TIFR/TH/91-47
October, 1991

GAUGE AND GENERAL COORDINATE INVARIANCE IN NON-POLYNOMIAL CLOSED STRING FIELD THEORY

Debashis Ghoshal and Ashoke Sen⋆

Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

ABSTRACT

An appropriate field configuration in non-polynomial closed string field theory is shown to correspond to a general off-shell field configuration in low energy effective field theory. A set of string field theoretic symmetries that act on the fields in low energy effective field theory as general coordinate transformation and anti-symmetric tensor gauge transformation is identified. The analysis is carried out to first order in the fields; thus the symmetry transformations in string field theory reproduce the linear and the first non-linear terms in the gauge transformations in the low energy effective field theory.

Prefitem

⋆ e-mail addresses: GHOSHAL@TIFRVAX.BITNET, SEN@TIFRVAX.BITNET

1. INTRODUCTION

It is by now well established that the low energy effective field theory describing any critical string theory contains gravity [1] [2]. The chain of results which lead to this conclusion is the following. First of all, one finds that the spectrum of free string theory contains a massless rank 2 symmetric tensor state; the physical states being in one to one correspondence with the physical states associated with the graviton field. Secondly, one finds that amplitudes involving this massless tensor state satisfy the same set of Ward identities as the ones satisfied by scattering amplitudes involving external graviton states in a generally covariant theory. Finally, the tree level amplitudes involving the massless tensor states agree, at low energies, to the tree level amplitudes calculated from Einstein's action.

More recently, a complete covariant field theory for closed bosonic strings has been constructed [3] [4]. (See also ref.[5].) This field theory is characterized by an infinite parameter gauge invariance [4] [6], and an infinite number of fields. Quantization of this theory has been carried out in refs.$[7-11]$. One expects that general coordinate transformation will emerge as a particular combination of the gauge transformations in string theory, and the metric will be related to the infinite component string field through a suitable functional relation. For previous work on this subject see ref.[12](also see ref.[13] for a review of linearized gauge invariance in string field theory).

In this paper we investigate this connection in detail. We show that at least to first non-linear order, it is possible to identify string field configurations which correspond to specific (off-shell) configuration of the fields that appear in the low energy effective field theory, namely the metric, the dilaton and the antisymmetric tensor gauge field. To this order we also identify specific symmetries of the string field theory which can be identified as gauge transformations of the low energy effective field theory. This contains general coordinate transformation and antisymmetric tensor gauge transformation. It turns out that the gauge symmetries of the low energy effective field theory are obtained by a combination of gauge transformation in string field theory and a ‘trivial’ symmetry of the form:

$$\delta\psi_r = K_{rs}(\{\psi\})\frac{\delta S}{\delta\psi_s} \tag{1.1}$$

where ψ_r are the string field components, K_{rs} is an antisymmetric matrix valued function of ψ_r, and S is the string field theory action.⋆

The plan of the paper is as follows. In Sect. 2 we explain our general approach to solve the problem of identifying the field configurations and gauge symmetries in low energy effective field theory to those in string field theory. We allow for the most general functional relation between string fields and the low energy fields, and also between the gauge transformations in string field theory and those in low energy effective field theory. We then derive constraints that must be satisfied in order that the gauge transformations in the low energy effective field theory are compatible with those in string field theory. In sect. 3 we show how a solution to these constraint equations may be obtained at the lowest level, so that the linearized gauge transformations involving the massless fields in the low energy effective field theory agree with those in string field theory. In sect. 4 we analyze the set of constraint equations derived in sect. 2 to the next order. We show that a solution to these set of equations can be obtained provided a certain set of consistency conditions are satisfied by the interaction vertices in string field theory. Appendix A contains a verification of the fact that these consistency conditions are indeed satisfied by the vertices of string field theory. This completes the proof that to first non-linear order, suitable functional relations between the fields and gauge transformation parameters appearing in string field theory and those appearing in low energy effective field theory may be found so that gauge invariance in low energy effective field theory can be derived as a consequence of the symmetries of string field theory. We conclude in sect. 5 with some speculations and implications of our result for the closure of the gauge algebra in string field theory.

⋆ This symmetry is called ‘trivial’ as it exists for all theories. However it is perhaps worth emphasizing that this is a genuine symmetry of the action.

2. GENERAL FORMALISM

Let $\mathcal{H}$ be the Hilbert space of the combined matter ghost conformal field theory describing first quantized string theory, and $\{|\Phi_{2,r}\rangle\}$ denote a basis of states in $\mathcal{H}$ with ghost number 2 and annihilated by b_0^- and L_0^-. (We shall work with the convention $c_0^\pm = (c_0 \pm \bar{c}_0)/\sqrt{2}$, $b_0^\pm = (b_0 \pm \bar{b}_0)/\sqrt{2}$, and $L_0^\pm = (L_0 \pm \bar{L}_0)/\sqrt{2}$.) Then the string field $|\Psi\rangle$ may be expanded as $b_0^-|\Psi\rangle = \sum_r \psi_r |\Phi_{2,r}\rangle$ and the string field theory action is given by [3] [4] [14]

$$S(\Psi) = \frac{1}{2}\langle\Psi|Q_B b_0^-|\Psi\rangle + \sum_{N=3}^{\infty} \frac{g^{N-2}}{N!}\{\Psi^N\} \equiv \sum_{N=2}^{\infty} \frac{1}{N}\tilde{A}^{(N)}_{r_1\ldots r_N}\psi_{r_1}\ldots\psi_{r_N}, \tag{2.1}$$

where Q_B is the BRST charge of the first quantized string theory, and $\{\ \ \}$ has been defined in refs.[4][14]. Throughout this paper we shall use the convention of ref.[14]. The coefficients $\tilde{A}^{(N)}$ are symmetric in all the indices and are given by,

$$\begin{aligned} \tilde{A}^{(2)}_{r_1 r_2} &= -\langle\Phi_{2,r_1}|c_0^- Q_B|\Phi_{2,r_2}\rangle, \\ \tilde{A}^{(N)}_{r_1\cdots r_N} &= \frac{g^{N-2}}{(N-1)!}\{(c_0^-\Phi_{2,r_1})\cdots(c_0^-\Phi_{2,r_N})\}, \qquad N \geq 3. \end{aligned} \tag{2.2}$$

Let $|\Phi_{1,\alpha}\rangle$ denote a basis of states in $\mathcal{H}$ of ghost number 1 and annihilated by b_0^- and L_0^-. Then the gauge transformation parameter $|\Lambda\rangle$ in string field theory may be expanded as $b_0^-|\Lambda\rangle = \sum_\alpha \lambda_\alpha |\Phi_{1,\alpha}\rangle$, and the gauge transformation in string field theory takes the form:

$$\delta(b_0^-|\Psi\rangle) = Q_B b_0^-|\Lambda\rangle + \sum_{N=3}^{\infty} \frac{g^{N-2}}{(N-2)!}[\Psi^{N-2}\Lambda] \tag{2.3}$$

with $[\ \ ]$ as defined in refs.[4][14]. Let us introduce a basis of bra states $\langle\Phi^c_{3,r}|$ of ghost number 3, annihilated by b_0^- and L_0^-, and satisfying,

$$\langle\Phi^c_{3,r}|c_0^-|\Phi_{2,s}\rangle = \delta_{rs} \tag{2.4}$$

so that $\{\langle\Phi^c_{3,r}|\}$ form a basis conjugate to $\{|\Phi_{2,s}\rangle\}$. We can now write eq. (2.3) in terms of the component fields as follows:

$$\delta\psi_s = \sum_{N=2}^{\infty} A^{(N)}_{s\alpha r_1\ldots r_{N-2}} \lambda_\alpha \psi_{r_1} \ldots \psi_{r_{N-2}}, \tag{2.5}$$

where,

$$\begin{aligned} A^{(2)}_{s\alpha} &= \langle \Phi^c_{3,s} | c_0^- Q_B | \Phi_{1,\alpha} \rangle, \\ A^{(N)}_{s\alpha r_1 \cdots r_{N-2}} &= \frac{g^{N-2}}{(N-2)!} \langle \Phi^c_{3,s} | c_0^- [(c_0^- \Phi_{1,\alpha})(c_0^- \Phi_{2,r_1}) \cdots (c_0^- \Phi_{2,r_{N-2}})] \rangle \\ &= -\frac{g^{N-2}}{(N-2)!} \{(c_0^- \Phi^c_{3,s})(c_0^- \Phi_{1,\alpha})(c_0^- \Phi_{2,r_1}) \cdots (c_0^- \Phi_{2,r_{N-2}})\}, \qquad N \geq 3. \end{aligned} \tag{2.6}$$

Note that $A^{(N)}_{s\alpha r_1 \cdots r_{N-2}}$ is symmetric in the indices $r_1, r_2, \cdots, r_{N-2}$.

