Use of synchronization and adaptive control in parameter estimation from a time series

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Abstract

A technique is introduced for estimating unknown parameters when time series of only one variable from a multivariate nonlinear dynamical system is given. The technique employs a combination of two different control methods, a linear feedback for synchronizing system variables and an adaptive control, and is based on dynamic minimization of synchronization error. The technique is shown to work even when the unknown parameters appear in the evolution equations of the variables other than the one for which the time series is given. The technique not only establishes that explicit detailed information about all system variables and parameters is contained in a scalar time series, but also gives a way to extract it out under suitable conditions. Illustrations are presented for Lorenz and Rössler systems and a nonlinear dynamical system in plasma physics. Also it is found that the technique is reasonably stable against noise in the given time series and the estimated value of a parameter fluctuates around the correct value, with the error of estimation growing linearly with the noise strength, for small noise.

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I. INTRODUCTION

One of the objectives of time series analysis is to study the detailed structure of the equations of the underlying dynamical system which govern its temporal evolution. This includes the number of independent variables, the form of the flow functions, the nonlinearities in them and parameters of the system [1]. This paper concentrates on the last aspect, i.e., estimating the parameters of a nonlinear system from a single time series when partial information about the system dynamics is available [2–4].

Assuming that the number of independent variables and the structure of underlying dynamical evolution equations for a nonlinear system is known, we address the problem of determining the values of the parameters. In particular, given a time series for a single variable (a scalar time series), we suggest a simple method which enables us to determine values of the unknown parameters dynamically. The unknown parameters may or may not appear in the evolution equation of the variable for which the time series is given. For this purpose, we employ a combination of two techniques, namely synchronization and adaptive control.

Owing mainly to the extreme sensitivity to initial conditions, engineering and controlling a nonlinear chaotic system requires a careful analysis. Feedback based synchronization techniques are investigated in this context to force a chaotic system, to go to a desired periodic or chaotic orbit. Such control mechanisms were suggested by Pecora and Carroll [5,6] and many others [7–11,13,14] with an aim to synchronize two chaotic orbits and to stabilize unstable periodic orbits or fixed points. In such mechanisms, some of the independent variables are used as drive variables and the remaining variables are found to synchronize with the desired trajectory under suitable conditions. There have been many other important attempts in controlling chaotic systems using synchronization [15–21].

The other method that we use is that of adaptive control which is used to bring back a system, deviated from a stable fixed point due to changes in parameters and variables, to its original state. This mechanism was suggested by Huberman and Lumer [12]. It was generalized for an unstable periodic orbit and a chaotic orbit by John and Amritkar [13] where it was shown that it is possible to synchronize with an unstable periodic orbit or a chaotic orbit starting from a random initial condition and different value of the parameter.

In this paper, we show that a simple combination of synchronization and adaptive control methods similar to that described by John and Amritkar [13,14] can be used for extracting information contained in a scalar time series.

We approach the problem by considering a dynamical system, in which the number of independent variables and the structure of evolution equations are assumed to be known. A linear feedback function is added to the variable corresponding to that for which the time series is given. This acts as a *drive* variable. The feedback serves the purpose of synchronization of all the system variables. The feedback function in our case is proportional to the difference between the new and the old values of the drive variable.

The system variables respond to this feedback by synchronizing with the corresponding values in the original system. In the context of application of synchronization techniques to telecommunications, the new system to be reconstructed is often referred to as the *receiver* whereas the old system, from which the time series is made available is termed the *transmitter*. We will borrow the terminology, although the meanings of terms in the two cases are not exactly identical.

The synchronization as described above becomes exact when the receiver parameters are set equal to those of the transmitter and takes place whenever the *Conditional Lyapunov Exponents* (CLE's) as defined in the next section are all negative. Now assume that precise values of only a few of the transmitter parameters are known to the receiver system. We show that, in such a case it is possible to write simple evolution equations for the unknown parameters (initially set to arbitrary values), which when coupled with the system equations, yield precise values of all the state variables and the unknown parameters asymptotically to any desired accuracy.

Our method comprises of raising the unknown parameters to the status of variables of a higher dimensional dynamical system which evolve according to a simple set of evolution equations. The receiver forms a subsystem of this higher dimensional system which in addition contains the evolution equation for the unknown parameters. The input to this higher dimensional system is a scalar time series obtained from the transmitter system. Thus our method uses a dynamical algorithm to estimate the parameters which are obtained asymptotically. We note that the method of estimating parameters using synchronization and minimization as proposed in Ref. [2], is essentially a static method. The problem of estimating model parameters was also handled in Ref. [3], in which starting with an ansatz, the optimal equations for parameter evolution are obtained. Our method gives a simpler and a systematic derivation of the parameter control loop and in many cases, a better convergence rate.

It is well known that a great deal of information about a chaotic system is contained in the time series of its variables. Techniques like embedding the time series in a space with chosen dimensionality are available for studying the universality class and other *global* features of the system. Our results suggest that a scalar time series, in addition to the information about the universality class also contains information about the exact values of the parameters of the underlying dynamical system, including the ones which appear in the evolution of other variables.

The method and the required notation is developed in section II. Section III consists of illustrations for Lorenz and Rössler systems and a set of equations in plasma physics. The effect of noise in the transmitter system is studied in Section IV. Finally we conclude in section V with a brief summary of results along with a few remarks.

II. FORMALISM

A. Description of the method

In this section, we will describe our method of parameter estimation, for a general system with n variables and m parameters. We will first consider the case when only one parameter is unknown to the receiver.

Consider an autonomous, nonlinear dynamical system with evolution equations,

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \{\mu_j\}\right), \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an n-dimensional state vector whose evolution is described by the function $\mathbf{f} = (f_1, \ldots, f_n)$. The dot represents time differentiation and $\{\mu_j\}, j = 1, 2, \ldots, m$, are the parameters of the system.

Now suppose a time series for one variable, which without loss of generality can be taken as x_1, is given as an output of the above system and in addition suppose the functional form of $\mathbf{f}$, and the values of all the parameters $\mu_j, j = 1, \ldots, l-1, l+1, \ldots, m$, are known while the time evolution of the remaining variables and value of μ_l, the l^{th} parameter are not known, then formally the problem at hand consists of writing a set of evolution equations which will yield the information about the unknown parameter and also other variables. With the unknown parameter μ_l written explicitly for convenience we rewrite Eq. (1) as,

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \{\mu_j | j \neq l\}, \mu_l\right). \tag{2}$$

Now we introduce a new system of variables $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_n)$ whose evolution equations have identical form to that of $\mathbf{x}$. We fix x'_1 as the drive variable and a feedback is introduced in the evolution of x'_1. The parameters are also the same except the one corresponding to the unknown parameter which will be set to an arbitrary initial value denoted by μ'_l. Thus the receiver system will have the structure,

$$\begin{aligned} \dot{x}'_1 &= g\left(\mathbf{x}', \{\mu_j | j \neq l\}, \mu'_l\right) \\ &= f_1\left(\mathbf{x}', \{\mu_j | j \neq l\}, \mu'_l\right) - w_f\left(x'_1, x_1(t)\right), \end{aligned} \tag{3}$$

$$\dot{x}'_i = f_i\left(\mathbf{x}', \{\mu_j | j \neq l\}, \mu'_l\right), \quad i = 2, \ldots, n, \tag{4}$$

where $w_f(x'_1, x_1(t))$ is a feedback function which depends upon the drive variable x'_1 and the variable x_1. The feedback function can be most simply chosen to be proportional to the difference $(x'_1 - x_1)$ and the evolution for the drive variable x'_1 can be written as,

$$\dot{x}_1' = f_1\Big(\mathbf{x}', \{\mu_j | j \neq l\}, \mu_l'\Big) - \epsilon\left(x_1' - x_1(t)\right), \tag{5}$$

where ϵ is called the feedback constant. More general forms of the feedback function are also possible and give similar results.

The receiver system is formed by Eqs. (4) and (5). If the parameter μ_l' in these equations is set precisely equal to μ_l then the two sets of variables, $\mathbf{x}$ and $\mathbf{x}'$, after a transient time, evolve in tandem and show exact synchronization under suitable conditions, but because the value of μ_l is *unknown* to the receiver system, this does not happen.

The solution is to set the parameter μ_l' to an arbitrary initial value, while all others are set to the known values μ_j, and *adapt* it through a suitable evolution equation. The resulting $(n+1)$-dimensional system then evolves all the receiver variables to correct values of the corresponding transmitter variables and simultaneously settles the value of μ_l' to that of μ_l provided all the CLE's as defined in the next subsection, are negative.

