θ. The invariant

$$\beta(Y) = \dim H^2(Y(\mathbf{C}); \mathbf{Z}/2)^\theta - \dim(H^2(Y(\mathbf{C}); \mathbf{Z}) \otimes \mathbf{Z}/2)^\theta.$$

By (3–4–1), we have $0 \le \beta(Y) \le 1$.

Besides, we need some new invariants of σ from [N4]. In [N4] for a triplet (L, S, σ) (see §3.3) the invariants $\delta_{\sigma S_+}$ and $\delta_{\sigma S_-}$ were defined. These invariants are necessary for us for $S = L^\tau, S_+ = L^{\tau,\sigma}, S_- = L_\sigma^\tau$. We use Notation 3.3.2 to explain these invariants.

Let A be a 2-elementary abelian group with a symmetric bilinear form $b : A \times A \to (1/2)\mathbf{Z}/\mathbf{Z}$. An element $v \in A$ is called characteristic if $b(x,x) = b(x,v)$ for any $x \in A$. Since the discriminant bilinear form $b(\sigma)$ (or bilinear form of $q(\sigma)$) on the 2-elementary group $A_{q(\sigma)}$ is non-degenerate, there exists the unique characteristic element $v_{q(\sigma)} \in A_{q(\sigma)}$ for $b(\sigma)$.

The invariant δ … 0 if we simultaneously have:

(a) the characteristic element $v_{q(\sigma)} \in H(\sigma)_+$,

(b) The element $v_{q(\sigma)} \in H(\sigma)_+$ is a characteristic element of the form $b_{L^{\tau,\sigma}} \mid A^{(1)}_{L^{\tau,\sigma}}$ (see (3–3–13)).

Otherwise, the invariant $\delta_{\sigma L^{\tau,\sigma}} = 1$.

The definition of $\delta_{\sigma L^{\tau}_{\sigma}}$ is similar: one should consider $H(\sigma)_-$ and L^{τ}_{σ} instead $H(\sigma)_+$ and $L^{\tau,\sigma}$.

By Lemma 3.3.1, we have the following properties:

(3–4–11) $$\delta_{\sigma L^{\tau,\sigma}} = 0 \ iff \ v_{q(\sigma)} \in \Gamma_{\pm}.$$

And

(3–4–12) $$\delta_{\sigma L^{\tau}_{\sigma}} = 0 \ iff \ v_{q(\sigma)} \in \Gamma_{\pm}.$$

In particular, at any case,

(3–4–13) $$\delta_{\sigma L^{\tau,\sigma}} = \delta_{\sigma L^{\tau}_{\sigma}}.$$

By definition the invariant $\delta(\sigma)$ (see (3–3–3)), the characteristic element $v_{q(\sigma)} = 0$ iff $\delta(\sigma) = 0$. Thus, we have the following important property too:

(3–4–14) $$\delta_{\sigma L^{\tau,\sigma}} = \delta_{\sigma L^{\tau}_{\sigma}} = 0 \ \ if \ \ \delta(\sigma) = 0.$$

We mention another definition of these invariants. An element $v(\sigma) \in L/2L$ is called characteristic if

$$x \cdot \sigma(x) \equiv x \cdot v(\sigma) \mod 2.$$

for any $x \in L$. The element $v(\sigma)$ is defined uniquely since L is unimodular. From [N4, §1] and the remark above, it follows that

(3–4–15) $$\delta_{\sigma L^{\tau,\sigma}} = \delta_{\sigma L^{\tau}_{\sigma}} = 0 \ \ iff \ \ v(\sigma) \in L^{\tau,\sigma} \cap L^{\tau}_{\sigma} \mod 2L$$

The geometrical sense of $v(\sigma)$ is that

(3–4–16) $$X_{\sigma}(\mathbf{R}) \sim v(\sigma) \mod 2 \ in \ H_2(X(\mathbf{C}); \mathbf{Z}).$$

Now we can formulate the basic Theorem.

Theorem 3.4.5. *Let Y be a real Enriques surface and suppose there are two liftings σ and $\tau\sigma$ of the antiholomorphic involution θ of Y to antiholomorphic involutions of the $K3$-surface $X(\mathbf{C})$ (for example this is true if $Y(\mathbf{R}) \neq \emptyset$).*

Then

$$\dim H^2(Y(\mathbf{C}); \mathbf{Z}/2)^{\theta} = \dim(H^2(Y(\mathbf{C}); \mathbf{Z}) \otimes \mathbf{Z}/2)^{\theta} = 10 - a(\theta) \ and \ \beta(Y) = 0$$

if $\alpha(\sigma) = 1$ and $\delta_{\sigma L^{\tau,\sigma}} = \delta_{\sigma L^{\tau}_{\sigma}} = 0$.

Otherwise (if either $\alpha(\sigma) = 0$ or $\delta_{\sigma L^{\tau,\sigma}} + \delta_{\sigma L^{\tau}_{\sigma}} > 0$),

$$\dim(H^2(Y(\mathbf{C}); \mathbf{Z}) \otimes \mathbf{Z}/2)^{\theta} = 11 - a(\theta)$$

and

$$\dim H^2(Y(\mathbf{C});\mathbf{Z}/2)^\theta = 11 - a(\theta) + \beta(Y), \quad \textit{where} \ \ 0 \le \beta(Y) \le 1.$$