Closed bosonic string theory at low energies is described by the effective field theory involving the graviton, the dilaton, and the antisymmetric tensor gauge fields. Let $\{\phi_i\}$ denote the set of all the dynamical degrees of freedom of the low energy effective field theory, where the index i stands for the field index, as well as space-time coordinates (or, equivalently, momenta if we are working with Fourier transforms of the fields). Also, let $\{\eta_\kappa\}$ denote the set of gauge transformation parameters of the low energy effective field theory, where the index κ again includes coordinate (momentum) index. The set $\{\eta_\kappa\}$ contains general coordinate transformation and antisymmetric tensor gauge transformation. Let the general form of the gauge transformation in the low energy effective field theory be:

$$\delta\phi_i = \sum_{N=2}^{\infty} B^{(N)}_{ij_1\ldots j_{N-2}\kappa} \eta_\kappa \phi_{j_1} \ldots \phi_{j_{N-2}} \tag{2.7}$$

The coefficients $B^{(N)}$ are all known from the low energy effective field theory. We now ask the following question: Can this low energy effective field theory be obtained (after some possible field redefinition) by integrating out the massive modes of the string field theory? If so, then any off shell configuration described by some arbitrary choice of the variables ϕ_i should correspond to some configuration of the string fields ψ_r. (This configuration certainly involves massive modes of the string. However the values they take are specified by their equations of motion.) We should also note that the configuration may not be unique — a given off shell configuration of the low energy effective field theory may correspond to more than one string field configurations which are related to each other by some gauge symmetry associated with higher level states (or by gauge symmetries at the massless level that have no counterpart in the effective field theory). Let us call $\psi_r(\phi_i)$ to be one of the string field configurations that correspond to a given configuration of the low energy fields ϕ_i, and allow for the most general form of this function:

$$\psi_r(\phi_i) = \sum_{N=0}^{\infty} \frac{1}{(N+1)} C^{(N)}_{ri_1\ldots i_{N+1}} \phi_{i_1} \ldots \phi_{i_{N+1}} \tag{2.8}$$

The gauge transformation (2.7) of ϕ_i induces a transformation on ψ_r. The first few terms take the form:

$$\delta\psi_r(\phi_i) = C^{(0)}_{ri} B^{(2)}_{i\kappa} \eta_\kappa + C^{(0)}_{ri} B^{(3)}_{ij\kappa} \phi_j \eta_\kappa + C^{(1)}_{rij} \phi_i B^{(2)}_{j\kappa} \eta_\kappa + \mathcal{O}(\phi^2) \tag{2.9}$$

The question we are interested in can be formulated as follows. Can we identify a gauge transformation parameter $b_0^-|\Lambda(\eta,\phi)\rangle = \sum_\alpha \lambda_\alpha(\eta,\phi)|\Phi_{1,\alpha}\rangle$ in string field theory such that the transformation (2.9) may be regarded as a gauge transformation of the string field with this parameter? If the answer is in the affirmative, then we could say that we have been able to identify the off-shell gauge transformations of low energy effective field theory to specific gauge transformations in string field theory. Again, we shall allow for the most general dependence of λ_α on ϕ_i and η_κ (keeping terms linear in η_κ only, since we are considering infinitesimal gauge transformation):

$$\lambda_\alpha(\eta,\phi) = \sum_{N=0}^{\infty} D^{(N)}_{\alpha\kappa i_1\ldots i_N} \eta_\kappa \phi_{i_1} \ldots \phi_{i_N} \tag{2.10}$$

Using eqs.(2.5), (2.10) and (2.8) we get,

$$\delta\psi_r = A^{(2)}_{r\alpha} D^{(0)}_{\alpha\kappa}\eta_\kappa + A^{(2)}_{r\alpha} D^{(1)}_{\alpha\kappa i}\phi_i\eta_\kappa + A^{(3)}_{r\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{si}\phi_i\eta_\kappa + \mathcal{O}(\phi^2) \tag{2.11}$$

Comparing eqs.(2.9) and (2.11) we get,

$$\begin{aligned} C^{(0)}_{ri} B^{(2)}_{i\kappa} &= A^{(2)}_{r\alpha} D^{(0)}_{\alpha\kappa} \\ C^{(0)}_{ri} B^{(3)}_{ij\kappa} + C^{(1)}_{rji} B^{(2)}_{i\kappa} &= A^{(2)}_{r\alpha} D^{(1)}_{\alpha\kappa j} + A^{(3)}_{r\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj} \\ &\cdots \\ &\cdots \end{aligned} \tag{2.12}$$

Thus the question is, can we find appropriate $C^{(N)}$'s and $D^{(N)}$'s such that the set of equations (2.12) are satisfied? As we shall see, these equations cannot be satisfied beyond lowest order, implying that off-shell general coordinate and antisymmetric tensor gauge transformations cannot be identified to pure gauge transformations in string field theory. Note, however, that besides gauge invariance, the string field theory action is also trivially invariant under the transformation:

$$\delta_{extra}\psi_r = K_{rs}(\{\psi_t\})\frac{\delta S}{\delta\psi_s} \tag{2.13}$$

for any K_{rs} which is antisymmetric in r and s. We shall show that a combination of gauge transformation in string field theory given in eq.(2.5) and the transformation given in eq.(2.13) can indeed be used to generate general coordinate and antisymmetric tensor gauge transformations in the low energy effective field theory, at least to first order in ψ_r. For this we allow for the most general choice of K_{rs} as a function of ϕ_i and η_κ (keeping terms linear in η_κ only):

$$K_{rs} = \sum_{N=1}^{\infty} K^{(N)}_{rs\kappa i_1\ldots i_{N-1}}\eta_\kappa\phi_{i_1}\ldots\phi_{i_{N-1}} \tag{2.14}$$

Using eqs.(2.13), (2.14), (2.1) and (2.8) we get:

$$\delta_{extra}\psi_r = K^{(1)}_{rs\kappa}\tilde{A}^{(2)}_{st} C^{(0)}_{ti}\phi_i\eta_\kappa + \mathcal{O}(\phi^2) \tag{2.15}$$

Using eqs.(2.11) and (2.15) we get the net transformation of ψ_r as,

$$\delta_{tot}\psi_r = A^{(2)}_{r\alpha} D^{(0)}_{\alpha\kappa}\eta_\kappa + (A^{(2)}_{r\alpha} D^{(1)}_{\alpha\kappa i} + A^{(3)}_{r\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{si} + K^{(1)}_{rs\kappa}\tilde{A}^{(2)}_{st} C^{(0)}_{ti})\phi_i\eta_\kappa + \mathcal{O}(\phi^2) \tag{2.16}$$

Comparing eqs.(2.9) and (2.16) we get,

$$\begin{aligned} C^{(0)}_{ri} B^{(2)}_{i\kappa} =& A^{(2)}_{r\alpha} D^{(0)}_{\alpha\kappa} \\ C^{(0)}_{ri} B^{(3)}_{ij\kappa} + C^{(1)}_{rji} B^{(2)}_{i\kappa} =& A^{(2)}_{r\alpha} D^{(1)}_{\alpha\kappa j} + A^{(3)}_{r\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj} + K^{(1)}_{rs\kappa}\tilde{A}^{(2)}_{st} C^{(0)}_{tj} \\ & \cdots \\ & \cdots \end{aligned} \tag{2.17}$$

Thus we now need to show that one can find appropriate $C^{(N)}$'s, $D^{(N)}$'s and $K^{(N)}$'s so as to satisfy eq.(2.17). We shall show in the next two sections that this can be done at least for the first and second equations in eqs.(2.17).

3. LINEARIZED OFF-SHELL GAUGE TRANSFORMATIONS

In this section we shall obtain solution to the first of eqs.(2.17), thereby reproducing linearized general coordinate invariance and antisymmetric tensor gauge invariance from string field theory. Although such analysis has been carried out in the past (see, for example ref.[13]), we shall repeat the analysis here for the sake of completeness, and also to set up the notations that we shall be using in the next section.

