Scaling properties of composite information measures and shape complexity for hydrogenic atoms in parallel magnetic and electric fields

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Abstract

The scaling properties of various composite information-theoretic measures (Shannon and Rényi entropy sums, Fisher and Onicescu information products, Tsallis entropy ratio, Fisher-Shannon product and shape complexity) are studied in position and momentum spaces for the non-relativistic hydrogenic atoms in the presence of parallel magnetic and electric fields. Such measures are found to be invariant at the fixed values of the scaling parameters given by $s_1 = \frac{B\hbar^3(4\pi\epsilon_0)^2}{Z^2m^2e^3}$ and $s_2 = \frac{F\hbar^4(4\pi\epsilon_0)^3}{Z^3e^5m^2}$. Numerical results which support the validity of the scaling properties are shown by choosing the representative example of the position space shape complexity. Physical significance of the resulting scaling behaviour is discussed.

Key words: Atoms under external fields, Shannon entropy, Rényi entropy, Fisher information, Shape complexity, Avoided crossings
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1. Introduction

The quantum-mechanical uncertainty principle, first formulated [1] in terms of the standard deviations of the position and momentum probability densities which characterize the quantum-mechanical states of one-dimensional single-particle systems, is fundamental to the understanding the electronic structure and properties of atoms and molecules. The position-momentum Heisenberg uncertainty relation has been extensively tested for many three-dimensional systems [2], and some interesting properties have been found for central potentials; namely, the Heisenberg uncertainty product (i) does not depend on the potential strength for the bound states of homogeneous power-type potentials [3], and (ii) has a lower bound which has a quadratic dependence on the orbital quantum number [4]. There exist formulations of the position-momentum uncertainty principle based on uncertainty measures other than the standard deviation, which are more stringent than the Heisenberg relation. They are the uncertainty-like relationships based

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on, e.g. the Shannon [5], Rényi [6] and Tsallis [7] entropies, the Fisher information [8] and the modified LMC or shape complexity, which are found and discussed in Ref. [9, 10, 11, 12, 13], respectively.

The scaling properties of the position-momentum uncertainty relations mentioned above for single particle systems with a wide variety of central potentials have recently been examined by using of the dimensional analysis of their associated Schrödinger equation [3, 14]. In this letter we present the first comprehensive information-theoretic study on the hydrogenic-like atoms in the presence of external parallel magnetic and electric fields. In particular, we have considered the scaling properties of the Heisenberg uncertainty measure (i.e. the standard variation), the Shannon, Rényi, Tsallis, Fisher information measures, and the shape complexity [15, 16]. The numerical validity of these scaling properties are presented. The predictive power of the presently obtained results on this statistical complexity is illustrated by taking the example of the most distinctive non-linear spectroscopic phenomenon, the avoided crossing of two energy levels with the same energy [17] of a hydrogenic system in the presence of intense parallel magnetic and electric fields.

This paper is organized as follows. We analyze the scaling transformation of the energies and eigenfunctions which characterize the quantum-mechanical states of a hydrogenic atom in the presence of parallel magnetic and electric fields in Section 2, and the dimensional properties of their position and momentum Heisenberg uncertainty measure in Section 3. In Section 4, we examine the scaling properties of the uncertainty relations associated with the following information-theoretic measures: Shannon, Rényi and Tsallis entropies and the Fisher and Onicescu informations as well as the shape complexity. Finally, in Section 5, we compute the shape complexity for two different pairs of energy levels of a hydrogen atom under intense parallel magnetic and electric fields, which show avoided crossing phenomena. Moreover, we check the validity of the corresponding scaling law obtained for this information measure in the previous section, and, most important, we show that this measure presents a peculiar mirror symmetry through the avoided crossing region. The latter implies that the shape complexity is a good indicator of this highly non-linear phenomena at the same level as the energy [17, 18, 19] and the Shannon and Fisher informations [20].

