EXISTENCE OF UNIVERSAL CONNECTIONS II.*

By M. S. Narasimhan and S. Ramanan.

1. Introduction. In an earlier paper [3] we proved the existence of universal connections for connections in bundles with a compact Lie group as structure group. In this paper we extend this result to the case of an arbitrary connected Lie group (Theorem 1). The proof of this theorem does not depend on [3]. However, this result does not include Theorem 1 of [3] which is more precise in that it asserts that the canonical connections in the Stiefel bundles themselves are universal for connections in unitary or orthogonal bundles. The latter is useful in some applications.

Since any two connections on a principal bundle differ by a 1-form of the adjoint type, one can reduce the problem of finding a universal connection to one of finding a universal 1-form of the adjoint type (§ 3). Regarding the latter problem we prove the following more general result (Theorem 2): if ρ is a finite dimensional representation of a connected Lie group G and n and p are non-negative integers, then there exists a n-universal p-form of type ρ. (For the definition of forms of type ρ see § 2.) This problem is essentially one for compact Lie groups (§ 6) since the structure group of a G-bundle P can be reduced to a maximal compact subgroup K of G, and forms of type ρ on P are precisely forms on P obtained by extending forms of type $\rho | K$ (restriction of ρ to K) on the reduced bundle (§ 2). It should be remarked, however, that the existence of universal connections for a connected Lie group does not follow immediately from that for compact Lie groups, since not every connection on a G-bundle is the extension of a connection on a reduced K-bundle. (For instance, the holonomy group of a connection got by extension will have to be contained in K). In the case of connections, Theorem 2 seems to be necessary for passing from the compact to the general case. In our procedure, however, Theorem 2 implies at once Theorem 1 without passing through the compact case.

In the case when G is the orthogonal group $O(k)$ and ρ is the natural representation, an n-universal p-form is constructed explicitly (§ 4). If G is compact we may suppose that ρ is a representation of G by orthogonal matrices. This enables one to reduce the compact to the orthogonal case (§ 5).

* Received May 7, 1962.

For the notions relating to connections in principal bundles we refer to [1], [2], [5].

2. Preliminaries. In this section, we first fix our notation and terminology, and then give a canonical way of extending p-forms of a certain type to bundles obtained by extension of structure group.

By 'differentiable' we always mean 'indefinitely differentiable.' All manifolds, groups, bundles, maps and forms are assumed differentiable. We assume all our manifolds are paracompact. By a p-form on a manifold we mean a covariant tensor of degree p. If $f\colon M \to M'$ is a map and α a p-form on M', $f^*(\alpha)$ will denote the inverse image of α by f.

By a G-bundle we always mean a *principal* bundle with structure group G. If $f\colon H \to G$ is a homomorphism of groups, P_1 a H-bundle, and P_2 a G-bundle, a map $h\colon P_1 \to P_2$ will be called a f-morphism if $h(\xi s) = h(\xi)f(s)$ for every $\xi \in P_1$, $s \in H$. When $H = G$ and f is the identity map, a f-morphism will be called a G-morphism. If ρ is a representation of G in a finite dimensional vector space V, a p-form on a G-bundle with values in V is said to be of type ρ if i) it is equivariant for the action of G and ii) it annihilates any p-tuple of tangent vectors one of which is vertical [2]. If $h\colon P_1 \to P_2$ is a f-morphism and α a p-form of type ρ on P_2, then $h^*\alpha$ is clearly a p-form on P_1 of type $(\rho \circ f)$.

