

BPS States on a Three Brane Probe

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Abstract

Recently Banks, Douglas and Seiberg have shown that the world volume theory of a three brane of the type IIB theory in the presence of a configuration of four Dirichlet seven branes and an orientifold plane is described by an N=2 supersymmetric SU(2) gauge theory with four quark flavours in 3+1 dimensions. In this note we show how the BPS mass formula for N=2 supersymmetric gauge theory arises from masses of open strings stretched between the three brane and the seven brane along appropriate geodesics.

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Recent work by Banks, Douglas and Seiberg has shown that the world-volume theory of a three brane, moving in the background of an orientifold seven plane and a set of four Dirichlet seven branes is described by an N=2 supersymmetric SU(2) gauge theory in 3+1 dimensions with four hypermultiplets in the fundamental representation of SU(2)[1]. There is a dual description of this background[2] known as F-theory[3], where the three brane moves in the background of six seven branes. In the weak coupling limit, four of these seven branes can be regarded as conventional D-branes, and the other two are related to it by SL(2,Z) S-duality transformations of the type IIB theory. In both descriptions the moduli space is spanned by the complex coordinate of the three brane in the plane transverse to the seven-branes. The first description represents the semiclassical limit of the moduli space of N=2 supersymmetric SU(2) gauge theory with four quark flavours, with the orientifold plane representing the point where SU(2) gauge symmetry is restored semiclassically, and the locations of the four Dirichlet seven branes representing the points where the hypermultiplets become massless. On the other hand the F-theory description gives the full quantum corrected moduli space of the N=2 supersymmetric SU(2) gauge theory[4], with the six seven branes representing the locations in the moduli space where one of the four elementary hypermultiplets, the monopole or the dyon becomes massless.

Given the fact that the three brane world volume theory is an N=2 supersymmetric SU(2) gauge theory with four hypermultiplets, one can ask: what are the analogs of the BPS states on the three brane world volume? Qualitatively, the answer to this question is simple, – these are just the open string states stretched between the three brane and a seven brane.³⁴ To be more specific, let us define a (p, q) string as the bound state of p elementary type IIB strings and q D-strings[6]. We shall also define a (p, q) seven-brane to be the seven-brane on which a (p, q) string can end[7]. Thus, for example an $(1,0)$ string is just an elementary type IIB string, and an $(1,0)$ seven brane is just the Dirichlet seven brane of type IIB theory. More general (p, q) strings or seven-branes are obtained from $(1,0)$ strings / seven-branes via S-duality transformation. With this convention the F-theory background under consideration in the weak coupling limit can be taken to consist of four $(1,0)$ seven branes, one $(0,1)$ seven brane and one $(2,1)$ seven brane. Note

³There are also BPS states corresponding to open strings starting and ending on the three brane. These correspond to massive gauge bosons and states that are related to them via SL(2,Z) transformation, and can be treated in a similar manner, but we shall not discuss them further in this paper.

⁴The possibility of representing these BPS states as stretched (self-dual) strings has also been discussed in ref.[5]; however there the string stretches along the Riemann surface associated with the Seiberg-Witten curve instead of along the moduli space.

however that there is non-trivial $SL(2,Z)$ monodromy as we go around the various seven branes, and so when ‘viewing’ a seven brane from a fixed point (say at infinity) we can find complicated paths, along which a $(1,0)$ (or $(0,1)$ or $(2,1)$) seven-brane may appear as a (p,q) seven brane.

While a particular type of seven-brane can only have a particular kind of string ending on it, the three brane in the type IIB theory can have any (p,q) string ending on it. This is related to the fact that the three brane is invariant under an $SL(2,Z)$ transformation. Thus we can have a state represented by a (p,q) string starting on the three brane and ending on a (p,q) seven brane.⁵ From the three brane world-volume point of view this will correspond to a state carrying p units of electric charge and q units of magnetic charge[8]. We shall refer to this state as a (p,q) state. As the three brane approaches the location of a (p,q) seven-brane, the open string stretched between the three brane and the seven brane becomes massless. Thus the location of a (p,q) 7-brane can be interpreted as the point in the moduli space where a (p,q) state becomes massless. Note however that due to non-trivial $SL(2,Z)$ monodromy in the moduli space, the values of (p,q) associated with such a point depend on the path in the moduli space that we take to approach such a point.