The equation for evolution of the μ_l' is chosen similar to those used in adaptive control mechanisms [13,14], and quite generally can have the form,

$$\dot{\mu}_l' = h\left(\left(x_1' - x_1(t)\right), \frac{\partial g}{\partial \mu_l'}\right). \tag{6}$$

The form of the function h that we have chosen is

$$\dot{\mu}_l' = -\delta(x_1' - x_1(t))\frac{\partial g}{\partial \mu_l'}, \tag{7}$$

where δ is another parameter in the combined $(n+1)$ -dimensional system formed by Eqs. (4), (5) and (7). We call it the *stiffness constant*. The values of ϵ and δ together control the convergence rates involved in synchronization and adaptive evolution. Towards the end of this subsection, we will show that the above form of function h (Eq. (7)) is obtained as a result of dynamic minimization of the synchronization error.

The last factor in the Eq. (7), $(\partial g/\partial \mu_l')$, needs some elaboration. In general the parameter μ_l' may or may not explicitly appear in the evolution function $g(\mathbf{x}', \{\mu_j | j \neq l\}, \mu_l')$ in Eq. (3). This stresses a need for identification of two separate cases.

If the function g explicitly depends on μ_l', then the calculation of $(\partial g/\partial \mu_l')$ is straightforward.

In case μ_l' does not appear in the function g explicitly, it still indirectly affects the evolution of x_1'. The information about the value of μ_l is contained in the given time series $x_1(t)$. Function $(\partial g/\partial \mu_l')$ "taps" this dependence. The calculation of $(\partial g/\partial \mu_l')$ in this case, when function g does not explicitly depend on μ_l' needs to be done carefully. This is done as follows :

Consider the system formed by (4) and (5) in which, a change in the variable x_1' in one time step due to a change in the parameter μ_l' can be estimated as follows.

$$\begin{aligned}\Delta x_1' &\approx \Delta g \; dt \\ &\approx \frac{\partial g}{\partial x_s'} \Delta x_s' \; dt \\ &\approx \frac{\partial g}{\partial x_s'} \Delta f_s \; (dt)^2 \\ &\approx \frac{\partial g}{\partial x_s'} \frac{\partial f_s}{\partial \mu_l'} \Delta \mu_l' \; (dt)^2,\end{aligned}$$

where x_s' is the s^{th} variable of the receiver, such that its evolution contains the parameter μ_l' explicitly. Thus the last of the above equations gives us,

$$\frac{\partial g}{\partial \mu_l'} \approx \frac{\partial g}{\partial x_s'} \frac{\partial f_s}{\partial \mu_l'}. \tag{8}$$

A further complication arises if the variable x_s' itself does not appear in the function g explicitly. In such a case further dependences appearing in more time steps may be considered. Note that here, x_s' may appear in more than one flow functions and a summation over all such functions becomes necessary. In this case we can write,

$$\begin{aligned}\Delta x_1' &\approx \Delta g \; dt \\ &\approx \frac{\partial g}{\partial x_s'} \Delta x_s' \; dt \\ &\approx \frac{\partial g}{\partial x_s'} \Delta f_s \; (dt)^2 \\ &\approx \left\{ \sum_k \frac{\partial g}{\partial x_k'} \frac{\partial x_k'}{\partial x_s'} \right\} \frac{\partial f_s}{\partial \mu_l'} \Delta \mu_l' \; (dt)^2.\end{aligned}$$

$$\approx \left\{ \sum_k \frac{\partial g}{\partial x'_k} \frac{\partial f_k}{\partial x'_s} \right\} \frac{\partial f_s}{\partial \mu'_l} \Delta\mu'_l \, (dt)^3.$$

Thus the last factor in Eq. (7) takes the form,

$$\frac{\partial g}{\partial \mu'_l} \approx \left\{ \sum_k \frac{\partial g}{\partial x'_k} \frac{\partial f_k}{\partial x'_s} \right\} \frac{\partial f_s}{\partial \mu'_l}. \tag{9}$$

One such case appears in the example of Lorenz system, which will be discussed in the next section.

Now, when more than one parameters of the transmitter are to be estimated, one may use a set of equations similar in form to that of Eq. (7). We will use such a set when we discuss Lorenz system where it will be assumed that two or three parameters of the Lorenz system are unknown to the receiver system. We note that a parameter estimation algorithm as described in Ref. [3] can also be used in the estimation of more than one unknown parameters. It uses autosynchronization method based on an Active Passive Decomposition (APD) of a dynamical system [4] and starts from an ansatz for the parameter control. In contrast, our method is a dynamical minimization for the synchronization error. This can be seen as follows :

Let us define the dynamical synchronization error $e(\mu'_l, t)$ as,

$$e(\mu'_l, t) = (x'_1 - x_1)^2, \tag{10}$$

where μ'_l is the receiver parameter corresponding to the unknown parameter and x'_1 is the drive variable.

We note that if μ'_l takes precisely the value of μ_l, then the transmitter and receiver synchronize, which makes the error as defined by Eq. (10) minimum, i.e. zero. To go to this minimum, we want to evolve μ'_l such that it will go to a value making $e(\mu'_l, t)$ minimum. With an analogy to an equation in mechanics, where an overdamped particle goes to a mimimum of a potential, we write the following,

$$\dot{\mu}'_l \propto -\frac{\partial e(\mu'_l, t)}{\partial \mu'_l}, \tag{11}$$

which leads to,

$$\dot{\mu}_l' \propto -(x_1' - x_1)\frac{\partial x_1'}{\partial \mu_l'}, \tag{12}$$

Further, to the lowest order in dt, $\Delta x_1' = \frac{\partial \dot{x}_1'}{\partial \mu_l'}\Delta\mu_l'\, dt$. Hence Eq. (12) can be written as,

$$\dot{\mu}_l' = -\delta(x_1' - x_1)\frac{\partial \dot{x}_1'}{\partial \mu_l'}\ , \tag{13}$$

where δ is a proportionality constant. This equation is same as Eq. (7).

In the next subsection we will define the *conditional Lyapunov exponents* (CLE's) for the newly reconstructed receiver system and state the condition for the combination of synchronization and adaptive control to work convergently such that parameter estimation is possible.

B. Condition for convergence

Consider the transmitter equations (Eq. (2)) and the receiver equations (Eqs. (4), (5) and (7)). Convergence between two trajectories of these systems means that the receiver variables evolve such that the differences $(x_k' - x_k), (k = 1, \ldots, n)$ and $(\mu_l' - \mu_l)$ all evolve to zero. In the $(n+1)$-dimensional space formed by these differences, origin acts as a fixed point and the condition for the algorithm to work is the same as the stability condition for this fixed point.

If the above differences are considered to form an $(n+1)$-dimensional vector $\mathbf{z} = (z_1, \ldots, z_{n+1}) = (x_1' - x_1, \ldots, x_n' - x_n, \mu_l' - \mu_l)$ then the differential $d\mathbf{z}$ evolves as,

$$d\dot{\mathbf{z}} = J d\mathbf{z}, \tag{14}$$

where the Jacobian matrix J is given by,

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial z_1} - \epsilon & \frac{\partial f_1}{\partial z_2} & \cdots & \frac{\partial f_1}{\partial z_n} & \frac{\partial f_1}{\partial \mu_l'} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \cdots & \frac{\partial f_2}{\partial z_n} & \frac{\partial f_2}{\partial \mu_l'} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial f_n}{\partial z_1} & \frac{\partial f_n}{\partial z_2} & \cdots & \frac{\partial f_n}{\partial z_n} & \frac{\partial f_n}{\partial \mu_l'} \\ \frac{\partial h}{\partial z_1} & \frac{\partial h}{\partial z_2} & \cdots & \frac{\partial h}{\partial z_n} & \frac{\partial h}{\partial \mu_l'} \end{pmatrix}, \tag{15}$$

where the function h describes the evolution of the parameter μ_l' as in Eq. (6) and the derivatives in the matrix J are evaluated at $\mathbf{z} = 0$ which is a fixed point. The condition for the convergence of our procedure is that the real part of the eigenvalues of the matrix J or *conditional Lyapunov exponents* (CLE's) are all less than zero.

It can be seen from the above matrix equation that choices of the feedback constant and the stiffness constant affect the values of conditional Lyapunov exponents. Thus the method will work convergently only for suitably chosen ϵ and δ. When these are chosen such that the largest of the CLE's become positive, the algorithm does not work due to diverging trajectories.

In the next section we will illustrate the method using the examples of Lorenz and Rössler flows and a set of equations in plasma physics.

