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Spin Field Correlators on An Arbitrary Genus Riemann Surface and Non-Renormalization Theorems in String Theories*

JOSEPH J. ATICK

Institute of Theoretical Physics, Department of Physics
Stanford University, Stanford, California 94305

and

ASHOKE SEN

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

We calculate the n-point correlation functions of spin fields on an arbitrary genus Riemann surface. We also calculate the corresponding correlators for the spin fields associated with the local supersymmetry ghosts using a specific ansatz for screening the background ghost charge. Using these results we show by explicit calculation that to all orders in the string perturbation theory all n point amplitudes involving massless fields in the superstring and the heterotic string theory vanish identically for $0 \leq n \leq 3$.

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String perturbation theory a la Polyakov [1] entails calculating correlation functions of vertex operators on Riemann surfaces of successively increasing genera. In a previous paper [2] we have calculated correlation functions of spin operators on a torus. These are relevant in string theory for calculation of one loop amplitudes involving fermion emission vertices. Using these results we demonstrated that one loop two and four fermion amplitudes in superstring theory vanish when all external momenta are set to zero. In this paper we extend our analysis to higher genus Riemann surfaces. In particular we calculate n-point correlation functions of spin fields on arbitrary genus Riemann surfaces. For the ghost system there are some new subtleties that one encounters at genus two and higher. These have to do with the now nonvanishing ghost background charge and with the presence of the supermoduli [3,4]. We propose a way for handling these subtleties which makes the analysis relatively simple. Within this framework it is shown that all n-point amplitudes in superstring theory and the heterotic string theory vanish for $0 \leq n \leq 3$, for arbitrary momenta of the external particles and to any order in perturbation theory. Thus our analysis provides an explicit verification of the non-renormalization theorems [5].

We start by analysing a system of one complex Weyl fermion ψ on a Riemann surface of genus g. Let us denote by $S^{\pm}(z)$ the spin fields associated with this system. The fields ψ, $\bar{\psi}$, $S^{\pm}$ obey the following operator product expansions:

$$\begin{aligned}
&\bar{\psi}(z)\,S^{+}(w) \sim (z-w)^{-\frac{1}{2}}\,S^{-}(w),\\
&\bar{\psi}(z)\,S^{-}(w) \sim (z-w)^{\frac{1}{2}}\,\hat{S}^{-}(w),\\
&\psi(z)\,S^{+}(w) \sim (z-w)^{\frac{1}{2}}\,\hat{S}^{+}(w),\\
&\psi(z)\,S^{-}(w) \sim (z-w)^{-\frac{1}{2}}\,S^{+}(w),\\
&\bar{\psi}(z)\,\bar{\psi}(w) \sim (z-w),\\
&\psi(z)\,\psi(w) \sim (z-w),\\
&\bar{\psi}(z)\,\psi(w) \sim (z-w)^{-1},\\
&S^{+}(z)\,S^{-}(w) \sim (z-w)^{-\frac{1}{4}},
\end{aligned} \tag{1}$$

where $\hat{S}^{\pm}$ are excited spin fields of conformal dimension $\frac{3}{2}$. These operator products may be realized explicitly by using bosonization [3]. The global issues involved in bosonization on higher genus surfaces are, however, more subtle, and at no stage of our analysis we shall use bosonization.

We shall be interested in calculating a correlation function of the form,

$$F(y_i, z_i, u_i, v_i) \equiv \left\langle \prod_{i=1}^{N_1} S^+(y_i) \prod_{i=1}^{N_2} S^-(z_i) \prod_{i=1}^{N_3} \bar{\psi}(u_i) \prod_{i=1}^{N_4} \psi(v_i) \right\rangle , \tag{2}$$

on a genus g Riemann surface. $\frac{1}{2}(N_1 - N_2) + (N_4 - N_3)$ must vanish in order to conserve the total fermionic charge. In order to calculate this we start from another Green's function:

$$G(y, z, y_i, z_i, u_i, v_i) = \frac{\langle \bar{\psi}(y)\psi(z) \prod_{i=1}^{N_1} S^+(y_i) \prod_{i=1}^{N_2} S^-(z_i) \prod_{i=1}^{N_3} \bar{\psi}(u_i) \prod_{i=1}^{N_4} \psi(v_i)\rangle}{\langle \prod_{i=1}^{N_1} S^+(y_i) \prod_{i=1}^{N_2} S^-(z_i) \prod_{i=1}^{N_3} \bar{\psi}(u_i) \prod_{i=1}^{N_4} \psi(v_i)\rangle} . \tag{3}$$

