Orthogonal and spin bundles over hyperelliptic curves

By

S. RAMANAN

1. Introduction

Let $\Sigma x_i^2 = 0$, $\Sigma \lambda_i x_i^2 = 0$ be two quadrics in the projective space of dimension $(2g + 1)$. Assume that $\lambda_i$'s are distinct so that these two quadrics determine a generic pencil of quadrics. One can associate to this pencil, in a natural way, a hyperelliptic curve $X$ of genus $g$, namely, the double of the projective line parameterising the pencil with ramification at the points corresponding to the singular quadrics. One can easily check that the variety $M_d$ of $d$ dimensional vector subspaces which are isotropic for both the quadrics is nonsingular. $M_d$ is nonempty if $d \leq g$ and in [1] we gave an interpretation of $M_d$ and $M_{g-1}$ as the Jacobian and the moduli space of vector bundles of rank 2 and fixed determinant of odd degree over $X$ respectively. One might naturally ask if one could obtain similar interpretation for $M_d$ for other $d$ as well. In particular, does $M_1$ = the intersection of two quadrics have an interpretation as a moduli-space of some sort of bundles over $X$? This paper answers, among other things, this question. In fact, we give a solution of a more general question, namely, the question of interpreting $M_d$ modulo $\Pi SO(n_d)$, when the $\lambda_i$'s are allowed to have multiplicities $n_i$.

The idea of associating a hyperelliptic curve to a pencil of quadrics is nothing new and goes back, at least to Weil [9], who considered the relationship between the intersection of two quadrics ($M_1$ in our notation) and the curve $X$ from the point of view of diophantine equations over finite fields. He computed the zeta function of $M_1$ in terms of that of $X$ and verified the Weil conjectures for this variety. Incidentally he also raised the specific question of the geometric relationship between $M_1$ and $X$, which we have investigated here. It must also be mentioned that Gauthier [2] has already noticed the identity of the space $M_g$ and the Jacobian of $X$, although as far as the present author is aware, no proof has been published till recently [6, 1]. The author is thankful to A. Weil for bringing to his attention these references.

The method of proof is quite simple and is not basically different from that in [1]. However, the greater generality leads to conceptual clarity and we have also made a few technical simplifications. If $E$ is a bundle to which the action of $i$ lifts, the bundle does not nevertheless go down to a bundle on $X/i = \mathbb{P}^1$. The obstruction to this is the lack of descent data at the fixed points of $i$, namely, the

*Supported in part by NSF grant MCS77-18723.
Weierstrass points of $X$. In other words, $i$ may not act as identity on the fibres of $E$ at the Weierstrass points. To rectify this, one makes a modification of $E$ at these points to obtain a new bundle $E'$ which does go down. If one keeps track of the additional structures needed to recover $E$ from this bundle on $\mathbb{P}^1$ obtained by descent from $E'$, one can hope for a classification of the bundles started with.

Another description of the method is as follows. Suppose we wish to classify vector bundles of fixed large degree. Consider pairs $(E, \pi)$ where $E$ is such a vector bundle and $\pi$ is a trivialisation of $E$ on a sufficiently large divisor $D$. Then the restriction $\mathcal{H}^0(X, E) \to \mathcal{H}^0(D, E)$ is injective and using the trivialisation of $E$ on $D$, we associate to $E$, a point of a suitable Grassmannian. Thus one could even construct the moduli space of vector bundles by passing to the quotient of the Grassmannian in $k^a$ by $\Pi GL(d)$. The special situation in our case then allows us to describe the image precisely.

2. Clifford bundles

If $V$ is a vector space of finite dimension and $Q$ is a non-degenerate quadratic form on $V$, then the Clifford group $\Gamma(Q)$ is the subgroup of the group $C(Q)^x$ of units in the Clifford algebra $C(Q)$, defined by

$$\Gamma(Q) = \{ x \in C(Q)^x : xVx^{-1} \subset V \}.$$ 

Let $\Gamma^+(Q)$ be the group of even elements in $\Gamma(Q)$. The multiplicative group $k^x$ is clearly contained in $\Gamma^+(Q)$. It is actually the centre of $\Gamma(Q)$ if $V$ is of even dimension. Then the spinor norm gives a surjective homomorphism $Nm : \Gamma(Q) \to k^x$ whose kernel is called the reduced Clifford group and denoted $\Gamma_0(Q)$. Let $\Gamma_0^+(Q) = \Gamma_0(Q) \cap \Gamma^+(Q)$. By definition, we have a natural (vector) representation of all these groups in the orthogonal group of $Q$. The kernel of this representation of $\Gamma$ is $k^x$. We first notice

**Lemma 2.1.** If $E$ is an $O(2n)$-bundle on a scheme $X$, then the obstruction to its liftability to a $\Gamma_0(2n)$-bundle (resp. $\Gamma(2n)$-bundle) is an element of $H^2(X, \mathbb{Z}/2)$ (resp. $H^2(X, \mathcal{O}_X^\times)$). In particular, any $O(2n)$ bundle on a projective nonsingular curve can be lifted to a $\Gamma(2n)$-bundle. Similar statements are valid for the odd dimensional case when $O$ is replaced by $SO$ and $\Gamma$ is replaced by $\Gamma^+$.

**Proof.** This is a trivial consequence of the exact sequences

$$(2.1) \quad 1 \to \mathbb{Z}/2 \to \Gamma_0(2n) \to O(2n) \to 1$$

and the induced exact sequence of sheaves

$$(2.2) \quad 1 \to \mathbb{Z}/2 \to \Gamma_0(2n) \to O(2n) \to 1$$

and the obstruction to its liftability to a $\Gamma_0(2n)$-bundle (resp. $\Gamma(2n)$-bundle) is an element of $H^2(X, \mathbb{Z}/2)$ (resp. $H^2(X, \mathcal{O}_X^\times)$). In particular, any $O(2n)$ bundle on a projective nonsingular curve can be lifted to a $\Gamma(2n)$-bundle. Similar statements are valid for the odd dimensional case when $O$ is replaced by $SO$ and $\Gamma$ is replaced by $\Gamma^+$. 

**Proof.** This is a trivial consequence of the exact sequences

$$(2.1) \quad 1 \to \mathbb{Z}/2 \to \Gamma_0(2n) \to O(2n) \to 1$$

and the induced exact sequence of sheaves

$$(2.2) \quad 1 \to \mathbb{Z}/2 \to \Gamma_0(2n) \to O(2n) \to 1$$
in view of [7]. In fact, the obstruction to the \( \Gamma(2n) \)-lifting is actually in the image of \( H^3(X, \mathbb{Z}/2) \) in \( H^2(X, \mathcal{O}_X^\times) \). Hence our assertion.

If \( P \) is a \( \Gamma(n) \)-bundle and \( \xi \) is a line bundle, then using the action of \( k^* \) on \( \Gamma(n) \), we get a new \( \Gamma(n) \)-bundle which we will denote \( \xi \circ P \). From the fact that \( \text{Nm}(\lambda a) = \lambda^2 \text{Nm}(A) \) for \( \lambda \in k^* \), \( A \in \Gamma(n) \) we obtain

\[
(2.3) \quad \text{Nm}(\xi \circ P) = \xi^2 \text{NmP}.
\]

**Lemma 2.4.** If \( P, P' \) are \( \Gamma(2n) \)-bundles on \( X \), giving rise to isomorphic \( O(2n) \)-bundles, then there exists a line bundle \( \xi \) such that \( P' \cong \xi \circ P \). If \( E \) is an \( O(2n) \)-bundle and \( P \) any lift of \( E \) to a \( \Gamma(2n) \)-bundle, then \( \text{Nm}P \) is of even or odd degree according as \( E \) can or cannot be lifted to a \( \Gamma_0(2n) \)-bundle. Similar results hold for the odd dimensional case with \( O(n) \) replaced by \( SO(n) \).