III. ILLUSTRATIVE EXAMPLES

A. Lorenz system

As a first example, we study the Lorenz system. We divide the discussion in two parts. In the first, we present the results when only a single parameter is estimated in a Lorenz system. Three different cases are discussed in detail. In the later part, we extend our method for the case when more parameters are to be estimated.

1. Single parameter estimation

The Lorenz system is given by,

$$\dot{x} = f_1(x, y, z) = \sigma(y - x),$$
$$\dot{y} = f_2(x, y, z) = rx - y - xz,$$
$$\dot{z} = f_3(x, y, z) = xy - bz, \tag{16}$$

where (x, y, z) form the state space and (σ, r, b) form the three dimensional parameter space. Now assume that the time series for x is given, and two of the three parameters are also known. We consider the following cases.

Case 1: When the unknown parameter appears in the evolution of x:

Here assuming σ to be the unknown parameter, we create a receiver system as described in the Section I, given by,

$$\dot{x}' = g(x', y', z') = \sigma'(y' - x') - \epsilon(x' - x(t)),$$
$$\dot{y}' = f_2(x', y', z') = rx' - y' - x'z',$$
$$\dot{z}' = f_3(x', y', z') = x'y' - bz', \tag{17}$$

where (x', y', z') are the new state variables and (σ', r, b) are the parameters, r and b being the same as those in the transmitter while σ' is initially set to an arbitrary value. ϵ is the the feedback constant. These constitute the receiver system. x' is the *drive* variable.

The parameter σ', which is initially set to an arbitrary value, is made to evolve through an equation similar the equation (Eq. (7)). Here we can use only the sign of the last factor in Eq. (7) since there is a single equation involving parameter evolution.

$$\dot{\sigma}' = -\delta\,(x' - x(t))\,\mathrm{sign}(y' - x'). \tag{18}$$

This equation along with the receiver system (Eq. (17)), can achieve required synchronization as well as parameter estimation since, a randomly chosen initial vector (x', y', z') evolves to (x, y, z) and $\sigma' \to \sigma$ as time $t \to \infty$.

Figure 1 displays the manner in which the synchronization takes place and how the parameter σ', initially set to an arbitrary value finally evolves towards the precise "unknown" value σ. In Fig. 1(a), (b) and (c) we show the differences $x' - x, y' - y, z' - z$ as functions of time and we observe that they eventually settle down to zero after an initial transient. In Fig. 1(d) we plot $\sigma' - \sigma$ as a function of time which also goes to zero simultaneously.

The synchronization as shown in Fig. 1 occurs when conditional Lyapunov exponents for the receiver system coupled to the parameter evolution are all negative or at most zero. This restricts the suitable choices for ϵ and δ. The Jacobian matrix J, for the evolution of the vector $(x' - x, y' - y, z' - z, \sigma' - \sigma)$ is given by (Eq. (14)),

$$J = \begin{pmatrix} -\sigma - \epsilon & \sigma & 0 & y' - x' \\ r - z' & -1 & -x' & 0 \\ y' & x' & -b & 0 \\ -\delta \mathrm{sign}(y' - x') & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

Figure 2 shows the curve along which the largest CLE becomes zero, in the (ϵ, δ) plane. In region I, all nontrivial CLE's are negative and the method works convergently, while in region II, the largest CLE becomes positive and no convergence takes place. Nevertheless note that for any positive value of δ, there can always be a suitably chosen ϵ such that the convergence occurs. On the other hand, there is a critical value of ϵ below which the method does not work.

Case 2 a : When the unknown parameter appears in the evolution of y variable :

Here, we consider the case of r as the unknown parameter (16) and reconstruct the receiver as,

$$\begin{aligned} \dot{x}' &= g(x', y', z') = \sigma(y' - x') - \epsilon(x' - x(t)), \\ \dot{y}' &= f_2(x', y', z') = r'x' - y' - x'z', \\ \dot{z}' &= f_3(x', y', z') = x'y' - bz', \end{aligned} \tag{20}$$

while the evolution of r' takes the form, (Eqs. (7) and (8)). Similar to Eq. (18) we use only the sign of the derivative involved.

$$\dot{r'} = -\delta\,(x' - x(t))\,\mathrm{sign}(\sigma x'). \tag{21}$$

When a time series for x from (16) is fed into the these equations, setting x', y', z' and r' to arbitrary initial condition, they finally evolve to the corresponding values of x, y, z and r. The associated Jacobian matrix (Eq. (14) is given by,

$$J = \begin{pmatrix} -\sigma - \epsilon & \sigma & 0 & 0 \\ r - z' & -1 & -x' & x' \\ y' & x' & -b & 0 \\ -\delta \mathrm{sign}(\sigma x') & 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

Figure 3 shows the curve along which the largest CLE becomes zero, in the (ϵ, δ) plane. In region I, all nontrivial CLE's are negative and the method works convergently, while in region II, the largest CLE is positive.

Let τ denote the time required for the convergence to the correct value of the parameter within a given accuracy, defined as $A = (r' - r)/r$. In Fig. 4 we plot (τ) as a function of the feedback constant ϵ, when the stiffness constant δ is held fixed. On the other hand, τ may be plotted as a function of δ for a fixed value of ϵ. This is plotted in Fig. 5. In both Fig. 4 and Fig. 5, r is assumed to be unknown and a time series for x is assumed to be given. The chosen accuracy for convergence was 10^{-7}.

In Fig. 6 we plot the time required for convergence of r' to r to within a given accuracy as a function of logarithm of the accuracy, which is 0 with respect to the initial value. The straight line shows that the time required to achieve better accuracy grows exponentially. The slope of the line in Fig. 6 corresponds to the Lyapunov exponent. It was compared with the Lyapunov exponent computed using a numerical algorithm and a fair agreement was observed.

Case 2 b : When the unknown parameter appears in the evolution of z variable :

The case where the parameter b appearing in the evolution of z, (Eq. (16)) is unknown, while the given time series is for x is a particularly interesting case. Since the variable z does not appear explicitly in the evolution equation for x, the calculation of sign in Eq. (7) has to be done using Eq. (9). Thus with the evolution for b', the complete receiver system becomes,

$$\dot{x}' = g(x', y', z') = \sigma(y' - x') - \epsilon(x' - x(t)),$$
$$\dot{y}' = f_2(x', y', z') = rx' - y' - x'z',$$
$$\dot{z}' = f_3(x', y', z') = x'y' - b'z', \tag{23}$$
$$\dot{b}' = -\delta\,(x' - x(t))\,\mathrm{sign}(\sigma x'z'). \tag{24}$$

An initial vector (x', y', z', b') in the above system goes to (x, y, z, b) and thus makes the estimation of the value of b possible. Here the matrix J takes the form, (Eq. (14))

$$J = \begin{pmatrix} -\sigma - \epsilon & \sigma & 0 & 0 \\ r - z' & -1 & -x' & 0 \\ y' & x' & -b & -z' \\ -\delta \mathrm{sign}(\sigma x'z') & 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

Figure 7 shows the curve along which the largest CLE becomes zero in the $\epsilon - \delta$ plane. In region(I), all CLE's are negative and the condition of convergence is satisfied.

Finally we note that in all the three cases discussed above, since the time series for x in Eq. (16) is assumed to be known, x' acts as a drive variable. A similar procedure is possible when a time series for y in Eq. (16) is given as an input. Here y' can be chosen as a drive variable which drives the evolution of the remaining variables as well as the unknown parameter. Thus it is possible to know an unknown value of any of the parameters of the Lorenz system from a single time series for x or y.

2. Extension to many parameters' estimation

Here we will consider the estimation of two or three parameters for the Lorenz system (16).

We have applied our method for estimation of two parameters of the Lorenz system (16), taking x or y as drive variables. A typical receiver system, taking x as the drive and (σ, r) as the unknown parameters, is constructed as,

$$\dot{x}' = \sigma'(y' - x') - \epsilon(x' - x(t)),$$

$$\dot{y}' = rx' - y' - x'z',$$

$$\dot{z}' = x'y' - bz',$$

$$\dot{\sigma}' = -\delta\,(x' - x(t))\,(y' - x'),$$

$$\dot{r}' = -\delta\,(x' - x(t))\,(\sigma x'). \tag{26}$$

Note that the same stiffness constant is used in controlling both the unknown parameters.

We have found that with similar receiver structure to that in Eq. (26), it is possible to estimate any two of the three parameters σ, r and b, when a time series for either of x or y is given. In Fig. 8 (a) we plot the difference $(\sigma' - \sigma)$ while Fig. 8 (b) shows $(r' - r)$ as functions of time, when the drive is x and two parameters σ and r are assumed unknown to the receiver. We see that the differences converge to zero, indicating that it is possible to estimate two parameters simultaneously.

