

## UNRAMIFIED COHOMOLOGY AND WITT GROUPS OF ANISOTROPIC PFISTER QUADRICS

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*Dedicated to my father on his sixtieth birthday*

ABSTRACT. The unramified Witt group of an anisotropic conic over a field  $k$ , with  $\text{char } k \neq 2$ , defined by the form  $\langle 1, -a, -b \rangle$  is known to be a quotient of the Witt group  $W(k)$  of  $k$  and isomorphic to  $W(k)/\langle 1, -a, -b, ab \rangle W(k)$ . We compute the unramified cohomology group  $H_{nr}^3, k(C)$ , where  $C$  is the three dimensional anisotropic quadric defined by the quadratic form  $\langle 1, -a, -b, ab, -c \rangle$  over  $k$ . We use these computations to study the unramified Witt group of  $C$ .

Let  $k$  be a field of characteristic not two and  $K/k$  be a finitely generated field extension. The  $n$ -th unramified cohomology group of  $K$  over  $k$  with coefficients in  $\mathbb{Z}/2$  (cf. [CT]) is denoted  $H_{nr}^n(K/k)$  (or just  $H_{nr}^n(K)$  when the ground field being considered is clear). It is a subgroup of the Galois cohomology group  $H^n(K, \mathbb{Z}/2)$  (abbreviated to  $H^n(K)$ ) and is defined as

$$H_{nr}^n(K/k) = \bigcap_{v \in \mathcal{V}(K)} H_{\acute{e}t}^n(\text{Spec } \mathcal{O}_v, \mu_2).$$

Here  $\mathcal{V}(K)$  is the set of all rank one discrete valuations on  $K$  that are trivial on  $k$ . For  $v \in \mathcal{V}(K)$ ,  $\mathcal{O}_v$  denotes the corresponding discrete valuation ring; it is well-known that the group  $H_{\acute{e}t}^n(\text{Spec } \mathcal{O}_v, \mu_2)$  injects into  $H^n(K)$ . If  $X/k$  is a smooth projective geometrically integral variety and  $k(X)$  is its function field, then – by abuse of terminology – we refer to the groups  $H_{nr}^n(k(X))$  as the *unramified cohomology groups of  $X$* .

For a field  $F$ , let  $W(F)$  be the Witt group of quadratic forms over  $F$  [Sc]. For  $k$  and  $K/k$  as above, we can similarly define the *unramified Witt group of  $K/k$* , denoted  $W_{nr}(K/k)$ , as the subgroup of  $W(K)$  defined by

$$W_{nr}(K/k) = \bigcap_{v \in \mathcal{V}(K)} W(\mathcal{O}_v).$$

Here  $W(\mathcal{O}_v)$  is the Witt group of the discrete valuation ring  $\mathcal{O}_v$ , which is a subgroup of  $W(K)$  [Sc, Chapter VI]. If  $X/k$  is smooth projective, then we refer to  $W_{nr}(k(X))$  as the *unramified Witt group of  $X$* .

Recall that a variety  $X$  is said to be  $k$ -rational if  $X$  is  $k$ -birational to the projective space. Then the natural maps  $H^n(k) \rightarrow H^n(k(X))$  (resp.  $W(k) \rightarrow W(k(X))$ ) induce isomorphisms  $H^n(k) \simeq H_{nr}^n(k(X))$  (resp.  $W(k) \simeq W_{nr}(k(X))$ ). Thus the unramified cohomology groups of a  $k$ -rational variety (resp. the unramified Witt

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group) coincide with the Galois cohomology groups of  $k$  (resp. the Witt group of  $k$ ). In particular, this is true for smooth projective quadrics  $X/k$  such that  $X$  has a  $k$ -rational point. A natural question, therefore, is the nature of these groups when  $X/k$  is a smooth projective quadric with no  $k$ -rational point.

For any field  $F$ , let  $\langle\langle a_1, a_2, \dots, a_n \rangle\rangle$  denote the  $n$ -fold Pfister form  $\langle 1, -a_1 \rangle \otimes \langle 1, -a_2 \rangle \otimes \dots \otimes \langle 1, -a_n \rangle$ . For  $a \in F^*$ , let  $(a)$  denote its class in  $H^1(F) \simeq F^*/F^{*2}$ . Given  $(a_1), \dots, (a_n)$  in  $H^1(F)$ , let  $(a_1) \cup \dots \cup (a_n)$  denote the element in  $H^n(F)$  obtained by taking cup-products. In this paper, we compute the unramified cohomology group  $H_{nr}^3(k(C))$ , where  $C$  is a smooth projective anisotropic quadric of dimension three defined by the quadratic form  $\langle\langle a, b \rangle\rangle \perp \langle -c \rangle$  over  $k$ . We prove that the group  $H_{nr}^3(k(C))$  (cf. §1) is a quotient of  $H^3(k)$  if  $(a) \cup (b) \cup (c) \cup (-1)$  is not trivial in  $H^4(k)$ . If the element  $(a) \cup (b) \cup (c) \cup (-1)$  is trivial in  $H^4(k)$ , then we show that the natural map  $H^3(k) \rightarrow H_{nr}^3(k(C))$  has cokernel  $\mathbb{Z}/2$ . We also prove a “descent” result for special symbols in  $H_{nr}^3(k(C))$ . More precisely, suppose  $\theta \in [(a) \cup (b) \cup H^1(k(C))] \cap H_{nr}^3(k(C))$ . We then show that  $\theta$  is “extended” from  $H^3(k)$  i.e.  $\theta = (a) \cup (b) \cup (e)$ , for some  $(e) \in H^1(k)$ . For special ground fields  $k$ , (e.g.  $k$  a global field), we compute higher unramified cohomology groups of  $C$ .

These results are then used to study the unramified Witt group of the quadric  $C$ . For anisotropic conics over  $k$ , it is a result of Parimala [P, Theorem 4.2] that the unramified Witt group is a quotient of  $W(k)$ . This result was extended by Colliot-Thélène and the author to real anisotropic quadrics of dimension three [CT-S]. It was shown that if  $C/\mathbb{R}$  is defined by  $\langle\langle -1, -1 \rangle\rangle \perp \langle 1 \rangle$ , then  $W_{nr}(\mathbb{R}(C)) \simeq W(\mathbb{R})/\langle\langle -1, -1, -1 \rangle\rangle$ . In this paper, we consider the case when  $k$  is a global field which is not totally imaginary. (Note that over a totally imaginary global field, any quadratic form of dimension five or more is isotropic.)

Suppose  $k$  is a global field as above. To transfer the results obtained in cohomology to Witt groups, we need to compare the quotients of the induced filtration on  $W_{nr}(k(C))$  with the unramified cohomology groups. In order to do this, we need Rost’s announcement that the norm residue homomorphism in degree four is an isomorphism and the map  $e_5$  (cf. §1) is well-defined. One can then use results of Arason-Elman-Jacob to carry out the above mentioned comparison. Using Jannsen’s higher dimensional Hasse principle, we see that the induced filtration on the unramified Witt group is finite. One then deduces that  $W_{nr}(k(C)) \simeq W(k)/\langle\langle a, b, c \rangle\rangle$ . This has the interesting corollary that any unramified quadratic form over  $k(C)$  is in fact *defined* over  $k$ .

Note that it is not *in general* true that the unramified Witt group of an arbitrary anisotropic quadric over a field  $k$  is a quotient of  $W(k)$ . Bruno Kahn [Ka] has constructed an example of an unramified Witt class that is not in the image of  $W(k)$  for an anisotropic “Albert quadric”, i.e. a quadric defined by an Albert form over  $k$ . However, we wonder if the unramified Witt group of “Pfister quadrics” over  $k$  (i.e. quadrics defined by Pfister forms and their neighbours) is a quotient of  $W(k)$  for arbitrary fields  $k$ . We show that this is indeed the case for a quadric defined by three-fold Pfister forms if the following statement is true:

**(A):** The subgroup  $[\langle\langle a, b \rangle\rangle I(k(C))] \cap I_{nr}^3(k(C))$  is generated by elements of the form  $\langle\langle a, b, f \rangle\rangle \in I_{nr}^3(k(C))$ , where  $I_{nr}^3(k(C)) = W_{nr}(k(C)) \cap I^3(k(C))$ .

Here,  $I^n(k(C))$  denotes the natural filtration on the Witt group (cf. §1) and  $I_{nr}^n(k(C))$  denotes the induced filtration on the unramified Witt group  $W_{nr}(k(C))$ .

The paper consists of four sections. In section 1, we list various results and notation that will be used in the later sections. Section 2 contains relevant results

on the unramified cohomology groups of the quadric  $C$  and some of its subgroups. In section 3, we specialise to the case when the ground field  $k$  is a global field. These results are used in section 4 to study the unramified Witt group  $W_{nr}(k(C))$ .

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Throughout the paper, the ground field  $k$  is assumed to be of characteristic not 2. For a more detailed exposition of various preliminary results, we refer the reader to §§1 and 2 of [CT-S], and to Scharlau's book [Sc] for all results and terminology from quadratic form theory.

## 1. PRELIMINARY RESULTS

In this section, we list various results that will be needed in the later sections.