At the linearized level the metric $G_{\mu\nu}$, the antisymmetric tensor field $B_{\mu\nu}$ and the dilaton D appearing in the low energy effective field theory transform as,

$$\begin{aligned} \delta G_{\mu\nu} =& \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu \\ \delta B_{\mu\nu} =& \partial_\mu\xi_\nu - \partial_\nu\xi_\mu \\ \delta D =& 0 \end{aligned} \tag{3.1}$$

where ϵ_μ is the parameter labelling infinitesimal general coordinate transformation and ξ_μ is the parameter labelling infinitesimal antisymmetric tensor gauge transformation. On the other hand, if we take a gauge transformation parameter in string field theory

$$b_0^-|\Lambda\rangle = \int d^D k [i\tilde{\epsilon}_\mu(k)(c_1\alpha^\mu_{-1} - \bar{c}_1\bar{\alpha}^\mu_{-1}) + i\tilde{\xi}_\mu(k)(c_1\alpha^\mu_{-1} + \bar{c}_1\bar{\alpha}^\mu_{-1}) + \sqrt{2}\tilde{\xi}c_0^+]|k\rangle \quad (3.2)$$

and define the string field components at the massless level as,⋆

$$\begin{aligned} b_0^-|\Psi\rangle = \int d^D k [&\frac{1}{2}\tilde{h}_{\mu\nu}(k)c_1\bar{c}_1(\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1} + \bar{\alpha}^\mu_{-1}\alpha^\nu_{-1}) + i\sqrt{2}\tilde{P}_\mu(k)c_0^+(c_1\alpha^\mu_{-1} - \bar{c}_1\bar{\alpha}^\mu_{-1}) \\ &- \tilde{F}(k)(c_1c_{-1} - \bar{c}_1\bar{c}_{-1}) + \frac{1}{2}\tilde{b}_{\mu\nu}(k)c_1\bar{c}_1(\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1} - \bar{\alpha}^\mu_{-1}\alpha^\nu_{-1}) \\ &+ i\sqrt{2}\tilde{S}_\mu(k)c_0^+(c_1\alpha^\mu_{-1} + \bar{c}_1\bar{\alpha}^\mu_{-1}) - \tilde{E}(k)(c_1c_{-1} + \bar{c}_1\bar{c}_{-1})]\,|k\rangle \end{aligned} \quad (3.3)$$

then the linearized gauge transformation $\delta(b_0^-|\Psi\rangle) = Q_B b_0^-|\Lambda\rangle$ takes the form:

$$\begin{aligned} &\delta\tilde{h}_{\mu\nu} = -i(\tilde{\epsilon}_\mu k_\nu + k_\mu\tilde{\epsilon}_\nu), \qquad \delta\tilde{P}_\mu = \frac{k^2}{2}\tilde{\epsilon}_\mu, \qquad \delta\tilde{F} = ik^\mu\tilde{\epsilon}_\mu \\ &\delta\tilde{b}_{\mu\nu} = i(k_\mu\tilde{\xi}_\nu - k_\nu\tilde{\xi}_\mu), \qquad \delta\tilde{S}_\mu = \frac{k^2}{2}\tilde{\xi}_\mu + ik_\mu\tilde{\xi}, \qquad \delta\tilde{E}(k) = i\tilde{\xi}_\mu k^\mu - 2\tilde{\xi} \end{aligned} \quad (3.4)$$

Here α^μ_n denote the modes of the world-sheet scalar fields X^μ. Let us define,

$$\begin{aligned} h_{\mu\nu} &= \tilde{h}_{\mu\nu} \\ P_\mu &= \tilde{P}_\mu - \frac{i}{2}k^\nu\tilde{h}_{\mu\nu} - \frac{i}{2}k_\mu\tilde{F} \\ F &= \tilde{F} + \frac{1}{2}h^\mu_\mu \\ b_{\mu\nu} &= \tilde{b}_{\mu\nu} \\ S_\mu &= \tilde{S}_\mu - \frac{i}{2}k^\nu\tilde{b}_{\mu\nu} + \frac{i}{2}k_\mu\tilde{E} \\ E &= \tilde{E} \end{aligned} \quad (3.5)$$

for the fields; and similarly for the gauge parameters

$$\begin{aligned} \hat{\epsilon}_\mu &= -\tilde{\epsilon}_\mu \\ \hat{\xi}_\mu &= \tilde{\xi}_\mu \\ \hat{\xi} &= -\frac{i}{2}\tilde{\xi}_\mu k^\mu + \tilde{\xi} \end{aligned} \quad (3.6)$$

⋆ At the linearized level we would expect an off-shell configuration in low energy effective field theory to correspond to a string field configuration in which only the massless modes are excited, but beyond the linearized order all the modes will be present.

Transformation laws of various fields now take the form:

$$\begin{aligned}\delta h_{\mu\nu} =& i(\hat{\epsilon}_\mu k_\nu + \hat{\epsilon}_\nu k_\mu)\\ \delta P_\mu =& 0, \qquad\qquad \delta F = 0\\ \delta b_{\mu\nu} =& i(k_\mu \hat{\xi}_\nu - k_\nu \hat{\xi}_\mu)\\ \delta S_\mu =& 0, \qquad\qquad \delta E = -2\hat{\xi}\end{aligned} \tag{3.7}$$

Comparing with eq.(3.1) we see that at this level we can make the identification:

$$\begin{aligned}G_{\mu\nu}(x) =& \eta_{\mu\nu} - \sqrt{2}g \int d^D k h_{\mu\nu}(k) e^{ik.x}\\ B_{\mu\nu}(x) =& -\sqrt{2}g \int d^D k b_{\mu\nu}(k) e^{ik.x}\\ D(x) =& -\sqrt{2}g \int d^D k F(k) e^{ik.x}\end{aligned} \tag{3.8}$$

and,

$$\begin{aligned}\epsilon_\mu(x) =& -\sqrt{2}g \int d^D k \hat{\epsilon}_\mu(k) e^{ik.x}\\ \xi_\mu(x) =& -\sqrt{2}g \int d^D k \hat{\xi}_\mu(k) e^{ik.x}\end{aligned} \tag{3.9}$$

The proportionality factor of $-\sqrt{2}g$ was worked out in ref.[15]. (See also ref.[4].)

Note that there are three extra set of fields: P_μ, S_μ and E, and one extra gauge transformation parameter $\hat{\xi}$. Eq.(3.7) shows that E corresponds to a pure gauge deformation generated by the parameter $\hat{\xi}$, hence we can set E to be 0 by adjusting $\hat{\xi}$. This also removes the spurious gauge degrees of freedom associated with the parameter $\hat{\xi}$. The fields P_μ and S_μ, on the other hand, can be identified as auxiliary fields, as can be easily seen from the linearized equations of motion $Q_B b_0^- |\Psi\rangle = 0$. The equations involving the fields P_μ and S_μ take the form:

$$P_\mu = 0, \qquad\qquad S_\mu = 0 \tag{3.10}$$

Since these equations are purely algebraic, we can set these fields to 0 to this order by using their equations of motion. The remaining degrees of freedom are then in one to one correspondence with the physical degrees of freedom of the low energy effective field theory, and the remaining gauge transformation reproduces exactly the linearized (but off-shell) gauge transformation of the low energy effective field theory.

Using eqs.(3.2), (3.3), (3.5) and (3.6) we get,

$$b_0^-|\Lambda\rangle = i\int d^Dk\Big[-\hat{\epsilon}_\mu(k)(c_1\alpha^\mu_{-1}-\bar{c}_1\bar{\alpha}^\mu_{-1})+\hat{\xi}_\mu(k)(c_1\alpha^\mu_{-1}+\bar{c}_1\bar{\alpha}^\mu_{-1}+\frac{1}{\sqrt{2}}k^\mu c_0^+)\Big]\,|k\rangle \tag{3.11}$$

$$\begin{aligned} b_0^-|\Psi\rangle = \int d^Dk\Big[&h_{\mu\nu}(k)c_1\bar{c}_1\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1} \\ &+\sqrt{2}(-\frac{1}{2}k^\nu h_{\nu\mu}(k)+\frac{1}{4}k_\mu h^\nu_\nu(k)-\frac{1}{2}k_\mu F(k))c_0^+(c_1\alpha^\mu_{-1}-\bar{c}_1\bar{\alpha}^\mu_{-1}) \\ &-(F(k)-\frac{1}{2}h^\mu_\mu(k))(c_1c_{-1}-\bar{c}_1\bar{c}_{-1})+b_{\mu\nu}(k)c_1\bar{c}_1\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1} \\ &-\frac{1}{\sqrt{2}}k^\nu b_{\mu\nu}(k)c_0^+(c_1\alpha^\mu_{-1}+\bar{c}_1\bar{\alpha}^\mu_{-1})\Big]\,|k\rangle \end{aligned} \tag{3.12}$$

where we have set P_μ and S_μ to 0 by their equations of motion, and E to zero by using the gauge invariance generated by $\hat{\xi}(k)$. Using eqs.(3.8) and (3.9) we see that $h_{\mu\nu}(k)$, $b_{\mu\nu}(k)$ and $F(k)$ may be identified to the dynamical variables ϕ_i that appear in the low energy effective action, whereas the parameters $\hat{\epsilon}_\mu(k)$ and $\hat{\xi}_\mu(k)$ may be identified to the gauge transformation parameters η_κ that appear in the low energy effective action. Comparing with eqs.(2.8) and (2.10) of sect. 2 we see that eqs.(3.11) and (3.12) give the expressions for $|\Lambda(\eta,\phi)\rangle$ and $|\Psi(\phi)\rangle$ to order ϕ^0 and ϕ respectively such that the gauge transformations in string field theory match those in the low energy effective field theory. In other words, it gives expressions for $C^{(0)}_{ri}$ and $D^{(0)}_{\alpha\kappa}$ satisfying the first of eqs.(2.17):

$$C^{(0)}_{ri}B^{(2)}_{i\kappa} = A^{(2)}_{r\alpha}D^{(0)}_{\alpha\kappa} \tag{3.13}$$