Let G_1, G_2 be two Lie groups and $f\colon G_1 \to G_2$ a homomorphism. Let T_f be the functor which associates to every G_1-bundle P the G_2-bundle $T_f(P)$ over the same base obtained by extension of the structure group by f. We recall that the total space of $T_f(P)$ is the orbit space of $P \times G_2$ under the action of G_1 given by $(\xi, g_2)g_1 = (\xi g_1, f(g_1)^{-1}g_2)$ for $\xi \in P$, $g_1 \in G_1$, $g_2 \in G_2$. The action of G_2 on $P \times G_2$ defined by $(\xi, g_2)g_2' = (\xi, g_2 g_2')$ for $\xi \in P$, $g_2, g_2' \in G_2$ commutes with the above action of G_1 and hence G_2 operates on $T_f(P)$ and makes of it a G_2-bundle. Moreover, if $\Phi\colon P \to P'$ is a G_1-morphism, the G_2-morphism $T_f(\Phi)\colon T_f(P) \to T_f(P')$ is induced by the map $(\xi, g_2) \to (\Phi(\xi), g_2)$ of $P \times G_2$ into $P' \times G_2$.

Let q be the projection $P \times G_2 \to T_f(P)$ and i_f the map $\xi \to q(\xi, e)$ of P into $T_f(P)$. We then have $i_f(\xi s) = i_f(\xi)f(s)$, $\xi \in P$, $s \in G_1$; i.e., i_f is a f-morphism.

Let ρ be a finite dimensional representation of G_2. If β is a p-form of type ρ on $T_f(P)$, $i_f^*\beta$ is a p-form on P of type $\rho \circ f$. Conversely, to every p-form α on P of type $\rho \circ f$ we can associate in a natural way a p-form $T_f(\alpha)$ of type ρ on $T_f(P)$ with $i_f^* T_f(\alpha) = \alpha$ in the following way. It is easy to check that the form α' on $P \times G_2$ defined by $\alpha'_{(\xi, g_2)} = \rho(g_2)^{-1}(p_1^*\alpha)_{(\xi, g_2)}$

(p_1 being the projection $P \times G_2 \to P$) is invariant under the action of G_1 and annihilates any p-tuple of tangent vectors one of which is vertical. Hence there exists a p-form $T_f(\alpha)$ of type ρ on $T_f(P)$ such that $q^*T_f(\alpha) = \alpha'$ and we have $i_f{}^*T_f(\alpha) = \alpha$. Moreover, if β is a p-form of type ρ on $T_f(P)$, we have $T_f(i_f{}^*\beta) = \beta$.

Finally, the correspondence $\alpha \to T_f(\alpha)$ is 'functorial' in the following sense: if $\Phi\colon P \to P'$ is a G_1-morphism and if α' is a p-form on P' of type $\rho \circ f$, then we have

$$T_f(\Phi^*\alpha') = (T_f\Phi)^*(T_f\alpha').$$

3. Statement of the theorems. Proof of Theorem 1.

THEOREM 1. *Let G be a connected Lie group and n a positive integer. Then there exist a principal G-bundle B and a connection form γ_0 on B such that any connection form on a principal G-bundle P with base of dimension $\leqq n$ is the inverse image of γ_0 by a G-morphism of P in B.*

We deduce Theorem 1 from the following theorem which seems to be of independent interest.

THEOREM 2. *Let G be a connected Lie group and ρ a finite dimensional representation of G. Let n and p be two non-negative integers. Then there exist a principal G-bundle E and a p-form α_0 of type ρ on E such that any p-form of type ρ on a principal G-bundle P with base of dimension $\leqq n$ is the inverse image of α_0 by a G-morphism of P into E. Moreover, the bundle E can be chosen to be classifying for dimension $\leqq n$.*

Remarks. 1. A p-form (resp. a connection) which possesses the property stated in Theorem 2 (resp. Th. 1) will be called n-universal.

2. Theorem 2 is also valid with "p-form" replaced by "exterior p-form." A universal exterior p-form is obtained by alternating a universal p-form.