**1.1. Unramified cohomology and the Bloch-Ogus spectral sequence.** Let  $X/k$  be a smooth, geometrically integral variety and let  $k(X)$  be its function field. For any variety  $Y$  over  $k$ , we denote the étale cohomology groups  $H_{\acute{e}t}^n(Y, \mu_2)$  of  $Y$  with coefficients in the étale sheaf  $\mu_2$  of square roots of unity by  $H^n(Y)$ . Let  $\mathcal{H}^q$  be the Zariski sheaf associated to the presheaf  $U \mapsto H^n(U)$ , where  $U$  is Zariski open in  $X$ . By the results of Bloch-Ogus [B-O], the group of global sections of the sheaf  $\mathcal{H}^q$ , denoted  $H^0(X, \mathcal{H}^q)$ , is a subgroup of the Galois cohomology group  $H^q(k(X))$ . Here, for a field  $F$  with  $\text{char } F \neq 2$ ,  $H^n(F) := H^n(\text{Gal}(F_s/F), \mathbb{Z}/2)$  is the Galois cohomology group with coefficients in  $\mathbb{Z}/2$ , and  $F_s$  denotes a separable closure of  $F$ . Further, if  $X$  is proper, the group  $H^0(X, \mathcal{H}^q)$  is a birational invariant of  $X$  [CT-O, Remarque 1.1.3] and is also denoted by  $H_{nr}^q(k(X))$ . These groups are called the *unramified cohomology groups of  $X$*  (cf. [CT-O, 1.1.1]). For instance, if  $X$  is smooth and proper, the group  $H_{nr}^2(k(X)) = H^0(X, \mathcal{H}^2)$  is the group of 2-torsion elements in the Brauer group of  $X$ . We will often use the notation  $H_{nr}^q(k(X))$  or  $H^0(X, \mathcal{H}^q)$  to denote the unramified cohomology groups in the case that  $X$  is smooth and proper.

Let  $H^p(X, \mathcal{H}^q)$  denote the Zariski cohomology groups of  $\mathcal{H}^q$ . Then there is a spectral sequence [B-O]

$$(1) \quad E_2^{p,q} = H^p(X, \mathcal{H}^q) \implies H^n(X) = E^n.$$

The filtration on  $H^n(X)$  coincides with the filtration given by the codimension of support. Further, it follows from the Bloch-Ogus resolution of the sheaf  $\mathcal{H}^q$  [B-O, Theorem 4.2] that  $H^p(X, \mathcal{H}^q) = 0$  for  $p > q$ . Another well-known result that we will need is the isomorphism [B-O, Theorem 7.7]

$$(2) \quad H^p(X, \mathcal{H}^p) \simeq CH^p(X)/2$$

where  $CH^p(X)$  denotes the Chow group of codimension  $p$  cycles on  $X$  modulo rational equivalence. The spectral sequence (1) thus gives an exact sequence

$$(3) \quad \cdots \rightarrow H^3(X) \rightarrow H^0(X, \mathcal{H}^3) \xrightarrow{d} CH^2(X)/2 \rightarrow H^4(X) \rightarrow \cdots.$$

The map  $CH^2(X)/2 \rightarrow H^4(X)$  is in fact the *cycle map* (cf. [CT-S, §1]).

**1.2. Galois cohomology, quadratic forms and function fields of quadrics.**

We recall some results from quadratic form theory and Galois cohomology. Given a field  $F$ ,  $\text{char } F \neq 2$ , let  $H^n(F)$  denote the  $n$ -th Galois cohomology group. Let  $W(F)$  denote the Witt group of quadratic forms over  $F$ . There is a standard filtration on  $W(F)$ , denoted  $(I^n(F))_{n \geq 0}$ , where  $I(F)$  is the fundamental ideal of even dimensional forms and  $I^n(F)$  its  $n$ -th power. Recall that the set  $P^n(F)$  of  $n$ -fold Pfister forms additively generates  $I^n(F)$ . There are maps  $e_n : P^n(F) \rightarrow H^n(F)$  which send an  $n$ -fold Pfister form  $\langle\langle a_1, a_2, \dots, a_n \rangle\rangle$  to the element  $(a_1) \cup (a_2) \cup \dots \cup (a_n)$  [AEJ]. These maps are known to extend to well-defined homomorphisms on  $I^n(F)$  for all fields  $F$  when  $n \leq 4$  [A], [J-R], [Sz]. Further, the maps

$$\bar{e}_n : I^n(F)/I^{n+1}(F) \rightarrow H^n(F)$$

are known to be isomorphisms for  $n \leq 3$  [Me], [Me-S1], [R]. Rost has announced a proof that  $\bar{e}_4$  is an isomorphism and that  $e_5$  is well-defined [R1]. If the maps  $e_i$  are well-defined homomorphisms for all fields  $F$  and for  $i \leq m$ , then it was observed by Parimala [P2, §1], that one can compare the filtrations on the unramified Witt group with the unramified cohomology groups. Indeed, there are homomorphisms  $I_{nr}^m(F) \rightarrow H_{nr}^m(F)$  induced by the homomorphisms  $e_m$ , where  $F$  is now a finitely generated extension of the ground field  $k$ .

We mention the following results about function fields of quadrics. Let  $Q/k$  be the smooth projective quadric defined by an  $n$ -fold Pfister form  $\langle\langle a_1, a_2, \dots, a_n \rangle\rangle$  over  $k$  and let  $k(Q)$  be the function field. The following conjecture is referred to as *Conjecture  $A_n$* :

$$\text{Ker}(H^n(k) \rightarrow H^n(k(Q))) = \{0, (a_1) \cup (a_2) \cup \dots \cup (a_n)\}.$$

This conjecture is known to be true for  $n \leq 4$ . Rost [R1] has announced a proof for  $n = 5$ . The cases  $n = 1, 2$  are classical, the case  $n = 3$  was proved by Arason [A, Satz 5.4] and the case  $n = 4$  was proved by Jacob-Rost [J-R] and independently, by Szyjewski [Sz].

For Witt groups, we have an analogous result, namely [A], [Pf]

$$(4) \quad \text{Ker}(W(k) \rightarrow W(k(Q))) = \langle\langle a_1, a_2, \dots, a_n \rangle\rangle W(k).$$

Finally, recall that a form  $\psi$  over  $F$  is said to be a *Pfister neighbour* of an  $n$ -fold Pfister form  $\phi$  over  $F$  if  $\dim \psi \geq 2^{n-1} + 1$  and  $\psi$  is isometric to a subform of  $\phi$ . Let  $\phi$  be an  $n$ -fold Pfister form and  $\psi$  a Pfister neighbour of  $\phi$ . We denote the quadrics defined by  $\phi$  and  $\psi$  by  $X_\phi$  and  $X_\psi$  respectively. Let  $F(X_\phi)$  (resp.  $F(X_\psi)$ ) denote the corresponding function fields. We record the following results for later use. We have [J-R, Lemma 1.2] for  $m \geq 0$ ,

$$(5) \quad \text{Ker}(H^m(F) \rightarrow H^m(F(X_\phi))) = \text{Ker}(H^m(F) \rightarrow H^m(F(X_\psi)))$$

and

$$(6) \quad \text{Ker}(W(F) \rightarrow W(F(X_\phi))) = \text{Ker}(W(F) \rightarrow W(F(X_\psi))).$$

(This follows from the observation that  $X_\phi$  has a  $k(X_\psi)$ -rational point (and vice-versa), namely the generic point. Thus the field  $k(X_\phi)(X_\psi)$  is a purely transcendental extension of both  $k(X_\phi)$  and  $k(X_\psi)$ .) We record that if  $m = 2$  (resp.  $m = 3$ ) and  $\phi = \langle\langle a, b \rangle\rangle$ ,  $a, b \in F^*$  (resp.  $\phi = \langle\langle a, b, c \rangle\rangle$ ,  $a, b, c \in F^*$ ), then the kernel in (6) is the subgroup consisting of elements of the form  $\{(a) \cup (b) \cup H^1(F)\}$  [A] (resp.  $\{(a) \cup (b) \cup (c) \cup H^1(F)\}$  [J-R] [Sz]).