Finally we mention that, if the time series for y is given, estimation of all the three parameters is possible though in this case, the convergence is very slow. The method fails to estimate all the three parameters σ, r and b, when time series for x is given.

We thus note that the detailed information about all the parameters of Lorenz system is contained in a time series for either x or y variables and can be extracted as above.

It should be however mentioned that when a time series for z is given from a Lorenz system, the eigenvalues of the associated matrix J (Eq. (15)) do not satisfy the condition of convergence for any choice of ϵ and δ. Thus the method fails when a time series for z is known.

B. Rössler system

We next consider the Rössler system of equations given by,

$$\dot{x} = -y - z,$$

$$\dot{y} = x + ay,$$

$$\dot{z} = b + z(x - c). \tag{27}$$

which contains the three parameters (a, b, c).

We have applied our procedure to estimate any of these parameters, when unknown, assuming the knowledge of a time series for the variable y in the Rössler system. The corresponding variable y', which acts as a drive variable for (x', y', z') and the evolution of the unknown parameter, then evolves through,

$$\dot{y}' = x' + ay' - \epsilon(y' - y(t)), \tag{28}$$

while the unknown parameter evolves adaptively.

Thus with the given time series $y(t)$, fed into the evolution of the drive variable y', we find that the convergence condition can be satisfied by a suitable choice of feedback constant and the stiffness constant.

In Fig. 9 we show the convergence of $(x' - x, y' - y, z' - z, a' - a)$ to (0, 0, 0, 0) when the parameter a is unknown. Thus our algorithm of parameter estimation works for y' as a drive variable and any of the three parameters can be estimated. We have however found that the convergence is not possible for x or z as the drive variables.

Finally, we have also applied our method to estimate two or three parameters of the Rossler system with y as a drive variable. It was seen that no choice of the feedback constant and the stiffness constant lead to convergences required for estimation.

C. An example from plasma physics

As our final example, we present a set of nonlinear equations appearing in plasma physics. This is the so called resonant three-wave coupling equations when high frequency wave is unstable and the remaining two are damped [23]. These equations are,

$$\dot{a}_1 = a_1 + {a_2}^2 \cos\phi,$$

$$\dot{a}_2 = -a_2(\gamma + a_1 \cos\phi),$$

$$\dot{\phi} = -\delta + a_1^{-1}(2{a_1}^2 - {a_2}^2)\sin\phi, \tag{29}$$

where γ and δ are the system parameters.

We find that with time series given for either a_1 or a_2, it is possible to know an unknown parameter γ or δ using synchronization and adaptive control. The method fails when a time series for ϕ is known.

Figure 10 displays the evolution of the differences between the transmitter and receiver variables as well as the evolution of $\delta' - \delta$ as functions of time, when the time series for a_1 is known. As expected, the differences go to zero asymptotically.

IV. EFFECT OF NOISE

In this section we will study the effect of noise present in the transmitter system. We will take the example of the Lorenz system (Eq. (16) for this purpose, where σ is assumed to be the unknown parameter and x acts as a drive variable.

Assuming that there is a small additive noise present in the time series given for x, we feed the noisy time series into the receiver system (Eq. (17) and carry out the parameter estimation as described.

We find that for weak noise, the asymptotically estimated value of the parameter fluctuates around the correct value with a small amplitude. Thus the estimation is possible using our method. The error in the estimation can be reduced by a suitable averaging over the time evolution of σ' in the asymptotic limit. For increasing strengths of noise, the fluctuations in the estimated value grow larger and precise estimation becomes difficult. Figure 11 shows the convergence of σ' to σ when additive noise is present in the evolution of x, the drive variable of the Lorenz system, for which the time series is given.

We define the accuracy (A) in the estimation of σ as $(\sigma' - \sigma)/\sigma$ while w denotes the strength of noise with uniform distribution ranging from $-w$ to w. In Fig.12 we plot the asymptotic value of A, the accuracy of the estimation of σ, against the strength w of noise in x. It can be seen from the curve that the accuracy grows linearly as the noise increases to a value of $w = 2$ which corresponds to about 12% of the range of x values. The plot thus shows that our method is quite robust for weak noise in x, while it can fail as the noise strength increases to a larger value.

V. CONCLUSIONS

To summarize, we have shown that a combination of synchronization based on linear feedback given into only a single receiver variable with an *adaptive* evolution for parameters unknown to the receiver, enables the estimation of the unknown parameters. The feedback comes from a scalar time series. We have also shown that our procedure corresponds to dynamic minimization of the synchronization error.

We have presented examples of Lorenz and Rössler systems taking different candidate parameters to be unknown to the receiver as well as that of a plasma system obeying resonant three-wave coupling equation. In the Lorenz system (Eq. (16)), any of the three parameters can be estimated when a time series is given for either of x and y, but the method fails when the known time series is for the variable z. Extensions to estimation of more than one parameters of the Lorenz system are also presented as a representative case. Estimation of two parameters is possible for both x or y as drive variables while estimation of all the three parameters is possible only when time series for y is given.

In the case of Rössler system (Eq. (27)) the method works only when the time series is given for the variable y where it is possible to estimate any of the three parameters. We find that in case of the plasma system, the parameters can be estimated with the feedback in the evolution for either a_1 or a_2.

We have thus numerically demonstrated that the explicit detailed information about the parameters of a nonlinear chaotic system is contained in the time series data of a variable and can be extracted under suitable conditions. This information includes the particular values of the parameters of the system which can be estimated even if they appear in the evolution of variables other than the one for which the time series is given.

We have also checked the robustness of the method against the noise and it shows reasonable robustness against small noise though the error of estimation becomes larger as the noise strength is increased.

The possibility of improving the efficiency of the method needs to be explored. This can be done, for example, by optimizing the choices of newly introduced parameters ϵ and δ or by trying to estimate initial values of variables of the transmitter system, corresponding to response variables and thereby starting from a "better" initial point. Work in these directions is under progress.

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Figure Captions

FIG.1. Figures (a), (b), (c) and (d) show the differences $(x' - x, y' - y, z' - z, \sigma' - \sigma)$ respectively as functions of time, for the Lorenz system (Eqs.(16), (17) and (18)). The unknown parameter is σ and the drive variable is x. The figures show that the differences tend to zero asymptotically. σ' which is set to an arbitrary initial value finally evolves to σ facilitating the parameter estimation to any desired accuracy in the asymptotic limit.

FIG.2. The curve along which the largest conditional Lyapunov exponent (computed using Eq. (19)) becomes zero in the (ϵ, δ)-plane for the Lorenz system with σ as the unknown parameter (Eqs. (17) and (18) is plotted. In region (I), the CLE's are all negative and parameter estimation works convergently. Region (II) corresponds to a positive largest CLE, where the method does not work. Note that there is a critical ϵ below which the method does not work. Nevertheless for any δ, an ϵ can be chosen so that the method works.

FIG.3. The figure shows the curve along which the largest conditional Lyapunov exponent for Lorenz system with the parameter r as unknown and x as drive variable (Eqs. (17) and (21) becomes zero in the (ϵ, δ)-plane. In region (I) all the CLE's are negative and the parameter estimation can be achieved. In region (II) the the largest Lyapunov exponent is positive.

FIG.4. The plot shows the time (τ) required for convergence of r' to r to a given accuracy with a fixed value of the stiffness constant (δ), as a function of the feedback constant, ϵ, for Lorenz system. The drive is x while the unknown parameter is r (Eq. (17)). It can be seen that the synchronization time tends to infinity when the largest CLE becomes zero.

FIG.5. The plot shows the time (τ) required for convergence of r' to r 2 a given accuracy, with a fixed value of the feedback constant (ϵ), as a function of the stiffness constant, δ, for the Lorenz system. The drive is x while the unknown parameter is r (Eq. (17)). It can be seen that the synchronization time tends to infinity as δ approaches a value so as to make the largest CLE zero.

FIG.6. The graph shows the time, t, required to achieve the parameter estimation to within a given accuracy as a function of the accuracy, A, (logarithmic scale) normalized with respect to the initial deviation of the parameter from the correct value for Lorenz system (Eq. (17)). The time series for x is assumed to be known while the value of r is unknown. The straight line shows that the time required for a better accuracy grows exponentially.

FIG.7. The curve along which the largest conditional Lyapunov exponent (computed using Eq. (19)) becomes zero in the (ϵ, δ)-plane for the Lorenz system with b as the unknown parameter and x as drive (Eqs. (23) and (24) is plotted. In region (I), the CLE's are all negative and parameter estimation works convergently. Region (II) corresponds to a positive largest CLE, where the method does not work. Similar to other cases, there is a critical ϵ below which the method does not work.