Proof of Theorem 1. We now prove Theorem 1 assuming Theorem 2. Let F be a differentiable G-bundle which is n-universal, and γ_1 any connection on F. On the other hand, let E be a G-bundle and α_0 a 1-form on E of the adjoint type which is n-universal for such forms. Consider the G-bundle $B = F \times E$, the action of G on B being given by $(f, e)g = (fg, eg)$, $f \in F$, $e \in E$, $g \in G$. Let $q_1\colon B \to F$, $q_2\colon B \to E$ be the canonical projections, which are clearly G-morphisms. The differential form $\gamma_0 = q_1{}^*\gamma_1 + q_2{}^*\alpha_0$ is a connection form on B since $q_1{}^*\gamma_1$ is a connection form and $q_2{}^*\alpha_0$ is a 1-form of the adjoint type. We assert that γ_0 is n-universal for connections in G-bundles.

In fact let P be any G-bundle with base of dimension $\leqq n$ and γ any connection on P. Since F is a n-universal bundle, there exists a G-morphism $\Psi_1 : P \to F$. Then $\gamma - \Psi_1^*(\gamma_1)$ is a 1-form of the adjoint type since γ and $\Psi_1^*(\gamma_1)$ are connection forms on P. Let $\Psi_2 : P \to E$ be a G-morphism such that $\Psi_2^*(\alpha_0) = \gamma - \Psi_1^*(\gamma_1)$. Consider the G-morphism $\Phi : P \to B$ defined by $q_1 \circ \Phi = \Psi_1$; $q_2 \circ \Phi = \Psi_2$. Then we have

$$\begin{aligned}\Phi^*(\gamma_0) &= \Phi^*(q_1^*\gamma_1 + q_2^*\alpha_0)\\ &= (q_1 \circ \Phi)^*\gamma_1 + (q_2 \circ \Phi)^*\alpha_0\\ &= \Psi_1^*(\gamma_1) + \Psi_2^*(\alpha_0)\\ &= \Psi_1^*(\gamma_1) + (\gamma - \Psi_1^*(\gamma_1))\\ &= \gamma.\end{aligned}$$

Remark. In the above construction, it is clear that the bundle B is n-classifying if the G-bundles E and F are n-classifying, so that the maps induced on the bases by two G-morphisms $P \to B$ are homotopic. Thus the theorem of A. Weil on connections [1] is an immediate consequence of Theorem 1. However, Weil's theorem can be proved in a simpler way; for, all one requires for the proof is that any *two* given connections γ_1 and γ_2 on a bundle P can be obtained as the inverse images of the *same* connection γ_0 on a bundle B by morphisms whose projections on the base are homotopic. This problem is considerably simpler as can be seen by taking $B = P \times I$ and $(\gamma_0)_{(\xi,t)} = tp^*(\gamma_2) + (1-t)p^*\gamma_1$, $\xi \in P$, $t \in I$ where p is the projection $P \times I \to P$ (I is the open interval $[-2, 2]$). The inclusions $P \to P \times I$ given by $\xi \to (\xi, 0)$ and $\xi \to (\xi, 1)$ induce γ_1, γ_2 respectively and their projections to the base are clearly homotopic, the homotopy being induced by the identity mapping of $P \times I$ into itself [3, § 6].

4. The orthogonal case. In this section, we prove Theorem 2 in the case where G is the real orthogonal group $O(k)$ and ρ is the natural representation in $\boldsymbol{R}^k$. We identify $O(k)$ with the group of (k, k) real matrices A such that $A'A = I_k$ (A' being the transpose of A) and $\boldsymbol{R}^k$ with $(k, 1)$ real matrices. ρ then corresponds to left multiplication of $(k, 1)$ matrices by (k, k) orthogonal matrices.

Let $W(N, k)$, $N \geqq k$, be the Stiefel bundle of (N, k)-real matrices A such that $A'A = I_k$ ([3, § 2]). $O(k)$ acts on $W(N, k)$ by multiplication on the right. If $A \in W(N, k)$ is of the form $\begin{pmatrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_N \end{pmatrix}$ where each A_i is a $(1, k)$

matrix, the function σ_i on $W(N,k)$ which assigns to each A the $(k,1)$ matrix A_i' is of type ρ. For

$$\sigma_i(As)=\sigma_i\begin{bmatrix}A_1s\\ \vdots\\ A_Ns\end{bmatrix}=(A_is)'=s'A_i'=s^{-1}\sigma_i(A)$$

for $A\in W(N,k)$, $s\in O(k)$.