**1.3. Chow groups and étale cohomology groups of quadrics.** We now recall some results of Karpenko, Merkurjev and Szyjewski. Let  $C/k$  be the smooth projective anisotropic quadric of dimension three, defined by the anisotropic quadratic form  $\langle\langle a, b \rangle\rangle \perp \langle -c \rangle$  over  $k$ . Thus  $C \subset \mathbb{P}_k^4$  is defined by  $X_0^2 - aX_1^2 - bX_2^2 + abX_3^2 - cX_4^2 = 0$ . Let  $B$  be the hyperplane section of  $C$  defined by  $X_4 = 0$  and let  $X$  be the plane section of  $C$  defined by  $X_4 = 0, X_3 = 0$ . Then  $[X]$  is a cycle of codimension one (respectively of codimension two) in  $B$  (resp. in  $C$ ). Let  $[T_1]$  be the *double line cycle* (of codimension one in  $B$  and codimension two in  $C$ ), defined by the equations  $X_0^2 - aX_1^2 = X_0X_3 - X_1X_2 = X_4 = 0$ . It is known (cf. [K-Me, 1.8]) that  $CH^1(B) \simeq Pic(B) \simeq \mathbb{Z}[X] \oplus \mathbb{Z}[T_1]$  and that  $CH^2(C) \simeq \mathbb{Z}[T_1] \oplus \mathbb{Z}/2[X - T_1]$  (cf. [Sz, Proposition 3.3.6]). The natural map

$$g : CH^1(B) \rightarrow CH^2(C)$$

maps the class  $[T_1]$  to the same class considered now as an element of  $CH^2(C)$  and similarly  $[X - T_1]$  to its corresponding class, which is now the 2-torsion element in  $CH^2(C)$ . Thus the map  $g$  is surjective and the induced map

$$\bar{g} : CH^1(B)/2 \rightarrow CH^2(C)/2$$

is an isomorphism, being a surjective map between two spaces of the same dimension. It is clear from the classical localisation sequence (cf. [K-Me, Proposition 2.3]) and (2) that the groups  $Pic(V)$  and  $CH^2(V)$  are trivial, where  $V$  is the open set  $C \setminus B$ . Further, the cycle class map  $cl_B : CH^1(B)/2 \rightarrow H^2(B)$  is injective. The images of the generators are denoted by  $cl_B[X]$  and  $cl_B[X - T_1]$ . The images under the cycle map  $cl_C : CH^2(C)/2 \rightarrow H^4(C)$  of the elements  $[T_1]$  and  $[X - T_1]$  are denoted by  $cl_C[T_1]$  and  $cl_C[X - T_1]$ . We have the commutative diagram below (cf. [CT-S, §1], observing that the statements there are valid for a general field of  $char \neq 2$ ):

$$(7) \quad \begin{array}{ccc} CH^1(B)/2 & \xrightarrow{\sim} & CH^2(C)/2 \\ cl_B \downarrow & & \downarrow cl_C \\ H^2(B) & \xrightarrow{i_*} & H^4(C). \end{array}$$

Szyjewski proved [Sz, Corollary 5.3.4] that the element  $cl_C[T_1] = (a) \cup (b) \cup cl_C[B]$ , where  $cl_C[B]$  is the image in  $H^2(C)$  of the non-trivial element under the Gysin map  $H^0(B) \rightarrow H^2(C)$  and  $(a) \cup (b) \in H^2(k) \hookrightarrow H^2(C)$  [Sz, Lemma 5.2.1 (b)]. Further,  $cl_C[X - T_1] \in H^4(k)$  and generates the kernel subgroup  $Ker(H^4(k) \rightarrow H^4(V))$  [Sz, Corollary 5.3.4], where  $V$  is the open set  $C \setminus B$ .

## 2. UNRAMIFIED COHOMOLOGY AND ÉTALE COHOMOLOGY

In this section, we use the results of Szyjewski and compute the structure of the unramified cohomology group  $H_{nr}^3(k(C))$ . We continue with the notation in §1. We have

**Theorem 1.** *Let  $C/k$  be the smooth projective anisotropic quadric defined by the form  $\langle\langle a, b \rangle\rangle \perp \langle -c \rangle$  over  $k$ . Let  $k(C)$  be the function field of  $C$  and  $H_{nr}^3(k(C))$  denote the unramified cohomology group. Then*

$$H_{nr}^3(k(C)) \simeq H^3(k)/(a) \cup (b) \cup (c) \text{ if } (a) \cup (b) \cup (c) \cup (-1) \neq 0 \text{ in } H^4(k)$$

and

$H_{nr}^3(k(C)) \simeq (H^3(k)/(a) \cup (b) \cup (c)) \oplus \mathbb{Z}/2$  if  $(a) \cup (b) \cup (c) \cup (-1) = 0$  in  $H^4(k)$ .

*Proof.* Let  $(H^3(C))^{(i)}_{i \geq 0}$  denote the filtration on  $H^3(C)$  given by the Bloch-Ogus spectral sequence (1.1). Then  $H^3(C)^{(1)} = \text{Ker}(H^3(C) \rightarrow H^3(k(C)))$ . Using the Hochschild-Serre spectral sequence (cf. [Sz, Lemma 5.2.1 (a)]), we have

$$H^3(C) \simeq H^3(k) \oplus H^1(k) \cup cl_C[B].$$

Note that the edge maps  $H^n(k) \rightarrow H^n(C)$  in the Hochschild-Serre spectral sequence are in fact induced by the structure morphism. Clearly, the second summand in the direct sum maps to zero in  $H^3(k(C))$ . By [A], the group  $\text{Ker}(H^3(k) \rightarrow H^3(k(C))) = (a) \cup (b) \cup (c)$ . Thus we have (cf. [Sz, Corollary 5.2.2]),

$$H^3(C)^{(1)} \simeq (a) \cup (b) \cup (c) \oplus H^1(k) \cup cl_C[B].$$

Hence the cokernel of the natural inclusion  $H^3(C)^{(1)} \hookrightarrow H^3(C)$  is given by

$$(8) \quad H^3(C)/H^3(C)^{(1)} \simeq H^3(k)/(a) \cup (b) \cup (c).$$

On the other hand we have an exact sequence (1.1)

$$(9) \quad \cdots H^3(C) \rightarrow H^0(C, \mathcal{H}^3) \rightarrow CH^2(C)/2 \rightarrow H^4(C)$$

where the map at the extreme left is the cycle map. We study the kernel of this map.

Assume that  $(a) \cup (b) \cup (c) \cup (-1) \neq 0$  in  $H^4(k)$ . We claim then that the cycle map is injective. By [Sz, Corollary 5.3.4, Proposition 5.4.6], we have  $(a) \cup (b) \cup (c) \cup (-1) = cl_C[X - T_1]$ . Recall (1.3) that  $CH^2(C)/2 \simeq \mathbb{Z}/2[T_1] \oplus \mathbb{Z}/2[X - T_1]$  and  $cl_C[T_1] = (a) \cup (b) \cup cl_C[B]$ . By the Hochschild-Serre spectral sequence,  $H^4(k) \oplus H^2(k).cl_C[B]$  injects into  $H^4(C)$  [Sz, Lemma 5.2.1 (c)]. Thus, considering this description of the cycle map  $cl_C$  on the generators, we see that  $cl_C : CH^2(C)/2 \rightarrow H^4(C)$  is injective. Therefore by (9), we see that  $H^3(C)$  surjects onto  $H^0(C, \mathcal{H}^3) = H_{nr}^3(k(C))$ . We thus have an isomorphism  $H^3(C)/H^3(C)^{(1)} \simeq H^0(C, \mathcal{H}^3)$ , and hence

$$(10) \quad H_{nr}^3(k(C)) \simeq H^3(k)/(a) \cup (b) \cup (c).$$

We now consider the case  $(a) \cup (b) \cup (c) \cup (-1) = 0$  in  $H^4(k)$ . Then, since  $cl_C[X - T_1] = (a) \cup (b) \cup (c) \cup (-1)$  [Sz, Proposition 5.4.6], we see that the cycle map  $cl_C$  is not injective and that  $\text{Ker}(cl_C)$  is in fact isomorphic to  $\mathbb{Z}/2(cl_C[X - T_1])$ . Using the exact sequence (9) along with (8), we get an exact sequence

$$(11) \quad 0 \rightarrow H^3(k)/(a) \cup (b) \cup (c) \rightarrow H_{nr}^3(k(C)) \rightarrow \text{Ker}(cl_C) \rightarrow 0.$$

Hence in this case  $H_{nr}^3(k(C)) \simeq (H^3(k)/(a) \cup (b) \cup (c)) \oplus \mathbb{Z}/2$ , and the theorem is proved.  $\square$

We highlight the following:

**Corollary.** *Let  $C/k$  be as above. Then the cycle map*

$$CH^2(C)/2 \rightarrow H^4(C)$$

is injective if and only if  $(a) \cup (b) \cup (c) \cup (-1) \neq 0$  in  $H^4(k)$ . If  $(a) \cup (b) \cup (c) \cup (-1) \neq 0$  in  $H^4(k)$ , then the kernel is isomorphic to  $\mathbb{Z}/2(\text{cl}_C[X - T_1])$ .

We now consider the subgroup of  $H_{nr}^3(k(C))$  which consists of elements that are of the form  $(a) \cup (b) \cup H^1(k(C))$ . We prove that this subgroup in fact lies in the subgroup  $H^3(k)/(a) \cup (b) \cup (c)$  of  $H_{nr}^3(k(C))$  even when  $(a) \cup (b) \cup (c) \cup (-1)$  is trivial in  $H^4(k)$ . We first need the following fact.