Proof. All homomorphisms (3–4–1) —(3–4–6) are equivariant with respect to the action of θ and its lifting σ. Thus, by (3–4–6)

$$(H^2(Y(\mathbf{C});\mathbf{Z}) \otimes \mathbf{Z}/2)^\theta \cong (\mathcal{L}/\mathcal{F})^\sigma$$

and we should calculate

$$\dim(\mathcal{L}/\mathcal{F})^\sigma.$$

The lattice L is unimodular and defines the natural exact sequence

$$0 \to L^\tau \to L \to L^*_\tau \to 0$$

and the corresponding exact sequence

$$0 \to \mathcal{L}^{(\tau)} \to \mathcal{L} \to \mathcal{L}^*_{(\tau)} \to 0.$$

Thus, we have the canonical isomorphism $\mathcal{L}/\mathcal{L}^{(\tau)} \cong \mathcal{L}^*_{(\tau)}$ and the corresponding filtration of subgroups

$$F = \mathcal{F}/\mathcal{L}^{(\tau)} \subset H = \mathcal{L}_{(\tau)}/\mathcal{L}^{(\tau)} \subset \mathcal{L}^*_{(\tau)},$$

where $\dim F = 1$ and $\dim H = 2$. We should calculate $\dim(\mathcal{L}^*_{(\tau)}/F)^\sigma$. Let $F = \{0, f\}$, where f is some linear function on $\mathcal{L}_{(\tau)}$. Let $\phi \in \mathcal{L}^*_{(\tau)}$. Then $\phi + F \in (\mathcal{L}^*_{(\tau)}/F)^\sigma$ iff $\sigma^*\phi + \phi \in F$. Thus, for ϕ we have two possibilities.

Case 1. $\sigma^*\phi + \phi = 0$ (or $\phi \in (\mathcal{L}^*_{(\tau)})^\sigma$). Thus, equivalently $\phi((id+\sigma)(\mathcal{L}_{(\tau)})) = 0$. From the exact sequence

$$0 \to (\mathcal{L}_{(\tau)})^\sigma \to \mathcal{L}_{(\tau)} \xrightarrow{id+\sigma} \mathcal{L}_{(\tau)}$$

we then have that $\dim(\mathcal{L}^*_{(\tau)})^\sigma = \dim(\mathcal{L}_{(\tau)})^\sigma$.

Case 2. $\sigma^*\phi + \phi = f$. Obviously, such a ϕ does exist iff $f \in Im(id+\sigma)^*$. Equivalently, by the exact sequence above, $f((\mathcal{L}_{(\tau)})^\sigma) = 0$.

Thus, by Proposition 3.4.2,

$$\dim(\mathcal{L}/\mathcal{F})^\sigma = 11 - a(\theta) - a(\sigma) + \dim H(\sigma)_+ + \dim H(\sigma)_-$$

if $f((\mathcal{L}_{(\tau)})^\sigma) \neq 0$, and

$$\dim(\mathcal{L}/\mathcal{F})^\sigma = 12 - a(\theta) - a(\sigma) + \dim H(\sigma)_+ + \dim H(\sigma)_-$$

if $f((\mathcal{L}_{(\tau)})^\sigma) = 0$. Then the statements of the Theorem about $\dim(H^2(Y(\mathbf{C});\mathbf{Z}) \otimes \mathbf{Z}/2)^\theta$ follow from

Lemma 3.4.6. *If $\alpha(\sigma)=0$, then $f((\mathcal{L}_{(\tau)})^\sigma)\neq 0$.*
If $\alpha(\sigma)=1$ and $\delta_{\sigma L^{\tau,\sigma}}+\delta_{\sigma L_\sigma^\tau}>0$, then $f((\mathcal{L}_{(\tau)})^\sigma)=0$.
If $\alpha(\sigma)=1$ and $\delta_{\sigma L^{\tau,\sigma}}=\delta_{\sigma L_\sigma^\tau}=0$, then $f((\mathcal{L}_{(\tau)})^\sigma)\neq 0$.

Proof. From [N4, §1, formula (8.9)], it follows that

$$\text{rk } L_\tau^\sigma - \text{rk } A_{L_\tau^\sigma} = (r(\sigma)-a(\sigma)) - (-2\dim H(\sigma)_+ + 2r(\theta)) \tag{3–4–17}$$

and

$$\text{rk } L_{\tau,\sigma} - \text{rk } A_{L_{\tau,\sigma}} = (2\dim H(\sigma)_- + 2r(\theta)+2) - (r(\sigma)+a(\sigma)). \tag{3–4–18}$$

Let $a(\sigma)=\dim H(\sigma)_+ + \dim H(\sigma)_- + 1$ (equivalently, $\alpha(\sigma)=1$). Then both inequalities (3–4–8) and (3–4–9) are equalities (see the proof of Lemma 3.4.4) and from (3–4–17) and (3–4–18) we then get that $\text{rk } L_\tau^\sigma = \text{rk } A_{L_\tau^\sigma}$ and $\text{rk } L_{\tau,\sigma} = \text{rk } A_{L_{\tau,\sigma}}$.

By Proposition 3.4.2, $\dim(\mathcal{L}^{(\tau)})^\sigma = 10-a(\theta), \dim(\mathcal{L}_{(\tau)})^\sigma = 11-a(\theta)$. It follows that

$$(\mathcal{L}_{(\tau)})^\sigma + \mathcal{L}^{(\tau)} \subset \mathcal{L}_{(\tau)}/\mathcal{L}^{(\tau)} = H$$

is a subspace of dimension 1 (over $\mathbf{Z}/2$). By Proposition 3.4.2, $(\mathcal{L}_{(\tau)})^\sigma = (L_\tau^\sigma \oplus L_{\tau,\sigma})/2L_\tau$. From Lemma 3.4.3 and earlier considerations, we then get that $f((\mathcal{L}_{(\tau)})^\sigma)\neq 0$ iff

$$(L_\tau^\sigma)^2/2 \subset 2\mathbf{Z} \textit{ and } (L_{\tau,\sigma})^2/2 \subset 2\mathbf{Z}. \tag{3–4–19}$$

Since $\text{rk } L_\tau^\sigma = \text{rk } A_{L_\tau^\sigma}$ and $\text{rk } L_{\tau,\sigma} = \text{rk } A_{L_{\tau,\sigma}}$ and from the decomposition of 2-adic lattices as an orthogonal sum of elementary lattices of rank 1 or 2, we get that (3–4–19) is equivalent to the facts

$$q_{L_\tau^\sigma} \neq q_{\gamma_1}^{(2)}(2)\oplus q_1 \quad (\gamma_1 \in \mathbf{Z}_2^*/(\mathbf{Z}_2^*)^2), \tag{3–4–20}$$

and

$$q_{L_{\tau,\sigma}} \neq q_{\gamma_2}^{(2)}(2)\oplus q_2 \quad (\gamma_2 \in \mathbf{Z}_2^*/(\mathbf{Z}_2^*)^2), \tag{3–4–21}$$

where $q_\gamma^{(2)}(2)$ is a non-degenerate quadratic form on a group of order 2. From [N4, §1, (8.8)—(8.10)], it follows that (3–4–20) is equivalent to $\delta_{\sigma L^{\tau,\sigma}}=0$ and (3–4–21) is equivalent to $\delta_{\sigma L_\sigma^\tau}=0$. Thus, we get the statement of Lemma 3.4.6 for $\alpha(\sigma)=1$.