We now construct a n-universal p-form of type ρ. For the rest of this section, N will denote the integer $(n+1)n^p+(n+k)$. Let $V_1,\cdots,V_{n+1}$ be $(n+1)$ copies of $\boldsymbol{R}^n$ and $V_0=\boldsymbol{R}$. Consider the $O(k)$-bundle

$$E=W(N,k)\times V_0\times V_1\times\cdots\times V_{n+1}$$

where the action of $O(k)$ on E is given by

$$(w,v_0,\cdots,v_{n+1})g=(wg,v_0,\cdots,v_{n+1}),\ w\in W(N,k),\ v_i\in V_i,\ g\in O(k).$$

Let π (resp. π_i) denote the projection of E onto $W(N,k)$ (resp. V_i). Let further $(x_i^1,\cdots,x_i^n)$ be the coordinate functions in V_i, $i>0$. For each multi-index $I=(i_1,\cdots,i_r,\cdots,i_p)$, $1\leqq i_r\leqq n$ and $1\leqq j\leqq n+1$, we shall denote by ω_j^I the p-form $\pi_j^*(dx_j^{i_1}\otimes\cdots\otimes dx_j^{i_p})$ on E. For convenience of notation, let us choose a bijection λ of the set of indices (I,j) with $I=(i_1,\cdots,i_p)$ $1\leqq i_r\leqq n$ and $1\leqq j\leqq n+1$, onto the set of integers $[1,(n+1)n^p]$. Obviously $\tau_j^I=\sigma_{\lambda(I,j)}\circ\pi$ is a function on E with values in $(k,1)$-matrices of type ρ. π_0 being a real-valued function on E, the form $(\pi_0\cdot\tau_j^I)\omega_j^I$ is a $(k,1)$-matrix valued form which is the product of the vector valued function $\pi_0\tau_j^I$ and the real-valued form ω_j^I. The form

$$\alpha_0=\sum_{I,j}(\pi_0\tau_j^I)\omega_j^I$$

is of type ρ. For, clearly α_0 annihilates any p-tuple of vectors one of which is vertical since each ω_j^I has this property. Moreover if $X_1,\cdots,X_p$ are vectors at $\xi\in P$ and $s\in O(k)$ we have

$$\begin{aligned}\alpha_0(X_1s,\cdots,X_ps)&=\sum\pi_0(\xi s)\tau_j^I(\xi s)\omega_j^I(X_1s,\cdots,X_ps)\\&=s^{-1}\{\pi_0(\xi)\sum\tau_j^I(\xi)\omega_j^I(X_1,\cdots,X_p)\}\\&=s^{-1}\alpha_0(X_1,\cdots,X_p)\end{aligned}$$

where the X_is are the vectors at ξs which are images of X_i by the differential of the map $\xi\to\xi s$ of P into itself.

We now proceed to prove that α_0 is n-universal for p-forms of type ρ.

Proof. Let P be a $O(k)$-bundle over a base M of dimension $\leqq n$ and

$q: P \to M$ be the projection. Let (U_i) be a covering of M by relatively compact open sets such that

i) each $\bar{U}_i$ is contained in a coordinate cell

ii) the U_i's can be divided into $(n+1)$ classes $\mathcal{C}_j$ in such a way that no two U_i's of the same class intersect [4, p. 61].

Let W_i be a shrinking of this covering, i.e., an open covering W_i such that $\bar{W}_i \subset U_i$. Let $D_j\{j=1,\cdots,(n+1)\}$ be the union of the open sets $q^{-1}(W_i)$ for those i's for which U_i belongs to $\mathcal{C}_j$. Let ζ_j be a partition of unity with respect to this covering, consisting of non-negative differentiable functions ζ_j invariant under the action of G with support of $\zeta_j \subset D_j$ and $\sum \zeta_j = 1$.