FIG.8. Plots (a) and (b) show the differences $(\sigma' - \sigma)$ and $(r' - r)$ respectively as functions of time in the Lorenz system (Eq.(16). The unknown parameters are σ and r and the drive variable is x. The plots show that the differences go to zero, and hence indicate that a simultaneous estimation of more than one unknown parameters is possible.

FIG.9. Plots (a), (b), (c) and (d) show the differences $(x' - x, y' - y, z' - z, a' - a)$ as functions of time respectively, in the Rössler system (Eq.(27)). The unknown parameter is a and the drive variable is y. The figures show that the differences tend to zero asymptotically. a' which is set to an arbitrary initial value finally evolves to a facilitating the parameter estimation.

FIG.10. Plots (a), (b), (c) and (d) show the differences $(a_1' - a_1, a_2' - a_2, \phi' - \phi, \delta' - \delta)$ as functions of time respectively, in the plasma system (Eq.(29)). The unknown parameter is δ and the drive variable is a_1. The figures show that the differences tend to zero asymptotically. δ' which is set to an arbitrary initial value finally evolves to δ facilitating the parameter estimation.

FIG.11. The graph shows the evolution of $\sigma' - \sigma$ as a function of time, in the presence of an additive noise ($w = 0.1$) in the given time series for x for Lorenz system (16). The value of σ is assumed unknown. The plot shows that the difference $\sigma' - \sigma$ fluctuates around

zero with a small amplitude after an initial transient and a reasonably good estimation is possible using a suitable averaging over these fluctuations.

FIG.12. The plot of asymptotic accuracy of parameter estimation ($A = (\sigma' - \sigma)/\sigma$), as a function of strength of the noise, w, in the given time series of x in Lorenz system (Eq. (16)). The noise with strength w takes uniformly distributed values from $-w$ to $+w$. The drive is x and the unknown parameter is σ. It is seen that the estimation of σ is stable for a range of noise strength growing from zero to about 2 which corresponds to about 12 % of the range of x values.

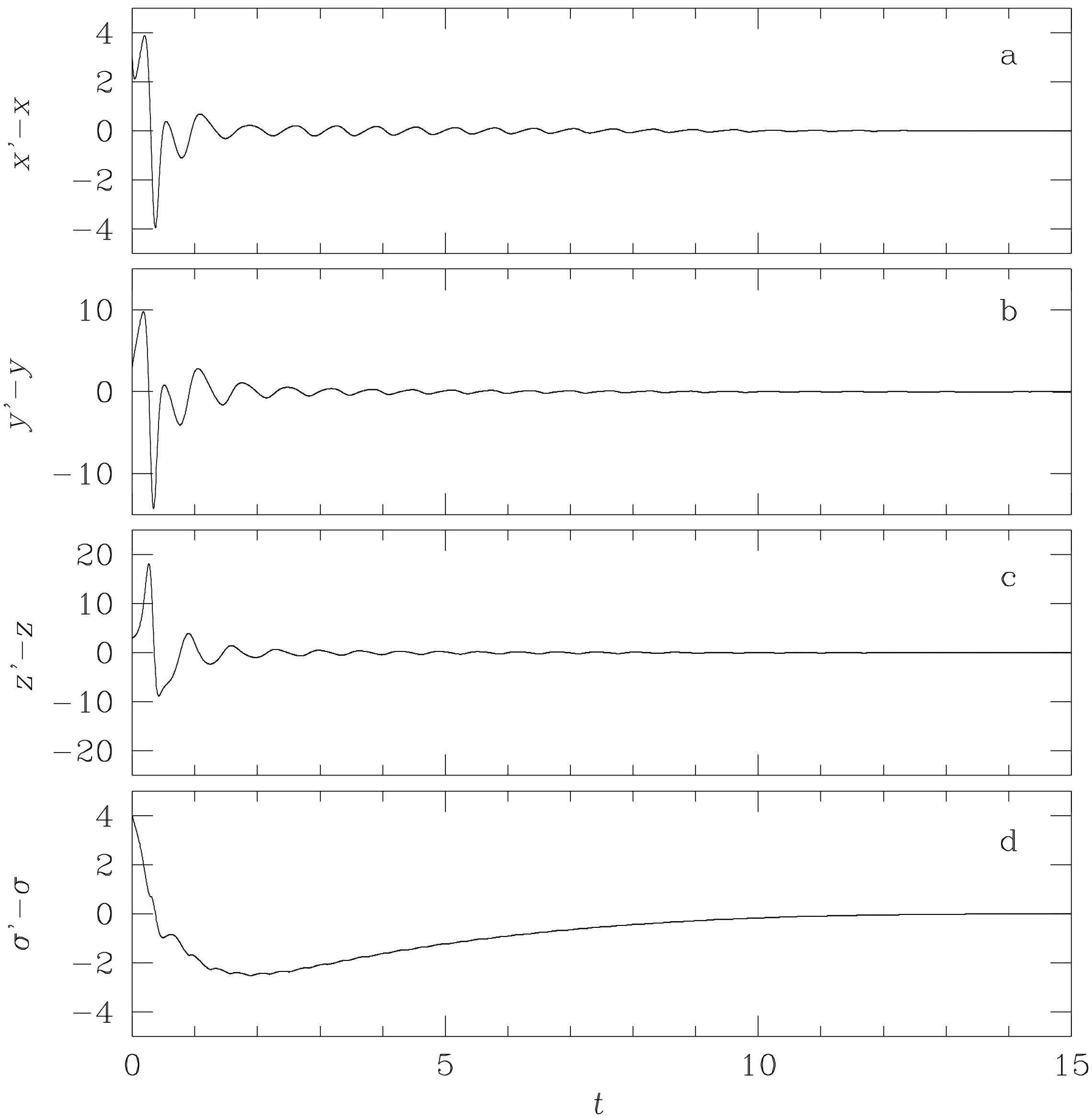
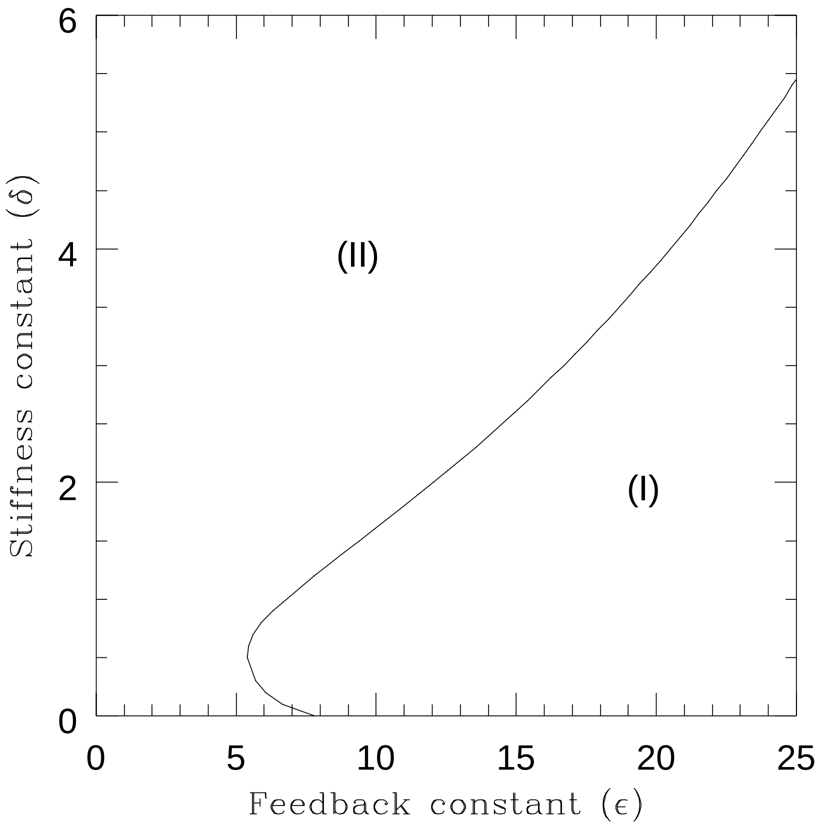