Now let $a(\sigma)=\dim H(\sigma)_+ + \dim H(\sigma)_-$ (equivalently, $\alpha(\sigma)=0$). Then, from Lemma 3.4.4, (3–4–17) and (3–4–18), we get that either $2\geq \text{rk } L_\tau^\sigma - \text{rk } A_{L_\tau^\sigma}\geq 1$ or $2\geq \text{rk } L_{\tau,\sigma} - \text{rk } A_{L_{\tau,\sigma}}\geq 1$. For example, let $\text{rk } L_\tau^\sigma - \text{rk } A_{L_\tau^\sigma}\geq 1$. Since the lattice L_τ^σ is even, $\text{rk } L_\tau^\sigma = \text{rk } A_{L_\tau^\sigma}+2$. It follows that $L_\tau^\sigma\otimes \mathbf{Z}_2 \cong K\oplus M$, where K is an unimodular 2-adic lattice of the rank 2. Then $L_\tau^\sigma \mod 2L_\tau = \mathcal{L}_{(\tau)}/\mathcal{L}^{(\tau)} = H$, since $\mathcal{L}^{(\tau)}$ is a kernel of the form on $\mathcal{L}_{(\tau)}$ and H has the dimension 2. Since $(\mathcal{L}_{(\tau)})^\sigma = (L_\tau^\sigma\oplus L_{\tau,\sigma})/2L_\tau$, we get $f((\mathcal{L}_{(\tau)})^\sigma)\neq 0$, since $(\mathcal{L}_{(\tau)})^\sigma \mod \mathcal{L}^{(\tau)}$ contains elements g_1, g_2 (see considerations before Lemma 3.4.3) and $f\cdot g_1 = f\cdot g_2 = 1 \mod 2$.

To finish the proof of Theorem 3.4.5, we should now show that $\beta(Y)=0$ if simultaneously $a(\sigma)=\dim H(\sigma)_+ + \dim H(\sigma)_- + 1$ (or $\alpha(\sigma)=1$) and $\delta_{\sigma L^{\tau,\sigma}} = \delta_{\sigma L_\sigma^\tau} = 0$.

From (3–4–4), $(H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)/\mathrm{Tor}\ H^2(Y(\mathbf{C});\mathbf{Z})\cong\mathcal{L}^{(\tau)}$. From Proposition 3.4.2, we then get that

$$\dim((H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)/\mathrm{Tor}\ H^2(Y(\mathbf{C});\mathbf{Z}))^\theta = 10 - a(\theta).$$

Besides, we had proved that

$$\dim(H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)^\theta = 10 - a(\theta).$$

It follows that

$$\dim H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)^\theta = \dim((H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)/\mathrm{Tor}\ H^2(Y(\mathbf{C});\mathbf{Z}))^\theta.$$

We now show that from this equality, we have

$$H^2(Y(\mathbf{C});\mathbf{Z}/2)^\theta = (H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)^\theta.$$

Let $0\neq t\in\mathrm{Tor}\ H^2(Y(\mathbf{C});\mathbf{Z})$. Since t is a torsion element, the intersection pairing $t\cdot x=0$ for any integral class $x\in H^2(Y(\mathbf{C});\mathbf{Z})$, because $t\cdot x\in H^4(Y(\mathbf{C});\mathbf{Z})\cong\mathbf{Z}$. Thus, $t\perp H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2$ with respect to the intersection pairing. Since $H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2\subset H^2(Y(\mathbf{C});\mathbf{Z}/2)$ has codimension 1 and the intersection pairing on $H^2(Y(\mathbf{C});\mathbf{Z}/2)$ is non-degenerate, it follows that $t\cdot u=1\mod 2$ for any

$$u\in H^2(Y(\mathbf{C});\mathbf{Z}/2)-H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2.$$

Suppose that

$$\beta(Y)=\dim H^2(Y(\mathbf{C});\mathbf{Z}/2)^\theta-\dim(H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)^\theta>0.$$

Then, there exists

$$u\in H^2(Y(\mathbf{C});\mathbf{Z}/2)^\theta-(H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)^\theta.$$

We have seen that $u\cdot t\neq 0\mod 2$. Since $\theta(u)=u$ and $\dim\mathrm{Tor}\ H^2(Y(\mathbf{C});\mathbf{Z})=1$, we then get the θ-invariant decomposition

$$H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2=\mathrm{Tor}\ H^2(Y(\mathbf{C});\mathbf{Z})\oplus\Delta$$

where $t\cdot\Delta=0\mod 2$. Also, we have

$$\dim(H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)^\theta=1+\dim\Delta^\theta,$$

where $\Delta\cong(H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)/\mathrm{Tor}\ H^2(Y(\mathbf{C});\mathbf{Z})$. Thus,

$$\dim\Delta^\theta=\dim((H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)/\mathrm{Tor}\ H^2(Y(\mathbf{C});\mathbf{Z}))^\theta.$$

Thus, we get that

$$\dim(H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)^\theta=1+\dim((H^2(Y(\mathbf{C});\mathbf{Z})\otimes\mathbf{Z}/2)/\mathrm{Tor}\ H^2(Y(\mathbf{C});\mathbf{Z}))^\theta.$$

This gives rise to a contradiction and finishes the proof of Theorem 3.4.5.