An expression for G may be written down by examining its singularities and periodicities as a function of z and w using the operator products in Eqn.(1). The result is*

$$\begin{aligned} &G_{\vec{a}\vec{b}}(y, z; y_i, z_i, u_i, v_i) \\ &= \left(\frac{\prod_i \vartheta \left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right] (\int_{y_i}^{z} \vec{\omega}) \prod_i \vartheta \left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right] (\int_{z_i}^{y} \vec{\omega})}{\prod_i \vartheta \left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right] (\int_{y_i}^{y} \vec{\omega}) \prod_i \vartheta \left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right] (\int_{z_i}^{z} \vec{\omega})} \right)^{\frac{1}{2}} \left(\frac{\prod_i \vartheta \left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right] (\int_{v_i}^{z} \vec{\omega}) \prod_i \vartheta \left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right] (\int_{u_i}^{y} \vec{\omega})}{\prod_i \vartheta \left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right] (\int_{v_i}^{y} \vec{\omega}) \prod_i \vartheta \left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right] (\int_{u_i}^{z} \vec{\omega})} \right) \\ &\frac{\vartheta \left[\begin{smallmatrix}\vec{a}_0+\vec{a}\\ \vec{b}_0+\vec{b}\end{smallmatrix}\right] (\int_{y}^{z} \vec{\omega} + \frac{1}{2}\sum_i \int_{P}^{y_i} \vec{\omega} - \frac{1}{2}\sum_i \int_{P}^{z_i} \vec{\omega} + \sum_i \int_{P}^{v_i} \vec{\omega} - \sum_i \int_{P}^{u_i} \vec{\omega})}{\vartheta \left[\begin{smallmatrix}\vec{a}_0+\vec{a}\\ \vec{b}_0+\vec{b}\end{smallmatrix}\right] (\frac{1}{2}\sum_i \int_{P}^{y_i} \vec{\omega} - \frac{1}{2}\sum_i \int_{P}^{z_i} \vec{\omega} + \sum_i \int_{P}^{v_i} \vec{\omega} - \sum_i \int_{P}^{u_i} \vec{\omega})} \frac{h(z)h(y)}{\vartheta \left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right] (\int_{y}^{z} \vec{\omega})} . \end{aligned} \tag{4}$$

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* This generalizes the result obtained by Sonoda [6] in the absence of spin fields.

Here $(\vec{a}, \vec{b})$ are each g dimensional real vectors with entries zero or half characterizing the spin structure of the fermion fields on the genus g Riemann surface. $(\vec{a}_0, \vec{b}_0)$ denote some reference spin structure which is chosen to be odd.[†] $\vartheta\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}$ denotes generalized ϑ-function as defined in Mumford [7]. $\vec{\omega}$ is a g dimensional complex vector whose entries are g linearly independent holomorphic one forms of the genus g Riemann surface. P is an arbitrary point on the Riemann surface which has been introduced only for convenience, the argument of $\vartheta\begin{bmatrix}\vec{a}_0+\vec{a}\\ \vec{b}_0+\vec{b}\end{bmatrix}$ does not depend on P. Finally $h(y)$ is the holomorphic half differential associated with the reference spin structure $(\vec{a}_0, \vec{b}_0)$, satisfying [7]

$$\left(h(y)\right)^2 = \vec{\omega}(y).\frac{\partial}{\partial \vec{e}}\,\vartheta\begin{bmatrix}\vec{a}_0\\ \vec{b}_0\end{bmatrix}(\vec{e})\,|_{\vec{e}=0}\,. \tag{5}$$

$\vartheta\begin{bmatrix}\vec{a}_0\\ \vec{b}_0\end{bmatrix}(\int_z^y \vec{\omega})$ has a zero at $z = y$ as well as at $(g-1)$ other points independent of y. We shall denote these points by $R_1, ..., R_{g-1}$. But $h(z)$ has $(g-1)$ simple zeros at the same points [7]. Using these properties it is not difficult to check that expression (4) as a function of y and z has all the correct periodicities and the correct singularities dictated by (1) with no additional spurious singularities. Now, from Eqn (4) we may derive a differential equation for F as follows. Let us define the stress tensor as:

$$T(z) = \lim_{y\to z}\left\{\frac{1}{2}\,\left[\partial_y\bar{\psi}(y)\psi(z) - \bar{\psi}(y)\partial_z\psi(z)\right] + \frac{1}{(z-w)^2}\right\}\,. \tag{6}$$

Given a primary field $\phi(z)$ of conformal dimension h, $T(z)$ satisfies the following operator product expansion with $\phi(z)$,

$$T(z)\,\phi(w) \sim \frac{h}{(z-w)^2}\,\phi(w) + \frac{1}{z-w}\,\partial_w \phi(w). \tag{7}$$

Let us define

$$\begin{aligned} H_{\vec{a}\vec{b}}(z; y_i, z_i, u_i, v_i) \;=\; \lim_{y\to z} \Bigg\{ \frac{1}{2} \Big[\partial_y G_{\vec{a}\vec{b}}(y, z; y_i, z_i, u_i, v_i) \\ - \partial_z G_{\vec{a}\vec{b}}(y, z; y_i, z_i, u_i, v_i) \Big] + \frac{1}{(z-w)^2} \Bigg\}, \end{aligned} \tag{8}$$

which is the expectation value of the stress tensor in the presence of the fermion fields and their spin operators. If we now consider the limit $z \to y_i$, the singular part of $H_{\vec{a},\vec{b}}$ may be identified with

$$\frac{1}{8}\frac{1}{(z-y_i)^2} + \frac{1}{z-y_i}\frac{\frac{\partial}{\partial y_i}F(y_i, z_i, u_i, v_i)}{F(y_i, z_i, u_i, v_i)}, \tag{9}$$