Substituting eq.(3.12) into the expression (2.1) for the string field theory action we get, to quadratic order:

$$S \propto \int d^D k \big[h^{\mu\nu}(-k) (\frac{k^2}{2} h_{\mu\nu}(k) - k_\mu k^\sigma h_{\nu\sigma}(k) + \frac{1}{2} k_\mu k_\nu h^\sigma_\sigma(k) - k_\mu k_\nu F(k)) \\ - 2\big(F(-k) - \frac{1}{2} h^\mu_\mu(-k)\big)\big(k^2 F(k) - \frac{1}{2}(k^2 \eta^{\rho\sigma} - k^\rho k^\sigma) h_{\rho\sigma}(k)\big) \\ + b^{\mu\nu}(-k)(\frac{1}{2} k^2 b_{\mu\nu}(k) - k_\nu k^\rho b_{\mu\rho}(k))\big] \tag{3.14}$$

which agrees with the low energy effective action.

Finally, note that the identification of the low energy fields from their gauge transformation laws can be made only up to field redefinitions which do not change the gauge transformation laws of various fields. At the linearized level such field redefinitions will take the form $h_{\mu\nu} \to h_{\mu\nu} + \Delta h_{\mu\nu}$, $F \to F + \Delta F$ and $b_{\mu\nu} \to b_{\mu\nu} + \Delta b_{\mu\nu}$ where $\Delta h_{\mu\nu}$, ΔF and $\Delta b_{\mu\nu}$ are gauge invariant linear functions of $h_{\mu\nu}$, $b_{\mu\nu}$, F and k^μ. In addition, if we want to express the action in a form such that the quadratic terms in the action contain only two derivatives [16], then one should be careful while adding momentum dependent terms in $\Delta h_{\mu\nu}$, $\Delta b_{\mu\nu}$ and ΔF. In the present situation this leaves us with a field redefinition of the form $h_{\mu\nu} \to h_{\mu\nu} + aF\eta_{\mu\nu}$ where a is an arbitrary constant. In the context of low energy effective field theory this corresponds to a field redefinition of the form $G_{\mu\nu} \to f(D) G_{\mu\nu}$ where $f(D)$ is an arbitrary function of D.

4. FIRST ORDER NON-LINEAR TERMS IN THE GAUGE TRANSFORMATION

In this section we shall show that the second of eq.(2.17),

$$C^{(0)}_{ri} B^{(3)}_{ij\kappa} + C^{(1)}_{rji} B^{(2)}_{i\kappa} = A^{(2)}_{r\alpha} D^{(1)}_{\alpha\kappa j} + A^{(3)}_{r\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj} + K^{(1)}_{rs\kappa} \tilde{A}^{(2)}_{st} C^{(0)}_{tj} \tag{4.1}$$

can be satisfied by appropriately choosing $C^{(1)}_{rji}$, $D^{(1)}_{\alpha\kappa j}$ and $K^{(1)}_{rs\kappa}$.

In analyzing these equations we shall first try to look for possible obstructions to solving these equations. Such obstructions are caused by equations which follow from eq.(4.1) and are completely independent of $C^{(1)}_{rji}$, $D^{(1)}_{\alpha\kappa j}$ and $K^{(1)}_{rs\kappa}$. These would then correspond to equations involving known constants and would have to be satisfied identically. For this, let us choose a complete set of gauge transformations $\{\eta^{(\rho)}_\kappa\}$ and a complete set of field configurations $\{\phi^{(m)}_i\}$ in the low energy effective field theory, and rewrite eq.(4.1) as:

$$\begin{aligned} C^{(0)}_{ri}B^{(3)}_{ij\kappa}\phi^{(m)}_j\eta^{(\rho)}_\kappa + C^{(1)}_{rji}B^{(2)}_{i\kappa}\phi^{(m)}_j\eta^{(\rho)}_\kappa =& A^{(2)}_{r\alpha}D^{(1)}_{\alpha\kappa j}\phi^{(m)}_j\eta^{(\rho)}_\kappa + A^{(3)}_{r\alpha s}D^{(0)}_{\alpha\kappa}C^{(0)}_{sj}\phi^{(m)}_j\eta^{(\rho)}_\kappa \\ &+ K^{(1)}_{rs\kappa}\tilde{A}^{(2)}_{st}C^{(0)}_{tj}\phi^{(m)}_j\eta^{(\rho)}_\kappa, \qquad \text{for all } r, m, \rho. \end{aligned} \tag{4.2}$$

We now divide the set $\{\phi^{(m)}_j\}$ into two linearly independent sets, $\{\hat{\phi}^{(\hat{m})}_j\}$ and $\{\tilde{\phi}^{(\tilde{m})}_j\}$, satisfying,

$$\begin{aligned} \tilde{A}^{(2)}_{st}C^{(0)}_{tj}\hat{\phi}^{(\hat{m})}_j &= 0, \qquad \text{for all } s; \\ \tilde{A}^{(2)}_{st}C^{(0)}_{tj}\tilde{\phi}^{(\tilde{m})}_j &\neq 0, \qquad \text{for some } s. \end{aligned} \tag{4.3}$$

In other words $\hat{\phi}^{(\hat{m})}_j$ denote field configurations which are solutions of the linearized equations of motion, $\tilde{\phi}^{(\tilde{m})}_j$ denote those which are not. Similarly we divide the set $\{\eta^{(\rho)}_\kappa\}$ into linearly independent sets $\{\hat{\eta}^{(\hat{\rho})}_\kappa\}$ and $\{\tilde{\eta}^{(\tilde{\rho})}_\kappa\}$ satisfying:

$$\begin{aligned} B^{(2)}_{i\kappa}\hat{\eta}^{(\hat{\rho})}_\kappa &= 0, \qquad \text{for all } i; \\ B^{(2)}_{i\kappa}\tilde{\eta}^{(\tilde{\rho})}_\kappa &\neq 0, \qquad \text{for some } i. \end{aligned} \tag{4.4}$$

In other words, $\hat{\eta}^{(\hat{\rho})}_\kappa$ denote the set of gauge transformations for which the field independent components in the expression for $\delta\phi_i$ vanishes. $\{\hat{\eta}^{(\hat{\rho})}_\kappa\}$ thus includes rigid translation and rigid antisymmetric tensor gauge transformations, as well as antisymmetric tensor gauge transformations of the form $\xi_\mu = \partial_\mu\chi$ for some χ.⋆

⋆ Note that global rotation is not included in the set $\{\hat{\eta}^{(\hat{\rho})}_\kappa\}$, since the gauge transformation parameters for global rotation blow up at infinity. Another way of saying this is that the gauge transformation parameters are linear in the space-time coordinates X^μ, and hence the corresponding states in $\mathcal{H}$ do not correspond to well defined local fields in the conformal field theory.