2. Hydrogenic systems in parallel magnetic and electric fields: scaling properties

Let us consider an electron moving in a Coulombic potential due to a nucleus with charge $+Ze$, in the presence of parallel magnetic and electric fields oriented in the z direction. The effective potential in spherical coordinates is

$$V(\mathbf{r}) = -\frac{Ze^2}{4\pi\epsilon_0 r} + \frac{eB}{m}L_z + \frac{e^2B^2}{2m}r^2\sin^2\theta + eFr\cos\theta. \tag{1}$$

where B and F are the constant magnetic and electric fields strengths, m the mass of the electron, ϵ_0 the electric constant, and L_z the z-component of the angular momentum. The corresponding Schrödinger equation for this potential is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r})\psi = E\psi. \tag{2}$$

Note that we have neglected the relativistic corrections [25] and the spin-orbit coupling [26] and assumed that the nuclear mass is infinity [27], because they do not affect the main results of

this work. Here, we shall study the scaling properties of the eigensolutions of this equation of motion under the transformation $\mathbf{r} = \lambda \mathbf{r}'$, where the scaling factor is specified later on. Then the transformed Schrödinger equation takes on the form

$$-\frac{\hbar^2}{2m\lambda^2}\nabla'^2\psi + \left[-\frac{Ze^2}{4\pi\epsilon_0\lambda r'} + \frac{eB}{m}L_{z'} + \frac{e^2B^2\lambda^2}{2m}r'^2\sin^2\theta' + eF\lambda r'\cos\theta'\right]\psi = E\psi.$$

Now, we multiply this equation by the factor $\frac{m\lambda^2}{\hbar^2}$ and fix λ by imposing that the factor in the Coulomb term is equal to unity. It turns out that

$$\lambda = \frac{4\pi\epsilon_0\hbar^2}{Ze^2m}, \tag{3}$$

and the scaled Schrödinger equation reads as

$$\left[-\frac{1}{2}\nabla'^2 - \frac{1}{r'} + \frac{s_1}{\hbar}L_{z'} + \frac{s_1^2}{2}r'^2\sin^2\theta' + s_2 r'\cos\theta'\right]\psi = E_1\psi,$$

where

$$s_1 = \frac{B\hbar^3(4\pi\epsilon_0)^2}{Z^2m^2e^3}, \quad \text{and} \quad s_2 = \frac{\mathrm{F}\hbar^4(4\pi\epsilon_0)^3}{\mathrm{Z}^3\mathrm{e}^5\mathrm{m}^2}. \tag{4}$$

Note that λ has length units and the new coordinate is dimensionless, as wanted. Moreover, the parameter s_1 and s_2 are also dimensionless, and the energy $E(\hbar^2/m, Z, B, F)$ rescales into $E_1 = E(1, 1, s_1, s_2)$ as

$$E(\hbar^2/m, Z, B, F) = \frac{e^4Z^2m}{\hbar^2(4\pi\epsilon_0)^2}E(1, 1, s_1, s_2).$$

Consequently the wavefunction $\psi(\mathbf{r}; \hbar^2/m, Z, B, F)$ will change as

$$\psi(\mathbf{r}; \hbar^2/m, Z, B, F) = \lambda^{-3/2}\psi(\mathbf{r}'; 1, 1, s_1, s_2),$$

because of the normalization to unity, and the associated probability density $\rho(\mathbf{r}) = |\psi(\mathbf{r})|^2$ as

$$\rho(\mathbf{r}; \hbar^2/m, Z, B, F) = \lambda^{-3}\rho(\mathbf{r}'; 1, 1, s_1, s_2). \tag{5}$$

To obtain the scaling of the wavefunction in momentum space, $\tilde{\psi}(\mathbf{p}; \hbar^2/m, Z, B, F)$, under the transformation $\mathbf{p}' = \lambda\mathbf{p}$, with λ given by (3), we take into account that $\psi(\mathbf{r})$ and $\tilde{\psi}(\mathbf{p})$ are mutually Fourier-transformed as

$$\tilde{\psi}(\mathbf{p}; \hbar^2/m, Z, B, F) = \frac{1}{(2\pi\hbar)^{3/2}}\int e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\psi(\mathbf{r}; \hbar^2/m, Z, B, F)d\mathbf{r}.$$