Let α be a p-form of type ρ on P. It is clear that there exist functions $(f_j^1,\cdots,f_j^n)$ on M whose restrictions to each W_i, for those i's for which U_i belongs to $\mathcal{C}_j$, form a coordinate system on W_i. Since α is of type ρ, α can be expressed in D_j in the form $\sum \alpha_j^I q^*(df_j^I)$, where α_j^I are functions of type ρ on D_j and $df_j^I = df_j^{i_1}\otimes\cdots\otimes df_j^{i_p}$. Then it is easy to see that $\alpha = \sum \beta_j^I q^*(df_j^I)$, where $\beta_j^I = \zeta_j\alpha_j^I$ are now differentiable functions on P of type ρ. Let h be a strictly positive invariant differentiable function on P such that $h(\xi)^2 > 2\,\|\,\beta_j^I(\xi)\beta_j^I(\xi)'\,\|$ for every $\xi \in P$, where $\|\;\|$ denotes the norm as a linear operator. (The existence of such an h follows for instance from the fact that $\|\,\beta_j^I(\xi)\beta_j^I(\xi)'\,\|$ is an invariant function on P). We have

$$\alpha = \sum h\eta_j^I q^*(df_j^I)$$

where $\eta_j^I = \dfrac{1}{h}\,\beta_j^I$. Obviously

$$\|\,\eta_j^I(\xi)\eta_j^I(\xi)'\,\| = \|\,\frac{1}{h(\xi)^2}\sum \beta_j^I(\xi)\beta_j^I(\xi)'\,\| \leqq \tfrac{1}{2}.$$

Therefore $R(\xi) = I_k - \sum \eta_j^I(\xi)\eta_j^I(\xi)'$ is a function on P with values in positive definite matrices. Moreover, for $s \in O(k)$ and $\xi \in P$,

$$R(\xi s) = s^{-1}R(\xi)s.$$

For,

$$\begin{aligned} R(\xi s) &= I_k - \sum \eta_j^I(\xi s)\eta_j^I(\xi s)' \\ &= I_k - \sum s^{-1}\eta_j^I(\xi)\eta_j^I(\xi)'(s^{-1})' \\ &= s^{-1}(I_k - \sum \eta_j^I(\xi)\eta_j^I(\xi)')s \\ &= s^{-1}R(\xi)s. \end{aligned}$$

Let $S(\xi)$ be the differentiable positive matrix-valued function on P such that

$S(\xi)^2 = R(\xi)$. It is clear from the uniqueness of the positive square root of a positive definite matrix that $S(\xi s) = s^{-1}S(\xi)s$ for $\xi \in P$, $s \in O(k)$.

Let $\psi\colon P \to W(n+k, k)$ be a G-morphism, the existence of which is assured by the universal bundle theorem [6, § 19]. Consider the matrix

$$\psi_1(\xi) = \begin{bmatrix} \eta_1(\xi)' \\ \vdots \\ \eta_i(\xi)' \\ \vdots \\ \eta_{(n+1)n^p}(\xi)' \\ \psi(\xi)S(\xi) \end{bmatrix}$$

where $\eta_i = \eta_j^I$ with $\lambda(I, J) = i$. Each η_i' is a $(1, k)$ matrix and $\psi(\xi)S(\xi)$ is a $(n+k, k)$ matrix so that $\psi_1(\xi)$ is a $((n+1)n^p + n + k, k)$ matrix. The map $\xi \to \psi_1(\xi)$ is a map of P into $W(N, k)$. For,

$$\begin{aligned} \psi_1'(\xi)\psi_1(\xi) &= \sum_{i=1}^{(n+1)n^p} \eta_i(\xi)\eta_i(\xi)' + S(\xi)'\psi(\xi)'\psi(\xi)S(\xi) \\ &= \sum_{(I,j)} \eta_j^I(\xi)\,(\eta_j^I(\xi))' + S(\xi)^2 \\ &= I_k - R(\xi) + S(\xi)^2. \end{aligned}$$

Hence $\psi_1'(\xi)\psi_1(\xi) = I_k$.