Recall that $\{\langle\Phi^c_{3,r}|\}$ form a basis conjugate to $\{|\Phi_{2,s}\rangle\}$. We now divide the set of states $\{\langle\Phi^c_{3,r}|\}$ into two sets $\{\langle\hat{\Phi}^c_{3,\hat{r}}|\}$ and $\{\langle\tilde{\Phi}^c_{3,\tilde{r}}|\}$ such that,

$$\langle\hat{\Phi}^c_{3,\hat{r}}|Q_B = 0, \qquad \langle\tilde{\Phi}^c_{3,\tilde{r}}|Q_B \neq 0. \tag{4.5}$$

Let us now consider eq.(4.2) with the free indices m, ρ and r restricted to $\hat{m}$, $\hat{\rho}$ and $\hat{r}$ respectively. Thus in this case the terms involving $C^{(1)}$ and $K^{(1)}$ vanish by eq.(4.4) and (4.3) respectively. Furthermore, from eqs. (2.6) and (4.5) it follows that,

$$A^{(2)}_{\hat{r}\alpha} = 0. \tag{4.6}$$

In deriving the above equation we have used the fact that $\langle\hat{\Phi}^c_{3,\hat{r}}|\mathcal{O}|\Phi_{1,\alpha}\rangle = 0$ if the operator $\mathcal{O}$ does not contain the mode c_0^-. Thus the term involving $D^{(1)}$ in eq.(4.2) also vanishes when we choose the index r to be $\hat{r}$. This equation may then be written as:

$$C^{(0)}_{\hat{r}i} B^{(3)}_{ij\kappa} \hat{\phi}^{(\hat{m})}_j \hat{\eta}^{(\hat{\rho})}_\kappa = A^{(3)}_{\hat{r}\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj} \hat{\phi}^{(\hat{m})}_j \hat{\eta}^{(\hat{\rho})}_\kappa \tag{4.7}$$

Note that the undetermined coefficients $C^{(1)}$, $D^{(1)}$ and $K^{(1)}$ have disappeared from the above equation, whereas $C^{(0)}$ and $D^{(0)}$ have already been determined in the previous section. Hence unless the above equation is satisfied identically, there is a genuine obstruction to solving the set of equations (4.2). We show in appendix A that eq.(4.7) is satisfied identically. For the time being we assume this to be true and look for other possible obstructions of this kind.

Next, let us restrict the index r to be of type $\hat{r}$, and take $\phi^{(m)}_j$ to be of the form $B^{(2)}_{j\kappa'}\eta^{(\rho')}_{\kappa'}$. This causes the term involving $D^{(1)}$ in eq.(4.2) to vanish, since $A^{(2)}_{\hat{r}\alpha} = 0$. On the other hand, using eqs.(3.13) the term involving $K^{(1)}$ in eq.(4.2) may be brought to the form:

$$K^{(1)}_{rs\kappa} \tilde{A}^{(2)}_{st} C^{(0)}_{tj} B^{(2)}_{j\kappa'} \eta^{(\rho')}_{\kappa'} \eta^{(\rho)}_\kappa = K^{(1)}_{rs\kappa} \tilde{A}^{(2)}_{st} A^{(2)}_{t\alpha} D^{(0)}_{\alpha\kappa'} \eta^{(\rho')}_{\kappa'} \eta^{(\rho)}_\kappa \tag{4.8}$$

From eqs.(2.2) and (2.6) we get,

$$
\begin{aligned}
\tilde{A}^{(2)}_{st} A^{(2)}_{t\alpha} &= -\langle \Phi_{2,s} | c_0^- Q_B | \Phi_{2,t} \rangle \langle \Phi^c_{3,t} | c_0^- Q_B | \Phi_{1,\alpha} \rangle \\
&= -\langle \Phi_{2,s} | c_0^- Q_B b_0^- c_0^- Q_B | \Phi_{1,\alpha} \rangle \\
&= 0
\end{aligned}
\tag{4.9}
$$

In deriving the above equation, we have used the completeness relation that follows from eq.(2.4), and the nilpotence of the BRST charge $(Q_B)^2 = 0$. Thus the term involving $K^{(1)}$ in eq.(4.2) also vanishes in this case. The equation then takes the form:

$$
C^{(0)}_{\hat{r}i} B^{(3)}_{ij\kappa} B^{(2)}_{j\kappa'} \eta^{(\rho')}_{\kappa'} \eta^{(\rho)}_{\kappa} + C^{(1)}_{\hat{r}ji} B^{(2)}_{i\kappa} B^{(2)}_{j\kappa'} \eta^{(\rho')}_{\kappa'} \eta^{(\rho)}_{\kappa} = A^{(3)}_{\hat{r}\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj} B^{(2)}_{j\kappa'} \eta^{(\rho')}_{\kappa'} \eta^{(\rho)}_{\kappa} \tag{4.10}
$$

From the definition of $C^{(1)}_{rji}$ given in eq.(2.8) we see that it is symmetric under the exchange of i and j. (We have used this symmetry property to get eq.(2.9).) Thus the term involving $C^{(1)}$ in eq.(4.10) is symmetric under the exchange of ρ and ρ'. If we exchange ρ and ρ' in eq.(4.10) and subtract from the original equation, we get,

$$
C^{(0)}_{\hat{r}i} B^{(3)}_{ij\kappa} B^{(2)}_{j\kappa'} \eta^{(\rho')}_{\kappa'} \eta^{(\rho)}_{\kappa} - (\rho \leftrightarrow \rho') = A^{(3)}_{\hat{r}\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj} B^{(2)}_{j\kappa'} \eta^{(\rho')}_{\kappa'} \eta^{(\rho)}_{\kappa} - (\rho \leftrightarrow \rho') \tag{4.11}
$$

Note that all the undetermined constants $C^{(1)}$, $D^{(1)}$ and $K^{(1)}$ have dropped out of the above equation, and hence it again represents a possible obstruction to solving the set of equations given in eq.(4.2).

Finally, let us choose $\eta^{(\rho)}_{\kappa}$ in eq.(4.2) to be of the form $\hat{\eta}^{(\hat{\rho})}_{\kappa}$ and multiply both sides of the equation by $\tilde{A}^{(2)}_{rt'} C^{(0)}_{t'j'} \phi^{(m')}_{j'}$. In this case the term involving $C^{(1)}$ vanishes since $B^{(2)}_{i\kappa} \hat{\eta}^{(\hat{\rho})}_{\kappa} = 0$. The term involving $D^{(1)}$ is given by,

$$
D^{(1)}_{\alpha\kappa j} \phi^{(m)}_{j} \hat{\eta}^{(\hat{\rho})}_{\kappa} A^{(2)}_{r\alpha} \tilde{A}^{(2)}_{rt'} C^{(0)}_{t'j'} \phi^{(m')}_{j'} \tag{4.12}
$$

which vanishes by eq.(4.9) and the fact that $\tilde{A}^{(2)}_{rs}$ is symmetric in the indices r and

s. Thus the equation takes the form:

$$\begin{aligned} C^{(0)}_{ri} B^{(3)}_{ij\kappa} \phi^{(m)}_j \hat\eta^{(\hat\rho)}_\kappa \tilde A^{(2)}_{rt'} C^{(0)}_{t'j'} \phi^{(m')}_{j'} = & A^{(3)}_{r\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj} \phi^{(m)}_j \hat\eta^{(\hat\rho)}_\kappa \tilde A^{(2)}_{rt'} C^{(0)}_{t'j'} \phi^{(m')}_{j'} \\ & + K^{(1)}_{rs\kappa} \tilde A^{(2)}_{st} C^{(0)}_{tj} \phi^{(m)}_j \hat\eta^{(\hat\rho)}_\kappa \tilde A^{(2)}_{rt'} C^{(0)}_{t'j'} \phi^{(m')}_{j'} \end{aligned} \tag{4.13}$$

As we have seen in sect. 2, the quantity K_{rs} and hence $K^{(1)}_{rs\kappa}$ defined in eq.(2.14) must be antisymmetric in r and s. Hence the term involving $K^{(1)}$ in eq.(4.13) in antisymmetric in the indices m and m'. Thus if we symmetrize both sides of the equation in m and m', the term involving $K^{(1)}$ drops out and we are left with the equation:

$$\begin{aligned} & C^{(0)}_{ri} B^{(3)}_{ij\kappa} \phi^{(m)}_j \hat\eta^{(\hat\rho)}_\kappa \tilde A^{(2)}_{rt'} C^{(0)}_{t'j'} \phi^{(m')}_{j'} + (m \leftrightarrow m') \\ & \quad = A^{(3)}_{r\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj} \phi^{(m)}_j \hat\eta^{(\hat\rho)}_\kappa \tilde A^{(2)}_{rt'} C^{(0)}_{t'j'} \phi^{(m')}_{j'} + (m \leftrightarrow m') \end{aligned} \tag{4.14}$$

Since the above equation involves only the known quantities, this represents a third set of obstructions to solving the set of equations (4.2).