**Lemma 2.** *Let  $C, B$  be as above and  $V$  be the open subvariety  $C \setminus B$ . Then the natural restriction map  $H^0(C, \mathcal{H}^3) \rightarrow H^0(V, \mathcal{H}^3)$  is an isomorphism.*

*Proof.* Clearly this map is injective. Further, the group  $CH^2(V)$  is zero (1.3). Thus, in the Bloch-Ogus spectral sequence for  $V$ , we see that the map  $H^3(V) \rightarrow H^0(V, \mathcal{H}^3)$  is surjective. The Hochschild-Serre spectral sequence for  $V$  gives an exact sequence (cf. [Sz, 5.4])

$$(12) \quad 0 \rightarrow H^3(k) \rightarrow H^3(V) \rightarrow H^0(k) \xrightarrow{d} H^4(k) \rightarrow H^4(V).$$

Szyjewski proves that the differential coincides with multiplication by  $(a) \cup (b) \cup (c) \cup (-1)$  [Sz, Corollary 5.3.4(b), Proposition 5.4.6]. The edge map  $H^3(k) \rightarrow H^3(V)$  is in fact the one induced by the structure morphism. We consider the following cases:

Case (i).  $(a) \cup (b) \cup (c) \cup (-1) \neq 0$  in  $H^4(k)$ .

By the remarks above, we see that the natural map  $H^3(k) \rightarrow H^3(V)$  is indeed an isomorphism. Hence  $H^3(k)$  surjects onto  $H^0(V, \mathcal{H}^3) \subseteq H^3(k(C))$  with kernel  $(a) \cup (b) \cup (c) \cup (-1)$ .

Case (ii).  $(a) \cup (b) \cup (c) \cup (-1) = 0$  in  $H^4(k)$ .

In this case, by Theorem 1, the natural map  $H^3(k) \rightarrow H_{nr}^3(k(C)) = H^0(C, \mathcal{H}^3)$  has kernel  $(a) \cup (b) \cup (c)$  and cokernel  $\mathbb{Z}/2$ . Further, by our assumption on the non-triviality of  $(a) \cup (b) \cup (c) \cup (-1)$ , using the sequence (12), one sees that the natural map  $H^3(k) \rightarrow H^3(V)$  has cokernel  $\mathbb{Z}/2$ . Consider the commutative diagram

$$\begin{array}{ccc} H^3(k) & \longrightarrow & H^3(V) \\ & & \downarrow \\ & & H^0(V, \mathcal{H}^3). \end{array}$$

We have  $\text{Ker}(H^3(V) \rightarrow H^0(V, \mathcal{H}^3)) = \mathbb{Z}/2$  [Sz, Lemma 5.4.3]. Thus we have  $\text{Ker}(H^3(k) \rightarrow H^0(V, \mathcal{H}^3)) = \text{Ker}(H^3(k) \rightarrow H^3(k(C))) = (a) \cup (b) \cup (c)$  and the cokernel is  $\mathbb{Z}/2$ . Hence the lemma is proved.  $\square$

In order to prove that the subgroup of  $H^0(C, \mathcal{H}^3)$  consisting of elements of the form  $(a) \cup (b) \cup H^1(k(C))$  lies in the subgroup  $H^3(k)/(a) \cup (b) \cup (c)$  of  $H^0(C, \mathcal{H}^3)$ , by the above lemma, it suffices to prove the same for  $H^0(V, \mathcal{H}^3)$ . In fact, we prove the stronger

**Lemma 3.** *Let  $C, V$  be as above. Then we have an inclusion*

$$[(a) \cup H^2(k(C))] \cap H_{nr}^3(k(C)) \subseteq H^3(k)/(a) \cup (b) \cup (c).$$

*Proof.* As mentioned above, we are reduced to proving

$$[(a) \cup H^2(k(V))] \cap H^0(V, \mathcal{H}^3) \subseteq H^3(k)/(a) \cup (b) \cup (c).$$

Consider the quadratic extension  $l = k(\sqrt{a})$  which splits  $C$ , and let  $V_l$  be the variety  $V \times_k l$ . Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & H^3(V_l)^{(1)} & \xrightarrow{h_*} & H^3(V)^{(1)} \\
 & & & & \downarrow \rho_1 & & \downarrow \rho_2 \\
 H^2(V) & \xrightarrow{\cup(a)} & H^3(V) & \xrightarrow{i_*} & H^3(V_l) & \xrightarrow{h_*} & H^3(V) \\
 (13) \quad \downarrow \wr & & \downarrow & & \downarrow & & \downarrow \\
 H^0(V, \mathcal{H}^2) & \xrightarrow{\cup(a)} & H^0(V, \mathcal{H}^3) & \longrightarrow & H^0(V_l, \mathcal{H}^3) & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & & 0 & & 0 & & 
 \end{array}$$

Here the middle row is exact and the columns are also exact. The vertical sequences are induced by the Bloch-Ogus spectral sequence and the maps  $\rho_1$  and  $\rho_2$  are the natural inclusions. The middle exact row is the long exact cohomology sequence associated to a quadratic extension of the ground field. The homomorphism  $h_*$  is the transfer map, and the vertical arrow at the extreme left is an isomorphism since  $Pic(V)/2 \simeq CH^1(V)/2$  is zero. The map  $H^3(V) \rightarrow H^0(V, \mathcal{H}^3)$  is surjective because  $CH^2(V)$  is zero.

Suppose  $(a) \cup (\zeta) \in H^0(V, \mathcal{H}^3)$ , with  $\zeta \in H^2(k(C)) = H^2(k(V))$ . Let  $\theta \in H^3(V)$  be an element which maps to  $(a) \cup (\zeta)$ . There are two possibilities: either  $i_*(\theta) = 0$  in  $H^3(V_l)$  or  $i_*(\theta) \neq 0$  in  $H^3(V_l)$ . We consider these cases independently.

*Case (i).* Suppose  $i_*(\theta) = 0$  in  $H^3(V_l)$ . Then there is an element  $\theta' \in H^2(V)$  such that  $\theta' \cup (a) = \theta$ . The natural map  $H^2(k) \rightarrow H^2(V)$  is in fact an isomorphism. This is an easy consequence of the Hochschild-Serre spectral sequence, noting that the groups  $H^1(V_s)$  and  $H^2(V_s)$  ( $V_s = V \times k_s$ ,  $k_s$  a separable closure of  $k$ ) are trivial. The latter statement is easily established from the long exact localisation sequence for the triple  $B_s \hookrightarrow C_s \hookrightarrow V_s$ . Hence we see that  $\theta' \in H^2(k)$  and  $\theta = \theta' \cup (a)$  in  $H^3(k) \subseteq H^3(V)$ . Now the result follows since  $\theta$  maps to  $(a) \cup (\zeta)$ .

*Case (ii).* Suppose  $i_*(\theta)$  is not zero in  $H^3(V_l)$ . We first observe that  $(a) \cup (\zeta)$  maps to zero in  $H^0(V_l, \mathcal{H}^3)$ . This follows from the following commutative diagram:

$$\begin{array}{ccc}
 H^0(V, \mathcal{H}^3) & \longrightarrow & H^0(V_l, \mathcal{H}^3) \\
 \downarrow & & \downarrow \\
 H^2(k(V)) & \xrightarrow{\cup(a)} & H^3(k(V)) \longrightarrow H^3(l(V)).
 \end{array}$$

Here the vertical maps are inclusions and the bottom exact row is part of the long exact sequence associated to a degree two extension [A, Corollary 4.6]. Thus, we see that  $i_*(\theta) \in H^3(V_l)^{(1)}$ . By our assumption,  $i_*(\theta)$  is not zero. Szyjewski proves that for both  $V$  and  $V_l$ , the groups  $H^3(V)^{(1)}$  and  $H^3(V_l)^{(1)}$  are isomorphic to  $\mathbb{Z}/2$  [Sz, Lemma 5.4.2 (a)]. Further, the transfer map  $H^3(V_l) \rightarrow H^3(V)$  induces an isomorphism  $H^3(V_l)^{(1)} \simeq H^3(V)^{(1)}$  [Sz, Lemma 5.4.3]. Thus we see that  $i_*(\theta) = \alpha_l$



is the generator of  $H^3(V_l)^{(1)}$  and  $h_*(\alpha_l) = \alpha$ , where  $\alpha$  is the generator of  $H^3(V)^{(1)} \simeq \mathbb{Z}/2$  [Sz, Lemma 5.4.3]. But  $h_*(\alpha_l) = h_*(i_*(\theta)) = 0$ , since the middle row of (13) is exact. This is a contradiction and hence this case does not occur. Therefore the lemma is proved.  $\square$

We refine the above lemma further.