It is straightforward that the momentum wavefunction scales as

$$\tilde{\psi}(\mathbf{p}; \hbar^2/m, Z, B, F) = \lambda^{3/2}\tilde{\psi}(\mathbf{p}'; 1, 1, s_1, s_2)$$

and the associated density $\gamma(\mathbf{p}) = |\tilde{\psi}(\mathbf{p})|^2$ as

$$\gamma(\mathbf{p}; \hbar^2/m, Z, B, F) = \lambda^3\gamma(\mathbf{p}'; 1, 1, s_1, s_2). \tag{6}$$

3. Dimensionality properties of the Heisenberg uncertainty measure

For a hydrogenic system with a potential $V(\mathbf{r})$ given by Eq. (1), a pure dimensional analysis of the standard deviation of its position wavefunction $\psi(\mathbf{r})$ defined by

$$\sigma_{\mathbf{r}}^2 = \int \psi^*(\mathbf{r})(\mathbf{r} - \langle \mathbf{r} \rangle)^2 \psi(\mathbf{r}) d\mathbf{r},$$

allows us to write down in a straightforward maner that

$$\sigma_{\mathbf{r}} = \lambda f_1(s_1, s_2), \tag{7}$$

where $f_1(s_1, s_2)$ is a fixed function of the dimensionless parameters s_1 and s_2 given by Eq. (4). Moreover, taking into account the reciprocity of the position and momentum spaces, a similar dimensional analysis for the standard deviation in momentum space

$$\sigma_{\mathbf{p}}^2 = \int \tilde{\psi}^*(\mathbf{p})(\mathbf{p} - \langle \mathbf{p} \rangle)^2 \tilde{\psi}(\mathbf{p}) d\mathbf{p},$$

leads to the expression

$$\sigma_{\mathbf{p}} = \hbar \lambda^{-1} f_2(s_1, s_2). \tag{8}$$

Hence, the Heisenberg uncertainty product is

$$\sigma_{\mathbf{r}} \, \sigma_{\mathbf{p}} = \hbar \; f_1(s_1) f_2(s_2). \tag{9}$$

Expressions (7), (8) and (9) allow us to state for hydrogenic systems under parallel magnetic and electric fields, that (i) the position and momentum spreadings around the corresponding centroids in position and momentum space depend only on the nuclear charge Z and the dimensionless parameters s_1 and s_2, and (ii) the Heisenberg uncertainty product depends only on s_1 and s_2.

4. Scaling of hydrogenic information-theoretic uncertainty measures

Here we examine the scaling properties of the information-theoretic-based uncertainty measures of Shannon, Fisher, Onicescu and Tsallis types, as well as their mutual relations, under the coordinate transformation $\mathbf{r} = \lambda \mathbf{r}'$ (where the scaling λ is given in Eq. (3)) for a hydrogenic system in the presence of parallel magnetic and electric fields. In particular, we show that the Shannon entropy sum, the Fisher and Onicescu information products, the Tsallis entropy ratio, the Fisher-Shannon measure and the shape complexity of this system depend only on the dimensionless parameters s_1 and s_2 for given values of the nuclear charge Z and the strengths (B, F) of the external fields as described by Eqs. (11), (13), (15), (16), (17), (18)-(19), and (20)-(21), respectively, later on.

4.1. Shannon entropy sum

The Shannon entropies [5] in the position space and momentum space, are

$$S_r = -\int \rho(\mathbf{r}) \ln \rho(\mathbf{r}) d\mathbf{r}, \qquad S_p = -\int \gamma(\mathbf{p}) \ln \gamma(\mathbf{p}) d\mathbf{p}.$$

Using the relations in Eqs. (5) and (6), we get for these entropies the scaling properties

$$S_r(\hbar^2/m, Z, B, F) = 3\ln\lambda + S_r(1, 1, s_1, s_2), \tag{10}$$
$$S_p(\hbar^2/m, Z, B, F) = -3\ln\lambda + S_p(1, 1, s_1, s_2),$$

which imply that the Shannon entropy sum $S_T = S_r + S_p$ satisfies the relation

$$S_T(\hbar^2/m, Z, B, F) = S_T(1, 1, s_1, s_2). \tag{11}$$

4.2. Fisher information product

The Fisher information [8] measures for position and momentum are

$$I_r = \int \frac{[\nabla\rho(\mathbf{r})]^2}{\rho(\mathbf{r})} d\mathbf{r}, \qquad I_p = \int \frac{[\nabla\gamma(\mathbf{p})]^2}{\gamma(\mathbf{p})} d\mathbf{p}.$$