Moreover $\psi_1\colon P \to W(N, k)$ is a G-morphism. In fact,

$$\psi_1(\xi s) = \begin{bmatrix} \eta_1^I(\xi s)' \\ \vdots \\ \eta_i^I(\xi s)' \\ \vdots \\ \eta_{(n+1)n^p}{}^I(\xi s)' \\ \psi(\xi s)S(\xi s) \end{bmatrix} = \begin{bmatrix} (s^{-1}\eta_1(\xi))' \\ \vdots \\ (s^{-1}\eta_{(n+1)n^p}(\xi))' \\ \psi(\xi)s \cdot s^{-1}S(\xi)s \end{bmatrix} = \begin{bmatrix} \eta_1^I(\xi s)' \\ \vdots \\ \eta_{(n+1)n^p}{}^I(\xi)' \\ \psi(\xi)S(\xi) \end{bmatrix} s$$

$$= \psi_1(\xi)s.$$

Finally, we construct a G-morphism Φ of P into E such that $\Phi^*\alpha_0 = \alpha$ (E and α_0 are the bundle and p-form constructed in the beginning of this section). $\Phi\colon P \to E$ is defined by $\pi_0 \circ \Phi = h$, $\pi_j \circ \Phi = (f_j^1 \circ q, \cdots, f_j^n \circ q)'$, $(j = 1, \cdots, n+1)$ and $\pi \circ \Phi = \psi_1$. Φ is a G-morphism since ψ_1 is so and h, $f_j^i \circ \pi$ are invariant functions. We then have

$$\begin{aligned} \Phi^*(\alpha_0) &= \Phi^*(\textstyle\sum \pi_0 \tau_j^I \omega_j^I) \\ &= \textstyle\sum (\pi_0 \circ \Phi)(\tau_j^I \circ \Phi)\Phi^*(\omega_j^I) \end{aligned}$$

$$= \sum h(\sigma_{\lambda(I,j)} \circ \pi \circ \Phi)\Phi^*(\omega_j{}^I)$$
$$= \sum h\eta_{\lambda(I,j)}\Phi^*\pi_j{}^*(dx_j{}^{i_1}\otimes\cdots\otimes dx_j{}^{i_p})$$

where $I=(i_1,\cdots,i_p)$ with $1\leqq i_p\leqq n$. Hence

$$\Phi^*(\alpha_0)=\sum_{(I,j)} h\eta_j{}^I d(f_j{}^{i_1}\circ q)\otimes\cdots\otimes d(f_j{}^{i_p}\circ q)=\alpha.$$

5. The case of a compact group. In this section we prove Theorem 2 with G compact and ρ any k-dimensional representation of G. Since every representation of G is equivalent to an orthogonal representation, we may assume that $\rho=j\circ f$ where f is a homomorphism $G\to O(k)$ and j is the natural representation of $O(k)$ in $\boldsymbol{R}^k$. Let E_1 be a $O(k)$-bundle together with an n-universal p-form of type j (§ 4). Let F be a n-universal G-bundle. We let G act on $F\times E_1$ by $(v,e_1)g=(vg,e_1f(g))$, $v\in F$, $e_1\in E_1$, $g\in G$. This makes of $F\times E_1$ a G-bundle E. Let q_1 and q_2 be the projections of $F\times E_1$ onto F and E_1 respectively. The p-form $\alpha_0=q_2{}^*\alpha_1$ is of type ρ since $q_2\colon E\to E_1$ is a f-morphism and α_1 of type j. We now prove that α_0 is a n-universal p-form of type ρ. In fact, let P be a G-bundle over a base of dimension $\leqq n$ and α a p-form of type ρ on P. Then there exists a $O(k)$-morphism Φ_2 of $T_f(P)$ into E_1 such that $\Phi_2{}^*(\alpha_1)=T_f(\alpha)$. (For the definition of T_f see § 2). On the other hand, P admits a G-morphism Φ_1 into F, since F is n-universal. Let $\Phi\colon P\to E$ be the map defined by $q_1\circ\Phi=\Phi_1$, $q_2\circ\Phi=\Phi_2\circ i_f$ (i_f is the canonical map $P\to T_f(P)$, see § 2). Φ is a G-morphism. For, if $\xi\in P$, $s\in G$, we have