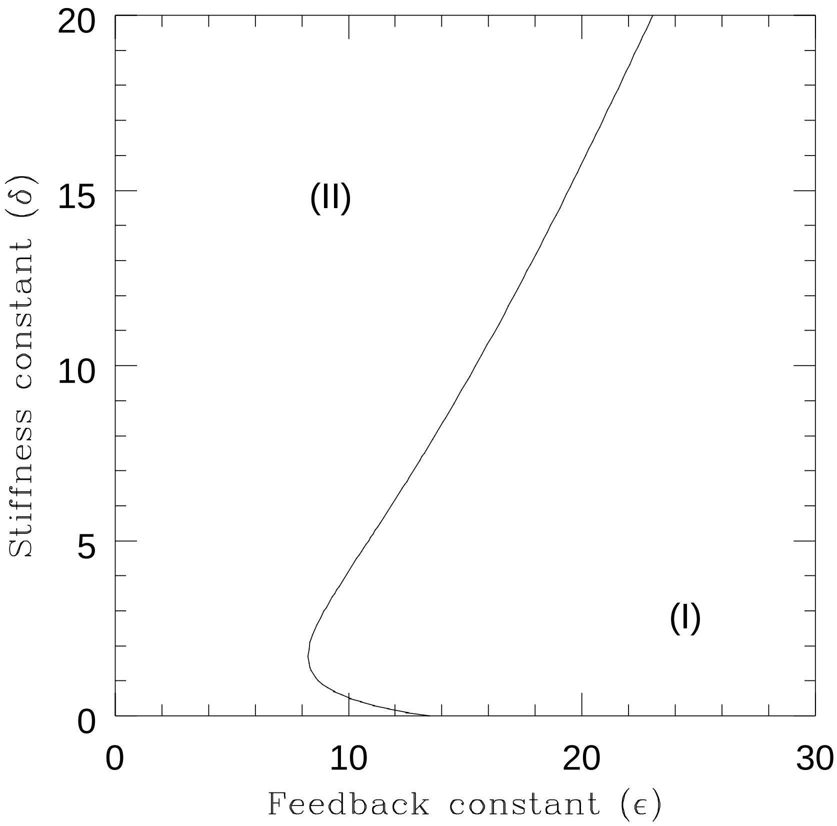
Fig.1

(AM & REA)



6
4
2
0
0
5
10
15
20
25
(II)
(I)
Stiffness constant (δ)
Feedback constant (ϵ)

Fig.2 (AM & REA)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15
10
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0
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10
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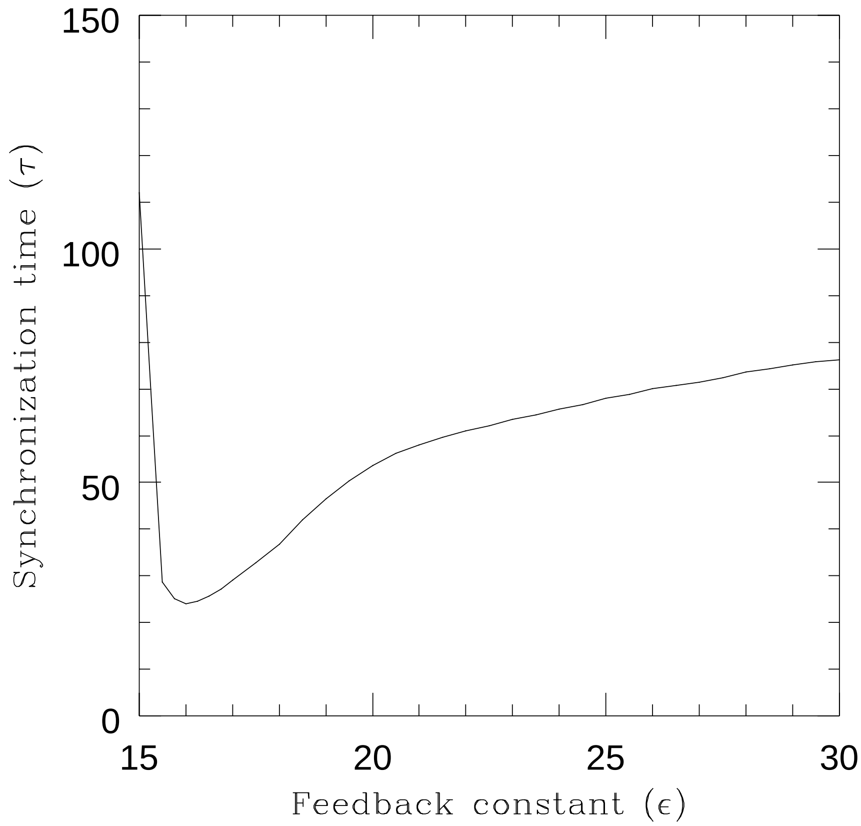
30
Stiffness constant (δ)
Feedback constant (ϵ)
(II)
(I)

Fig.3

(AM & REA)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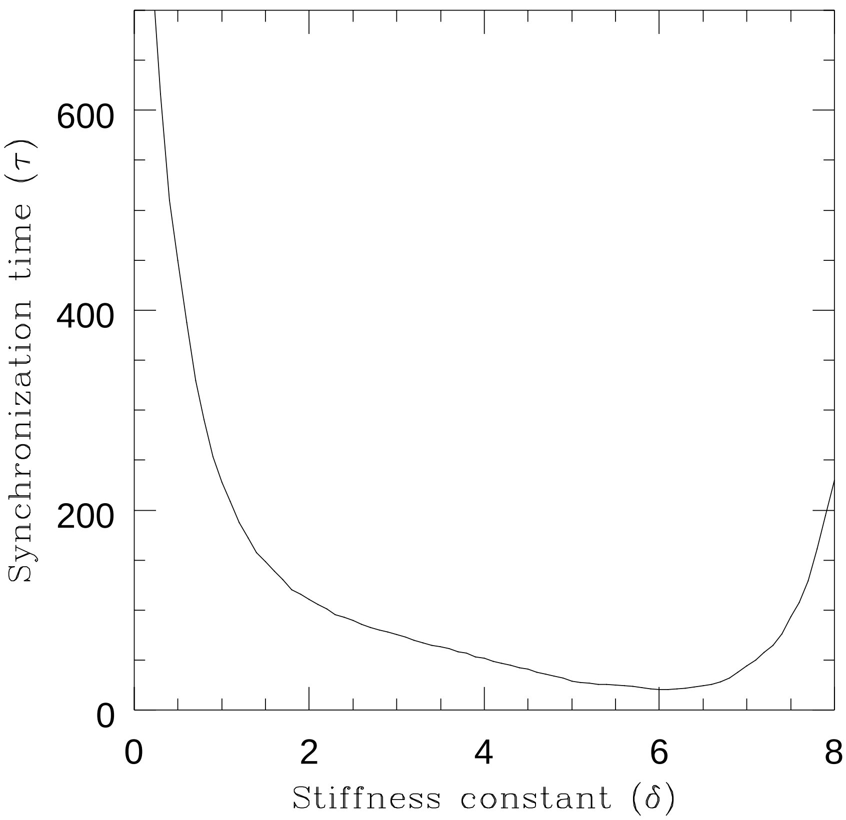
100
50
0
15
20
25
30
Synchronization time (τ)
Feedback constant (ϵ)

Fig.4 (AM & REA)



Synchronization time (τ)
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400
200
0
0
2
4
6
8
Stiffness constant (δ)

Fig.5 (AM & REA)