Using the relations in Eqs (5) and (6), one obtains the scaling properties

$$I_r(\hbar^2/m, Z, B, F) = \frac{1}{\lambda^2} I_r(1, 1, s_1, s_2), \tag{12}$$
$$I_p(\hbar^2/m, Z, B, F) = \lambda^2 I_p(1, 1, s_1, s_2),$$

which together imply that the Fisher information product $I_{rp} = I_r I_p$ satisfies the relation

$$I_{rp}(\hbar^2/m, Z, B, F) = I_{rp}(1, 1, s_1, s_2). \tag{13}$$

4.3. Onicescu information product

The Onicescu informations [28] in position and momentum spaces are

$$E_r = \int [\rho(\mathbf{r})]^2 d\mathbf{r}, \qquad E_p = \int [\gamma(\mathbf{p})]^2 d\mathbf{p}.$$

Using the relations in Eqs. (5) and (6), we get the scaling properties

$$E_r(\hbar^2/m, Z, B, F) = \frac{1}{\lambda^3} E_r(1, 1, s_1, s_2), \tag{14}$$
$$E_p(\hbar^2/m, Z, B, F) = \lambda^3 E_p(1, 1, s_1, s_2),$$

which imply that the Onicescu information product $E_{rp} = E_r E_p$ satisfies the relation

$$E_{rp}(\hbar^2/m, Z, B, F) = E_{rp}(1, 1, s_1, s_2). \tag{15}$$

4.4. Rényi entropy sum

The Rényi entropies [6] in position and momentum spaces are

$$H_\alpha^{(r)} = \frac{1}{1-\alpha} \ln \int [\rho(\mathbf{r})]^\alpha d\mathbf{r}, \qquad H_\alpha^{(p)} = \frac{1}{1-\alpha} \ln \int [\gamma(\mathbf{p})]^\alpha d\mathbf{p}.$$

With the relations in Eqs. (5) and (6), we get for these entropies the scaling properties

$$H_\alpha^{(r)}(\hbar^2/m, Z, B, F) = 3\ln\lambda + H_\alpha^{(r)}(1, 1, s_1, s_2),$$
$$H_\alpha^{(p)}(\hbar^2/m, Z, B, F) = -3\ln\lambda + H_\alpha^{(p)}(1, 1, s_1, s_2),$$

which imply that the Rényi entropy sum $H_\alpha^{(T)} = H_\alpha^{(r)} + H_\alpha^{(p)}$ satisfies the relation

$$H_\alpha^{(T)}(\hbar^2/m, Z, B, F) = H_\alpha^{(T)}(1, 1, s_1, s_2). \tag{16}$$

4.5. Tsallis entropy ratio

The Tsallis entropies [7] in position and momentum spaces are

$$T_n^{(r)} = \frac{1}{n-1}\Big[1 - J_n^{(r)}\Big], \quad T_q^{(p)} = \frac{1}{q-1}\Big[1 - J_q^{(p)}\Big], \quad \frac{1}{q} + \frac{1}{n} = 2.$$

where the integral terms are given by

$$J_n^{(r)} = \int [\rho(\mathbf{r})]^n d\mathbf{r}, \quad J_q^{(p)} = \int [\gamma(\mathbf{p})]^q d\mathbf{p}.$$

Using the relations in Eqs. (5) and (6), we get the scaling properties

$$J_n^{(r)}(\hbar^2/m, Z, B, F) = \lambda^{3-3n} J_n^{(r)}(1, 1, s_1, s_2),$$
$$J_q^{(p)}(\hbar^2/m, Z, B, F) = \lambda^{3q-3} J_q^{(p)}(1, 1, s_1, s_2).$$

Then one obtains for the ratio $J_{p/r} = \frac{(J_q^{(p)})^{1/2q}}{(J_n^{(r)})^{1/2n}}$ the following equality

$$J_{p/r}(\hbar^2/m, Z, B, F) = J_{p/r}(1, 1, s_1, s_2), \quad \frac{1}{n} + \frac{1}{q} = 2. \tag{17}$$