where $\frac{1}{8}$ is the conformal dimension of $S^+(y_i)$. Thus we may first calculate $H_{\vec{a}\vec{b}}$ using Eqs. (4) and (8), and then take the limit $z \to y_i$ to get a first order differential equation for F in the variable y_i. Studying other limits ($e.g.$, $z \to z_i$, u_i or v_i in H) furnishes differential equations for F in all other variables. The details of the analysis are very similar to the 1-loop case [2]. As in there, it turns out to be straightforward to integrate the set of first order differential equations for F. Here we only give the solution:

$$
\begin{aligned}
F_{\vec{a}\vec{b}}\,(y_i, z_i, u_i, v_i) = K_{\vec{a}\vec{b}} & \left\{ \prod_{i<j} \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \left(\int_{z_i}^{z_j} \vec{\omega} \right) \right\}^{\frac{1}{4}} \left\{ \prod_{i<j} \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \left(\int_{y_i}^{y_j} \vec{\omega} \right) \right\}^{\frac{1}{4}} \\
& \left\{ \prod_{i<j} \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \left(\int_{u_i}^{u_j} \vec{\omega} \right) \right\} \left\{ \prod_{i<j} \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \left(\int_{v_i}^{v_j} \vec{\omega} \right) \right\} \left\{ \prod_{i,j} \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \left(\int_{y_i}^{z_j} \vec{\omega} \right) \right\}^{-\frac{1}{4}} \\
& \left\{ \prod_{i,j} \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \left(\int_{y_i}^{u_j} \vec{\omega} \right) \right\}^{-\frac{1}{2}} \left\{ \prod_{i,j} \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \left(\int_{y_i}^{v_j} \vec{\omega} \right) \right\}^{\frac{1}{2}} \left\{ \prod_{i,j} \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \left(\int_{z_i}^{u_j} \vec{\omega} \right) \right\}^{\frac{1}{2}} \\
& \left\{ \prod_{i,j} \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \left(\int_{z_i}^{v_j} \vec{\omega} \right) \right\}^{-\frac{1}{2}} \left\{ \prod_{i,j} \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \left(\int_{u_i}^{v_j} \vec{\omega} \right) \right\}^{-1} \\
& \vartheta \begin{bmatrix} \vec{a}_0 + \vec{a} \\ \vec{b}_0 + \vec{b} \end{bmatrix} \left(\frac{1}{2} \sum_i \int_P^{y_i} \vec{\omega} - \frac{1}{2} \sum_i \int_P^{z_i} \vec{\omega} - \sum_i \int_P^{u_i} \vec{\omega} + \sum_i \int_P^{v_i} \vec{\omega} \right) \\
& \left[\prod_i h(y_i) \right]^{\frac{1}{4}} \left[\prod_i h(z_i) \right]^{\frac{1}{4}} \left[\prod_i h(u_i) \right] \left[\prod_i h(v_i) \right] ,
\end{aligned}
\tag{10}
$$

where $K_{\vec{a},\vec{b}}$ is a normalization factor to be determined later. Note that since the argument of $\vartheta\left[\begin{smallmatrix} \vec{a}_0+\vec{a} \\ \vec{b}_0+\vec{b} \end{smallmatrix}\right]$ contains $(1/2)\int_P^{y_i} \vec{\omega}$ and $(1/2)\int_P^{z_i} \vec{\omega}$, it changes to a ϑ-function with a different characteristic as we translate y_i or z_i once along any of the canonical homology cycles. As we shall show later, this will help us determine the relative phases and normalizations of the contributions from different spin structures.

Next we turn to the superconformal ghost system β, γ with the stress tensor [3]:

$$
T_g(z) = \lim_{z \to w} \left[-\frac{3}{2}\, \beta(z)\, \partial_w \gamma(w) - \frac{1}{2}\, \partial_z \beta(z)\, \gamma(w) - \frac{1}{(z-w)^2} \right] \tag{11}
$$

We can now introduce the spin fields $S_g^{\pm}$ for this system. The short distance behaviour of various operator product expansions may be understood by representing the various ghost fields as

$$\begin{aligned} &\gamma(z) \sim e^{+\phi(z)}\,\eta(z), \quad \beta(z) \sim e^{-\phi(z)}\,\partial\xi(z) \\ &S_g^{\pm}(z) \sim e^{\pm\frac{1}{2}\phi(z)} \end{aligned} \tag{12}$$

where ϕ is a scalar field and η, ξ are two fermionic fields [3]. The relevant operator product expansions are

$$\begin{aligned} &\beta(z)\,\gamma(w) \sim (z-w)^{-1} \\ &\beta(z)\,S_g^{+}(w) \sim (z-w)^{\frac{1}{2}}\,e^{-\frac{1}{2}\phi(w)}\,\partial\,\xi(w) \\ &\beta(z)\,S_g^{-}(w) \sim (z-w)^{-\frac{1}{2}}\,e^{-\frac{3}{2}\phi(w)}\,\partial\xi(w) \\ &\gamma(z)\,S_g^{+}(w) \sim (z-w)^{-\frac{1}{2}}\,e^{+\frac{3}{2}\phi(w)}\,\eta(w) \\ &\gamma(z)\,S_g^{-}(w) \sim (z-w)^{\frac{1}{2}}\,e^{+\frac{1}{2}\phi(w)}\,\eta(w) \end{aligned} \tag{13}$$