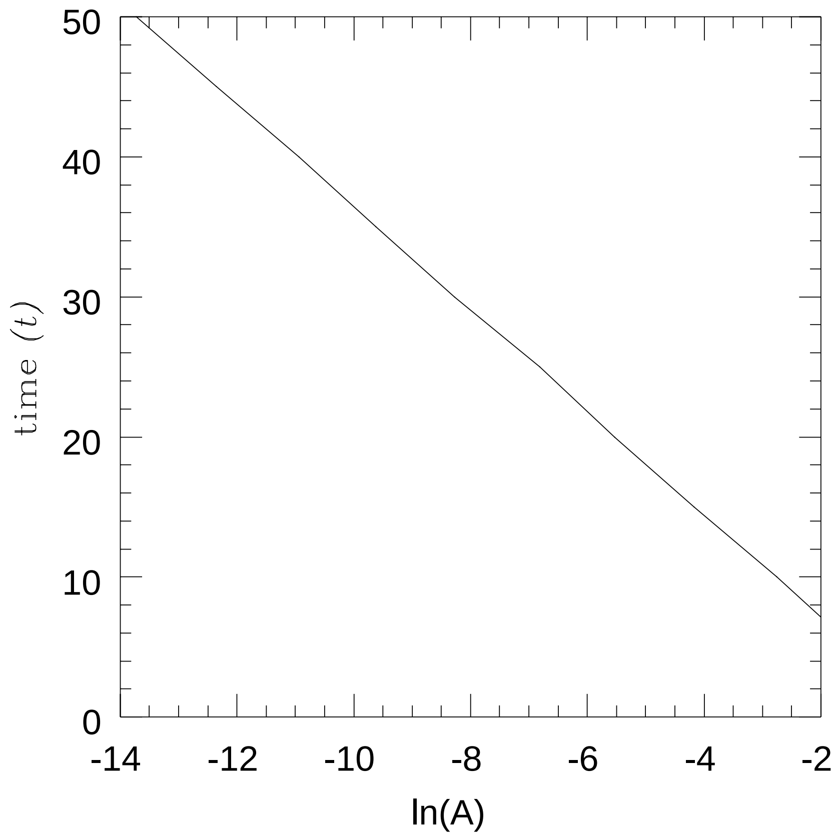
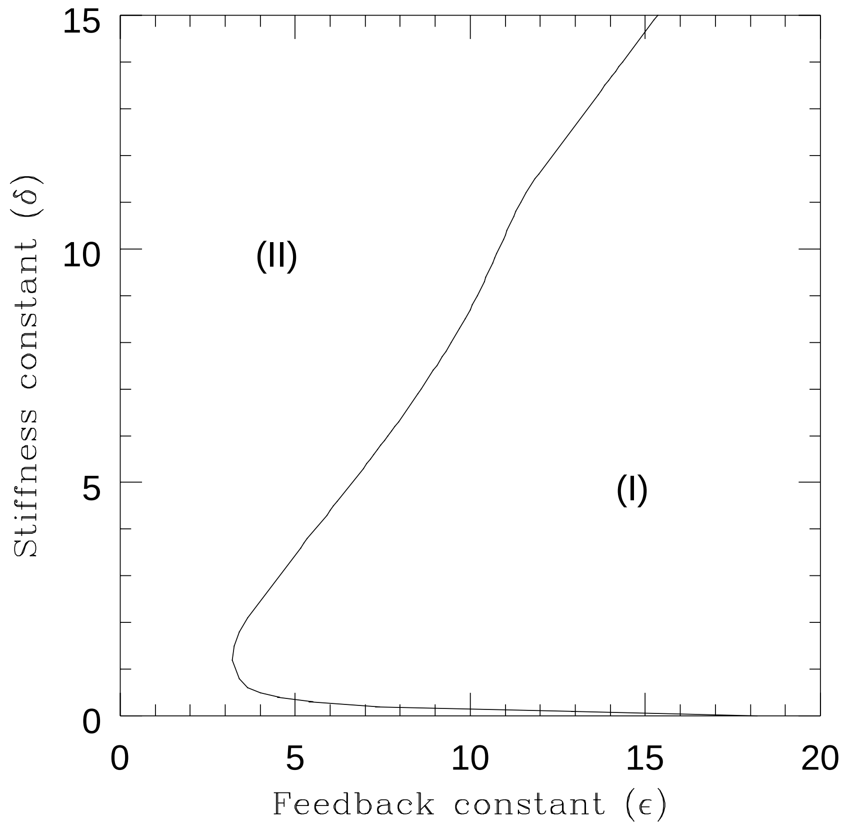


Fig.6 (AM & REA)



15
10
5
0
(II)
(I)
0
5
10
15
20
Stiffness constant (δ)
Feedback constant (ϵ)

Fig.7

(AM & REA)

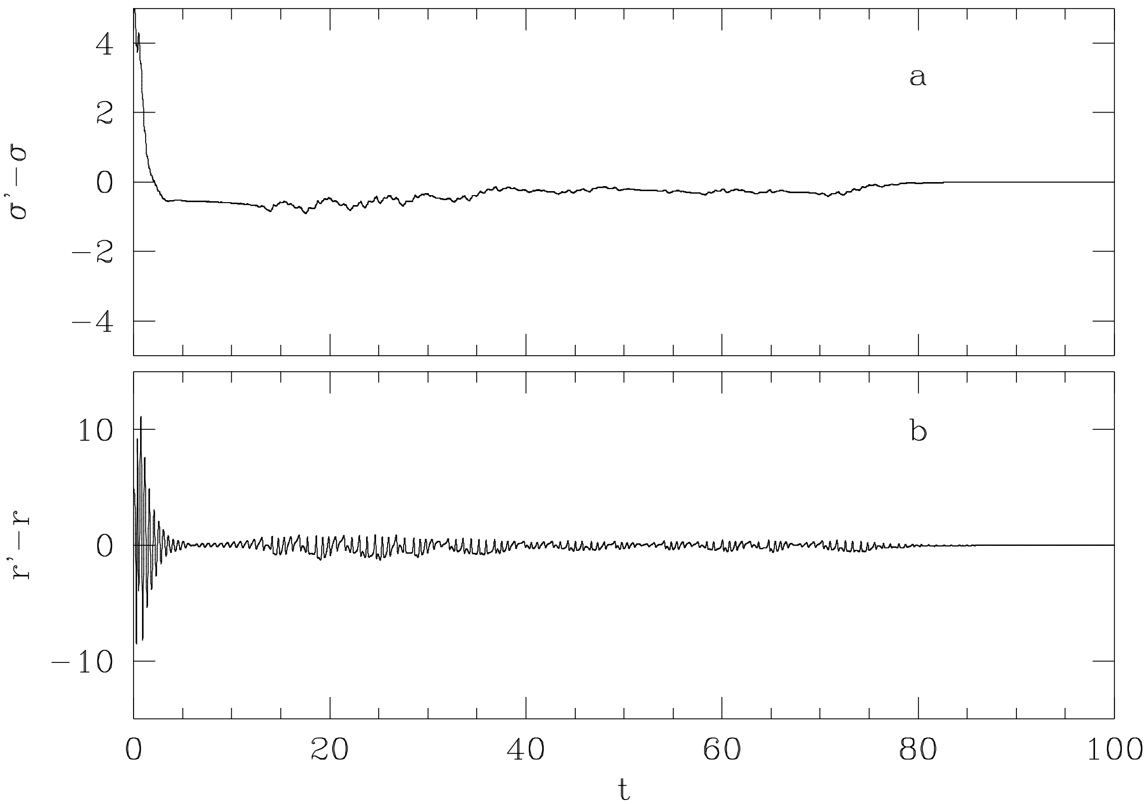


Fig.8 (AM & REA)

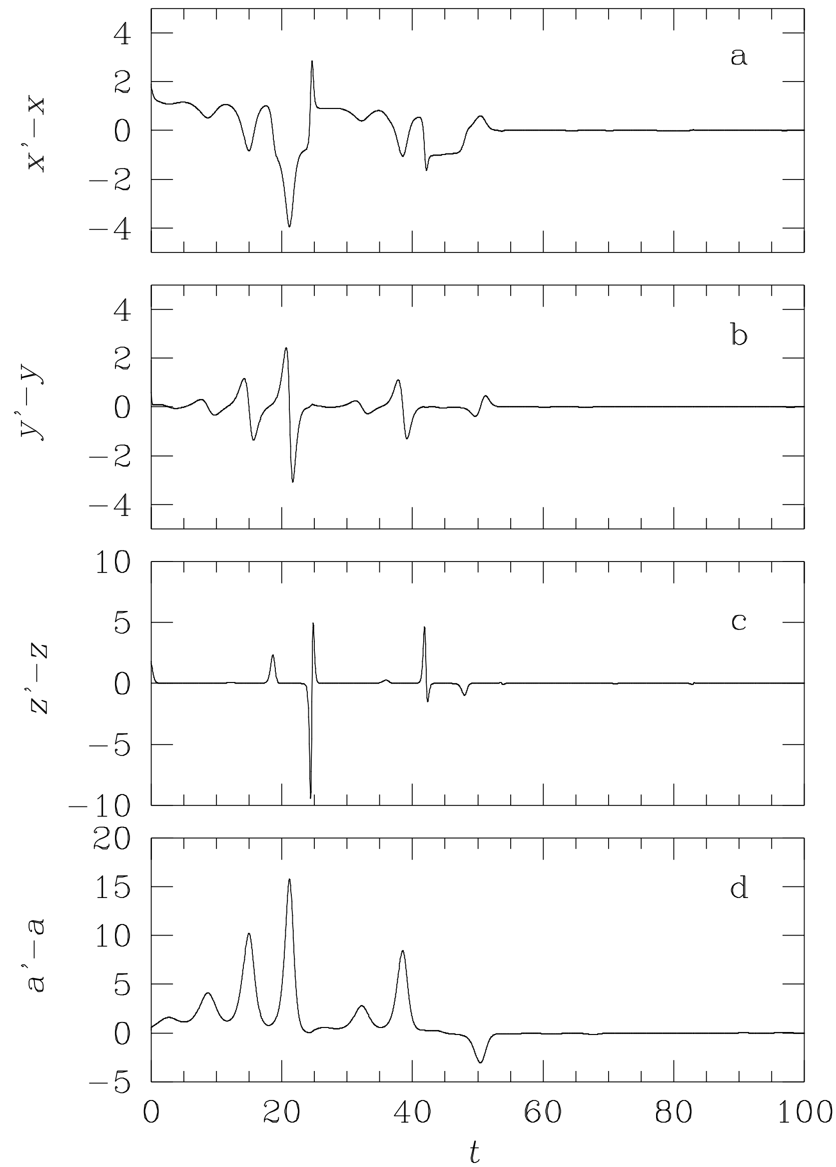


Fig.9 (AM & REA)