4.6. Fisher-Shannon measure

For the Shannon entropy power

$$N_r = \frac{1}{\pi e} e^{2S_r/3}, \qquad N_p = \frac{1}{\pi e} e^{2S_p/3}$$

in the two conjugated spaces, we obtain from Eqs. (10) the following scaling:

$$N_r(\hbar^2/m, Z, B, F) = \lambda^2 N_r(1, 1, s_1, s_2),$$

$$N_p(\hbar^2/m, Z, B, F) = \frac{1}{\lambda^2} N_p(1, 1, s_1, s_2),$$

Using these expressions and Eq. (12) for the Fisher information, we obtain for the Fisher-Shannon measure the scaling

$$N_r(\hbar^2/m, Z, B, F) I_r(\hbar^2/m, Z, B, F) = N_r(1, 1, s_1, s_2) I_r(1, 1, s_1, s_2), \tag{18}$$

$$N_p(\hbar^2/m, Z, B, F) I_p(\hbar^2/m, Z, B, F) = N_p(1, 1, s_1, s_2) I_p(1, 1, s_1, s_2) \tag{19}$$

in position and momentum spaces, respectively.

4.7. Shape complexity

For the shape complexity [15, 16] $C = e^{S_r} E_r$, with S_r and E_r being the Shannon entropy and the Onicescu information or disequilibrium, respectively, we use the relations in Eqs. (10) and (14) to obtain

$$e^{S_r(\hbar^2/m,Z,B,F)} E_r(\hbar^2/m, Z, B, F) = e^{S_r(1,1,s_1,s_2)} E_r(1, 1, s_1, s_2), \tag{20}$$

$$e^{S_p(\hbar^2/m,Z,B,F)} E_p(\hbar^2/m, Z, B, F) = e^{S_p(1,1,s_1,s_2)} E_p(1, 1, s_1, s_2), \tag{21}$$

for the scaling in the two reciprocal spaces.

Besides the scaling invariance shown by Eqs. (20) and (21) for the shape complexity, there are two noteworthy features: (i) the scaling properties are independent of the relative orientation of the external fields, and more interestingly, (ii) the functional dependence on s_1 and s_2 predicts the existence of extremum points when one of the fields is varied keeping fixed the other one. We note here that the functional form of the shape complexity is not obtained through the dimensional analysis and the number of maximum and minimum points in it depends upon the specific details. In the next section we shall discuss these features in some detail. These observations are equally valid for the other uncertainty-like products discussed in this work.

5. Hydrogenic shape complexity: numerical scaling test and avoided crossing indicator

We have successfully carried out extensive numerical tests of the scaling properties of the various uncertainty-like products discussed above. In this section, we will use atomic units ($m = \hbar = e = 4\pi\epsilon_0 = 1$) and take B in units of speed of light c. We will discuss the shape complexity, as a representative example, in the neighborhood of some typical avoided crossings of hydrogenic systems in parallel magnetic and electric field. The details of the computational approach used to solve the Schrödinger equation (2) can be found elsewhere [29]. In particular, we have considered the pair of levels $3p_0$ and $3d_0$ of the ($Z = 1$) hydrogen atom, for which the paramagnetic term does not contribute. Note that, for simplicity, the field-free quantum numbers are used to label these states. In the presence of the magnetic field the magnetic quantum number and the z-axis parity are good quantum numbers. Hence, these levels have different symmetry and as the magnetic field strength is varied they could have the same energy, which occurs at the magnetic field interval 0.087 a.u. $\leq B \leq$ 0.08825 a.u.

If an additional parallel electric field is also on, only the azimuthal symmetry remains so that both levels may have the same symmetry; then an avoided crossing is formed between them due to the Wigner-non-crossing rule [17]. This non-linear phenomenon is illustrated in Figs. 1a and b, which show the ionization energies and shape complexities, respectively, of these levels for a magnetic field with strength 0.087 a.u.$\leq B_1 \leq$ 0.08825 a.u. and a fixed electric field with strength $F_1 = 1.946 \times 10^{-6}$ a.u. An analogous result should be expected for the same pair of states in a hydrogenic atom with nuclear charge $Z = 2$ if the magnetic and electric field strengths are scaled according the rules discussed in the previous section. The corresponding energies and shape complexities are presented in Fig. 2a and b, as a function of the magnetic field strength in the range 0.348 a.u. $\leq B_2 \leq$ 0.353 a.u., and fixed electric field strength $F_2 = 1.557 \times 10^{-5}$ a.u. Note, that the scaling laws $F_2 = F_1 * (Z = 2)^3$ and $B_2 = B_1 * (Z = 2)^2$ are satisfied.