There are some subtleties in the calculation of the correlation functions involving the supersymmetry ghost fields due to the presence of the $2(g-1)$ ghost zero modes on a surface of genus $g \geq 2$. An associated problem is the integration over the supermoduli [3,4].* In general the ghost zero modes may be removed by insertion of operators proportional to $e^{\phi(P_i)}$ at $2(g-1)$ points $P_1, ..., P_{2g-2}$.† Following reference [10] we choose a basis for the holomorphic $\frac{3}{2}$ differentials $h_i^{z\theta}$

⋆ Here we only consider the effect of the supermoduli which appear due to the handles on the surface. The supermoduli appearing due to the punctures on the surface are used to the generate the picture changed vertex function along the lines discussed in Ref. [8,9].

† Here the operator e^{ϕ} refers to an operator with ghost charge 1 and conformal dimension $-\frac{3}{2}$, and need not be interpreted as the exponential of a bosonic field ϕ.

which may be expressed as

$$h_i^{z\theta} = \nabla^z v_i^\theta(z,\bar{z}) - \sum A_{ij}\, \delta^{(2)}(z - P_j) \tag{14}$$

where $v_i^\theta(z)$ is a half differential with poles of the form $\sum A_{ij}(z-P_j)^{-1}$ at $z = P_j$. The part of the action that depends on the supermodulii c_i may be written as,

$$\int d^2\, z\, c_i\, h_i^{z\theta}(z)\ T_F(z) \tag{15}$$

where T_F is the fermionic part of the super stress tensor. Since $h_i^{z\theta}$ does not have any singularity at P_i we can exclude a small region around each P_i from the z integral in Eq.(15). Substitution of (14) into (15) and integration by parts yields,

$$c_i\, A_{ij} \oint \frac{dz}{z - P_j}\ T_F(z) + c_i\, M_F^i \sim c_i \left[A_{ij} T_F(P_j) + M_F^i\right] \tag{16}$$

where M_F^i denote the contributions from the singularities of T_F. These will appear if we are calculating the correlation function of a set of vertex operators, in the presence of which T_F will develop singularities. Thus M_F^i will in general involve the contour integral of T_F around these vertex operators. The integral over c_i will now give several terms, each containing a product of $(2g-2)$ factors. One of them will contain the product of $T_F(P_i)$ for $1 \leq i \leq (2g-2)$, in the others some factors of $T_F(P_i)$ will be replaced by M_F^i.

Let us now take the limit $P_1 \to P_2 \equiv Q_1$, $P_3 \to P_4 \equiv Q_2$ etc. With the amplitudes properly normalized, only the most singular part will contribute. But the most singular part in the operator product $e^{\phi(P_1)}\, T_F(P_1)\, e^{\phi(P_2)}\, T_F(P_2)$ is $e^{2\phi(P_2)}(P_1 - P_2)^{-4}$. Only the term involving $(2g-2)$ factors of $T_F(P_i)$ will contribute, terms involving M_F^i will be non-leading. Thus the final prescription will be to insert factors of $e^{2\phi(Q_i)}$ at each of the (g-1) points Q_i in any correlation function.

In what follows we shall further choose the points $Q_1,..Q_{g-1}$ to coincide with $R_1,..R_{g-1}$, the position of the $(g-1)$ zeros of $\vartheta\left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right](\int_y^z \vec{\omega})$ in the z-plane which are y-independent. We believe the final result will be independent of the choice of these points, but the calculation becomes much simpler with this choice.* The total ghost charge carried by these operators adds up to $2(g-1)$ and hence we get a nonzero answer for the correlator provided the total ghost charge of all other fields in the correlator adds up to zero.‡

Thus we have to consider the correlation function

$$F^g(y_i,z_i) = \left\langle \prod_{k=1}^{g-1} e^{2\phi(R_k)} \prod_{i=1}^{N} \left[S_g^+(y_i)\ S_g^-(z_i)\right] \right\rangle \tag{17}$$

As before, we start with the function:

$$G^g(y,z;y_i,z_i) \equiv \frac{\langle \prod_{k=1}^{g-1} e^{2\phi(R_k)}\ \prod_{i=1}^{N} \left[S_g^+(y_i)\, S_g^-(z_i)\right] \beta(y)\, \gamma(z)\rangle}{\langle \prod_{k=1}^{g-1} e^{2\phi(R_k)}\ \prod_{i=1}^{N} \left[S_g^+(y_i)\, S_g^-(z_i)\right]\rangle} \tag{18}$$

The analytic properties of G^g as a function of y and z may be determined using the operator product expansion (13) as well as,

$$\begin{aligned} \beta(z)\, e^{2\phi(w)} &\sim (z-w)^2\, e^{\phi(w)}\, \partial\, \xi(w) \\ \gamma(z)\, e^{2\phi(w)} &\sim (z-w)^{-2}\, e^{3\phi(w)}\, \eta(w) \end{aligned} \tag{19}$$

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* The special role played by these points have also been noted in Refs. [11,12].