We shall now show that once eqs.(4.7), (4.11) and (4.14) are satisfied, we can always find a solution of the set of equations (4.1) (or, equivalently, eqs. (4.2)). In other words we can find appropriate $C^{(1)}$, $D^{(1)}$ and $K^{(1)}$ satisfying these equations. To start with, let us consider the case where the index r is of type $\tilde r$, i.e. $\langle\tilde\Phi^c_{3,\tilde r}|Q_B \neq 0$. In this case we may express eq.(4.1) as,

$$A^{(2)}_{\tilde r\alpha} D^{(1)}_{\alpha\kappa j} = C^{(0)}_{\tilde r i} B^{(3)}_{ij\kappa} + C^{(1)}_{\tilde r j i} B^{(2)}_{i\kappa} - A^{(3)}_{\tilde r\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj} - K^{(1)}_{\tilde r s\kappa} \tilde A^{(2)}_{st} C^{(0)}_{tj} \tag{4.15}$$

Let us now note that $A^{(2)}_{\tilde r\alpha}$ has no eigenvector with zero eigenvalue acting on the left. For, if there was such an eigenvector (say $x_{\tilde r}$) then it would give,

$$0 = \sum_{\tilde r} x_{\tilde r} A^{(2)}_{\tilde r\alpha} = \sum_{\tilde r} x_{\tilde r} \langle\tilde\Phi^c_{3,\tilde r}|c_0^- Q_B|\Phi_{1,\alpha}\rangle, \qquad \text{for all } \alpha \tag{4.16}$$

which, in turn, would imply that $\sum_{\tilde r} x_{\tilde r}\langle\tilde\Phi^c_{3,\tilde r}|$ is annihilated by Q_B. But we have already chosen the basis states such that all such states are included in the set

$\{\langle\hat{\Phi}^c_{3,\hat{r}}|\}$; the set $\{\langle\tilde{\Phi}^c_{3,\tilde{r}}|\}$ contains only those states which are not annihilated by Q_B. Thus there is no vector $x_{\tilde{r}}$ for which $\sum x_{\tilde{r}} A^{(2)}_{\tilde{r}\alpha} = 0$. This, in turn, shows that the matrix $A^{(2)}_{\tilde{r}\alpha}$ has a (non-unique) right inverse $M_{\alpha\tilde{s}}$, satisfying,⋆

$$A^{(2)}_{\tilde{r}\alpha} M_{\alpha\tilde{s}} = \delta_{\tilde{r}\tilde{s}} \tag{4.17}$$

We thus get a solution of eq.(4.15) of the form:

$$D^{(1)}_{\alpha\kappa j} = M_{\alpha\tilde{r}}[C^{(0)}_{\tilde{r}i} B^{(3)}_{ij\kappa} + C^{(1)}_{\tilde{r}ji} B^{(2)}_{i\kappa} - A^{(3)}_{\tilde{r}\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj} - K^{(1)}_{\tilde{r}s\kappa} \tilde{A}^{(2)}_{st} C^{(0)}_{tj}] \tag{4.18}$$

In other words, we can always adjust the coefficients $D^{(1)}_{\alpha\kappa j}$ so as to satisfy eq.(4.15).

Next we consider the case where the index r is taken to be of the type $\hat{r}$, but the index ρ (in eq.(4.2)) is taken to be of the type $\tilde{\rho}$. In this case, the term involving $D^{(1)}$ in eq.(4.2) vanishes, and we may rewrite this equation as,

$$C^{(1)}_{\hat{r}ji} B^{(2)}_{i\kappa} \phi^{(m)}_j \tilde{\eta}^{(\tilde{\rho})}_\kappa = (A^{(3)}_{\hat{r}\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj} + K^{(1)}_{\hat{r}s\kappa} \tilde{A}^{(2)}_{st} C^{(0)}_{tj} - C^{(0)}_{\hat{r}i} B^{(3)}_{ij\kappa}) \phi^{(m)}_j \tilde{\eta}^{(\tilde{\rho})}_\kappa \tag{4.19}$$

Let us consider the matrix $S_{i\tilde{\rho}} \equiv B^{(2)}_{i\kappa} \tilde{\eta}^{(\tilde{\rho})}_\kappa$. From our choice of basis described in eq.(4.4) it is clear that this matrix cannot have an eigenvector with zero eigenvalue while acting on the right; for if there was such an eigenvector (say $\{y_{\tilde{\rho}}\}$) then the combination $y_{\tilde{\rho}} \tilde{\eta}^{(\tilde{\rho})}_\kappa$ would have to be included in the set $\{\hat{\eta}^{(\hat{\rho})}_\kappa\}$, and the set $\{\hat{\eta}^{(\hat{\rho})}_\kappa\}$, $\{\tilde{\eta}^{(\tilde{\rho})}_\kappa\}$ will not together form a set of *linearly independent* basis vectors. This, in turn, shows that the matrix $S_{i\tilde{\rho}}$ must have a (non-unique) left inverse $N_{\tilde{\rho}j}$ satisfying,

$$N_{\tilde{\rho}i} B^{(2)}_{i\kappa} \tilde{\eta}^{(\tilde{\rho}')}_\kappa \equiv N_{\tilde{\rho}i} S_{i\tilde{\rho}'} = \delta_{\tilde{\rho}\tilde{\rho}'} \tag{4.20}$$

Thus we see that eq.(4.19) is satisfied if we choose,

$$C^{(1)}_{\hat{r}ji} = (A^{(3)}_{\hat{r}\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj} + K^{(1)}_{\hat{r}s\kappa} \tilde{A}^{(2)}_{st} C^{(0)}_{tj} - C^{(0)}_{\hat{r}l} B^{(3)}_{lj\kappa}) \tilde{\eta}^{(\tilde{\rho})}_\kappa N_{\tilde{\rho}i} \tag{4.21}$$

We must, however, make sure that the expression for $C^{(1)}_{\hat{r}ji}$ obtained this way is

⋆ Although $A^{(2)}$ is an infinite dimensional matrix, it is block diagonal in the basis where the states are taken to be eigenstates of the momentum operator and L^+_0, with each block being a finite matrix. Thus all the results for finite dimensional matrices can be applied here.

symmetric in i and j. To do this it is convenient to choose a specific basis for the variables $\{\phi_i\}$, which makes the matrix $S_{i\tilde{\rho}}$ look simple. Let us assume that $\tilde{\rho}$ takes N different values. (In general N is infinite, but if we work within a subspace of states with a given momentum, then N is finite.) Let us choose the basis for the variables ϕ_i such that[†]

$$S_{i\tilde{\rho}} = \begin{cases} \delta_{i\tilde{\rho}} & \text{for } 1 \le i \le N \\ 0 & \text{for } i > N \end{cases} \tag{4.22}$$

In this case

$$N_{\tilde{\rho}i} = \begin{cases} \delta_{\tilde{\rho}i} & \text{for } 1 \le i \le N \\ arbitrary & \text{for } i > N \end{cases} \tag{4.23}$$

Hence eq.(4.21) may be written as,

$$C^{(1)}_{\hat{r}ji} = \begin{cases} (A^{(3)}_{\hat{r}\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj} + K^{(1)}_{\hat{r}s\kappa} \tilde{A}^{(2)}_{st} C^{(0)}_{tj} - C^{(0)}_{\hat{r}l} B^{(3)}_{lj\kappa}) \tilde{\eta}^{(i)}_{\kappa}, & \text{for } 1 \le i \le N \\ \text{arbitrary} & \text{for } i \ge N+1. \end{cases} \tag{4.24}$$

Thus we see that if $i \le N$ and $j > N$, then the relation $C^{(1)}_{\hat{r}ij} = C^{(1)}_{\hat{r}ji}$ can be satisfied by setting $C^{(1)}_{\hat{r}ij}$ (which is undetermined otherwise) to be equal to $C^{(1)}_{\hat{r}ji}$. For $i, j > N$ one can always choose $C^{(1)}_{\hat{r}ij}$ to be symmetric in i and j, since it is not constrained at all. Thus the only possible source of discrepancy comes when both i and j are less than or equal to N. However, in this case, using eq.(4.22) we may write,

$$C^{(1)}_{\hat{r}ji} = C^{(1)}_{\hat{r}j'i} S_{j'j}, \qquad 1 \le i, j \le N. \tag{4.25}$$

Substituting the solution (4.21) on the right hand side of the above equation, using

† Such a choice of basis is possible through a linear field redefinition of the form $\phi_i \to W_{ij}\phi_j$, with

$$W_{ij} = \begin{cases} N_{ij} & \text{for } 1 \le i \le N \\ V_{ik}(\delta_{kj} - S_{k\tilde{\rho}'} N_{\tilde{\rho}'j}) & \text{for } i > N \end{cases}$$

where V is a matrix which should be chosen so as to make W non-singular, but is otherwise arbitrary.