We shall be interested in deriving the mass of a (p,q) state at a generic point in the moduli space when the three brane is away from the location of the (p,q) seven-brane. According to ref.[4] it is given by

$$m_{p,q}^{SW} = |pa(z) + qa_D(z) + \sum_i m_i S_i|, \quad (1)$$

where a and a_D are known functions of the gauge invariant coordinate z (called u in [4]) labelling the moduli space of the $3 + 1$ dimensional gauge theory; m_i are the bare masses of the hypermultiplets, and S_i are the global $U(1)$ charges carried by the state. In the present context z can be identified to the complex coordinate of the three brane in the plane transverse to the seven-branes and m_i^2 can be identified as the positions of the four Dirichlet seven branes in the orientifold description[2, 1]. We shall use a slightly different form of eq.(1). Let us focus on a (p,q) state carrying a given set of global $U(1)$ charges. If z_0 denotes the point on the moduli space where the particular (p,q) state

⁵In principle for any pair (p,q) relatively prime, we can find paths along which a given seven brane will appear as a (p,q) seven brane to the three brane, and the (p,q) open string can stretch along such a path. However, in general not all of them will represent stable BPS states.

under consideration becomes massless, then $m_{p,q}^{SW}$ must vanish at z_0 . This gives,

$$pa(z_0) + qa_D(z_0) + \sum_i m_i S_i = 0. \quad (2)$$

Hence eq.(1) can be rewritten as

$$m_{p,q}^{SW} = |pa(z) + qa_D(z) - pa(z_0) - qa_D(z_0)|. \quad (3)$$

We shall show that this expression can be reinterpreted as the mass of a (p, q) string stretched between the three brane and the (p, q) seven brane. To this end note that the string tension of a (p, q) string (measured in the canonical metric) is given by[9]

$$T_{p,q} = \frac{1}{\sqrt{\lambda_2}} |p + q\lambda|, \quad (4)$$

where

$$\lambda = a + ie^{-\Phi/2} \equiv \lambda_1 + i\lambda_2, \quad (5)$$

a being the massless scalar arising from the Ramond-Ramond sector of the type IIB theory and Φ being the dilaton field. Thus the mass of a (p, q) string state stretched along a curve C will be given by

$$\int_C T_{p,q} ds, \quad (6)$$

where ds is the line element along the curve C measured in the canonical metric.

Let us now apply this formula to the specific F -theory background that we have been considering. We shall follow the notation of ref.[2]. The F -theory background is given by[2]

$$\begin{aligned} \lambda(z) &= \tau(z) \equiv \tau_1 + i\tau_2, \\ ds^2 &= \tau_2 \eta(\tau)^2 \bar{\eta}(\bar{\tau})^2 \prod_{i=1}^6 (z - z_i)^{-1/12} (\bar{z} - \bar{z}_i)^{-1/12} dz d\bar{z}, \end{aligned} \quad (7)$$

where τ is determined from the equation

$$j(\tau) = \frac{4 \cdot (24f)^3}{4f^3 + 27g^2}, \quad (8)$$

f and g being arbitrary polynomials in z of degree 2 and 3 respectively. z_i represent the six zeroes of

$$\Delta = 4f^3 + 27g^2 \quad (9)$$

where $\tau \rightarrow i\infty$ up to an $SL(2,Z)$ transformation. Here $\eta(\tau)$ represents the Dedekind eta function, and $j(\tau)$ is the modular function of τ with a single pole at $\tau = i\infty$ and zero at $\tau = e^{i\pi/3}$.