‡ Again we shall never explicitly use the representation of the operator $e^{2\phi}$ in terms of bosonic fields. The only information we shall be using is that it carries ghost charge 2 and has conformal weight -4.

This gives,

$$G^g_{\vec{a}\vec{b}}(y,z;y_i,z_i) = \left[\frac{\prod_i \vartheta\left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right](\int_{y_i}^{y}\vec{\omega})\prod_i \vartheta\left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right](\int_{z_i}^{z}\vec{\omega})}{\prod_i \vartheta\left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right](\int_{y_i}^{z}\vec{\omega})\prod_i \vartheta\left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right](\int_{z_i}^{y}\vec{\omega})}\right]^{\frac{1}{2}}$$

$$\frac{\vartheta\left[\begin{smallmatrix}\vec{a}_0+\vec{a}\\ \vec{b}_0+\vec{b}\end{smallmatrix}\right](\int_{y}^{z}\vec{\omega}+\frac{1}{2}\sum_i\int_{y_i}^{z_i}\vec{\omega})}{\vartheta\left[\begin{smallmatrix}\vec{a}_0+\vec{a}\\ \vec{b}_0+\vec{b}\end{smallmatrix}\right](\frac{1}{2}\sum_i\int_{y_i}^{z_i}\vec{\omega})}\quad\frac{h^{-1}(y)h^3(z)}{\vartheta\left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right](\int_{y}^{z}\vec{\omega})} \tag{20}$$

It may be easily verified that $G^g_{\vec{a},\vec{b}}$, so defined has all the right poles and zeros.*

From this expression for $G^g_{\vec{a},\vec{b}}$ we can derive differential equations for $F^g_{\vec{a},\vec{b}}$ as before, using the stress tensor (11). Here we write down the solution:

$$\begin{aligned} F^g_{\vec{a}\vec{b}}(y_i,z_i) &= K^g_{\vec{a}\vec{b}}\left\{\prod_{i<j}\vartheta\begin{bmatrix}\vec{a}_0\\ \vec{b}_0\end{bmatrix}\left(\int_{z_i}^{z_j}\vec{\omega}\right)\right\}^{-\frac{1}{4}}\left\{\prod_{i<j}\vartheta\begin{bmatrix}\vec{a}_0\\ \vec{b}_0\end{bmatrix}\left(\int_{y_i}^{y_j}\vec{\omega}\right)\right\}^{-\frac{1}{4}} \\ &\left\{\prod_{i,j}\vartheta\begin{bmatrix}\vec{a}_0\\ \vec{b}_0\end{bmatrix}\left(\int_{y_i}^{z_j}\vec{\omega}\right)\right\}^{\frac{1}{4}}\left\{\vartheta\begin{bmatrix}\vec{a}_0+\vec{a}\\ \vec{b}_0+\vec{b}\end{bmatrix}\left(\frac{1}{2}\sum_i\int_{z_i}^{y_i}\vec{\omega}\right)\right\}^{-1} \\ &\prod_i\left\{[h(y_i)]^{-\frac{5}{4}}[h(z_i)]^{\frac{3}{4}}\right\} \end{aligned} \tag{21}$$

where $\hat{K}_{\vec{a},\vec{b}}$ is a normalization factor.

We are now ready to assemble all the above results in order to calculate the correlation functions of fermion emission vertex operators. Here we will only

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* For that one needs to use the fact that $h(z)$ possesses simple zeros at the points $R_1,...,R_{g-1}$ [7].

exhibit what is needed in the proof of the nonrenormalization theorems. We shall use two different forms of the fermion emission vertex, [3,13] namely:

$$
\begin{aligned}
V_{-\frac{1}{2}}(u,k,z) &= u^{\alpha}(k)\ S_{\alpha}(z)\ e^{ik.X(z)}\ S_g^{-}(z), \\
V_{\frac{1}{2}}(u,k,z) &= u^{\alpha}(k)\ e^{ik.X(z)}\ S_g^{+}(z)\ \Big\{ \partial X^{\mu}\ (\gamma_{\mu})_{\alpha\beta}\ S^{\beta} \\
&+\frac{i}{4}\ k_{\nu} \lim_{w\to z} \Big[(w-z)^{-\frac{1}{2}}\ \psi^{\nu}(w)\ S_{\alpha}(z) - (w-z)^{-1}\ (\gamma^{\nu})_{\alpha\beta}\ S^{\beta}(z) \Big] \Big\}\ ,
\end{aligned}
\tag{22}
$$

where we have defined a normal ordered operator by point splitting. The rule for calculating scattering amplitudes involving fermions is to choose a charge neutral combination of $V_{\frac{1}{2}}$ and $V_{-\frac{1}{2}}$. We shall also need the bosonic vertex operator

$$
V_0(\varsigma,k,z) = \varsigma_{\mu}\left[\partial X^{\mu} + ik_{\nu}\ \psi^{\mu}(z)\ \psi^{\nu}(z)\right]\ e^{ik.X(z)} \tag{23}
$$