From Proposition 3.4.1 and Theorem 3.4.5, we get the following basic result of this section (see Definitions 1.2 and 3.4.2):

Theorem 3.4.7. *Let Y be a real Enriques surface such that there are two liftings σ and $\tau\sigma$ of the antiholomorphic involution θ of Y to antiholomorphic involutions of the $K3$-surface $X(\mathbf{C})$ (this is true for example if $Y(\mathbf{R})\neq\emptyset$).*

Then

$$b(Y)=r(\theta)-a(\theta)\geq 0 \text{ and } \beta(Y)=0$$

if $\alpha(\sigma)=1$ and $\delta_{\sigma L^{\tau,\sigma}}=\delta_{\sigma L^{\tau}_{\sigma}}=0$.

Otherwise (if we have either $\alpha(\sigma)=0$ or $\delta_{\sigma L^{\tau,\sigma}}+\delta_{\sigma L^{\tau}_{\sigma}}>0$)

$$b(Y)=r(\theta)-a(\theta)+1+\beta(Y)\geq 1+\beta(Y).$$

where $\beta(Y)=0$ or 1.

We will remark that inequalities here follow from the evident inequality $r(\theta)\geq a(\theta)$. This holds since

$$r(\theta)=\text{rk } L^{\tau,\sigma}\geq \text{rk } (L^{\tau,\sigma}(1/2))^*/L^{\tau,\sigma}(1/2)=a(\theta).$$

We can unify the formulae of Theorem 3.4.7:

$$b(Y)=r(\theta)-a(\theta)+\max\{1-\alpha(\sigma),(\delta_{\sigma L^{\tau,\sigma}}+\delta_{\sigma L^{\tau}_{\sigma}})/2\}+\beta(Y) \quad (3–4–22)$$

where $\beta(Y)=0$ if $\alpha(\sigma)=1$ and $\delta_{\sigma L^{\tau,\sigma}}=\delta_{\sigma L^{\tau}_{\sigma}}=0$. We recall that by (3–4–13), $\delta_{\sigma L^{\tau,\sigma}}=\delta_{\sigma L^{\tau}_{\sigma}}$.

3.5. Connection between $b(Y)$ and $s(\sigma)+s(\tau\sigma)=s_{nor}+2s_{or}$.

From Theorems 3.3.4, 3.4.7 and the equality $a(\sigma)=\dim H(\sigma)_+ +\dim H(\sigma)_- + \alpha(\sigma)$, we get the following basic statements of this paper. We divide them into three parts according to the cases of Theorem 3.4.7.

Theorem 3.5.1. *Let Y be a real Enriques surface such that there are two liftings σ and $\tau\sigma$ of the antiholomorphic involution θ of Y to antiholomorphic involutions of the $K3$-surface $X(\mathbf{C})$ (this is true for example if $Y(\mathbf{R})\neq\emptyset$). Suppose that $\alpha(\sigma)=1$ and $\delta_{\sigma L^{\tau,\sigma}}=\delta_{\sigma L^{\tau}_{\sigma}}=0$.*

Then $\beta(Y)=0$ and:

If both $s(\sigma)>0$ and $s(\tau\sigma)>0$,

$$b(Y)=s(\sigma)+s(\tau\sigma)-2+\dim H(\sigma)_- -\dim(H(\sigma)_+)^\perp\cap H(\sigma)_-.$$

If either $s(\sigma)=0, s(\tau\sigma)>0$ or $s(\tau\sigma)=0, s(\sigma)>0$,

$$b(Y)=s(\sigma)+s(\tau\sigma)-1+\dim H(\sigma)_- -\dim(H(\sigma)_+)^\perp\cap H(\sigma)_-.$$

If $s(\sigma)=s(\tau\sigma)=0$,

$$b(Y)=s(\sigma)+s(\tau\sigma)+\dim H(\sigma)_- -\dim(H(\sigma)_+)^\perp\cap H(\sigma)_-.$$

Here $s(\sigma)+s(\tau\sigma)=s_{nor}+2s_{or}$.

Theorem 3.5.2. *Let Y be a real Enriques surface such that there are two liftings σ and $\tau\sigma$ of the antiholomorphic involution θ of Y to antiholomorphic involutions of the $K3$-surface $X(\mathbf{C})$ (this is true for example if $Y(\mathbf{R}) \neq \emptyset$). Suppose that simultaneously $\alpha(\sigma) = 1$ and $\delta_{\sigma L^{\tau,\sigma}} + \delta_{\sigma L^{\tau}_{\sigma}} > 0$.*

Then both $s(\sigma) > 0, s(\tau\sigma) > 0$, and

$$b(Y) = s(\sigma) + s(\tau\sigma) - 1 + \dim H(\sigma)_- - \dim(H(\sigma)_+)^{\perp} \cap H(\sigma)_- + \beta(Y).$$

Here $s(\sigma) + s(\tau\sigma) = s_{nor} + 2s_{or}$ and $0 \leq \beta(Y) \leq 1$.

Theorem 3.5.3. *Let Y be a real Enriques surface such that there are two liftings σ and $\tau\sigma$ of the antiholomorphic involution θ of Y to antiholomorphic involutions of the $K3$-surface $X(\mathbf{C})$ (this is true for example if $Y(\mathbf{R}) \neq \emptyset$). Suppose that $\alpha(\sigma) = 0$.*

Then:

If both $s(\sigma) > 0$ and $s(\tau\sigma) > 0$,

$$b(Y) = s(\sigma) + s(\tau\sigma) - 2 + \dim H(\sigma)_- - \dim(H(\sigma)_+)^{\perp} \cap H(\sigma)_- + \beta(Y) \geq 1 + \beta(Y).$$

If either $s(\sigma) = 0, s(\tau\sigma) > 0$ or $s(\tau\sigma) = 0, s(\sigma) > 0$,

$$b(Y) = s(\sigma) + s(\tau\sigma) - 1 + \dim H(\sigma)_- - \dim(H(\sigma)_+)^{\perp} \cap H(\sigma)_- + \beta(Y) \geq 1 + \beta(Y).$$

If $s(\sigma) = s(\tau\sigma) = 0$,

$$b(Y) = s(\sigma) + s(\tau\sigma) + \dim H(\sigma)_- - \dim(H(\sigma)_+)^{\perp} \cap H(\sigma)_- + \beta(Y) \geq 1 + \beta(Y).$$

Here $s(\sigma) + s(\tau\sigma) = s_{nor} + 2s_{or}$ and $0 \leq \beta(Y) \leq 1$.