**Lemma 4.** *Let  $C$  be as above. Then we have an isomorphism*

$$[(a) \cup (b) \cup H^1(k(C))] \cap H_{nr}^3(k(C)) \simeq (a) \cup (b) \cup H^1(k)/(a) \cup (b) \cup (c).$$

*Proof.* Consider the exact sequence (cf. (1.2))

$$(14) \quad 0 \rightarrow (a) \cup (b) \cup H^1(k(C)) \rightarrow H^3(k(C)) \xrightarrow{\bar{\phi}} H^3(k(C \times B))$$

where  $B$  is considered as a quadric defined over  $k(C)$ . Since  $C$  has a rational point over  $k(B)$ , the field  $k(C \times B) = k(B \times C)$  is a pure transcendental extension of  $k(B)$ . The unramified cohomology groups are invariant under pure transcendental extensions [CT-O, Proposition 1.2]. Hence, the restriction of the above exact sequence to unramified subgroups gives an exact sequence

$$0 \rightarrow [(a) \cup (b) \cup H^1(k(C))] \cap H_{nr}^3(k(C)) \rightarrow H_{nr}^3(k(C)) \xrightarrow{\bar{\phi}} H_{nr}^3(k(B)).$$

Consider the commuting diagram below, where the maps  $\phi_C$  and  $\phi_B$  are the natural maps induced by the inclusions  $k \hookrightarrow k(C)$  and  $k \hookrightarrow k(B)$  respectively. We have

$$\begin{array}{ccccc} & & & H^3(k) & \\ & & & \phi_C & \phi_B \\ 0 & \longrightarrow & [(a) \cup (b) \cup H^1(k(C))] \cap H_{nr}^3(k(C)) & \longrightarrow & H_{nr}^3(k(C)) & \xrightarrow{\bar{\phi}} & H_{nr}^3(k(B)). \end{array}$$

The commuting triangle in this diagram gives an exact sequence

$$0 \rightarrow Ker \phi_C \rightarrow Ker \phi_B \rightarrow Ker \bar{\phi} \rightarrow Coker \phi_C.$$

By a theorem of Arason (1.2), we have  $Ker \phi_B = (a) \cup (b) \cup H^1(k)$ . The group  $Ker \phi_C$  is isomorphic to  $\mathbb{Z}/2\langle (a) \cup (b) \cup (c) \rangle$  (1.2). Further, by Theorem 1, the map  $\phi_C$  is surjective if  $(a) \cup (b) \cup (c) \cup (-1) \neq 0$  in  $H^4(k)$  and  $Coker \phi_C$  is isomorphic to  $\mathbb{Z}/2$  if  $(a) \cup (b) \cup (c) \cup (-1) = 0$  in  $H^4(k)$ . Using Lemma 3 in the latter case, we see that the map  $Ker \bar{\phi} \rightarrow Coker \phi_C = \mathbb{Z}/2$  is trivial. Thus, in both cases, the map  $Ker \phi_B \rightarrow Ker \bar{\phi}$  is surjective, and hence we have  $Ker \bar{\phi} \simeq Ker \phi_B / Ker \phi_C$ . Therefore,

$$[(a) \cup (b) \cup H^1(k(C))] \cap H_{nr}^3(k(C)) \simeq (a) \cup (b) \cup H^1(k)/(a) \cup (b) \cup (c),$$

and the lemma is proved.  $\square$

We now compute the group  $H^0(V, \mathcal{H}^4)$ . We recall some results of Szyjewski that are used in the computation and briefly mention the main steps involved in proving these results. Let  $\mathcal{K}_n^M$  ( $1 \leq n \leq 3$ ) denote the Milnor  $K$ -sheaf on a smooth integral variety  $X/k$  [K]. The homology of the complex (cf. [K])

$$C_n^M : 0 \rightarrow K_n^M(k(X)) \rightarrow \bigoplus_{x \in X^{(1)}} K_{n-1}^M(k(x)) \rightarrow \bigoplus_{x \in X^{(2)}} K_{n-2}^M(k(x)) \rightarrow \dots$$

computes the sheaf cohomology groups of  $\mathcal{K}_n^M$ . Here  $X^{(i)}$  denotes the set of codimension  $i$  points on  $X$  and given  $x \in X^{(i)}$ ,  $k(x)$  denotes the residue field at  $x$ . The groups  $K_n^M$  denote the Milnor  $K$ -groups of the fields. Recall that for any field  $F$  ( $\text{char } F \neq 2$ ), and  $n \leq 3$ , the norm residue homomorphism gives an isomorphism [Me], [Me-S1] [R]

$$K_n^M(F)/2 \simeq H^n(F).$$

Thus, since  $H^{n-1}(F)$  is generated by symbols for  $n \leq 3$ , using the description of torsion in  $K_2$  for fields [Su, Theorem 1.8], we get a complex

$$H^{n-1}(F) \xrightarrow{\{-1\}} K_n^M(F) \xrightarrow{2} K_n^M(F).$$

Here the first arrow maps  $(a_1) \cup \dots \cup (a_{n-1})$  to the symbol  $\{a_1, \dots, a_{n-1}, -1\}$ , and this map is well-defined. This yields a complex (for  $n \leq 3$ )

$$(15) \quad C_{n-1}^H \xrightarrow{\{-1\}} C_n^M \xrightarrow{2} C_n^M$$

where  $C_p^H$  is the complex that computes the cohomology groups  $H^i(X, \mathcal{H}^p)$  (cf. [B-O, Theorem 4.2]). Note that since the norm residue homomorphism  $h_n : K_n^M(F)/2 \rightarrow H^n(F)$  is an isomorphism for  $n \leq 3$ , there is an isomorphism of complexes  $C_n^M/2C_n^M \simeq C_n^H$  for  $n \leq 3$ . Using the excision sequence (cf. [K-Me, Proposition 2.3]), Szyjewski proves that for the open set  $V$ , the groups  $H^2(V, \mathcal{K}_3^M)$  and  $H^3(V, \mathcal{K}_3^M)$  are trivial [Sz1, Proposition 12.1.1 (c)]. Further, he proves (*loc.cit.*) that  $H^0(V, \mathcal{K}_3^M) \simeq K_3^M(F)$  and  $H^1(V, \mathcal{K}_3^M) \simeq D_k(\langle\langle a, b, c \rangle\rangle)$ . Here, for a quadratic form  $q$  over  $k$ ,  $D_k(q)$  represents the group of non-zero values represented by the quadratic form over  $k$ . The bicomplex associated to (15) for  $V$  with  $n = 3$  gives a spectral sequence abutting to  $H^*(V, \mathcal{H}^3)$ . Using the above computations of the cohomology groups for  $\mathcal{K}_3^M$ , Szyjewski [Sz1, Proposition 12.2.1 (d)] proves that

$$(16) \quad H^1(V, \mathcal{H}^3) \simeq D_k(\langle\langle a, b, c \rangle\rangle)/(D_k(\langle\langle a, b, c \rangle\rangle))^2.$$

We now prove

**Lemma 5.** *Let  $V$  be as before. Then the natural map  $H^4(V) \rightarrow H^0(V, \mathcal{H}^4)$  is surjective.*

*Proof.* As recalled above, the groups  $H^i(V, \mathcal{K}_3^M)$  are trivial for  $i = 2, 3$ . Note however that for  $i = 1, 2$  these groups coincide with  $H^i(V, \mathcal{K}_3)$  ( $\mathcal{K}_3$  denotes the Quillen  $K$ -theory sheaf) [J-R, pp. 561-562]. Therefore the group  $CH^3(V)$  is trivial. We have an exact sequence [CT1, p. 17]

$$0 \rightarrow H^2(V, \mathcal{K}_3)/2 \rightarrow H^2(V, \mathcal{H}^3) \rightarrow {}_2CH^3(V) \rightarrow 0.$$

This sequence implies that  $H^2(V, \mathcal{H}^3)$  is trivial. The Bloch-Ogus spectral sequence gives an exact sequence

$$H^4(V) \rightarrow H^0(V, \mathcal{H}^4) \rightarrow H^2(V, \mathcal{H}^3)$$

and the lemma follows. □

Finally, we have the following:

**Lemma 6.** *Assume that  $(a) \cup (b) \cup (c) \cup (-1)$  is not zero in  $H^4(k)$ . Then the group  $H^0(V, \mathcal{H}^4)$  is isomorphic to*

$$H^4(k)/(a) \cup (b) \cup (c) \cup H^1(F) \oplus D_k(\langle\langle a, b, c, -1 \rangle\rangle)/D_k(\langle\langle a, b, c \rangle\rangle)$$

*with notation as above.*

*Proof.* The Hochschild-Serre spectral sequence for  $V$  gives an exact sequence (cf. [Sz, Corollary 5.4.1])

$$0 \rightarrow H^4(k)/(a) \cup (b) \cup (c) \cup (-1) \rightarrow H^4(V) \rightarrow L \rightarrow 0$$

where  $L \subseteq H^1(k)$  is the subgroup consisting of elements  $(\lambda) \in k^*/k^{*2}$  such that  $(\lambda) \cup (a) \cup (b) \cup (c) \cup (-1) = 0$  in  $H^5(k)$ .