We shall now illustrate the non-renomalization theorem for the fermion-fermion boson vertex. For this we need to calculate

$$
\Big\langle\ V_{-\frac{1}{2}}(u_1,k_1,z_1)\ V_{\frac{1}{2}}(u_2,k_2,z_2)\ V_0(\varsigma_3,k_3,z_3)\ \Big\rangle \tag{24}
$$

We shall show that the non-renormalization theorems hold only as a consequence of the vanishing of the correlators in the ψ space, hence we shall never need to calculate the correlators in the X space. Let us first consider the k independent part of the correlator. The relevant correlator is proportional to

$$
\Big\langle\ S_g^{-}(z_1)\, S_g^{+}(z_2)\ \Big\rangle\ \Big\langle\ S_{\alpha_1}(z_1)\, S^{\beta_2}(z_2)\ \Big\rangle\ \equiv\ A(z_1,z_2)\ \delta_{\alpha_1}^{\beta_2} \tag{25}
$$

where in writing down such an expression we should keep in mind that the ghost correlator is to be defined after soaking up the zero modes along the lines explained above. The function $A(z_1,z_2)$ may be calculated by choosing specific values of α_1,β_2 and representing the $SO(10)$ spin operators as a product of five $SO(2)$ spin operators. Thus we may choose $S_{\alpha_1}(z_1)$ and $S^{\beta_2}(z_2)$ to be $S_1^+(z_1)\,S_2^+(z_1)\,S_3^+(z_1)\,S_4^+(z_1)S_5^+(z_1)$ and $S_1^-(z_2)\,S_2^-(z_2)\,S_3^-(z_2)\,S_4^-(z_2)\,S_5^-(z_2)$ respectively. The final expression for $A(z_1, z_2)$ may be obtained using Eqn (10) and (21) and summing over spin structures. The result is,

$$A = \left\{ \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \left(\int_{z_1}^{z_2} \vec{\omega} \right)^{-1} \right\} [h(z_1)]^2 \sum_{\vec{a},\vec{b}} \eta(\vec{a},\vec{b}) \left\{ \vartheta \begin{bmatrix} \vec{a}_0 + \vec{a} \\ \vec{b}_0 + \vec{b} \end{bmatrix} \left(\frac{1}{2} \int_{z_1}^{z_2} \vec{\omega} \right) \right\}^4 , \tag{26}$$

where $\eta(\vec{a},\vec{b})$ denotes the relative normalization and phases of the contribution from different spin structures. At this stage we have to demand that (26) behave as a one form as a function of z_1 in order for the integral over z_1 of $V_{-\frac{1}{2}}$ to be well defined. This in turn implies that the combination of ϑ-functions multiplying $h^2(z_1)$ in (26) must be periodic as we shift z_1 around any of the canonical homology cycles. Note however that under such a shift $\vartheta\left[\begin{smallmatrix}\vec{a}_0+\vec{a}\\ \vec{b}_0+\vec{b}\end{smallmatrix}\right]\,[\,(1/2)\,\int_{z_1}^{z_2}\vec{\omega}\,]$ goes over to a theta function of a different characteristic. Hence the relative normalization and phases $\eta(\vec{a},\vec{b})$ are completely determined by demanding periodicity in z_1 [2,8].* We obtain

$$\eta(\vec{a},\vec{b}) = K \ \exp\left\{ 4\pi i \left(\vec{a}.\vec{b}_0 - \vec{b}.\vec{a}_0 \right) \right\} . \tag{27}$$

Substitution of this result in (26) shows that the answer vanishes identically due to the generalized Riemann ϑ-identity [7,14],

---

* In other words if we know the contribution from one spin structure, we know that from all others by translating z_1 around different cycles.

$$\sum_{\vec{a},\vec{b}} \exp\left\{4\pi i\left(\vec{a}.\vec{b}_0 - \vec{b}.\vec{a}_0\right)\right\}$$

$$\vartheta\begin{bmatrix}\vec{a}_0+\vec{a}\\ \vec{b}_0+\vec{b}\end{bmatrix}(\vec{x})\ \vartheta\begin{bmatrix}\vec{a}_0+\vec{a}\\ \vec{b}_0+\vec{b}\end{bmatrix}(\vec{y})\ \vartheta\begin{bmatrix}\vec{a}_0+\vec{a}\\ \vec{b}_0+\vec{b}\end{bmatrix}(\vec{u})\ \vartheta\begin{bmatrix}\vec{a}_0+\vec{a}\\ \vec{b}_0+\vec{b}\end{bmatrix}(\vec{v})$$

$$= 2^g\vartheta\begin{bmatrix}\vec{a}_0\\ \vec{b}_0\end{bmatrix}\left(\frac{\vec{x}+\vec{y}+\vec{u}+\vec{v}}{2}\right)\vartheta\begin{bmatrix}\vec{a}_0\\ \vec{b}_0\end{bmatrix}\left(\frac{\vec{x}-\vec{y}+\vec{u}-\vec{v}}{2}\right) \tag{28}$$

$$\vartheta\begin{bmatrix}\vec{a}_0\\ \vec{b}_0\end{bmatrix}\left(\frac{\vec{x}+\vec{y}-\vec{u}-\vec{v}}{2}\right)\vartheta\begin{bmatrix}\vec{a}_0\\ \vec{b}_0\end{bmatrix}\left(\frac{\vec{x}-\vec{y}-\vec{u}+\vec{v}}{2}\right)$$

and that $\vartheta\left[\begin{smallmatrix}\vec{a}_0\\ \vec{b}_0\end{smallmatrix}\right](0) = 0$.