**Proof.** The first assertion follows from the sequence 2.2 and [7]. From this it is clear that (a) the parity of \( \text{deg}(\text{Nm}P) \) is independent of the lift \( P \), and (b) if \( \text{deg} \text{Nm}P \) is even we might, in view of 2.3, as well assume that \( \text{Nm}P \) is trivial by replacing \( P \) by a suitable \( \xi \circ P \). If \( \text{Nm}P \) is trivial, then \( P \) is a \( \Gamma_0(2n) \)-bundle as is seen from the sequence

\[
1 \rightarrow \Gamma_0(2n) \rightarrow \Gamma(2n) \rightarrow \mathcal{O}_X^\times \rightarrow 1.
\]

**Definition 2.5.** An \( O(2n) \)-bundle on \( X \) is said to be of even or odd type according as it is or is not liftable to a \( \Gamma_0(2n) \)-bundle.

**Remark 2.6.** An \( SO(n) \)-structure on an \( O(n) \)-bundle \( E \) with trivial determinant is given by a trivialisation of \( \tilde{\Lambda}E \), such that its square is the trivialisation of \( (\tilde{\Lambda}E)^2 \) given by the ‘discriminant’, namely, the lift \( \tilde{\Lambda}E \rightarrow \tilde{\Lambda}E^* \) of the given quadratic form \( E \rightarrow E^* \). Accordingly, even when \( \tilde{\Lambda}E = \text{det} E \) is only an element of order 2, we could still talk of a special orthogonal structure, when we mean an isomorphism \( \text{det} E \rightarrow \alpha \) such that its square \( (\text{det} E)^2 \rightarrow \alpha^2 \simeq 1 \), is the trivialisation given by the discriminant. Also, if \( P \) is a \( \Gamma(n) \)-bundle, we talk of a special \( \Gamma(n) \)-structure when the associated \( O(n) \)-bundle \( E \) comes provided with a special orthogonal structure. If \( \text{det} E \) is trivial, this simply means that \( P \) is a \( \Gamma(n)^+ \)-bundle.

3. Orthogonal bundles over hyperelliptic curves

We will make the convention that when we deal with orthogonal bundles of odd rank, we assume that it has trivial determinant. Generally, we will deal with the case of even rank, the odd case being similar.

Let \( E \) be an orthogonal bundle over a hyperelliptic curve with a lift \( E \approx i^*E \) of the involution \( i \). If \( P \) is a lift of \( E \) to a \( \Gamma \)-bundle, then we must have \( P \approx a \circ i^*P \) for some line bundle \( a \) by lemma 2.4. Taking norms on both sides and using 2.3, we get \( a^2 \text{Nm}P \approx i^*\text{Nm}P \) and hence \( (a \text{Nm}P)^2 \approx \text{h}^{\text{deg} \text{Nm}P} \). In particular, \( a \) is of degree 0. Let us then assume given an isomorphism

\[
P \approx \beta \circ (\text{Nm}P)^{-1} \circ i^*P, \text{ with } \beta^2 = \text{h}^{\text{deg} \text{Nm}P}.
\]

It is clear that \( \beta \) is essentially unique, at least if \( \text{Nm}P \) is fixed. We now recall

P—11
Lemma 3.1. There is a one–one correspondence between line-bundles $\eta$ of fixed even (respectively odd) degree such that $\eta^2 \approx \eta^{\text{des}}$, and the set of partitions $W = S \cup T$ of the set of Weierstrass points into subsets of even (resp. odd) cardinality.

Proof. See ([1] lemma 2.1).

If an isomorphism $i^* E \rightarrow E$ is given, then at every fixed point of $i$, namely, a Weierstrass point, we obtain an involution of $E \cdot E$ preserving the quadratic form $g_\sigma$ on $E \cdot E$. Let $E_+^\sigma, E_-^\sigma$ be the eigenspaces corresponding to the eigenvalues $\pm 1$. Then $E_+^\sigma, E_-^\sigma$ are clearly orthogonal for $g_\sigma$. In particular, $g_\sigma$ is nondegenerate on both these spaces.

We now seek to prove that for such an $O (2n)$-bundle $E$, the line bundle $\beta$ with $i^* P \approx \beta^{-1} \otimes N m P \circ P$ is the unique line bundle with $i^* \beta \approx \beta$ whose associated partition of $W$ (lemma 3.1) is determined by the integers $r_\sigma = \dim E_+^\sigma$. Before we do this, let us normalise the isomorphism $\varphi : P \rightarrow a \circ i^* P$ given above. We will of course require that this should induce the given isomorphism $E \rightarrow i^* E$. This requirement fixes $\varphi$ up to an automorphism of the $\Gamma (2n)$-bundle $P$ inducing the identity map on $E$. This means that $\varphi$ is determined upto a nonzero scalar factor. We now wish to further specify $\varphi$ by requiring that it be an involution in the following sense. Firstly, we have an isomorphism $N m \varphi : N m P \rightarrow a^2 \circ i^* N m P$. Writing $a = \beta \otimes N m P$, this gives an isomorphism $\beta^{-1} \rightarrow N m P \otimes i^* N m P$. On the other hand $i^* \varphi$ gives an isomorphism $i^* P \rightarrow i^* \beta \otimes i^* N m P \circ P$ and hence $\beta \otimes N m P \circ i^* P \approx i^* \beta \otimes i^* \otimes N m P \otimes i^* N m P \circ P$ and using $N m \varphi$ also $\beta \otimes N m P \circ i^* P \approx \beta^{-1} \otimes i^* \beta \circ P$. Composing this with $\varphi$ we get an isomorphism $P \rightarrow \beta^{-1} \otimes i^* \beta \circ P$ which induces identity on $E$. We require that the involutive lift $\beta \approx i^* \beta$ is so chosen that the above composite is the Identity. We shall call this the normalised isomorphism.

Let then $\varphi : P \rightarrow \beta \circ N m P \circ i^* P$ be a normalised lift of the given involutive isomorphism $E \rightarrow i^* E$.

If we trivialise $\beta_\cdot$ at $w \in W$, we get a bijection $\varphi_\cdot : P_\cdot \rightarrow P_\cdot$ which satisfies $\varphi_\cdot (\xi) = \varphi_\cdot (\xi) (N m s) \cdot s$. Since $P_\cdot$ is a set on which $\Gamma (2n)$ acts simply transitively we see that $\varphi_\cdot$ gives rise to an element $g_\cdot$ of $\Gamma (2n)$ which is determined upto the equivalence relation

$g \sim g'$ if there exists $s \in \Gamma (2n)$ with $sgs^{-1} N m s = g'$.

Moreover $g_\cdot$ gives rise to a reflection in an $r_\cdot$-dimensional subspace in the vector representation, and satisfies in view of our normalisation $g_\cdot = e_\cdot N m g_\cdot$, where $e_\cdot = \pm 1$ according as the lift of $i$ to $\beta_\cdot$ is $\pm 1$ at $w$. Thus we have the following situation. $E_+^\cdot$ is an $r_\cdot$-dimensional subspace of $E_\cdot$ and $g_\cdot \in \Gamma (2n, g_\cdot)$ which represents the reflection in $E_+^\cdot$ satisfies $g_\cdot \cdot = e_\cdot N m g_\cdot$.

Lemma 3.2. If $g \in \Gamma (2n)$ represents the reflection in a nondegenerate subspace of dimension $r$, then $g^2 = N m g$ if $r \equiv 1, 2n$ and $g^2 = -N m g$ if $r \equiv 3, 2n - 2 \mod 4$.