Similar to (3–4–22), we can write down an unifying formula:

(3–5–1) $$b(Y) = s(\sigma) + s(\tau\sigma) - \sharp\{x \in \{s(\sigma), s(\tau\sigma)\} \mid x > 0\} +$$
$$+ \min\{\alpha(\sigma), (\delta_{\sigma L^{\tau,\sigma}} + \delta_{\sigma L^{\tau}_{\sigma}})/2\} + \dim H(\sigma)_- - \dim(H(\sigma)_+)^{\perp} \cap H(\sigma)_- + \beta(Y)$$

where $\beta(Y) = 0$ if $\alpha(\sigma) = 1$ and $\delta_{\sigma L^{\tau,\sigma}} = \delta_{\sigma L^{\tau}_{\sigma}} = 0$.

By these Theorems, it is important to estimate the invariant

(3–5–2) $$\gamma(\sigma) = \dim H(\sigma)_- - \dim(H(\sigma)_+)^{\perp} \cap H(\sigma)_-.$$

Here we have

Proposition 3.5.4. *We have inequalities: $0 \leq \gamma(\sigma) \leq 2$.*

Proof. We use Notation 3.3.2. By (3–3–10), either $\dim H(\sigma)_+ - \dim \Gamma_{\pm} \leq 2$ or $\dim H(\sigma)_- - \dim \Gamma_{\pm} \leq 2$ because $\dim H(\sigma)_+ \leq r(\theta)$ and $\dim H(\sigma)_- \leq 10 - r(\theta)$. By Lemma 3.3.1, $\Gamma_{\pm}$ is an isotropic subgroup of $A_{q(\sigma)}$ for a bilinear form of $q(\sigma)$. It follows the Proposition.

We mention another inequalities for items of the formula (3–5–1): Evidently,

(3–5–3) $$2 \geq \sharp\{x \in \{s(\sigma), s(\tau\sigma)\} \mid x > 0\} \geq 0,$$

(3–5–4) $$1 \geq \min\{\alpha(\sigma), (\delta_{\sigma L^{\tau,\sigma}} + \delta_{\sigma L^{\tau}_{\sigma}})/2\} \geq 0.$$

Besides, we have

(3–5–5) $$\sharp\{x \in \{s(\sigma), s(\tau\sigma)\} \mid x > 0\} - \min\{\alpha(\sigma), (\delta_{\sigma L^{\tau,\sigma}} + \delta_{\sigma L^{\tau}_{\sigma}})/2\} \geq 0,$$

and we have an equality here only if $s(\sigma) = s(\tau\sigma) = 0$. We explain the last inequality. If $s(\sigma) = 0$, then $\delta(\sigma) = 0$ and $\delta_{\sigma L^{\tau,\sigma}} = \delta_{\sigma L^{\tau}_{\sigma}} = 0$ (see (3–3–5) and (3–4–14)). It follows the (3–5–5).

Further, we give applications of our basic Theorems 1.2, 2.1, 3.4.7 and 3.5.1—3.5.3.

3.6. The case $Y(\mathbf{R}) \neq \emptyset$ and $b(Y) = 0$.

This case is interesting because, by Theorem 1.2 and Proposition 0.3, in this case we know precisely the dimension: $\dim\ {}_2Br(Y) = \epsilon(Y) = 1$.

The following Theorem describes completely this situation.

Theorem 3.6.1. *Let Y be a real Enriques surface such that $Y(\mathbf{R}) \neq \emptyset$.*

Then $b(Y) = 0$ iff $Y(\mathbf{R})$ is connected non-orientable and the invariants $r(\theta)$ and $a(\theta)$ are equal: $r(\theta) = a(\theta)$.

For this surface Y, the invariant $\epsilon(Y) = 1$ and $\dim\ {}_2Br(Y) = 1$.

Proof. We use Notation 3.3.2. Let $b(Y) = 0$. Since $Y(\mathbf{R}) \neq \emptyset$, by Theorem 2.1, we have $s = 1$. Thus $Y(\mathbf{R})$ is connected. By Lemma 3.2.1, one of the involutions σ or $\tau\sigma$ has empty set of real points. We can suppose that $s(\sigma) = 0$. By (3–3–5) and (3–4–14), then the invariants $\delta(\sigma) = 0$ and $\delta_{\sigma L^{\tau,\sigma}} = \delta_{\sigma L^{\tau}_{\sigma}} = 0$. By Theorems 3.5.1, 3.5.2 and 3.5.3, we then get that $\alpha(\sigma) = 1$, $s(\tau\sigma) = 1$ and $H(\sigma)_+ \perp H(\sigma)_-$. Since $s(\sigma) = 0$ and $s(\tau\sigma) = 1$, by Lemma 3.2.1, the surface $Y(\mathbf{R})$ is non-orientable. Since $b(Y) = 0$, from Theorem 3.4.7, $r(\theta) = a(\theta)$.

Now let $Y(\mathbf{R})$ be a connected non-orientable surface and $r(\theta) = a(\theta)$. By Lemma 3.2.1,

$$s(\sigma) + s(\tau\sigma) = 1. \tag{3–6–1}$$

We can suppose that $s(\sigma) = 0$ and $s(\tau\sigma) = 1$.