Next let us analyse the part involving k. It is easy to see that the relevant contributions are proportional to the following structures

$$I_1 = i(k_3)_\nu\, u^{\alpha_1}(k_1)\, u^{\alpha_2}(k_2)\, (\gamma_\rho)_{\alpha_2\beta_2}\, (\varsigma_3)_\mu$$
$$\left\langle S_{\alpha_1}(z_1)\, S^{\beta_2}(z_2)\, \psi^\mu(z_3)\, \psi^\nu(z_3)\right\rangle \left\langle S_g^-(z_1)\, S_g^+(z_2)\right\rangle \tag{29}$$

$$I_2 = \lim_{w_2\to z_2} u^{\alpha_1}(k_1)\, u^{\alpha_2}(k_2)\, (k_2)_\mu\, (k_3)_\rho\, (\varsigma_3)_\nu\, (w_2-z_2)^{-\frac{1}{2}}$$
$$\langle S_{\alpha_1}(z_1)\, S_{\alpha_2}(z_2)\, \psi^\mu(w_2)\, \psi^\nu(z_3)\, \psi^\rho(z_3)\rangle \left\langle S_g^-(z_1)\, S_g^+(z_2)\right\rangle \tag{30}$$

where we have dropped terms that would vanish by on-shell conditions, *e.g.*, terms of the form $(k_2)_\nu u^{\alpha_1}(k_1)u^{\alpha_2}(k_2)\langle S_{\alpha_1}(z_1)S_{\alpha_2}(z_2)\psi^\nu(w_2)\rangle$. Expression (29) has two tensor structures of the form

$$A(z_1,z_2,z_3)\delta_{\alpha_1}^{\beta_2}\,\delta^{\mu\nu} + B(z_1,z_2,z_3)\ (\Sigma^{\mu\nu})_{\alpha_1}^{\ \beta_2} \tag{31}$$

The tensor proportional to A does not contribute due to the on-shell condition $\varsigma \cdot k_3 = 0$. B on the other hand may be evaluated by setting $\mu = 1$, $\nu = 2$: $S_{\alpha_1} = S_1^- S_2^- S_3^+ S_4^+ S_5^+$ and $S^{\beta_2} = S_1^- S_2^- S_3^- S_4^- S_5^-$. The product of the spin correlators, after being summed over the spin structures gives,

$$B \propto \left\{ \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \left[\int_{z_1}^{z_3} \vec{\omega} \right] \right\}^{-1} \left\{ \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \left(\int_{z_2}^{z_3} \vec{\omega} \right) \right\}^{-1} \left[h(z_1)\right]^2 \left[h(z_2)\right]^2 \left[h(z_3)\right]^2$$

$$\sum_{\vec{a},\vec{b}} \left[\exp \left\{ 4\pi i (\vec{a}.\vec{b}_0 - \vec{b}.\vec{a}_0) \right\} \right. \tag{32}$$

$$\left. \left\{ \vartheta \begin{bmatrix} \vec{a}_0 + \vec{a} \\ \vec{b}_0 + \vec{b} \end{bmatrix} \left(\frac{1}{2} \int_{z_3}^{z_1} \vec{\omega} + \frac{1}{2} \int_{z_3}^{z_2} \vec{\omega} \right) \right\}^2 \left\{ \vartheta \begin{bmatrix} \vec{a}_0 + \vec{a} \\ \vec{b}_0 + \vec{b} \end{bmatrix} \left(\frac{1}{2} \int_{z_2}^{z_1} \vec{\omega} \right) \right\}^2 \right]$$

This again vanishes using Eq. (28). Finally we turn to the correlator appearing in Eq. (30). There are four independent tensor structures which we shall take to be,

$$A(\gamma^\nu)_{\alpha_1\alpha_2}\, \delta^{\rho\mu} + B(\gamma^\mu)_{\alpha_1\alpha_2}\, \delta^{\nu\rho} + C(\gamma^\rho)_{\alpha_1\alpha_2}\, \delta^{\mu\nu} + D(\gamma^\mu \Sigma^{\rho\nu})_{\alpha_2\alpha_1} \tag{33}$$