Let us first analyze the ionization energy. Looking at Figs. 1a and 2a, the ionization energy shows a qualitatively similar but quantitatively different behavior as a function of B in the two hydrogenic atoms. On the one hand, the typical avoided-crossing behavior is observed, i.e. they approach each other with increasing magnetic field, until they come close and strongly interact, splitting apart thereafter. For both systems, the ionization energy of the $3p_0$ ($3d_0$) state monotonically increases (decreases) as the magnetic field strength is enhanced, passes through a maximum (minimum), and decreases (increases) thereafter. However, major differences appear in the computed values of the energies, which differ by a factor Z^2, as expected by the scaling properties discussed above. The minimal energetic spacing $\Delta E = |E_{3p_0} - E_{3d_0}| = 3.35 \times 10^{-5}$ a.u. occurs at the field strength $B = 8.760038 \times 10^{-2}$ a.u. for the $Z = 1$ atom. For the $Z = 2$ atom, the avoided crossing is energetically much broader, being $\Delta E = |E_{3p_0} - E_{3d_0}| = 1.4 \times 10^{-4}$ a.u. the minimal energetic spacing at $B = 0.3504$ a.u. Please note the different energy scales in Figs. 1a and 2a.

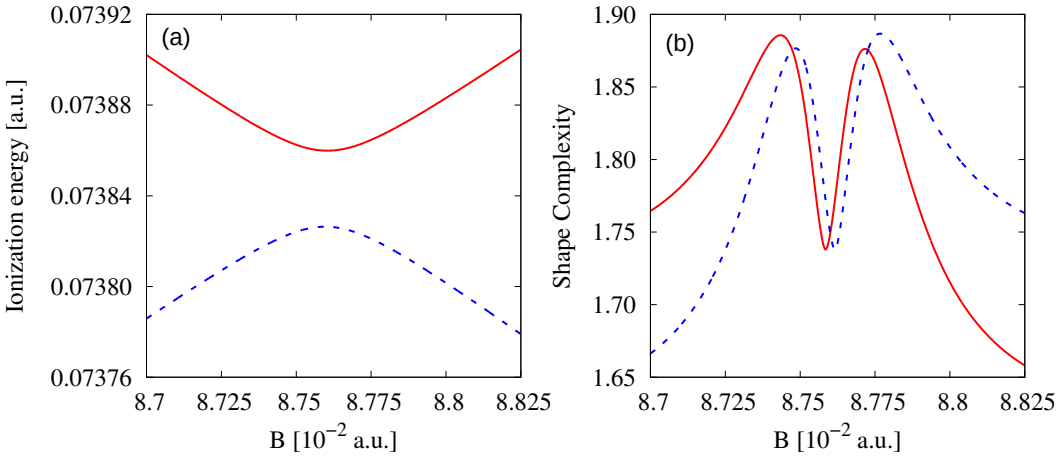

Figure 1: Color online. The ionization energies (a) and shape complexities (b) of the states $3p_0$ (dashed line) and $3d_0$ (solid line), of the hydrogen atom (so, with $Z = 1$) in parallel electric and magnetic fields as a function of the magnetic field strength, and with an electric field fixed to $F = 1.946 \times 10^{-6}$ a.u.