Proof. Let $e_1, \ldots, e_\gamma$ be an orthonormal basis for $g_\cdot$ such that $e_1, \ldots, e_\gamma$ is a basis for $E_+^\cdot$. All the elements $e_\gamma$ belong to $\Gamma (2n)$ and in particular the element $e_1, \ldots, e_\gamma$ as well. The induced transformation on $E_\cdot$ is $x \mapsto (e_1, \ldots, e_\gamma) x (e_1, \ldots, e_\gamma)^{-1}$.
Orthogonal and spin bundles over hyperelliptic curves

\( e_1 \ldots e_r x e_{r+1} \ldots e_l \) since \( e_i^2 = g_\sigma(e_i) = 1 \) by choice. If \( x \) is \( e_i \), \( i \geq r \), then \( x \mapsto (-1)^{y} x \) and if \( x \) is \( e_i \), \( i \leq r \), then \( x \mapsto (-1)^{r-1} x \). Thus \( g = e_1 \ldots e_r \) or \( e_{r+1} \ldots e_l \) \( \in \Gamma(2n) \) represents reflection in \( E_\sigma^+ \) according as \( r \) is odd or even. Now \( (e_1 \ldots e_r)^2 = (-1)^{r(r-1)/2} \), while \( Nm(e_1 \ldots e_r) = 1 \). If \( r \) is odd, \( g^2 = (-1)^{(r-1)/2} Nm g \). This proves the assertion.

We have now proved

Proposition 3.3. Let \( E \) be an orthogonal vector bundle of dimension \( 2n \) to which the action of \( i \) lifts. If \( P \) is a Clifford bundle lifting \( E \), then \( P \approx \beta \circ (NmP) \circ i^*P \), where \( \beta \) is the unique line bundle of degree \(-\) degree \( NmP \) with \( i^* \beta \approx \beta \) associated to the partition of \( W \) given by the subset

\[
S = \{ w \in W : r_w \equiv 1, 2n \mod 4 \},
\]

\[
T = \{ w \in W : r_w \equiv 3, 2n - 2 \mod 4 \},
\]

where \( r_w \) is the dimension of the fixed subspace of \( E_w \) under the action of \( i \). In particular, if \( E \) is of even (resp. odd) type, then \( \# S \) is even (resp. odd).

Remark 3.4. We call attention to a subtlety regarding special orthogonal structures. Suppose \( \alpha \) is a line bundle with an action of \( i \) on it. Let \( E \) be an orthogonal bundle of odd type with \( i^*E \approx E \). Then \( i^*E \) also comes with a special orthogonal structure whenever \( E \) is provided with one. However, we claim that the isomorphism \( \phi : i^*E \to E \) cannot preserve this special orthogonal structure. For, if it does, we would have a commutative diagram

\[
det \phi \quad i^* \quad \det E \quad \rightarrow \quad \det E \quad i^* \eta \quad \downarrow \quad i \quad \eta \quad \downarrow i^* \alpha \quad \rightarrow \quad \alpha
\]

Restricting to a fibre at \( w \in W \), we get a commutative diagram

\[
det \varphi_w \quad \det E_w \quad \rightarrow \quad \det E_w \quad \eta_w \quad \downarrow \quad \epsilon_w \quad \downarrow \eta_w \quad \downarrow \quad \alpha_w \quad \rightarrow \quad \alpha_w
\]

Now, \( \det \varphi_w = \pm 1 \) according as \( \dim E_w^\pm \) is even or odd and since \( E \) is of odd type, \( \{ w : \det \varphi_w = 1 \} \) is of odd cardinality, while \( \{ w : \epsilon_w = 1 \} \) is of even cardinality (lemma 3.1).

4. Stability of orthogonal bundles

Definition 4.1. Let \( E \) be an orthogonal bundle. We say that \( E \) is stable (resp. semistable), if every proper isotropic sub-bundle \( F \) of \( E \) has degree \( < 0 \) (resp. \( \leq 0 \)).

Proposition 4.2. \( E \) is semistable as an orthogonal bundle if and only if it is semistable as a vector bundle.
Proof. It is obvious that semistability (resp. stability) as a vector bundle implies semistability (resp. stability) as an orthogonal bundle. As for the converse, let \( F \) be a sub-bundle of \( E \). Consider the sheaf \( N = F \cap F^\perp \) where \( F^\perp \) is obtained from \( F \) by orthocomplementation. Now \( N \) may not be a vector sub-bundle of \( F \), but let \( N' \) be the sub-bundle generated by it. Then we have the exact sequence

\[
0 \to N' \to F \oplus F^\perp \to M' \to 0,
\]

where \( M' \) is the sub-bundle of \( E \) generated by \( F + F^\perp \). Obviously, \( M' = (N')^\perp \). Thus \( \deg (F \oplus F^\perp) = \deg N' + \deg (N')^\perp = 2 \deg N' \), since we have the exact sequence \( 0 \to (N')^\perp \to E \to (N')^\ast \to 0 \). Hence \( \deg F = \deg N' \leq 0 \), since \( N' \) is obviously isotropic.

Remarks 4.3.

(i) It is easy to see that our definition of stability and semi-stability coincides with the general notion introduced by Ramanathan [5], for principal bundles with reductive structure group.

(ii) \( E \) could be stable as an orthogonal bundle and yet be nonstable as a vector bundle. The reason is, with the notation of proposition 4.2, that the bundle \( F + F^\perp \) could be the whole of \( E \), and \( N' = 0 \). When this happens, the bundle \( E \) is a direct sum of two orthogonal bundles. Actually, this phenomenon does happen as the following example shows.

Example 4.4. Let \( E \) be a vector bundle over \( X \) of rank 2. Then \( \text{ad} E \), the bundle of endomorphisms of trace 0 is a special orthogonal bundle, the Killing form giving the orthogonal structure. It is easy to see, using the one-one correspondence between isotropic sub-bundles of \( \text{ad} E \) and sub-bundles of \( E \), that \( E \) is stable if and only if \( \text{ad} E \) is stable as an orthogonal bundle. On the other hand, if \( E \approx \alpha \otimes E \) for some line bundle with \( \alpha^2 = 1 \), then \( \text{ad} E \) contains \( \alpha \) as a proper nondegenerate sub-bundle. Indeed, one can even construct bundles (see [4]) with \( E \approx \alpha \otimes E \) and \( E \approx \beta \otimes E \) for two distinct elements \( \alpha, \beta \) with \( \alpha^2 = \beta^2 = 1 \), so that \( \text{ad} E \) becomes a direct sum \( \alpha \oplus \beta \oplus a\beta \).

Regarding this, we have

Proposition 4.5. (i) \( E \) is stable as an orthogonal bundle if and only if \( E \) is an orthogonal direct sum of sub-bundles \( E_i \), which are mutually nonisomorphic with each \( E_i \) stable as a vector bundle.

(ii) If \( \text{Aut} E \) is the bundle of orthogonal automorphisms of a stable bundle \( E \), then \( E \) is stable as a vector bundle if and only if \( \Gamma (\text{Aut} E) = \pm 1 \).