Consider the filtration  $H^4(V)^{(i)}$  on  $H^4(V)$  associated to the Bloch-Ogus spectral sequence. Since  $H^2(V, \mathcal{H}^3)$  is trivial, we have

$$H^0(V, \mathcal{H}^4) \simeq H^4(V)/H^4(V)^{(1)}.$$

Further,  $H^4(V)^{(1)}$  is isomorphic to  $H^1(V, \mathcal{H}^3)$  since  $H^4(V)^{(2)} \simeq CH^2(V)/2$  is trivial. By (16), the group  $H^1(V, \mathcal{H}^3)$  is isomorphic to  $D_k\langle\langle a, b, c \rangle\rangle / (D_k\langle\langle a, b, c \rangle\rangle)^2$ . We analyse this group by considering the following filtration on  $D_k\langle\langle a, b, c \rangle\rangle$ :

$$(D_k\langle\langle a, b, c \rangle\rangle)^2 \subseteq k^{*2} \subseteq D_k\langle\langle a, b, c \rangle\rangle.$$

The first quotient  $D_k\langle\langle a, b, c \rangle\rangle / k^{*2}$  is a natural subgroup of  $H^1(k)$ . The other quotient  $k^{*2} / (D_k\langle\langle a, b, c \rangle\rangle)^2$  fits in the exact sequence

$$0 \rightarrow k^{*2} / (D_k\langle\langle a, b, c \rangle\rangle)^2 \rightarrow D_k\langle\langle a, b, c \rangle\rangle / (D_k\langle\langle a, b, c \rangle\rangle)^2 \rightarrow D_k\langle\langle a, b, c \rangle\rangle / k^{*2} \rightarrow 0.$$

Consider the natural surjection

$$\begin{array}{ccc} \pi : & k^*/D_k\langle\langle a, b, c \rangle\rangle & \rightarrow & k^{*2} / (D_k\langle\langle a, b, c \rangle\rangle)^2, \\ & (\lambda) & \mapsto & (\lambda^2). \end{array}$$

Recall that the map  $e_4$  is an isomorphism on Pfister forms, this being a consequence of Merkurjev’s theorem and  $e_3$  being well-defined (cf. [J-R, p. 555]). Under the assumption that  $(a) \cup (b) \cup (c) \cup (-1)$  is not zero in  $H^4(k)$ , we see that the kernel of this map is isomorphic to  $\mathbb{Z}/2$ , the class of  $(-1)$  being the non-trivial element. The map

$$\begin{array}{ccc} \kappa : & k^*/D_k\langle\langle a, b, c \rangle\rangle & \rightarrow & (a) \cup (b) \cup (c) \cup H^1(k)/(a) \cup (b) \cup (c) \cup (-1), \\ & \lambda(mod\ D_k\langle\langle a, b, c \rangle\rangle) & \mapsto & (a) \cup (b) \cup (c) \cup (\lambda)(mod\ (a) \cup (b) \cup (c) \cup (-1)), \end{array}$$

along with the surjection  $\pi$  thus gives an isomorphism

$$k^{*2} / (D_k\langle\langle a, b, c \rangle\rangle)^2 \simeq (a) \cup (b) \cup (c) \cup H^1(k)/(a) \cup (b) \cup (c) \cup (-1).$$

Thus  $H^1(V, \mathcal{H}^3)$  is (non-canonically) isomorphic to

$$\begin{aligned} H^4(V)^{(1)} &\simeq H^1(V, \mathcal{H}^3) \\ &\simeq (a) \cup (b) \cup (c) \cup H^1(k)/(a) \cup (b) \cup (c) \cup (-1) \oplus D_k\langle\langle a, b, c \rangle\rangle / k^{*2}. \end{aligned}$$

Consider the two exact sequences

$$0 \rightarrow H^4(k)/(a) \cup (b) \cup (c) \cup (-1) \rightarrow H^4(V) \rightarrow L \rightarrow 0,$$

and

$$\begin{aligned} 0 &\rightarrow (a) \cup (b) \cup (c) \cup H^1(k)/(a) \cup (b) \cup (c) \cup (-1) \\ &\rightarrow H^4(V)^{(1)} \rightarrow D_k\langle\langle a, b, c \rangle\rangle / k^{*2} \rightarrow 0. \end{aligned}$$

Recall that the first sequence follows from the Hochschild-Serre spectral sequence. The first two groups of the second exact sequence embed naturally into the corresponding groups of the first exact sequence. To see this, observe that

$$\begin{aligned} H^4(V)^{(1)} \cap H^4(k)/(a) \cup (b) \cup (c) \cup (-1) \\ = (a) \cup (b) \cup (c) \cup H^1(k)/(a) \cup (b) \cup (c) \cup (-1). \end{aligned}$$

This is because  $\text{Ker}(H^4(k) \rightarrow H^4(k(V))) = (a) \cup (b) \cup (c) \cup H^1(k)$  (1.2). Consider the induced map  $D_k\langle\langle a, b, c \rangle\rangle/k^{*2} \rightarrow L$ . We claim that this map is also injective. To see this, consider the following commutative diagram

$$\begin{array}{ccc} & & H^4(k(V)) \\ & & \downarrow \\ H^4(k)/(a) \cup (b) \cup (c) \cup H^1(k) & \longrightarrow & H^4(V)/H^4(V)^{(1)} \end{array}$$

The vertical arrow and the slanted arrow are injective. This, along with the snake lemma applied to the two exact sequences above, proves the claim. Therefore we have an exact sequence

$$0 \rightarrow H^4(k)/(a) \cup (b) \cup (c) \cup H^1(k) \rightarrow H^4(V)/H^4(V)^{(1)} \rightarrow L/D_k\langle\langle a, b, c \rangle\rangle \rightarrow 0$$

and hence an isomorphism (non-canonical)

$$H^0(V, \mathcal{H}^4) \simeq H^4(V)/H^4(V)^{(1)} \simeq H^4(k)/(a) \cup (b) \cup (c) \cup H^1(k) \oplus L/D_k\langle\langle a, b, c \rangle\rangle.$$

Suppose  $(\lambda) \in L$ . Then  $(\lambda) \cup (a) \cup (b) \cup (c) \cup (-1) = 0$  in  $H^5(k)$ . This implies that  $e_5\langle\langle \lambda, a, b, c, -1 \rangle\rangle = 0$ . By injectivity of  $e_5$  on Pfister forms [J-R, Remark p. 555], we have  $\langle\langle \lambda, a, b, c, -1 \rangle\rangle = 0$ , which is equivalent to saying that  $\lambda \in D_k\langle\langle a, b, c, -1 \rangle\rangle$ . We have therefore proved

$$H^0(V, \mathcal{H}^4) \simeq H^4(k)/(a) \cup (b) \cup (c) \cup H^1(k) \oplus D_k\langle\langle a, b, c, -1 \rangle\rangle/D_k\langle\langle a, b, c \rangle\rangle.$$

□

Finally, we have

**Corollary 6'.** *Suppose  $k$  is a field such that  $(a) \cup (b) \cup (c) \cup (-1) \neq 0$ , and suppose  $D_k\langle\langle a, b, c, -1 \rangle\rangle = D_k\langle\langle a, b, c \rangle\rangle$ . Then  $H_{nr}^4(k(C)) \simeq H^4(k)/(a) \cup (b) \cup (c) \cup H^1(k)$ . In particular, the result is true for global fields.*

*Proof.* The assumption on  $k$  implies that  $H^0(V, \mathcal{H}^4) \simeq H^4(k)/(a) \cup (b) \cup (c) \cup H^1(k)$ . From the excision sequence, we get an injection  $H^0(C, \mathcal{H}^4) \hookrightarrow H^0(V, \mathcal{H}^4)$ . Since  $H^4(k)$  surjects onto  $H^0(V, \mathcal{H}^4)$ , the lemma follows. For global fields, we have isomorphisms  $H^i(k) \xrightarrow{\cup(-1)} H^{i+1}(k)$  for  $i \geq 3$ , and the graded Witt ring of  $k$  is isomorphic to the graded cohomology ring [AEJ]. This implies that the assumptions hold for global fields. □

*Remark.* It does not seem straightforward to compute  $H_{nr}^4(k(C))$  in a similar manner, as the sheaf cohomology groups  $H^p(C, \mathcal{H}^a)$  are difficult to compute. One of the obstacles is the difficulty in deciphering the kernel of the differential  $H^0(C, \mathcal{H}^4) \rightarrow H^2(C, \mathcal{H}^3)$  in the Bloch-Ogus spectral sequence.

3. THE CASE WHEN  $k$  IS A GLOBAL FIELD.

In this section we assume that  $k$  is a global field. Recall that we then have isomorphisms  $H^n(k) \xrightarrow{\cup(-1)} H^{n+1}(k)$  for  $n \geq 3$ . If  $k$  is totally imaginary, then  $C$  is necessarily isotropic and hence we have isomorphisms  $H^i(k) \simeq H_{nr}^i(k(C))$  for all  $i$ . We therefore assume that  $k$  is not totally imaginary and  $C/k$  is an anisotropic quadric. The natural map  $H^n(k) \rightarrow H_{nr}^n(k(C))$  is surjective for  $n \leq 4$ . For  $n \leq 2$ , this is obvious from the Hochschild-Serre spectral sequence. For  $n = 3$ , this follows from Theorem 1, noting that from the above remarks,  $(a) \cup (b) \cup (c) \cup (-1)$  is not zero. For  $n = 4$ , this follows from Corollary 6'. We prove the surjectivity for  $n = 5$ , assuming certain results which we outline below. We remark that proofs for these results have been announced. For  $n \geq 6$ , we prove the surjectivity without any restrictions.