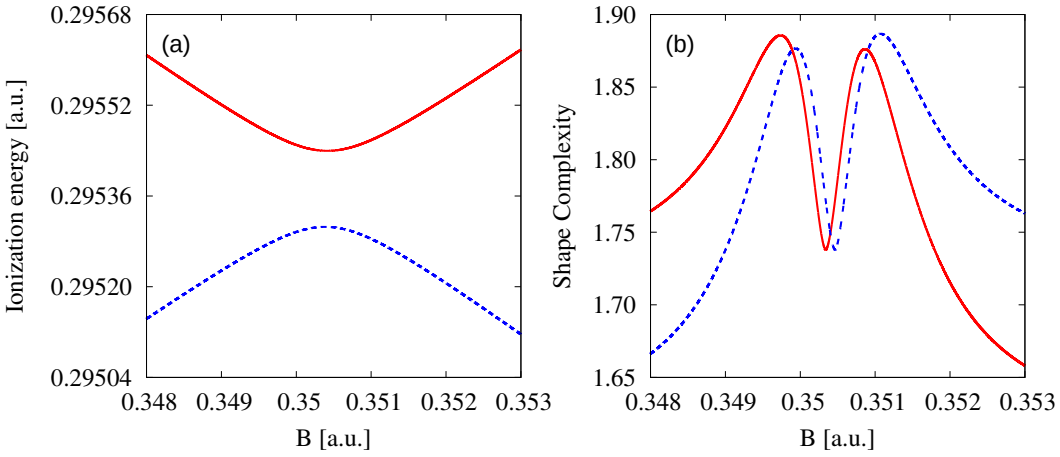

Figure 2: Color online. The same as Fig. 1, but for a hydrogenic atom with nuclear charge $Z = 2$ and $F = 1.557 \times 10^{-5}$ a.u.

The evolution of the shape complexities with the magnetic field, as can be seen from Figs. 1b and 2b, displays interesting features. They show a double-hump structure with a mirror symmetry as a function of the magnetic field strength. The computed values for the shape complexity are identical, although they are achieved at the different magnetic field strengths which are related by the scaling rules as derived above. Close to the magnetic field strength at which the minimal energetic spacing occurs, the shape complexities of both states achieve the same value, $C_{3p_0} = C_{3d_0} = 1.7492$, and this is at $B = 0.0876004$ a.u. and 0.350416 a.u. for the $Z = 1$ and 2 systems, respectively. The minimal values of the shape complexities are equal for both states, $C_{3p_0} = C_{3d_0} = 1.7380$, and are located at symmetric positions with respect to the critical magnetic field values B_c, i.e. the $3p_0$ and $3d_0$ minima are shifted to the left and to the right by 1.569×10^{-5} a.u. and 1.566×10^{-5} a.u. for the $Z = 1$ system, and by 6.3×10^{-5} a.u. and 6.2×10^{-5} a.u. for the $Z = 2$ atom, respectively. The first hump of C_{3d_0} and the second one of C_{3p_0}, also have the very similar value $C_{3d_0} = 1.8856$ and $C_{3p_0} = 1.8867$, and are shifted to the left and to the right by 1.6706×10^{-4} a.u. and 1.6632×10^{-4} a.u. for the $Z = 1$ atom, respectively, and by 6.64×10^{-4} a.u. and 6.69×10^{-4} a.u. for the $Z = 2$ system, respectively. Analogously, the second maxima of the $3p_0$ level, $C_{3p_0} = 1.8766$, and first one of the $3d_0$ state, $C_{3d_0} = 1.8762$, are identical for both systems, and are shifted for $Z = 1$ by 1.1591×10^{-4} a.u. and 1.1566×10^{-4} a.u., to the right and left, respectively; and for $Z = 2$ are shifted 4.64×10^{-4} a.u. and 4.62×10^{-4} a.u. to the right, respectively. It is interesting to remark that the presently calculated values of the shape complexity C obey the universal bound $C \geq 1$, which has been recently shown for general monodimensional [15] and D-dimensional ($D \geq 1$) probability densities [13].

Finally, let us point out here that in absence of the external fields, for the free hydrogenic atoms, C is a constant, independent of the nuclear charge Z. This is a consequence of the homogeneous character of the potential which leads to a parameter-free scaling property of the shape complexity [14]. In presence of the external fields, the shape complexity varies with the parameters of the potential which becomes inhomogeneous in character.

In conclusion, the existence of extremum points and the scaling behavior with the external fields is numerically verified for the shape complexity as given in Eq. (20). Further, according to the shape complexity analysis here described the scaling property can be used to predict the existence of avoided crossings for a heavy hydrogenic atom under strong external fields from the avoided crossings data on a lighter member and vice-versa. Similar results should be expected for the remaining composite information-measures analyzed in this work.

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