Proof. (1) We have proved above that if \( E \) is not stable as a vector bundle, then \( E \) is the orthogonal direct sum of sub-bundles \( E_1, E_2 \) of \( E \). Now clearly \( E_1, E_2 \) are stable orthogonal so that by induction on rank \( E \), we see that \( E \approx \Sigma E_i \) where \( E_i \) are stable as vector bundles. If such a direct sum has repeated summands, say \( E_1 \cong E_2 \), then the imbedding \( X \to (X, iX) \) of \( E_i \) in \( E \) gives an isotropic sub-bundle of degree zero, contradicting the stability of \( E \). Thus all the \( E_i \)'s are mutually nonisomorphic. Conversely, if \( E = \Sigma E_i \), \( E_i \) stable as vector bundles and \( E_i \not\sim E_i \), then any sub-bundle \( \neq 0 \) of degree 0 must be a sum of some of the \( E_i \) and hence cannot be isotropic.
Orthogonal and spin bundles over hyperelliptic curves

(2) If $E$ is stable as a vector bundle, then $\Gamma(\text{Aut } E) \subset k^*$ and hence must be $\pm 1$. The converse is a consequence of (1).

**Proposition 4.6.** If $E$ is an orthogonal bundle over a curve with an involution $i$ which lifts to $E$, then $E$ is semistable as an orthogonal bundle if and only if for every $i$-invariant, isotropic sub-bundle $F$ of $E$, we have $\deg F \leq 0$.

**Proof.** If $F$ is any sub-bundle of $E$, then as in proposition 4.2, we have the sequence

$$0 \to N' \to F \oplus i^*F \to M' \to 0,$$

where $N'$, $M'$ are sub-bundles generated by $F \cap i^*F$ and $F + i^*F$ respectively. Now $\deg N' \leq 0$ by assumption, while the fact that $\deg M' \leq 0$, follows from proposition 4.2. Since $\deg F = \deg i^*F$, this proves our claim.

Let $X$ be a hyperelliptic curve with rational involution $i$. Let $E$ be a vector bundle on $X$ with a lift of $i$. For every $w \in W$, the set of Weierstrass points of $X$, denote by $r_w$ the dimension of the space of fixed points of $i$ on the fibre $E_w$. Consider the exact sequence

$$(4.7) \quad 0 \to E' \to E \to \sum_{w \in W} E_w^{-} \otimes \mathcal{O}_w \to 0,$$

where $E_w^-$ is the eigenspace for $i_w$ corresponding to the eigenvalue $-1$. The maps $E \to E_w^- \otimes \mathcal{O}_w$ are given by the natural projections $E_w \to E_w^-$. The kernel $E'$ is clearly locally free. Moreover, the involution $i$ acts on $E'$.

**Lemma 4.8.** For every $w \in W$, the involution $i$ acts as Identity on $E_w^-$.\n
**Proof.** We have only to show that $i$ acts as Identity on the kernel $E_w' \to E_w$ since it does so on the image which is $E_w^+$. Now this kernel is $\text{Tor}_1(E_w^- \otimes \mathcal{O}_w, \mathcal{O}_w) = E_w^- \otimes \text{Tor}_1 \mathcal{O}_w(\mathcal{O}_w, \mathcal{O}_w)$. From the resolution

$$0 \to L_w^{-1} \to \mathcal{O}_w \to \mathcal{O}_w \to 0,$$

we conclude that $\text{Tor}_1 \mathcal{O}_w(\mathcal{O}_w, \mathcal{O}_w)$ is canonically isomorphic to the fibre $(L_w^{-1})_w = K_w$, the space of differentials at $w$. Since $i$ acts as Identity on $K_w$, it follows that $i$ acts as Identity on $E_w^- = E_w^- \otimes K_w$.

From lemma 4.8, the bundle $E'$ descends to a vector bundle $\tilde{E}$ on $X/4 \cong P^1$. Let us now make the assumption on $E$:

(A) $E$ satisfies $H^0(X, E \otimes h^{-}) = 0$ and $H^1(X, E) = 0$. Then we have $H^0(X, E' \otimes h^{-}) \subset H^0(X, E \otimes h^{-}) = 0$ by (A) and hence in particular $H^0(P^1, \tilde{E} \otimes h^{-}) = 0$. Consequently, $\tilde{E} \otimes h^{-}$ is a direct sum of line bundles, each of which is of degree $< 0$. This implies that $\tilde{E} \otimes h^{-(s-1)}$ is a direct sum of line bundles of degree $\leq 0$ and hence contained in a trivial bundle. Conically speaking, we have $\tilde{E} \otimes h^{-(s-1)} \subset H^1(\tilde{E} \otimes h^{-(s-1)}) P^1$. On the other hand, we have the isomorphism $H^1(P^1, \tilde{E} \otimes h^{-(s-1)}) \cong H^1(X, E')$ in view of

**Lemma 4.9.** For any vector bundle $V$ on $P^1$, we have the isomorphism $H^1(P^1, V) \cong H^1(X, \pi^*V \otimes h^{-(s-1)})$.\n
Proof. The left side is dual to \( H^0(\mathbb{P}^1, V^* \otimes h^{-2}) \approx H^0(\mathbb{P}, \pi^* V^* \otimes h^{-2} \|) \) while the right side is dual to \( H^0(\mathbb{P}, \pi^* V^* \otimes h^{-\sigma(1)} \otimes K_X) \). But the isomorphism \( h^{\sigma(1)} \otimes K_X \approx h^{-2} \) commutes with \( i \) only up to sign. This proves the assertion.

Thus \( \tilde{E} \otimes h^{-\sigma-1} \subset H^1(\mathbb{P}, E')_\mathbb{P} \) canonically. From the exact sequence (4.7), we obtain the exact sequence

\[
0 \to H^0(\mathbb{P}, E)^\perp \to \Sigma E^\perp \to H^1(\mathbb{P}, E')^\perp \to 0.
\]

To check the injectivity \( H^0(\mathbb{P}, E)^\perp \to \Sigma E^\perp \), we note that its kernel is contained in the kernel \( H^0(\mathbb{P}, E) \to \Sigma E^\perp \) which is \( H^0(\mathbb{P}, E \otimes L^{-1}) = H^0(\mathbb{P}, E \otimes h^{-\sigma-1}) \subset H^0(E \otimes h^{-2}) = 0 \) by assumption (A). As for the surjectivity \( \Sigma E^\perp \to H^1(\mathbb{P}, E')^\perp \), this follows from the fact that \( H^1(\mathbb{P}, E') \subset H^1(\mathbb{P}, E) = 0 \) by (A).

Let us return to orthogonal bundles. If \( F \) is a semistable orthogonal bundle, then \( E = F \otimes \alpha \) for some \( i \)-invariant line bundle \( \alpha \) of degree \( j = 2g - 1 \) satisfies (A). Thus we can associate to such a bundle the subspace \( H^0(\mathbb{P}, E)^\perp \) of \( \Sigma E^\perp \). The orthogonal structure on \( F \) gives rise to nondegenerate quadratic forms on \( F^\perp \) and \( F^\perp \) (see §3), and hence an \( a^2 \)-values quadratic form on \( E^\perp \). This fits in the following diagram

\[
\begin{array}{ccc}
H^0(\mathbb{P}, E)^\perp & \xrightarrow{q} & \Sigma E^\perp \\
\downarrow q & & \downarrow q^\perp \\
H^0(\mathbb{P}, \alpha^2) \perp & \approx H^0(\mathbb{P}, h^i) & \to \Sigma a^2 = h^i
\end{array}
\]

(4.10)

Now consider the exact sequence (on \( \mathbb{P}^1 \))

\[
0 \to h^{-2\sigma+2} \to h^i \to h^i | \Omega_\mathbb{P} \to 0.
\]

Then we get a linear map \( \Sigma h^i_\mathbb{P} \to H^1(\mathbb{P}, h^{-2\sigma+2}) \). Clearly, \( \dim H^1(\mathbb{P}, h^{-2\sigma+2}) = 1 \) or 2 according as \( j = 2g \) or \( 2g - 1 \).