$$\begin{aligned}\Phi(\xi s) &= (\Phi_1(\xi s),\Phi_2\circ i_f(\xi s))\\ &= (\Phi_1(\xi)s,\Phi_2(i_f(\xi)f(s)))\\ &= \Phi(\xi)s.\end{aligned}$$

Now

$$\begin{aligned}\Phi^*(\alpha_0)=\Phi^*(q_2{}^*\alpha_1) &= (q_2\circ\Phi)^*\alpha_1\\ &= (\Phi_2\circ i_f)^*\alpha_1=i_f{}^*\Phi_2{}^*\alpha_1=i_f{}^*T_f\alpha=\alpha.\end{aligned}$$

This completes the proof in the compact case.

6. Proof of Theorem 2. The general case. Let G be a connected Lie group and K a maximal compact subgroup of G. We denote by f the inclusion map $K\to G$. We seek to construct a n-universal p-form of type ρ, where ρ is any finite dimensional representation of G. From § 5, there exists a n-universal p-form α_1 of type $(\rho\circ f)$ on a K-bundle E_1. Then the p-form $\alpha_0=T_f(\alpha_1)$ on the G-bundle $E=T_f(E_1)$ is n-universal for p-forms of type ρ. In fact, let P be a G-bundle over a base of dimension $\leqq n$ and α a p-form of type ρ on P. It is well known that there exists a K-bundle P_1 such that $T_f(P_1)$ is G-isomorphic to P (reduction of structure group to K, see [6. § 12]). We identify P and $T_f(P_1)$ by such an isomorphism. Consider the form $i_f^*(\alpha)$ on P_1 which is of type $\rho \circ f$. Now let $\Phi_1 \colon P_1 \rightarrow E_1$ be a K-morphism such that $\Phi_1^*(\alpha_1) = i_f^*(\alpha)$. Consider the G-morphism $\Phi = T_f(\Phi_1)$ of P into E. Then we have

$$\begin{aligned}\Phi^*(\alpha) &= (T_f(\Phi_1))^*\alpha_0 = (T_f(\Phi_1))^*T_f(\alpha_1)\\ &= T_f(\Phi_1{}^*\alpha_1) = T_f(i_f{}^*\alpha)\\ &= \alpha.\end{aligned}$$

It is clear, referring to §§ 4, 5, 6, that the bundle E can be chosen to be n-classifying. This completes the proof of Theorem 2.

Remark. Theorems 1 and 2 hold even when G is a Lie group with a finite number of connected components; our proofs continue to be valid in this case.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH, BOMBAY.

REFERENCES.

[1] S. S. Chern, *Topics in differential geometry*, Mimeographed notes, the Institute for Advanced Study, Princeton, 1951.

[2] J. L. Koszul, *Lectures on fibre bundles and differential geometry*, The Tata Institute of Fundamental Research, Bombay, 1960.

[3] M. S. Narasimhan and S. Ramanan, "Existence of universal connections," *American Journal of Mathematics*, vol. 83 (1961), pp. 563-572.

[4] J. Nash, "The imbedding problem for Riemannian manifolds," *Annals of Mathematics*, vol. 63 (1956), pp. 20-63.

[5] K. Nomizu, *Lie groups and differential geometry*, Publications of Mathematical Society of Japan, 1956.

[6] N. Steenrod, *The Topology of fibre bundles*, Princeton, 1951.