We claim that if $r(\theta) = a(\theta)$, then $H(\sigma)_+ \perp H(\sigma)_-$. Actually, we have a subgroup $\Gamma(\sigma)_\pm \subset H(\sigma)_+$ where $\dim \Gamma(\sigma)_\pm = a(\theta)$ and $\dim H(\sigma)_+ \leq r(\theta)$. Since $r(\theta) = a(\theta)$, we then get that $\Gamma(\sigma)_\pm = H(\sigma)_+$. Thus, we have $\Gamma(\sigma)_\pm = H(\sigma)_+ \subset H(\sigma)_-$. By Lemma 3.3.1, the subgroup $\Gamma(\sigma)_\pm$ is isotropic for the bilinear form $q(\sigma)$ on $A_{q(\sigma)}$. Hence the statement we claimed follows. Then we get that

$$\gamma(\sigma) = \dim H(\sigma)_- - \dim(H(\sigma)_+)^\perp \cap H(\sigma)_- = 0. \tag{3–6–2}$$

Like above, $\delta(\sigma) = 0$ and $\delta_{\sigma L^{\tau,\sigma}} = \delta_{\sigma L^{\tau}_{\sigma}} = 0$. Thus, we can apply to our situation Theorems 3.5.1 or 3.5.3. If $\alpha(\sigma) = 0$, we should apply Theorem 3.5.3. Then from (3–6–1) and (3–6–2) we get the contradiction: $0 = b(Y) \geq 1 + \beta(Y) > 0$. Thus, $\alpha(\sigma) = 1$, and we should apply Theorem 3.5.1. Then, from (3–6–1) and (3–6–2), we get $b(Y) = 0$.

By Theorem 1.2 and Proposition 0.3, we have $\dim\ {}_2Br(Y) = \epsilon(Y) = 1$, since $b(Y) = 0$ and $Y(\mathbf{R}) \neq \emptyset$. This proves the Theorem.

3.7. Some geometric applications.

We want to prove the following result where we use the invariants described above.

Theorem 3.7.1. *Let Y be a real Enriques surface. Then we have the inequalities for the numbers s and s_{nor} or connected and non-orientable connected components of $Y(\mathbf{R})$:*

$$s \leq (2 + r(\theta) - a(\theta) + \max\{1 - \alpha(\sigma), (\delta_{\sigma L^{\tau,\sigma}} + \delta_{\sigma L^{\tau}_{\sigma}})/2\} + \beta(Y))/2 \leq 6;$$

$$s_{nor} \leq 2 - \sharp\{x \in \{s(\sigma), s(\tau\sigma)\} \mid x > 0\} + \min\{\alpha(\sigma), (\delta_{\sigma L^{\tau,\sigma}} + \delta_{\sigma L^{\tau}_{\sigma}})/2\}$$
$$+ \dim H(\sigma)_- - \dim(H(\sigma)_+)^\perp \cap H(\sigma)_- + \beta(Y) \leq 4.$$

Proof. The first inequality follows from (3–4–22), Theorem 2.1 and (3–3–10). The second inequality follows from the formula (3–5–1), Theorem 2.1, Proposition 3.5.4 and inequalities (3–5–2)—(3–5–5).

We should mention that the inequality $s \leq 6$ also follows from results of V. M. Harlamov [Ha2]

3.8. A remark about further results.

We want to mention further results which were obtained by the first author during the time this paper was considered for a publication.

In [N5], further results about Brauer groups of real Enriques surfaces were obtained. Many of them valid for an arbitrary real smooth projective algebraic surface. We cite these results below.

The following Theorem generalizes the result from [C-P] we mentioned in Introduction about the homomorphism (0–1).

Theorem 3.8.1. *Let $X/\mathbf{R}$ be an algebraic projective manifold with the antiholomorphic involution g, and $G = \{id, g\}$. Then the homomorphism (0–1) is epimorphic if $H^3(X(\mathbf{C})/G; \mathbf{Z}/2) = 0$. More generally, the homomorphism (0–1) is epimorphic if the kernel of the homomorphism*

$$i^* : H^3(X(\mathbf{C})/G; \mathbf{Z}/2) \to H^3(X(\mathbf{R}); \mathbf{Z}/2)$$

is equal to zero. Here $i : X(\mathbf{R}) \subset X(\mathbf{C})/G$ denote the embedding.

The following result valid for smooth surfaces.

Theorem 3.8.2. *Let X be a real smooth projective algebraic surface and $H^3(X(\mathbf{C})/G; \mathbf{Z}/2) = 0$.*

Then the Hochschild–Serre spectral sequence degenerates and

$$\dim\ {}_2Br(X) = 2s - 1 + h^{2,0}(X(\mathbf{C})) + h^{1,1}_{-}(X(\mathbf{C})) - \rho_{+}(X \otimes \mathbf{C}).$$

Here $h^{1,1}_{-}(X(\mathbf{C})) = \dim H^{1,1}_{-}(X(\mathbf{C}))$ where

$$H^{1,1}_{-}(X(\mathbf{C})) = \{x \in H^{1,1}(X(\mathbf{C})) \mid g(x) = -x\}$$

is the set of potentially real algebraic cycles of X. And $\rho_{+}(X \otimes \mathbf{C}) = \dim(Pic\ (X \otimes \mathbf{C}) \otimes \mathbf{C})^G$. The characteristic cycle map gives an injection of $(Pic\ (X \otimes \mathbf{C}) \otimes \mathbf{C})^G$ to $H^{1,1}_{-}(X(\mathbf{C}))$.

Both these Theorems work for "general" real Enriques surfaces, and we get for these surfaces exactly the same results as for rational surfaces with non-empty set of real points.

Theorem 3.8.3. *Let Y be a real Enriques surface.*

Then $H^3(Y(\mathbf{C})/G; \mathbf{Z}/2) = 0$ iff the both liftings σ and $\tau\sigma$ of the antiholomorphic involution θ of Y have non-empty sets $X_\sigma(\mathbf{R})$ and $X_{\tau\sigma}(\mathbf{R})$ of real points ($s(\sigma) > 0$ and $s(\tau\sigma) > 0$).