In either case, we have

**Proposition 4.11.** The quadratic form on \( E^\perp \) with values in \( h^i_\mathbb{P} \) gives rise to a form on \( \Sigma E^\perp \) or a quadratic map with values in a 2-dimensional space according as \( j = 2g \) or \( 2g - 1 \). In any case, the subspace \( H^0(\mathbb{P}, E) \subset \Sigma E^\perp \) is totally isotropic for the quadratic maps involved.

**Proof.** This is a simple consequence of the commutativity of 4.10, and the exactness of

\[
H^0(\mathbb{P}, h^i) \to \Sigma h^i_\mathbb{P} \to H^1(\mathbb{P}, h^{-2\sigma+2}).
\]

Let us further analyse the situation. If \( j = 2g \), a simple computation using ([1], proposition 2.2) shows that \( H^0(\mathbb{P}, E)^\perp \) has dimension \( \frac{1}{2} (\Sigma \dim E^\perp) \). In other words, it is a maximal isotropic space for the quadratic form. If \( j = 2g - 1 \), \( H^0(\mathbb{P}, E)^\perp \) has dimension \( \frac{1}{2} (\Sigma \dim E^\perp - rkE) \). For any \( t \in \mathbb{P}^1 \setminus W \), we get a linear map \( H^1(\mathbb{P}, h^{-2}) \to H^1(\mathbb{P}, h^{-2}) \) and hence a nondegenerate quadratic form on \( \Sigma E^\perp \). It is easy to determine the orthocomplement of \( H^0(\mathbb{P}, E)^\perp \) for this form. Indeed, it is \( H^0(\mathbb{P}, E \otimes h)^\perp \) where the latter space is imbedded by evaluation in \( \Sigma E^\perp \otimes h_{ij} \) and \( h_{ij} \) is trivialised by means of the section of \( h \) vanishing at the point of \( X \) lying over \( t \). To see this, we have only to show that the bilinear
product of an element in $H^0(X, E)$ and an element in $H^0(X, E \otimes h)^\sharp$ goes to zero in $H^1(P^1, h^{-2})$. From the diagram

$$
\begin{align*}
H^0(X, h^{2g-1}) &\to \Sigma h^{2g-1}_- \to H^1(X, h^{-3}) \\
\downarrow &\downarrow \\
H^0(X, h^{2g}) &\to \Sigma h^{2g}_+ \to H^1(X, h^{-5})
\end{align*}
$$

this is equivalent to showing that $\sigma_w$ maps it in the image of $H^0(X, h^{2g})$ in $\Sigma h^{2g}_+$. But this is obvious from the diagram

$$
\begin{align*}
H^0(X, E)^\sharp \otimes H^0(X, E \otimes \mathcal{h}) &\longrightarrow \Sigma E^\mathcal{h}_+ \\
\downarrow &\downarrow \\
H^0(X, \alpha^2 \otimes \mathcal{h})^\sharp = H^0(P^1, h^{2g}) &\longrightarrow \Sigma h^{2g-1}_-
\end{align*}
$$

In other words, we have shown

**Proposition 4.12.** The ortho complement of $V = H^0(X, E)^\sharp \subset \Sigma E^\mathcal{h}_+$ is $H^0(X, E \otimes \mathcal{h})^\sharp$ for the quadratic form at $t \in P^1$ and $H^0(X, E \otimes \mathcal{h})^\sharp / H^0(X, E)^\sharp$ is canonically the fibre of $\tilde{E} \otimes h^{-(g-1)}$ at $t$.

In other words, if the quadratic map $\Sigma E^\mathcal{h}_+ \to H^1(P^1, h^{-2})$ is regarded as a bundle with an $h$-valued quadratic form on the trivial bundle $\Sigma E^\mathcal{h}_+$ over $P^1$, then $V/V \cong \tilde{E} \otimes h^{-(g-1)}$.

**Proof.** We have only to trace the isomorphisms

$$
\begin{align*}
H^0(X, E \otimes \mathcal{h})^\sharp / H^0(X, E)^\sharp &\cong \ker \Sigma E^\mathcal{h}_+/H^0(X, E)^\sharp \to \Sigma E^\mathcal{h}_+/H^0(X, E \otimes \mathcal{h}) \\
&\cong H^1(X, E') \to H^1(X, E' \otimes \mathcal{h}) \\
&\cong \ker H^1(P^1, \tilde{E} \otimes h^{-(g-1)}) \to H^1(P^1, \tilde{E} \otimes h^{-g})
\end{align*}
$$

this map being given by the section of $h$ vanishing at $t$. From the exact sequence (on $P^1 \times P^1$)

$$
0 \to p^* \left( \tilde{E} \otimes h^{-(g+1)} \right) \to p^* \left( \tilde{E} \otimes h^{-(g+1)} \otimes L_{\triangle} \right) \to p^*_1 \left( \tilde{E} \otimes h^{-(g+1)} \right) \otimes \mathcal{O}_{\Delta} \to 0,
$$

where $\triangle$ is the diagonal divisor in $P^1 \times P^1$. On the other hand $L_{\triangle} \cong K_{\mathcal{E}}$, where $K_{\mathcal{E}}$ is the canonical line bundle. Taking direct images on $P^1$, we get

$$
E \otimes h^{-(g+1)} \otimes h^3 \cong \text{the kernel required}.
$$

5. Classification of $\iota$-invariant orthogonal bundles

**Theorem 1.** Let $X$ be a hyperelliptic curve of genus $\geq 2$. Then the space $U$ of semistable orthogonal bundles of rank $r$ with an $\iota$-action is isomorphic to the following space. Let $\Sigma \xi$ be an orthogonal direct sum of spaces $\xi$ with non-degenerate quadratic forms $Q_\xi$. Let $\Sigma Q_\xi$ and $\Sigma \lambda Q_\xi$ be two quadratic forms, where
\( \lambda \) are mutually distinct scalars determined by \( X \). Then the group \( \Pi O (Q_w) \) acts on the variety \( \mathcal{M} \) of subspaces of dimension \( \frac{1}{2} (\Sigma \dim \xi_w - r) \) isotropic for both these forms. The quotient (in the sense of geometric invariant theory) is isomorphic to \( U \).

**Proof:** Let us for convenience reinterpret the pencil of quadrics on \( \Sigma \xi_w \). First of all, we will assume the standard quadratic forms \( Q_w \) to have values, not in the field, but in the one-dimensional space \( h^{2g-1}_w \). Then the quadratic map \( \Sigma \xi_w \to \Sigma h^{2g-1}_w \) gives, on composition with the boundary homomorphism \( \Sigma h^{2g-1}_w \to H^0 (\mathbb{P}^1, \mathcal{O}(-r)) \) of the sequence
\[
0 \to h^{-3} \to h^{2g-1} \to h^{1-g-1} \otimes Q_w \to 0,
\]
a pencil of quadrics as required.

Fix a line bundle \( \alpha \) of degree \( 2g - 1 \) with an \( i \)-action on it, once for all. Then we have seen in § 4, that we have a natural inclusion \( H^0 (X, E) \subset \Sigma E_w \), where \( E = F \otimes \alpha, F \) an orthogonal bundle with an \( i \)-satisfying condition (A) in 4.8. Choosing quadratic isomorphisms \( E_w \to \xi_w \), we associate to \( E \), a subspace of \( \Sigma \xi_w \) of \( \dim = \dim H^0 (X, E)' = \frac{1}{2} (\dim \xi_w - n) \), where \( rkE = n \). By proposition 4.11, we see that this subspace is isotropic for the given pencil on \( \Sigma \xi_w \). Moreover, the intersection of this subspace with \( \xi_w \) is the kernel of
\[
H^0 (X, E)' \to \Sigma E_w',
\]
and is hence isomorphic to \( H^0 (X, E \otimes L^{-1} \otimes \xi_w) \subset H^0 (X, E \otimes L^{-1} \otimes h^{-n}) = 0 \) by (A). Thus the subspace \( V \) associated to \( F \) satisfies (A') \( V \) is a subspace of \( \Sigma \xi_w \), which is isotropic for the pencil of quadrics and \( V \cap \xi_w = 0 \) for every \( w \in W \).