Let  $k$  be a global field and  $X/k$  be a smooth, projective, geometrically integral variety of dimension  $d$ . Given a place  $v$  of  $k$ , let  $k_v$  denote the completion. Let  $k(X)$  (resp.  $k_v(X)$ ) denote the function field of  $X$  (resp.  $X_v := X \times k_v$ ). Consider the natural map (of Galois cohomology groups)

$$H^{d+2}(k(X), \mathbb{Q}/\mathbb{Z}(d+1)) \rightarrow \prod_v H^{d+2}(k_v(X), \mathbb{Q}/\mathbb{Z}(d+1))$$

where  $v$  varies over the places of  $k$ . The *cohomological Hasse principle*  $H_d$  is said to hold if this map is injective. For  $d = 1$  this was proved by Kato, and Jannsen gave a proof for  $d \leq 2$ . Jannsen [Ja] has announced a proof for arbitrary  $d$ . Suppose that the norm residue homomorphism  $h_4 : K_4^M(F)/2 \rightarrow H^4(F)$  is an isomorphism for any field  $F$ . From  $H_3$ , one can then deduce (cf. [Ra, p. 177]) that the natural map

$$H^5(k(X)) \rightarrow \prod_v H^5(k_v(X))$$

where  $v$  varies over all places, is injective. We now prove

**Theorem 7.** *Let  $C/k$  be as above. Assume that  $h_4$  is an isomorphism for all fields  $F$  and that the cohomological Hasse principle  $H_3$  holds. Then the natural map  $H^5(k) \rightarrow H_{nr}^5(k(C))$  is surjective. Further, there is an isomorphism  $H_{nr}^5(k(C)) \simeq \bigoplus_{v|\infty} H_{nr}^5(k_v(C))$ , the direct sum varying over the real places of  $k$ .*

*Remark.* We remark that Rost [R1] has announced a proof that  $h_4$  is an isomorphism. Jannsen [Ja] has announced a proof that  $H_d$  is true for any  $d$ .

*Proof.* By our hypothesis, the natural map  $H^5(k(C)) \rightarrow \prod_v H_{nr}^5(k_v(C))$  is injective, and hence, so is the restricted map  $H_{nr}^5(k(C)) \rightarrow \prod_v H_{nr}^5(k_v(C))$ . For a finite place  $v$ ,  $C(k_v) \neq \emptyset$ , since any quadratic form of dimension at least five is isotropic over a local field. Hence, we have isomorphisms  $H^n(k_v) \simeq H_{nr}^n(k_v(C))$  for all  $n$ . Since the 2-cohomological dimension  $cd_2(k_v)$  is two, we have  $H_{nr}^n(k_v(C)) = 0 = H^n(k_v)$  for  $n \geq 3$ . Similarly, if  $k_v = \mathbb{C}$ , the groups  $H_{nr}^n(k_v(C))$  are trivial for  $n \geq 1$ . If  $v$  is a real place such that  $C(k_v) \neq \emptyset$ , then  $H^n(k_v) \simeq H_{nr}^n(k_v(C)) \simeq \mathbb{Z}/2$  for all  $n$ . If  $v$  is a real place with  $C(k_v) = \emptyset$ , then  $H_{nr}^n(k_v(C))$  is trivial for  $n \geq 4$ , since  $cd_2(k_v(C)) = 3$ . Therefore we have an injective map

$$H_{nr}^5(k(C)) \hookrightarrow \prod_{v|\infty} H_{nr}^5(k_v(C)).$$

Consider the commutative diagram

$$\begin{array}{ccc}
 H^5(k) & \longrightarrow & \prod_{v|\infty} H^5(k_v) \\
 \downarrow & & \downarrow \\
 H_{nr}^5(k(C)) & \longrightarrow & \prod_{v|\infty} H_{nr}^5(k_v h(C)).
 \end{array}$$

The top horizontal map is an isomorphism the right vertical map is surjective, and the lower horizontal map is injective. Hence the map  $H^5(k) \rightarrow H_{nr}^5(k(C))$  is surjective and  $H_{nr}^5(k(C)) \rightarrow \bigoplus_{v|\infty} H_{nr}^5(k_v(C))$  is an isomorphism.  $\square$

**Theorem 8.** *Let  $C/k$  be as above. Then the natural map  $H^i(k) \rightarrow H_{nr}^i(k(C))$  is surjective for all  $i \geq 6$ .*

*Proof.* Let  $l = k(\sqrt{-1})$ . Since  $cd_2(l(C)) = 5$ , the groups  $H^i(k(C))$  and hence  $H_{nr}^i(k(C))$  are  $(-1)$ -torsion free for  $i \geq 6$  [AEJ, p. 29]. Recall that the  $(-1)$ -torsion subgroup of  $H^i(k(C))$  is precisely the group  $\text{Ker}(H^i(k(C)) \rightarrow \mathcal{C}(\text{Spec}_r(k(C)), \mathbb{Z}/2))$  [AEJ, p. 29]. Here  $\mathcal{C}(A, B)$  denotes the set of continuous maps from a space  $A$  into  $B$  and  $\text{Spec}_r$  denotes the real spectrum (cf. [C-R]). One checks that the  $(-1)$ -torsion subgroup of  $H_{nr}^i(k(C))$  is the group  $\text{Ker}(H_{nr}^i(k(C)) \rightarrow \mathcal{C}(\text{Spec}_r C, \mathbb{Z}/2))$ . Since  $k$  has only finitely many orderings corresponding to the real places, the natural map  $\text{Spec}_r C \rightarrow \prod_{v|\infty} \text{Spec}_r C_v$  is a homeomorphism [C-R, Proposition 4.3]. Consider the commutative diagram

$$\begin{array}{ccc}
 H_{nr}^i(k(C)) & \longrightarrow & \mathcal{C}(\text{Spec}_r C, \mathbb{Z}/2) \\
 \downarrow & & \downarrow \\
 \prod_{v|\infty} H_{nr}^i(k_v(C)) & \longrightarrow & \prod_{v|\infty} \mathcal{C}(\text{Spec}_r C_v, \mathbb{Z}/2)
 \end{array}$$

for  $i \geq 6$ . The right vertical map is an isomorphism and the top horizontal map is injective, since  $H_{nr}^6(k(C))$  is  $(-1)$ -torsion-free. Therefore the left vertical map is injective for  $i \geq 6$ . Arguing as in the proof of Theorem 7, the surjectivity follows for  $i \geq 6$ .  $\square$

Our next goal is to prove an analogous result for the groups  $I_{nr}^n(k(C))$ ,  $n \geq 5$ . Before we prove the result, we need to compare the filtration of the Witt group with the cohomology groups. Towards this end, we note that for a field of transcendence degree three over a global field or a local field, the graded Witt ring is isomorphic to the graded cohomology ring once we use Rost’s announced result that  $\bar{e}_4$  is an isomorphism and  $e_5$  is well-defined for all fields  $F$ . This follows by standard arguments of Arason-Elman-Jacob [AEJ3, Theorem 1.5]. We record this as

**Theorem 9.** *Let  $F$  be a field of transcendence degree at most three over a global (resp. local) field  $k$ . Suppose  $\bar{e}_4$  is an isomorphism and  $e_5$  is well-defined for  $F$ . Then the graded Witt ring is isomorphic to the graded cohomology ring of  $F$ .*

We also remark that for a field of transcendence degree three over a real-closed field (resp. algebraically closed field), it is known that the graded Witt ring is isomorphic to the graded cohomology ring [AEJ2, Corollary 3]. We have

**Corollary 10.** *Let  $C/k$  be as above. Assume that  $\bar{e}_4$  is an isomorphism and  $e_5$  is well-defined for the fields  $k(C)$  and  $k_v(C)$ ,  $v$  a finite place of  $k$ . Then the natural map  $I^n(k) \rightarrow I_{nr}^n(k(C))$  is surjective for  $n \geq 5$ .*

*Proof.* We first remark that an easy application of the ‘Normality Theorem’ due to Elman-Lam [E-L, Theorem 3.2] for  $k$  shows that the map  $I^n(k) \rightarrow \prod_v I^n(k_v)$  is surjective for  $n \geq 3$ . Since this map is injective, it is an isomorphism. Further, using Theorems 8 and 9 and arguing as in [AEJ2, Theorem 4], one sees that the map  $I_{nr}^n(k(C)) \rightarrow \prod_v I_{nr}^n(k_v(C))$  is injective for  $n \geq 5$ . Combining Theorem 9 with the Arason-Pfister Hauptsatz [Sc, p.156], one sees that the group  $I_{nr}^5(k_v(C))$  is trivial whenever  $H_{nr}^5(k(C))$  is. Further, if  $k_v$  is a real place with  $C(k_v) \neq \emptyset$ , then the specialisation map gives isomorphisms  $I_{nr}^n(k_v(C)) \simeq I^n(k_v)$  for all  $n$ . Arguing exactly along the lines of the proof of Theorem 9, we get the corollary.  $\square$

4. UNRAMIFIED WITT GROUPS OF ANISOTROPIC QUADRICS

In this section, we prove that the unramified Witt group  $W_{nr}(k(C))$  of the three dimensional quadric  $C$  is a quotient of the Witt group  $W(k)$  of  $k$  under the assumption **(A)** below. We continue with the notation of earlier sections. Recall that (1.2) we have an exact sequence

$$(17) \quad 0 \rightarrow \langle\langle a, b \rangle\rangle W(k(C)) \rightarrow W(k(C)) \rightarrow W(k(C \times B))$$

where the quadric surface  $B$  is now considered as being defined over  $k(C)$ . Now  $k(B \times C) = k(C \times B)$  is a pure transcendental extension of  $k(B)$  (1.2), and the unramified Witt group remains invariant under pure transcendental extensions [O, Proposition 7.2]. Hence, the restriction of (17) to the unramified subgroups gives an exact sequence

$$(18) \quad 0 \rightarrow [\langle\langle a, b \rangle\rangle W(k(C))] \cap W_{nr}(k(C)) \rightarrow W_{nr}(k(C)) \rightarrow W_{nr}(k(B))$$

with notation as before. But since the quadratic form defining the anisotropic conic  $X$  is a Pfister neighbour of the quadratic form defining  $B$ , we have [CT-S, Lemma 1.2]  $W_{nr}(k(B)) \simeq W_{nr}(k(X)) \simeq W(k)/\langle\langle a, b \rangle\rangle W(k)$  (1.2). We now state the assumption we need.