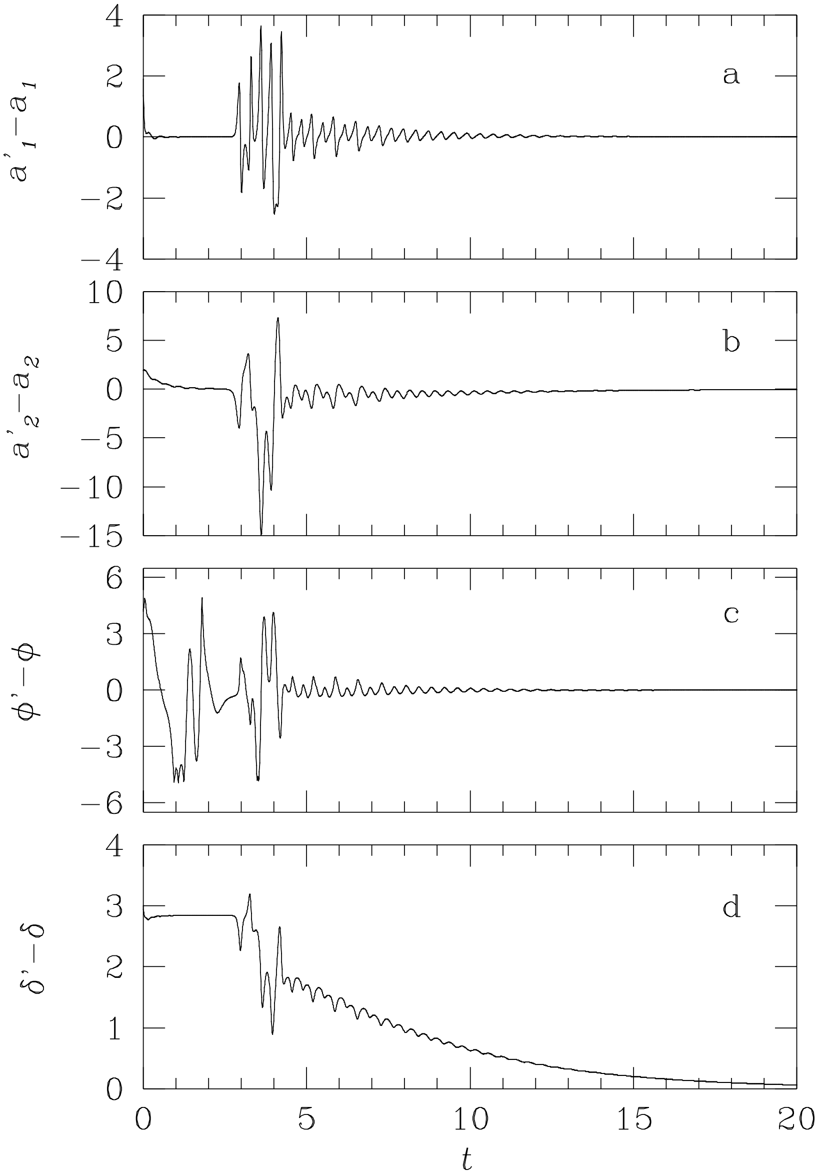


Fig.10

(AM & REA)

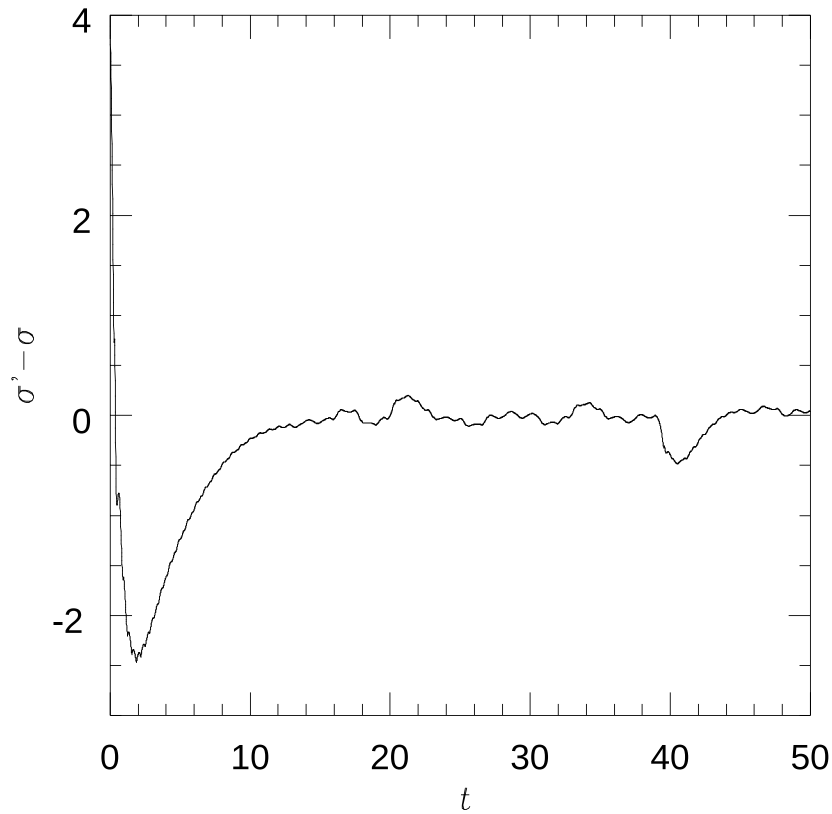


Fig.11

(AM & REA)

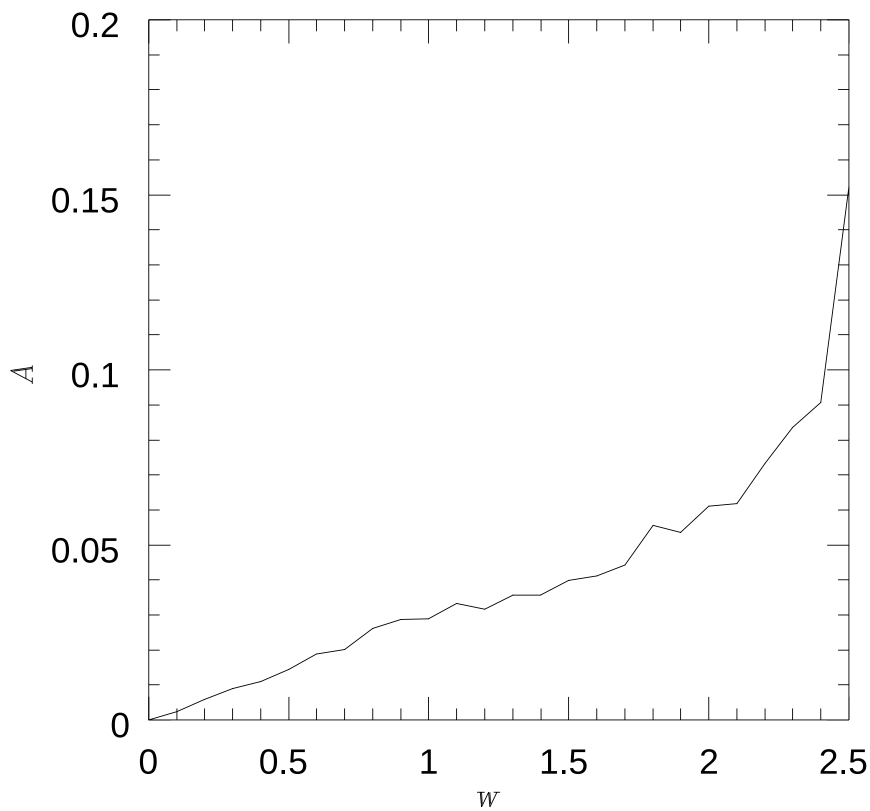


Fig.12 (AM & REA)