Conversely, if \( V \) is a subspace of \( \Sigma \xi_w \) satisfying (A'), then the ortho complement of \( V \) with respect to any quadric of the pencil is of the same dimension. Indeed this needs to be checked only for quadrics corresponding to points \( w \in W \subset \mathbb{P}^1 \). The nullity of this quadric is \( \xi_w \) and \( V \cap \xi = 0 \) implies that \( V^\perp \) is of the complementary dimension containing \( \xi_w \). This proves that \( V^\perp / V \) is a vector bundle on \( \mathbb{P}^1 \) which comes with a subspace of the fibre \( (V^\perp / V)_w \), namely, the isomorphic image of \( \xi_w \). Using this, we can construct a bundle \( E \) (on \( X \))
\[
0 \to V^\perp / V \otimes h^{-1} \to E \to \mathcal{J} \to 0,
\]
in which \( \mathcal{J} \) is a torsion sheaf concentrated on \( W \) with length \( (\mathcal{J}) = \dim \xi_w \), and the kernel \( (V^\perp / V \otimes h^{-1})_w \to E_w \) is \( \xi_w \otimes h^{-1} \). Then \( E = F \otimes \alpha \) with \( F \) an \( i \)-invariant orthogonal bundle. It is then easy to see that \( H^0 (E \otimes h^{-n}) = 0 \), so that we have

**Proposition 5.1.** Let \( \alpha \) be a line bundle on \( X \) of degree \( 2g - 1 \), invariant under \( i \).

There is then a bijection between the set of orthogonal bundles \( F \) of rank \( n \) together with (a) an \( i \)-action and (b) orthogonal isomorphisms \( \eta_w (F_w \otimes \alpha_w) \to \xi_w \) satisfying \( H^1 (X, F \otimes \alpha) = 0 \), and the space of subspaces \( V \) of \( \Sigma \xi_w \) of dimension \( \frac{1}{2} (\Sigma \dim \xi_w - n) \) which are isotropic for the pencil of quadrics \( \Sigma Q_w = 0 \), \( \Sigma \lambda_w Q_w = 0 \), and satisfy \( V \cap \xi = 0 \) for all \( w \).

Clearly, this bijection is compatible with the action of \( \Pi O (Q_w) \) on both the sets. The action on \( (F, \eta_w) \) is simply \( (F, g_w \circ \eta_w) \) by \( g = (g_w) \). On the other hand, clearly,
the group \( \Pi O(Q_w) \) acts on \( \Sigma \xi_w \) and leaves the pencil invariant and hence also acts on the set of isotropic subspaces in question.

To complete the proof of theorem 1, we have to investigate the correspondence between stable (resp. semistable) bundles and stable (resp. semistable) points under the action of \( \Pi O(Q_w) \).

We recall

**Definition 5.2.** Let \( G \) be a reductive group acting linearly on a vector space and hence also on the projective space. If \( M \) is a projective variety invariant under \( G \), a point \( m \in M \) is said to be semistable (resp. properly stable) for the action of \( G \), if it satisfies the following equivalent conditions.

1. There exists a \( G \)-invariant hypersurface not passing through \( m \) (resp. and in addition, the isotropy group at \( m \) is finite and the orbit of \( m \) in the complement of this hypersurface is closed).

2. For any one parameter group \( t \mapsto \lambda(t) \) of \( G \), consider \( \lim_{t \to 0} \lambda(t) m \in M \). This point is represented by a line in \( V \) on which the one parameter group acts as a character \( t \mapsto t^{s_m} \), for some \( s_m \in \mathbb{Z} \). Then \( s_m > 0 \) (resp. \( s_m > 0 \)).

The equivalence of these two conditions is proved in ([3], Chapter 2).

For the action of \( \Pi O(Q_w) \) on the space of isotropic subspaces, we have

**Proposition 5.3.** A subspace \( V \) of \( \Sigma \xi_w \) is semistable (resp. properly stable) if and only if for every proper family \( (N_t) \) of isotropic subspaces of \( \xi_w \), we have

\[
\dim (\Sigma N_t) \cap V + \dim (\Sigma N_t^\perp) \cap V \leq (\text{resp.} <) \dim V.
\]

**Proof.** We will use the equivalent condition (2) in Definition 5.2 to prove this. Any one-parameter group \( t \mapsto \lambda_t \) of \( \Pi O(Q_w) \) can be described as follows. There exist isotropic subspaces \( N_t \) of \( \Sigma \xi_w \), compatible with this orthogonal decomposition with

\[
0 = N_0 \subset N_1 \subset \ldots \subset N_r \subset N_{r+1}^t \subset \ldots \subset N_r^t \subset N_r^{t+1} = \Sigma \xi_w = \tilde{w}.
\]

\( \lambda_t \) leaves all \( N_i \) invariant and induces the character \( t \mapsto t^{a_i} \) on \( N_i/N_{i-1} \), \( i = 1, \ldots, r \) with \( a_1 \leq a_2 \leq \ldots \leq a_r \leq 0 \). The action on \( N_r^t/N_r \) is trivial, while on \( N_r^{t+1}/N_r^t \), it acts as \( t \mapsto t^{a_i} \). Now if \( V \) is any isotropic subspace of \( \Sigma \xi_w \), then \( \lim_{t \to 0} \lambda_t(V) \) is a space on which \( \lambda \) acts as on the associated graded space for the above filtration \( (N_t) \). In other words, we have

\[
s_V = \sum_{i=1}^{r'} a_i \dim \frac{N_i \cap V}{N_{i-1} \cap V} + \sum_{i=1}^{r'} (-a_i) \dim \frac{N_i^{t+1} \cap V}{N_i^t \cap V}
\]

\[
= \sum_{i=1}^{r'} a_i (\dim N_i \cap V - \dim N_{i-1} \cap V)
\]

\[
+ \sum_{i=1}^{r'} (-a_i) \left( \dim \text{Im} \ V \text{ in } \frac{W}{N_r^t} - \dim \text{Im} \ V \text{ in } \frac{W}{N_r^{t+1}} \right)
\]

\[
= \sum_{i=1}^{r'} (a_i - a_{i+1}) \left( \dim N_i \cap V - \dim \text{Im} \ V \text{ in } \frac{W}{N_r^t} \right)
\]
where we make the convention that $a_{r+1} = 0$. Since $a_r - a_{r+1}$ are arbitrary negative integers, we must have

$$\dim \left( N_i \cap V - \dim \text{Im } V \in \frac{W}{N_i^2} \right)$$

$$= \dim (N_i \cap V) + \dim (N_i^2 \cap V) - \dim V > 0,$$

in order that we may have $s_V < 0$. This proves our assertion.

We will now investigate the equivalence among semistable points which gives
the geometric invariant theoretic quotient. According to [3], two semistable
points are equivalent if their orbits have a common limit point. We claim

**Proposition 5.4.** Two semistable subspaces $V_1, V_2 \subset \Sigma \xi_w$ represent the same point
in the quotient if and only if there exist isotropic flags $(N_i), (N_i) \subset \Sigma \xi_w$ compatible
with the decomposition such that the associated graded spaces $V_1, V_2$ are in the
same $G$-orbit. Moreover, any semistable subspace $V$ is equivalent to a space
$\bigoplus V_i$ where each $V_i$ is stable in $\Sigma V_i \cap \xi_w$.