the relation $N_{\tilde{\rho}i} = \delta_{\tilde{\rho}i}$, and that $S_{j'j} = B^{(2)}_{j'\kappa}\tilde{\eta}^{(j)}_{\kappa}$ for $1 \le j \le N$, we get,

$$\begin{aligned} C^{(1)}_{\hat{r}ji} =& (A^{(3)}_{\hat{r}\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj'} + K^{(1)}_{\hat{r}s\kappa}\tilde{A}^{(2)}_{st} C^{(0)}_{tj'} - C^{(0)}_{\hat{r}l} B^{(3)}_{lj'\kappa})\tilde{\eta}^{(i)}_{\kappa} B^{(2)}_{j'\kappa'}\tilde{\eta}^{(j)}_{\kappa'} \\ & \text{for } 1 \le i, j \le N. \end{aligned} \tag{4.26}$$

Since $K^{(1)}_{\hat{r}s\kappa}\tilde{A}^{(2)}_{st} C^{(0)}_{tj} B^{(2)}_{j\kappa'}\tilde{\eta}^{(j)}_{\kappa'}$ vanishes (see eqs.(4.8) and (4.9)) the term involving $K^{(1)}$ vanishes in the above equation. Eq.(4.11) then implies that the right hand side of the equation is symmetric in i and j, showing that $C^{(1)}_{\hat{r}ji}$ calculated from eq.(4.21) in indeed symmetric in i and j.

Thus it remains to show that if in eq.(4.2) we take the index r to be of type $\hat{r}$, and ρ to be of the type $\hat{\rho}$, then this equation can still be satisfied. In this case the terms involving $C^{(1)}$ as well as $D^{(1)}$ drop out of the equation. The only other free index in this equation is m, which can either be of type $\tilde{m}$, or of type $\hat{m}$. If this index is of type $\hat{m}$ then eq.(4.7) guarantees that eq.(4.2) is automatically satisfied. Thus we now need to ensure that we can satisfy eq.(4.2) by adjusting $K^{(1)}$ when the indices r, ρ and m are of the type $\hat{r}$, $\hat{\rho}$ and $\tilde{m}$ respectively. In this case we can write eq.(4.2) as,

$$K^{(1)}_{\hat{r}s\kappa}\tilde{A}^{(2)}_{st} C^{(0)}_{tj}\tilde{\phi}^{(\tilde{m})}_{j}\hat{\eta}^{(\hat{\rho})}_{\kappa} = C^{(0)}_{\hat{r}i} B^{(3)}_{ij\kappa}\tilde{\phi}^{(\tilde{m})}_{j}\hat{\eta}^{(\hat{\rho})}_{\kappa} - A^{(3)}_{\hat{r}\alpha s} D^{(0)}_{\alpha\kappa} C^{(0)}_{sj}\tilde{\phi}^{(\tilde{m})}_{j}\hat{\eta}^{(\hat{\rho})}_{\kappa}. \tag{4.27}$$

Let us define,

$$T_{s\tilde{m}} = \tilde{A}^{(2)}_{st} C^{(0)}_{tj}\tilde{\phi}^{(\tilde{m})}_{j} \tag{4.28}$$

Using eq.(4.3) and arguments similar to the case of the matrix $S_{i\tilde{\rho}}$, it is clear that $T_{s\tilde{m}}$, acting on the right, does not have any eigenvector with zero eigenvalue. As a result, it has a left inverse; let us call this $U_{\tilde{m}r}$:

$$U_{\tilde{m}r} T_{r\tilde{n}} = \delta_{\tilde{m}\tilde{n}} \tag{4.29}$$

Also note that since $\{\eta^{(\rho)}\}$ form a complete set of linearly independent vectors,

$\eta_\kappa^{(\rho)}$, regarded as a matrix in (ρ,κ) space, is invertible. Thus if we define,

$$\mathcal{K}^{(1)}_{rs\rho} = K^{(1)}_{rs\kappa}\eta_\kappa^{(\rho)} \tag{4.30}$$

then we can freely obtain K from $\mathcal{K}$ and vice versa with the help of the matrix $\eta_\kappa^{(\rho)}$ or its inverse. A solution to eq.(4.27) is given by,

$$\begin{aligned} \mathcal{K}^{(1)}_{rs\hat\rho} &= (C^{(0)}_{ri}B^{(3)}_{ij\kappa} - A^{(3)}_{r\alpha s'}D^{(0)}_{\alpha\kappa}C^{(0)}_{s'j})\tilde\phi_j^{(\tilde m)}U_{\tilde m s}\hat\eta_\kappa^{(\hat\rho)} \\ \mathcal{K}^{(1)}_{rs\tilde\rho} &= \text{ arbitrary.} \end{aligned} \tag{4.31}$$

Note that only $\mathcal{K}^{(1)}_{\hat r s\hat\rho}$ is determined by eq.(4.27). Eq.(4.31) gives a specific solution of eq.(4.27), but in general $\mathcal{K}^{(1)}_{\tilde r s\hat\rho}$ can be chosen arbitrarily. Finally we need to verify that $\mathcal{K}^{(1)}$ determined this way is antisymmetric in r and s. For this we assume that the index $\tilde m$ runs from 1 to M and, as in the case of the matrix $S_{i\tilde\rho}$, choose a basis of states $|\Phi_{2,r}\rangle$ such that:

$$T_{r\tilde m} = \begin{cases} \delta_{r\tilde m} & \text{for } 1\le r\le M \\ 0 & \text{for } r > M \end{cases} \tag{4.32}$$

so that $U_{\tilde m r}$ can be taken to be,

$$U_{\tilde m r} = \begin{cases} \delta_{\tilde m r} & \text{for } 1\le r\le M, \\ \text{arbitrary} & \text{for } r > M. \end{cases} \tag{4.33}$$

In this basis $\mathcal{K}^{(1)}_{rs\hat\rho}$ given in eq.(4.31) takes the form

$$\mathcal{K}^{(1)}_{rs\hat\rho} = \begin{cases} (C^{(0)}_{ri}B^{(3)}_{ij\kappa} - A^{(3)}_{r\alpha s'}D^{(0)}_{\alpha\kappa}C^{(0)}_{s'j})\phi_j^{(s)}\hat\eta_\kappa^{(\hat\rho)}, & \text{for } 1\le s\le M; \\ \text{arbitrary} & \text{for } s\ge M+1. \end{cases} \tag{4.34}$$

Thus we see that as long as either s or r is larger than M, the antisymmetry of $\mathcal{K}^{(1)}_{rs\hat\rho}$ may be satisfied by judicious choice of $\mathcal{K}^{(1)}_{rs\hat\rho}$ in the region $s\ge M+1$, where it is undetermined otherwise. The only possible problem comes from the range where both r and s lie in the range between 1 and M. But note that in this basis eq.(4.14) takes the form:

$$(C^{(0)}_{ri}B^{(3)}_{ij\kappa}\phi^{(s)}_j\hat{\eta}^{(\hat{\rho})}_\kappa - A^{(3)}_{r\alpha s'}D^{(0)}_{\alpha\kappa}C^{(0)}_{s'j}\phi^{(s)}_j\hat{\eta}^{(\hat{\rho})}_\kappa) + (r \leftrightarrow s) = 0, \quad \text{for } 1 \leq r, s \leq M \tag{4.35}$$

which is precisely the statement that $\mathcal{K}^{(1)}_{rs\hat{\rho}}$ given in eq.(4.34) is antisymmetric in r and s.

This completes the proof that once eqs.(4.7), (4.11) and (4.14) are satisfied, we can choose appropriate $C^{(1)}$, $D^{(1)}$ and $K^{(1)}$ so as to satisfy eq.(4.1). We have shown explicitly in appendix A that these equations are indeed satisfied. This in turn means that it is possible to identify appropriate string field configurations to off shell field configurations in low energy effective field theory, and to identify appropriate symmetries in string field theory to gauge symmetries in low energy effective field theory, so as to get the correct transformation laws of various fields in low energy effective field theory to first non-linear order.