**(A):** The subgroup  $[\langle\langle a, b \rangle\rangle I(k(C))] \cap I_{nr}^3(k(C))$  is generated by elements of the form  $\langle\langle a, b, f \rangle\rangle \in I_{nr}^3(k(C))$ , where  $I_{nr}^3(k(C)) = W_{nr}(k(C)) \cap I^3(k(C))$ .

This assumption is to be viewed as a “quadratic-form” version of Lemma 3. We use this to prove the following:

**Theorem 11.** *Let  $C/k$  be the smooth projective anisotropic quadric of dimension three defined by the quadratic form  $\langle\langle a, b \rangle\rangle \perp \langle -c \rangle$ . Assume **(A)** is true. Then the unramified Witt group  $W_{nr}(k(C))$  is isomorphic to  $W(k)/\langle\langle a, b, c \rangle\rangle W(k)$ .*

*Proof.* We use the exact sequence (17), observing that by the remarks above, we have  $W_{nr}(k(B)) \simeq W(k)/\langle\langle a, b \rangle\rangle W(k)$ . Considering the natural maps, we have a commuting diagram with exact row as follows:

$$(19) \quad \begin{array}{ccc} & & W(k) \\ & \eta_C & \eta_B \\ 0 - [\langle\langle a, b \rangle\rangle W(k(C))] \cap W_{nr}(k(C)) - W_{nr}(k(C)) & \xrightarrow{\bar{\eta}} & W(k)/\langle\langle a, b \rangle\rangle W(k). \end{array}$$

Here  $\eta_C$  and  $\eta_B$  denote the natural maps, and clearly  $\bar{\eta}$  is surjective. To prove that  $\eta_C : W(k) \rightarrow W_{nr}(k(C))$  is surjective, note that by (19), it suffices to prove the following: Given an element  $\langle q_1 \rangle$  in  $[\langle\langle a, b \rangle\rangle W(k(C))] \cap W_{nr}(k(C))$ , there is an element  $\langle \tilde{q}_1 \rangle$  in  $W(k)$  such that

$$(*) \quad \eta_C \langle \tilde{q}_1 \rangle = \langle q_1 \rangle.$$

We now proceed to prove this. Consider the ideal  $I(k(C)) \subseteq W(k(C))$  of even dimensional forms. Then we have  $W(k(C))/I(k(C)) \simeq \mathbb{Z}/2$ , with the non-trivial element being the class  $\langle 1 \rangle$ , which clearly belongs to  $W(k)$ . Thus, it suffices to prove  $(*)$  for elements which are in the subgroup  $[\langle\langle a, b \rangle\rangle I(k(C))] \cap W_{nr}(k(C))$ . By assumption, this group is generated by elements of the form  $\langle\langle a, b, f \rangle\rangle$ . Since the homomorphisms  $e_3$  and  $e_2$  are well-defined, (1.2), we have  $e_3(\langle\langle a, b, f \rangle\rangle) = (a) \cup (b) \cup (f) \in H_{nr}^3(k(C))$  (1.2). Further, it is clear that the element  $(a) \cup (b) \cup (f) \in [(a) \cup (b) \cup H^1(k(C))] \cap H_{nr}^3(k(C))$ . By Lemma 4, we have

$$(a) \cup (b) \cup (f) = (a) \cup (b) \cup (d) + (a) \cup (b) \cup (c), \quad d \in k^*,$$

and this implies that

$$(20) \quad (a) \cup (b) \cup (f) = (a) \cup (b) \cup (e) \in H^3(k), \quad e \in k^*.$$

Consider the Pfister forms  $\langle\langle a, b, f \rangle\rangle$  and  $\langle\langle a, b, e \rangle\rangle$ . These elements belong to the subgroup  $[\langle\langle a, b \rangle\rangle I(k(C))] \cap W_{nr}(k(C))$ . Further, by (20), we have

$$e_3 \langle\langle a, b, f \rangle\rangle = (a) \cup (b) \cup (f) = (a) \cup (b) \cup (e) = e_3 \langle\langle a, b, e \rangle\rangle.$$

Since  $e_3$  is injective on Pfister forms [A2, Proposition 2], we have  $\eta_C \langle\langle a, b, e \rangle\rangle = \langle\langle a, b, f \rangle\rangle$ , and  $(*)$  is proved.

Thus the natural map  $\eta_C : W(k) \rightarrow W_{nr}(k(C))$  is surjective. Further, since the form defining  $C$  is a Pfister neighbour of the three-fold Pfister form  $\langle\langle a, b, c \rangle\rangle$ , by (1.2) we have,

$$\text{Ker} (W(k) \rightarrow W_{nr}(k(C))) = \text{Ker} (W(k) \rightarrow W(k(C)) = \langle\langle a, b, c \rangle\rangle W(k)).$$

Hence  $W_{nr}(k(C)) \simeq W(k)/\langle\langle a, b, c \rangle\rangle W(k)$ , and the theorem is proved. □

In general, we have

**Theorem 12.** *Let  $C/k$  be as above. Then the natural map  $W(k) \rightarrow W_{nr}(k(C))$  gives a surjection  $W(k)/I^4(k) \rightarrow W_{nr}(k(C))/I_{nr}^4(k(C))$ .*

*Proof.* Arguing as in the proof of Theorem 11, this is immediate from Theorem 1, since  $\bar{e}_3$  is a well-defined isomorphism (1.2). □

Assuming that  $k$  is a global field, we have

**Theorem 13.** *Let  $k$  be a global field and  $C$  an anisotropic three dimensional quadric over  $k$  defined by  $\langle\langle a, b \rangle\rangle \perp \langle -c \rangle$ . Assume that the norm residue homomorphism  $h_4$  is an isomorphism for all fields and that  $e_5$  is well-defined. Under the assumption that the cohomological Hasse principle  $H_3$  holds, there is an isomorphism*

$$W_{nr}(k(C)) \simeq W(k)/\langle\langle a, b, c \rangle\rangle W(k).$$

*Remark.* We remark that Rost [R1] has announced a proof that  $h_4$  is an isomorphism and  $e_5$  is well-defined. Jannsen [Ja] has announced a proof that  $H_d$  holds for all  $d$ .



*Proof.* The initial steps in the proof are analogous to those in the proof of Theorem 11. We now prove (\*) (cf. proof of Theorem 11) without assuming (A). As before, we reduce to proving (\*) for elements in the subgroup  $[\langle\langle a, b \rangle\rangle]I(k(C)) \cap W_{nr}(k(C))$ . Let  $q_1$  be an element of this group. Then,  $q_1 \in I_{nr}^3(k(C))$  and  $e_3(q_1) \in H_{nr}^3(k(C))$ . By Theorem 1,  $e_3(q_1) = (a) \cup (b) \cup (e)$ ,  $(e) \in H^1(k)$ . Thus  $\langle\langle a, b, e \rangle\rangle$  and  $q_1$  map to the same element under  $e_3$ , and hence  $q_1 = \langle\langle a, b, e \rangle\rangle \perp q_2$  where  $q_2 \in I_{nr}^4(k(C))$ . Again,  $e_4(q_2) \in H_{nr}^4(k(C))$  can be lifted to a symbol in  $H^4(k)$  under the natural map  $H^4(k) \rightarrow H_{nr}^4(k(C))$ , which is surjective by Corollary 6'. Therefore  $q_2 = \tilde{q}_2 \perp q_3$  where  $\tilde{q}_2 \in I^4(k)$  and  $q_3 \in I_{nr}^5(k(C))$ . By Corollary 10, there exists  $\tilde{q}_3 \in I^5(k)$  such that  $\tilde{q}_3$  maps to  $q_3$  under the natural map  $I^5(k) \rightarrow I_{nr}^5(k(C))$ , and hence (\*) is proved.  $\square$

We have the following corollary.

**Corollary 14.** *With hypotheses as in Theorem 13, any unramified quadratic form over  $k(C)$  is defined over  $k$ .*

*Proof.* Let  $q$  be an unramified quadratic form over  $k(C)$ . Without loss of generality, we may assume that  $q$  is anisotropic. By a result of Elman-Lam-Wadsworth [ELW, Corollary 6.4], the field  $k(C)$  is an excellent extension of  $k$ . The corollary now follows from Theorem 13.  $\square$

*Note added in proof.* By the recent results of Voevodsky, the norm-residue maps  $h_n$  are isomorphisms for all  $n$ . In particular, the maps  $\bar{e}_n$  are well-defined isomorphisms for all  $n$ .

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