**Proof.** If $V$ is not stable, then there exists an isotropic sub-bundle $N \subset \Sigma \xi_w$, com-
patible with the decomposition such that

$$\dim V \cap N + \dim V \cap N^\perp = \dim V.$$

Now using induction and replacing $V$ by $V \cap N^\perp \cap N \subset N^\perp/N$, we can prove
the last part of the statement. From the analysis of the stability condition, we
see that the closure of an orbit of $V$ under a suitable one-parameter group contains
the associated graded of $V$. Finally, it remains to prove that if $V_1, V_2$ have
different stable series, they map on distinct points in the quotient. This is clear,
if $V_1$ is stable since then, the orbit is closed. In general, we use induction on the
length of the stable series and semicontinuity of rank in a family of linear trans-
formations.

**Lemma 5.5.** If a point $V \subset \Sigma \xi_w$ is semistable for the action of $\Pi O (Q_w)$, then
$V \cap \xi_w = 0$ for all $w$.

**Proof.** Indeed, take $N_{w_0} = V \cap \xi_{w_0}$ and $N_w = 0$ for $w \neq w_0$ in proposition 5.3.
Then semistability of $V$ implies that

$$\dim (V \cap \xi_{w_0}) + \dim (V \cap (N_{w_0}^\perp + \sum_{w \neq w_0} \xi_w)) \leq \dim V.$$

Since $V$ is isotropic, any $v = \Sigma v_v \in V$ should satisfy $v_v \perp N_{w_0}$. Hence $V \subset N_{w_0}^\perp +
\sum_{w \neq w_0} \xi_w$. This leads to a contradiction if $N_{w_0} \neq 0$.

In view of lemma 5.5, we claim we have a natural morphism from the open
set of semistable points into $\tilde{M}$, where $\tilde{M}$ is the variety over the moduli space $M$ consisting of $i$-invariant orthogonal bundles $E$, together with isomorphisms $E_v^- \rightarrow \xi_v$ as in proposition 5.1. To see this, we need only check.

**Proposition 5.6.** A semistable (resp. stable) point $V$ corresponds to a semistable (resp. stable) bundle.
Proof. In fact, if \( F \) is such that \( H^1(X, F \otimes a) = 0 \), and if \( H^0(F \otimes a)^\perp \subset E_w \) is a semistable point, then we have to show that \( F \) is semistable. Let \( L \subset F \) be any \( i \)-invariant isotropic sub-bundle. Then \( H^0(X, L \otimes a)^\perp \subset H^0(X, F \otimes a)^\perp \) is a subspace contained in \( H^0(X, F \otimes a)^\perp \cap (\Sigma(L \otimes a)_w) \). Taking \( N_w = (L \otimes a)_w \) in the definition of stable points (proposition 5.3), we get

\[
\dim H^0(X, L \otimes a)^\perp + \dim H^0(X, L^\perp \otimes a)^\perp \leq \dim H^0(X, F \otimes a)^\perp.
\]

This implies that

\[
\dim H^0(X, L \otimes a)^\perp - \dim H^0(X, L \otimes a)^\perp - \dim H^0(X, L^\perp \otimes a)^\perp \leq \dim H^0(F \otimes a)^\perp.
\]

Using ([1], proposition 2.2), we get (setting \( (L \otimes a)_w = a_w \), \( rkL = r \))

\[
r + \frac{1}{2} \left( \deg L + (2g - 1) r - \Sigma (r - a_w) \right) + n - r + \frac{1}{2} \left( \deg L^\perp + (2g - 1)(n - r) - \Sigma (n - r - r_w + a_w) \right) \leq \frac{1}{2} \Sigma r_w - n.
\]

i.e.,

\[
n + \frac{1}{2} \left( \deg L + \deg L^\perp + (2g - 1) n - (2g + 2) n + \Sigma r_w \right) \leq \frac{1}{2} (\Sigma r_w a_w - n).
\]

This means that \( \deg L + \deg L^\perp \leq 0 \), proving our assertion.

We have thus obtained a morphism from the open subset of semistable points into the required moduli space. It is obvious that this morphism is \( \Pi O(Q_w) \)-invariant and hence goes down to a morphism of the quotient variety. Moreover, proposition 5.1 implies that this is an isomorphism of the open set in the quotient corresponding to points representing closed orbits in the set of semistable points (namely stable points), onto an open subset. Hence it follows that there is a surjective, birational morphism of this quotient into the moduli variety.

Now since the moduli variety in question is a component of the action of \( i \) on the moduli of orthogonal bundles and since the latter is normal [5], it follows that the former is also normal. Finally to prove the theorem we have only to check that the morphism is injective. Any point in the quotient can be represented by \( \Sigma V_i \) where \( V_i \) is stable in \( \Sigma V_i \cap \xi_w \). Thus \( \Sigma V_i \) is mapped on the bundle \( \Sigma \phi_i(V_i) \), where \( \phi_i \) are similar maps defined on \( \Sigma V_i \cap \xi_w \). If \( \phi(V) \) is \( S \)-equivalent to \( \phi(V) \) then it is necessary that after a permutation of the factors, we must have \( \phi_i(V_i) \sim \phi_i(V_i) \). But this implies that \( V_i \) and \( V_i \) are in the same orbit, under the group \( \Pi O(V_i \cap \xi_w) \). This proves that \( \phi \) is injective and hence that it is an isomorphism by Zariski's main theorem.

6. Spin structures and low dimensional cases

If \( E \) is an orthogonal bundle of odd type, then any lift \( P \) of \( E \) to a \( I \)-bundle has norm of odd degree. Now the lift of \( i \)-action to \( E \) gives rise to a line bundle \( \beta \) as in § 3, determined by the behaviour of \( i \) at \( w \). For each such lift \( P \) of \( E \) with fixed norm, we obtain a lift to \( \Gamma(q_w) \) of a reflection in the subspace \( E_w \) with respect to the quadratic form \( q_w \) induced from the quadratic structure on \( E \). As we have seen in proposition 3.3, this induces an orientation on each \( E_w \), and hence we have analogously the
Theorem 2. The moduli of $\Gamma^*$-bundles $P$ of fixed norm of odd degree with isomorphisms $i^* P \rightarrow a \circ P$ of $\Gamma^*$-bundles is isomorphic to the quotient of the variety of isotropic spaces for the pencil

$$\mathcal{Q}_w = 0, \Sigma \lambda_w \mathcal{Q}_w = 0,$$

where $\mathcal{Q}_w$ is a nondegenerate form on a space $\xi_w$, by the group $\Pi \text{SO} (\mathcal{Q}_w)$.