Thus (by Theorems 3.8.1, 3.8.2), in this case, the homomorphism (0–1) is epimorphic, Hochschild–Serre spectral sequence degenerates, and

$$b(Y) = 2s - 2, \quad \dim\ {}_2Br(Y) = 2s - 1$$

Now we want to show how Theorems 3.4.7 and 3.5.1—3.5.3 of our paper work together with Theorem 3.8.3. By Theorem 3.4.7, if $\alpha(\sigma)=1$ and $\delta_{\sigma L^{\tau,\sigma}}=\delta_{\sigma L_\sigma^\tau}=0$, then at any case, the invariant $\beta(Y)=0$.

Now suppose that either $\alpha(\sigma)=0$ or $\delta_{\sigma L^{\tau,\sigma}}+\delta_{\sigma L_\sigma^\tau}>0$. Then, by Theorem 3.4.7, $b(Y)=r(\theta)-a(\theta)+1+\beta(Y)$. Additionally, let us suppose that $s(\sigma)>0$ and $s(\tau\sigma)>0$. By Theorem 3.8.3, we then get that $b(Y)=2s-2\equiv 0 \mod 2$. Since $r(\theta)-a(\theta)\equiv 0 \mod 2$, we have $\beta(Y)=1$. Thus, we have

Theorem 3.8.4. *Let Y be a real Enriques surface and either $\alpha(\sigma)=0$ or $\delta_{\sigma L^{\tau,\sigma}}+\delta_{\sigma L_\sigma^\tau}>0$.*

Then if $s(\sigma)>0$ and $s(\tau\sigma)>0$, we have $\beta(Y)=1$.

We don't know now, does the same statement holds if one of $s(\sigma)$ or $s(\tau\sigma)$ is 0.

We continue to suppose that $s(\sigma)>0, s(\tau\sigma)>0$. For example, assume the case $\alpha(\sigma)=0$. Then by Theorem 3.5.3 and Theorems 3.8.3 and 3.8.4, we have the equality: $s_{nor}+2s_{or}-2+\dim\ H(\sigma)_- - \dim\ (H(\sigma)_+)^\perp\cap H(\sigma)_- +1 = 2s_{nor}+2s_{or}-2$. Thus, we have

$$(3\text{–}8\text{–}1)\qquad if\ \alpha(\sigma)=0,\ then\quad s_{nor}=1+\dim H(\sigma)_- - \dim\ (H(\sigma)_+)^\perp\cap H(\sigma)_-.$$

By the same considerations, from Theorems 3.8.3, 3.8.4 and Theorems 3.5.1, 3.5.2, we get

$$(3\text{–}8\text{–}2)\qquad if\ (\alpha(\sigma)=1,\ \delta_{\sigma L^{\tau,\sigma}}=\delta_{\sigma L_\sigma^\tau}=0),\ then\quad s_{nor}=\dim\ H(\sigma)_- - \dim\ (H(\sigma)_+)^\perp\cap H(\sigma)_-;$$

$$(3\text{–}8\text{–}3)\qquad if\ (\alpha(\sigma)=1,\ \delta_{\sigma L^{\tau,\sigma}}+\delta_{\sigma L_\sigma^\tau}>0),\ then\quad s_{nor}=2+\dim\ H(\sigma)_- - \dim\ (H(\sigma)_+)^\perp\cap H(\sigma)_-;$$

Statements (3–8–1), (3–8–2) and (3–8–3) are very important for a topological classification of real Enriques surfaces (see below). We recall that we claim these statements only if $s(\sigma)>0$ and $s(\tau\sigma)>0$.

We remark that Enriques surfaces of Theorem 3.6.1 does not satisfy to conditions of Theorem 3.8.3. For these surfaces one of $s(\sigma)$, $s(\tau\sigma)$ is zero. But there exists another generalization of the Theorem 3.6.1.

Theorem 3.8.5. *Let Y be a real Enriques surface.*

Then the inequality (of Theorem 2.1)

$$b(Y)\geq 2s-2$$

is an equality iff the Hochschild–Serre spectral sequence degenerates. In particular (by Theorem 1.2), we have

$$\epsilon(Y)=1\ and\ \dim\ {}_2Br(Y)=2s-1,\ if\ b(Y)=2s-2.$$

At any case, we have an inequality

$$\dim\ {}_2Br(Y)\geq 2s-1.$$

We mention that the inequality $\dim\ {}_2Br(Y)\geq 2s-1$ also gives another proof of the Corollary 2.2.

In [N6], there were obtained some results about a topological classification of real Enriques surfaces. We cite some of these results below.

All possible invariants of [N4] for triplets (L,σ,S) (see §3.3) were described. In particular, all possibilities for pairs $(X_\sigma(\mathbf{R}), X_{\tau\sigma}(\mathbf{R}))$ were obtained. Using formulae (3–8–1), (3–8–2) and (3–8–3) for the case $s(\sigma)>0$ and $s(\tau\sigma)>0$ and Theorems 2.1 and 3.5.1—3.5.3 for the case $s(\sigma)=0$ or $s(\tau\sigma)=0$, this permits to construct many topological types of real Enriques surfaces and describe all theoretical possibilities.

The topological classification of real Enriques surfaces Y with connected non-orientable $Y(\mathbf{R})$ was obtained. For these surfaces, $Y(\mathbf{R})$ is one of the following non-orientable connected surfaces (all these possibilities take place): $U_k,\ k=1,3,5,7,9$. Here U_k denotes a connected non-orientable surface of the Euler characteristic $1-k$; its 2-sheeted unramified orientable covering is a connected orientable surface T_k of the genus k. For all these surfaces the invariants $r(\theta)=a(\theta)$. Thus, by Theorem 3.6.1, for these surfaces ${}_2Br(Y)\cong\mathbf{Z}/2$. Besides, we can remove the condition $r(\theta)=a(\theta)$ from the formulation of Theorem 3.6.1.