Upon substitution in Eqn. (30) only the tensor structure proportional to A would contribute to the amplitude. A may be evaluated by setting $\rho = 1$ $\mu = \bar{1}$, $\nu = 2$, $S_{\alpha_1} = S_1^+ S_2^- S_3^- S_4^+ S_5^+$, $S_{\alpha_2} = S_1^- S_2^- S_3^+ S_4^- S_5^-$. This gives,

$$
\begin{aligned}
A \propto \lim_{w_2 \to z_2} \Bigg[& (w_2 - z_2)^{-\frac{1}{2}} \left\{ \vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \Big(\int_{z_2}^{z_1} \vec{\omega} \Big) \right\}^{-\frac{1}{2}} \left(\vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \Big(\int_{w_2}^{z_1} \vec{\omega} \Big) \right\}^{-\frac{1}{2}} \\
& \left(\vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \Big(\int_{z_3}^{z_2} \vec{\omega} \Big) \right\}^{-1} \left(\vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \Big(\int_{z_3}^{w_2} \vec{\omega} \Big) \right\}^{-1} \left(\vartheta \begin{bmatrix} \vec{a}_0 \\ \vec{b}_0 \end{bmatrix} \Big(\int_{w_2}^{z_2} \vec{\omega} \Big) \right\}^{\frac{1}{2}} \\
& \sum_{\vec{a},\vec{b}} \Big[exp\{4\pi i(\vec{a}.\vec{b}_0 - \vec{b}.\vec{a}_0)\} \left\{ \vartheta \begin{bmatrix} \vec{a}_0 + \vec{a} \\ \vec{b}_0 + \vec{b} \end{bmatrix} (\frac{1}{2} \int_{z_3}^{z_1} \vec{\omega} + \frac{1}{2} \int_{z_3}^{z_2} \vec{\omega}) \right\} \qquad (34) \\
& \left\{ \vartheta \begin{bmatrix} \vec{a}_0 + \vec{a} \\ \vec{b}_0 + \vec{b} \end{bmatrix} (\frac{1}{2} \int_{z_2}^{z_1} \vec{\omega}) \right\}^2 \left\{ \vartheta \begin{bmatrix} \vec{a}_0 + \vec{a} \\ \vec{b}_0 + \vec{b} \end{bmatrix} (\frac{1}{2} \int_{z_2}^{z_1} \vec{\omega} + \int_{w_2}^{z_3} \vec{\omega}) \right\} \Big] \\
& (h(z_1))^2 (h(z_3))^2 h(z_2) h(w_2) \Bigg]
\end{aligned}
$$

Since the prefactor does not diverge in the $\omega_2 \to z_2$ limit, we may set $\omega_2 = z_2$ in the sum. The result vanishes by (28).

So far we have discussed the two fermion one boson vertex. The advantage of having the fermion vertex operator in the correlator is that the relative normalization and phase between contributions from different spin structures is fixed by demanding periodicity in the argument of the fermion vertex operator. This is not the case for correlation functions of purely bosonic vertex operators, since the function is periodic in each sector separately. However one may obtain the correlation function involving the bosonic vertex operators by starting from the fermionic vertex operator and then using operator product expansion.* For example, if we start from the correlation function

$$\Big\langle\ V_{-\frac{1}{2}}(u_1, k_1, z_1)\ V_{\frac{1}{2}}(u_2, k_2, z_2)\ V_0(\varsigma_3, k_3, z_3)\ \Big\rangle$$

---

* We would like to thank E. Martinec for discussion on this point.

and take the limit $z_1 \to z_2$, the singular part of the expression is proportional to

$$(z_1 - z_2)^{-1} \langle\ V_0(\bar{u}_1 \gamma^\mu u_2,\ k_1 + k_2,\ z_1)\ V_0(\varsigma_3,\ k_3,\ z_3)\ \rangle \tag{35}$$

Thus the vanishing of the correlation function of the two fermion one boson vertex operators also implies the vanishing of the correlation function of the two bosonic vertex operators. If we now take the limit $z_1 \to z_3$ in (35), the most singular part proportional to $(z_1 - z_3)^{-2}$ is proportional to the identity operator, thus showing the vanishing of the expectation value of the identity operator. This in turn shows the vanishing of the cosmological constant.† Vanishing of the correlator $\langle V_{\frac{1}{2}}(z_1)\, V_{-\frac{1}{2}}(z_2)\rangle$ is a straightforward consequence of (28), in fact all the relevant correlators were evaluated in calculating two fermion one boson vertex. Taking the $z_1 \to z_2$ limit in this correlator one may show that the one point bosonic amplitude vanishes.

Calculation of the correlator of three bosonic vertex operators is somewhat more involved. We start from the correlator $\langle V_{\frac{1}{2}} V_{-\frac{1}{2}} V_0 V_0\rangle$, then study the limit when the arguments of $V_{\frac{1}{2}}$ and $V_{-\frac{1}{2}}$ approach each other. The result again vanishes as a consequence of the Riemann theta identity. We shall not give the details of the calculation here.

Thus in this paper we have derived a general expression for correlation functions of spin fields on arbitrary genus Riemann surfaces. Using a specific ansatz for screening the background ghost charge makes the calculation relatively simple. With this ansatz we have shown that all n-point amplitudes in the superstring and the heterotic string theory vanish for $0 \leq n \leq 3$, to all orders of string perturbation.

† Vanishing of the cosmological constant on higher genus Riemann surfaces have also been discussed in Refs. [12,15].

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