Proof. What we have to show is only that the space of $i$-invariant $\Gamma^*$-bundles are in (1, 1)-correspondence with $i$-invariant orthogonal bundles modulo the group $\Pi \text{SO} (E^w_w)$. Now the group $\Pi \text{SO} (E^w_w) / \Pi \text{O} (E^w_w)$ is isomorphic to $(\mathbb{Z}/2)^w$ and acts on the set of $\Gamma^*$-bundles of fixed norm with a fixed associated orthogonal bundle $E$ as follows. An element of $(\mathbb{Z}/2)^w$ can be identified (lemma 3.1) with line bundles whose square is 1 or $i^{\deg E}$. Now if $a^2 = 1$, it acts on a $\Gamma^*$-bundle $P$ by sending $P$ to $a \circ P$. If $a^2 \sim N$, we send $P$ to $a^{-1} N \circ i^* P$. It is easy to see that this action is simply transitive. Indeed, the subgroup $\{a : a^2 = 1\}$ clearly acts on the set of $\Gamma^*$-bundles with a fixed associated special orthogonal bundle $E$. Any element in the other coset twists $P$ into another $\Gamma^*$-bundle whose associated special orthogonal bundle is not isomorphic to $P$ (remark 3.4). This proves our assertion. Finally, to check that the morphism from the moduli of $\Gamma^*$-bundles to the variety of isotropic spaces is actually an isomorphism (instead of being just bijective), we have to show that the inverse map is an isomorphism. But we know that the composite of the inverse map with the morphism into the moduli of orthogonal bundles is a morphism. Now this follows from the fact that, given any orthogonal bundle in $T \times X$ and $T \in T$, there exists an etale neighbourhood $U \rightarrow T$ of $t$ such that the induced bundle on $U \times X$ can be lifted to a $\Gamma^*$-bundle.

By using lemma 2.1 and the Kunneth decomposition, we can pass to such a neighbourhood $U$ that the induced bundle on $U \times X$ has the obstruction to $\Gamma^*$-lifting in $H^1 (X, \mathbb{Z}/2)$. But its image in $H^2 (X, \Theta^*)$ is zero and hence the obstruction to $\Gamma^*$-liftable of the induced bundle is also zero, since it is in the image of $H^2 (X, \Theta^*) \rightarrow H^2 (U \times X, \Theta^*)$. This proves theorem 2.

We will now make a few remarks on the dimensions of $\xi_w$, although these are implicit.

Remark 6.1. When we deal with orthogonal bundles $F$ of even rank, then the corresponding integers (dim $F_w$) have the property: $\{w : \text{dim } F^w_w \text{ is odd}\}$ is of even cardinality. Thus we see that $F$ has even or odd rank according as $\Sigma \text{dim } \xi_w$ is even or odd. This also follows from ([1], proposition 2.2) since $\frac{1}{2} (2g - 1) n - \Sigma \text{dim } E^w_w$ is an integer, where $E = F \otimes a$, deg $a = 2g - 1$.

By construction, $\xi_w$ is isomorphic to $E^w_w$.

We shall now consider special cases. The simplest is when all the spaces $\xi_w$ are one-dimensional or 0. In that case, there are no proper isotropic subspaces contained in $\xi_w$. Moreover, the group $\Pi \text{O} (\mathcal{Q}_w)$ is a finite group. If we deal with $\Gamma(n)$-structures is such a case, then the moduli space is actually isomorphic to the variety of isotropic spaces. In other words, we have

Theorem 3. The moduli space of $i$-invariant orthogonal bundles $E$ of rank $2n$ with $\Gamma^*$-structures such that dim $(E \otimes a^w_w) = 1$ for all $w$, where $a$ is an $i$-invariant line bundle of degree $2g - 1$, is isomorphic to the variety of $(g + 1 - n)$-dimensional subspaces of $\mathbb{k}^{2g+1}$ which are isotropic to the quadrics.
Orthogonal and spin bundles over hyperelliptic curves

\[ \Sigma X_i^2 = 0, \Sigma \lambda_4 X_i^4 = 0. \]

In particular, the intersection of the above two quadrics is itself a moduli space of \( \Gamma^+ (2n) \)-bundles of the above type.

6.2. Case of rank 2. Let us further specialise to the case rank = 2. In this case, \( i \)-invariance of orthogonal bundles of determinant 1 is automatic. Thus we get: The Jacobian of \( X \) is isomorphic to the variety of \( g \)-dimensional subspaces of \( k^{2g+2} \), isotropic for \( \Sigma X_i^2 = 0 \) and \( \Sigma \lambda_4 X_i^4 = 0 \). See [1, 2 and 6].

6.3. Case of rank 4. In this case, \( \Gamma^+ (4) \) is isomorphic to the subgroup of \( GL(2) \times GL(2) \) consisting of elements \((A, B)\) with \( \det A \cdot \det B = 1 \). The homomorphism into \( SO(4) \) is given by \((A, B) \mapsto A \otimes B \). On the other hand, \( \Gamma(4) \) is the split extension of \( Z(2) \) by \( \Gamma^+ (4) \) given by the action \((A, B) \mapsto (B, A)\), by the non-trivial element of \( Z(2) \). Thus a \( \Gamma^+(4) \)-bundle is essentially a pair of bundles \( M, N \) with \( \det M \otimes \det N = \) trivial bundle. But a \( \Gamma(4) \)-bundle does not distinguish between \( M \) and \( N \). Firstly, we have: The moduli space of vector bundles of rank 2 and fixed determinant of odd degree is isomorphic to the variety of \((g - 1)\)-dimensional subspaces of \( k^{2g+2} \), isotropic for \( \Sigma X_i^2 = 0 \) and \( \Sigma \lambda_4 X_i^4 = 0 \). The group of even changes of sign in the coordinates correspond to tensorisation by line bundles of order 2.

6.4. Case of rank 6. If \( R \) is a four-dimensional vector space, \( \Lambda^2 R \) is a six-dimensional space equipped with a \( \Lambda^4 R \)-valued quadratic form. Thus we get a homomorphism of the subgroup \( G = \{ (\lambda, g) : \lambda^2 = \det g \} \) of \( k^2 \times GL(R) \) into \( SO(6) \) by sending \((\lambda, g) \) to \( \lambda^{-1} \circ \Lambda^2 g \). It is easy to see that \( G = \Gamma^+ (6) \). Thus a \( \Gamma^+(6) \)-bundle is the same as the data consisting of a line bundle \( \eta \) and a vector bundle \( V \) of rank 4 with \( \eta^2 \approx \det V \). Also \( Nm(\eta, V) = \eta \) so that \( (\eta, V) \) is of odd type if and only if \( V \) has degree \( \equiv 2 \pmod{4} \). If \( \alpha \) is any line bundle, \( \alpha \circ (\eta, V) = (\alpha^2 \otimes \eta, \alpha \otimes V) \). Two \( \Gamma^+(6) \)-bundles \( (\eta, V), (\eta', V') \) are isomorphic as \( \Gamma(6) \)-bundles if and only if \( \eta = \eta' \) and \( V = V' \) or \( \eta' = \eta^{-1} \) and \( V' = \eta^{-1} \otimes V^* \). Hence an isomorphism \((\eta, V) \approx i^* (\eta, V) \circ Nm(\eta, V) \circ \beta^{-1} \) for some \( \beta \) with \( \beta^2 = h^{-deg} V \) yields \( V \approx i^* V \otimes \eta \otimes \beta^{-1} \), or \( V \approx i^* V^* \otimes \beta^{-1} \). Thus we deduce: Consider stable vector bundles of rank 4 and fixed determinant of degree 2. Assume given an isomorphism \( i^* V \approx V^* \otimes \beta^{-1} \) with \( \beta^2 = h \). The moduli of such objects can thus be described as the variety of subspaces isotropic for a pencil of quadrics.

Remark 6.5. The constructions involved in the proof of the main theorems seem to be of a very general nature. For one thing, one can obtain results similar to [1, Theorem 3] for spin bundles with an \( i \)-action. Furthermore, it is also possible to get results for symplectic bundles, etc. However one ought to be able to understand these results in a group scheme theoretic set-up. In other words, given a bundle with an \( i \)-action, we modify the bundle at the Weierstrass points so that it goes down to \( P^1 \). By keeping track of the additional structures on the quotient bundle, one would expect to classify bundles on \( X \) with \( i \)-action.
References


Institute for Advanced Study, Princeton, New Jersey, USA and Tata Institute of Fundamental Research, Bombay 400 005