Using eqs.(4), (6) and (7) we see that the mass of a (p, q) string stretched between a (p, q) seven-brane located at z_0 and a three brane located at z along a curve C is given by

$$m_{p,q} = \int_C |\eta(\tau)^2 \prod_{i=1}^6 (z - z_i)^{-1/12} (p + q\tau) dz|. \quad (10)$$

Introducing a new coordinate $w_{p,q}$ through the relation

$$dw_{p,q} = \eta(\tau)^2 \prod_{i=1}^6 (z - z_i)^{-1/12} (p + q\tau) dz, \quad (11)$$

we can rewrite eq.(10) as

$$m_{p,q} = \int_C |dw_{p,q}|. \quad (12)$$

In order to get a BPS state, we need to choose C so as to minimize the mass of the open string stretched between z_0 and z . By looking at eq.(12) it is clear that in the $w_{p,q}$ coordinate system this corresponds to the straight line from the point $w_{p,q}(z_0)$ to $w_{p,q}(z)$. Thus the mass of the BPS state represented by the (p, q) open string stretched between the point z_0 and z is given by

$$m_{p,q}^{BPS} = |w_{p,q}(z) - w_{p,q}(z_0)|. \quad (13)$$

We now want to compare this with eq.(3). To do this we use the equation (which we shall prove later)

$$da = \eta(\tau)^2 \prod_{i=1}^6 (z - z_i)^{-1/12} dz. \quad (14)$$

In that case, using eq.(11) and the fact that[4]

$$da_D = \tau da, \quad (15)$$

we get

$$dw_{p,q} = pda + qda_D. \quad (16)$$

Eq.(13) can now be rewritten as

$$m_{p,q}^{BPS} = |pa(z) + qa_D(z) - pa(z_0) - qa_D(z_0)|. \quad (17)$$

This agrees with the BPS formula (3) for N=2 supersymmetric SU(2) gauge theory derived by Seiberg and Witten.

Thus it remains to prove eq.(14). Let us define

$$F(z) = \frac{da}{dz} \eta(\tau)^{-2} \prod_{i=1}^6 (z - z_i)^{1/12}. \quad (18)$$

We shall show that $F(z)$ has no singularity in the z plane and asymptotically goes to a constant (which can be set to unity by a rescaling of z). This would prove that $F(z) = 1$ and hence establish eq.(14). The possible singularities of F can arise at one of the z_i 's. Let us focus on one of them, say z_1 , and let us suppose it denotes the location of an (m, n) seven brane in the z plane. Then near z_1 we get[4],

$$\begin{aligned} ma + na_D &\simeq c(z - z_1), \\ ra + sa_D &\simeq \frac{c}{2\pi i}(z - z_1) \ln(z - z_1) + \text{constant}, \\ \frac{s\tau + r}{n\tau + m} &\simeq \frac{1}{2\pi i} \ln(z - z_1), \end{aligned} \quad (19)$$

where $\begin{pmatrix} m & n \\ r & s \end{pmatrix}$ is an SL(2,Z) matrix and c is a constant. This gives

$$a \simeq -\frac{c}{2\pi i} n(z - z_1) \ln(z - z_1) + \text{constant}, \quad (20)$$

so that

$$\frac{da}{dz} \simeq -\frac{c}{2\pi i} n \ln(z - z_1). \quad (21)$$

We also get,

$$\eta^2(\tau) \simeq (z - z_1)^{1/12} \frac{n}{2\pi i} \ln(z - z_1) \quad (22)$$

up to a phase. Substituting this asymptotic behaviour of (da/dz) and η in the right hand side of eq.(18) we see that $F(z)$ is non-singular at $z = z_1$. Similarly $F(z)$ can be shown to be non-singular near the other z_i 's as well. Finally, as $z \rightarrow \infty$, $a \propto \sqrt{z}$ [4], and τ approaches a constant; hence $F(z)$ approaches a constant. This shows that $F(z)$ must be a constant independent of z , thereby proving the desired result.

To summarize, we have derived the mass formula for the BPS states in N=2 supersymmetric gauge theory by examining the masses of open strings stretched between the three brane and the seven brane for the configuration studied in ref.[1]. This result is likely to be relevant in the study of a three brane dynamics in the presence of more complicated

seven brane configurations, notably those which correspond to the presence of more than one orientifold plane. Representing BPS states as stretched open strings should also be useful in determining the full spectrum of BPS states at a given point in the moduli space, a question that has already been addressed by other methods before [4, 10, 5].

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