Also, real Enriques surfaces Y are constructed with $Y(\mathbf{R})=U_k\amalg U_0$, where $k=2,4,6,8,10$ (at first, the type $Y(\mathbf{R})=U_{10}\amalg U_0$ was constructed by R. Silhol [Si]) and one of invariants $s(\sigma), s(\tau\sigma)$ equal to zero. By this construction, using Theorems 2.1 and 3.5.1–3.5.2, for these surfaces we have $b(Y)=2$. Thus, by Theorem 3.8.5, for these Enriques surfaces, Hochschild–Serre spectral sequence degenerates and $\dim\ {}_2Br(Y)=3$. But it is not known if the homomorphism (0–1) epimorphic for these cases.

Enriques surfaces with 6 real connected components were constructed. For example, there exists a real Enriques surface Y with $Y(\mathbf{R})=U_1\amalg 5T_0$. By Theorem 3.7.1, it is the maximum number of real connected components for real Enriques surfaces. For this surface $s(\sigma)>0$ and $s(\tau\sigma)>0$. Thus, by Theorem 3.8.3, $\dim\ {}_2Br(Y)=11$.

Enriques surfaces with 4 real connected non-orientable components were constructed. For example, there exists a real Enriques surface Y with $Y(\mathbf{R})=2U_2\amalg 2U_0$. By Theorem 3.7.1, it is the maximum number of real connected non-orientable components for real Enriques surfaces.

We send the reader to the paper [N6] for further examples.

At last, we want to mention the most interesting problem (from our point of view) connected with real Enriques surfaces: To construct an example of a real Enriques surface such that the homomorphism (0–1) is not an epimorphism. By Theorem 3.8.3 (from [N5]), one can construct this example only if one of $s(\sigma),\ s(\tau\sigma)$ is zero.

References

[A] I. R. Shafarevich (ed.), *Algebraic surfaces*, Proc. Steklov Math. Inst. Vol 75, 1965; English transl. by A.M.S. 1969.

[Br] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York and London, 1972.

[CT-P] J.-L. Colliot-Thélène and R. Parimala, *Real components of algebraic varieties and étale cohomology*, Invent. math. **101** (1990), 81–99.

[C-D] F. R. Cossec and I. Dolgachev, *Enriques surfaces.I*, Progress in Mathematics, vol. 76., Birkhäuser, 1989.

[Ha1] V. M. Harlamov, *Topological types of nonsingular surfaces of degree 4 in* RP^3, Funktcional. Anal. i Prilozhen. **10** (1976), 55–68; English transl. in Functional Anal. Appl..

[Ha2] ———, *Real algebraic surfaces*, Proceedings of the International Congress of Mathematicians, Helsinki, I, 1978, pp. 421-428.

[Ho] E. Horikawa, *On the periods of Enriques surfaces, I, II*, Math. Ann. **234 and 235** (1978), 73–108 and 217–246.

[Kr] V. A. Krasnov, *Harnack-Thom inequalities for mappings of real algebraic varieties*, Izv. Akad. Nauk SSSR Ser. Mat. **47** (1983), 268–297; English transl. in Math. USSR Izv. **22** (1984), 247–275.

[Ku] Vik. S. Kulikov, *Degenerations of K3-surfaces and Enriques surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), 1008–1042; English transl. in Math. USSR Izv. **11** (1978), 957–989.

[Ma] Y. I. Manin, *Le groupe de Brauer-Grothendieck en Géometrie diophantienne*, Actes du Congrès Intern. Math. Nice (1970), vol. 1, Gauthier-Villars, Paris, 1971, pp. 401-411.

[Mi] J. Milne, *Étale cohomology*, Princeton Univ. Press, 1980.

[M-N] Sh. Mukai and Yu. Namikawa, *Automorphisms of Enriques surfaces which act trivially on the cohomology groups*, Invent. math. **77** (1984), 383–397.

[N0] V. V. Nikulin, *On Kummer surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 278–293; English transl. in Math. USSR Izv. **9** (1976), 261–275.

[N1] ———, *Finite groups of automorphisms of Kählerian surfaces of type K3*, Trudi Moscow Mat. Ob-va **38** (1979), 75–138; English transl. in Moscow Math. Soc. **38** (1980).

[N2] ———, *On the quotient groups of the automorphism groups of hyperbolic forms by the subgroups generated by 2-reflections, Algebraic-geometric applications*, Current Problems in Math. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow **18** (1981), 3-114; English transl. in J. Soviet Math. **22** (1983), 1401-1476.

[N3] ———, *Integral symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), 111–177; English transl. in Math. USSR Izv. **14** (1980), 103–167.

[N4] ———, *Involutions of integral quadratic forms and their application to real algebraic geometry*, Izv. Akad. Nauk SSSR Ser. Mat. **47** (1983), 109–188; English transl.in Math. USSR Izv. **22** (1984).

[N5] ———, *Lectures on the Brauer group of real algebraic surfaces*, Preprint of University of Notre Dame, College of science, Dept. of Math. # **179** (January, 1992).

[N6] ———, *On the topological classification of real Enriques surfaces.*, Preprint (1993).

[N-S] V. V. Nikulin and R. Sujatha, *On Brauer groups of real Enriques surfaces*, Preprint IHES /M/91/33 (1991).

[PŠ-Š] I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič, *A Torelli theorem for algebraic surfaces of type K3*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 530–572; English transl. in Math. USSR Izv. **5** (1971).

[Si] R. Silhol, *Real algebraic surfaces*, Lecture Notes Math., vol. 1392, 1989, pp. 215.

[Sp] E. H. Spanier, *Algebraic topology*, McGraw-Hill Book Company, 1966.

[Su] R. Sujatha, *Witt groups of real projective surfaces*, Math. Ann. **28** (1990), 89–101.

Steklov Mathematical Institute, ul.Vavilova 42, Moscow 117966, GSP-1, Russia

E-mail address: slava@nikulin.mian.su

School of Mathematics, TATA Institute of Fundamental Research, Homi Bhabha Road, Bombay-400-005, India

E-mail address: sujatha@tifrvax.bitnet