Before we conclude this section, we would like to show that a solution to eq.(4.1) (or eq.(4.2)) cannot be obtained if we set $K^{(1)}_{rs\kappa} = 0$. For this let us take $K^{(1)}_{rs\kappa} = 0$, and choose $\eta^{(\rho)}_\kappa$, and $\langle\Phi^c_{3,r}|$ to be of the forms $\hat{\eta}^{(\hat{\rho})}_\kappa$ and $\langle\hat{\Phi}^c_{3,\hat{r}}|$ respectively. As a result, terms involving $C^{(1)}$ and $D^{(1)}$ drop out of eq.(4.2) and it takes the form:

$$C^{(0)}_{\hat{r}i}B^{(3)}_{ij\kappa}\phi^{(m)}_j\hat{\eta}^{(\hat{\rho})}_\kappa = A^{(3)}_{\hat{r}\alpha s}D^{(0)}_{\alpha\kappa}C^{(0)}_{sj}\phi^{(m)}_j\hat{\eta}^{(\hat{\rho})}_\kappa \tag{4.36}$$

Since this equation involves only known coefficients, this is a consistency equation. In appendix A we show one example of a specific choice of $\langle\hat{\Phi}^c_{3,\hat{r}}|$, $\hat{\eta}^{(\hat{\rho})}_\kappa$ and $\phi^{(m)}_j$ for which this equation is not satisfied. This, in turn, shows that it is not possible to obtain solutions of eq.(4.2) with $K^{(1)} = 0$.

5. DISCUSSION

In this paper we have studied how off-shell general coordinate transformations and antisymmetric tensor gauge transformations arise in string field theory. Working to first non-linear order, we have shown that it is possible to identify specific string field configurations with off-shell field configurations in low energy effective field theory, and specific symmetry transformations in string theory with off-shell gauge transformations in low energy effective field theory, so that the symmetry transformations in the former theory are compatible with those in the latter theory.

One of the specific results of our analysis is that the off-shell gauge symmetries of low energy effective field theory cannot be identified to just a combination of off-shell gauge symmetries alone of string field theory. Instead, they can be identified to a combination of off-shell gauge symmetries of string field theory and the trivial symmetries of the form given in eq. (1.1).

The gauge algebra of string field theory is characterized by two important features. The first is that it closes only on-shell; the second is that the algebra has field dependent structure constants. More specifically, if $|\Lambda_1\rangle$ and $|\Lambda_2\rangle$ are two independent gauge transformation parameters, the commutator of these two gauge transformation parameters, acting on the string field $|\Psi\rangle$ gives*:

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}]\psi_r = \delta_\Lambda \psi_r + M_{rs}(\{\psi_t\})\frac{\delta S}{\delta \psi_s} \tag{5.1}$$

where,

$$b_0^-|\Lambda\rangle = \sum_{N=0}^{\infty} \frac{g^{N+1}}{N!}[\Lambda_2 \Lambda_1 \Psi^N] \tag{5.2}$$

and M_{rs} is an antisymmetric matrix, given by,

$$M_{rs}(\Psi, \Lambda_1, \Lambda_2) = \sum_{N=0}^{\infty} \frac{g^{N+2}}{N!}\{(c_0^- \Phi^c_{3,s})(c_0^- \Phi^c_{3,r})\Psi^N \Lambda_1 \Lambda_2\} \tag{5.3}$$

Antisymmetry of M_{rs} follows from the property of $\{\ \}$ [4] [14]. On the other hand,

---

$\star$ This observation has been made independently by Zwiebach [17] the gauge symmetries of the low energy effective Lagrangian closes off-shell, and have field independent structure constants. Thus if our analysis can be extended to all orders in the string field Ψ, this would imply that the string field theory contains a symmetry (sub-)algebra which does close even when the massless field in the theory are off-shell (whereas the massless auxiliary fields and the massive fields are eliminated by their equations of motion). It is therefore natural to ask how to reconcile these apparently different features of string field theory and low energy effective field theory.

Let us first address the question of closure of the algebra. Note that although the commutator of two gauge transformations in string field theory contains trivial symmetry transformations given in eq.(1.1), and hence the gauge algebra closes only on-shell, it is conceivable that one can redefine the gauge transformation laws by adding appropriate combination of the trivial symmetry transformations to each gauge transformation, so that the resulting algebra (or some subalgebra of the resulting algebra) closes off-shell. We expect that this is precisely what happens in this case. In fact, from our analysis, we have already seen that the gauge transformations of low energy effective field theory indeed correspond to combinations of gauge transformation and the trivial symmetry in string field theory.

On the other hand, the structure constants of the algebra can be changed by appropriate redefinition of gauge transformation parameters. (A somewhat contorted example is of a $U(1)$ gauge theory, where we could have defined the gauge transformation law of the gauge field A_μ to $\delta A_\mu = \partial_\mu((1+f(A))\epsilon)$ where f is some function of A_μ. This would, in general, give field dependent structure constants for the gauge group, although the standard $U(1)$ algebra has field independent structure constants.) Thus it is not surprising that one can obtain suitable (field dependent) combination of gauge transformations in string field theory to get a subalgebra of the gauge group with field independent structure constants. To show that such combinations can really be obtained we need to extend the analysis of the paper to higher orders in the fields.

We expect that the extension of our analysis to higher orders in Ψ can be carried out using a method of induction, where we assume that a solution of the set of first N equations appearing in eq.(2.17) have been obtained, and then prove that the $(N+1)$th equation in that set can also be solved. We hope to come back to this question in the future.

APPENDIX A

EXPLICIT VERIFICATION OF THE CONSISTENCY CONDITIONS

In this appendix we shall show that the consistency conditions represented by eqs.(4.7), (4.11) and (4.14) are indeed satisfied by the vertices of string field theory. We also demonstrate by one example that eq. (4.36) is in general not satisfied.

We begin with eq.(4.7). Let us define,

$$b_0^-|\hat{\Psi}^{(\hat{m})}\rangle = C^{(0)}_{sj}\hat{\phi}^{(\hat{m})}_j|\Phi_{2,s}\rangle \tag{A.1}$$

and,

$$b_0^-|\hat{\Lambda}^{(\hat{\rho})}\rangle = D^{(0)}_{\alpha\kappa}\hat{\eta}^{(\hat{\rho})}_\kappa|\Phi_{1,\alpha}\rangle \tag{A.2}$$

From this we see that,

$$\langle\Phi_{2,r}|c_0^-Q_Bb_0^-|\hat{\Psi}^{(\hat{m})}\rangle = -C^{(0)}_{sj}\hat{\phi}^{(\hat{m})}_j\tilde{A}^{(2)}_{rs} = 0 \tag{A.3}$$

and,

$$\langle\Phi^c_{3,r}|c_0^-Q_Bb_0^-|\hat{\Lambda}^{(\hat{\rho})}\rangle = D^{(0)}_{\alpha\kappa}\hat{\eta}^{(\hat{\rho})}_\kappa A^{(2)}_{r\alpha} = C^{(0)}_{ri}B^{(2)}_{i\kappa}\hat{\eta}^{(\hat{\rho})}_\kappa = 0 \tag{A.4}$$

using eq.(4.3) and eqs.(3.13) and (4.4) respectively. This, in turn, shows that $Q_Bb_0^-|\hat{\Psi}^{(\hat{m})}\rangle$ and $Q_Bb_0^-|\hat{\Lambda}^{(\hat{\rho})}\rangle$ vanish identically.

Using eq.(2.6) and the definition of { } [4] [14] we get

$$A^{(3)}_{\hat{r}\alpha s} = g\langle f_1 \circ \hat{\Phi}^c_{3,\hat{r}}(0) f_2 \circ \Phi_{1,\alpha}(0) f_3 \circ \Phi_{2,s}(0)\rangle \tag{A.5}$$

where f_i are known conformal maps [18] [3] [4] [14]. From this we see that the right hand side of eq.(4.7) takes the form:

$$g\langle f_1 \circ \hat{\Phi}^c_{3,\hat{r}}(0) f_2 \circ b_0^- \hat{\Lambda}^{(\hat{\rho})}(0) f_3 \circ b_0^- \hat{\Psi}^{(\hat{m})}(0)\rangle \tag{A.6}$$

where $b_0^- \hat{\Lambda}^{(\hat{\rho})}$ and $b_0^- \hat{\Psi}^{(\hat{m})}$ are the local fields in conformal field theory which create the states $b_0^-|\hat{\Lambda}^{(\hat{\rho})}\rangle$ and $b_0^-|\hat{\Psi}^{(\hat{m})}\rangle$ acting on the vacuum [19]. As we have seen, $\hat{\Phi}^c_{3,\hat{r}}$, $b_0^- \hat{\Lambda}^{(\hat{\rho})}$ and $b_0^- \hat{\Psi}^{(\hat{m})}$ are all BRST invariant fields. Hence if any of them is a BRST trivial field, then expression (A.6) vanishes identically, $-$ the only contribution comes from the term when $\hat{\Phi}^c_{3,\hat{r}}$, $\hat{\Lambda}^{(\hat{\rho})}$ and $\hat{\Psi}^{(\hat{m})}$ all correspond to (non-zero) elements of BRST cohomology.

We thus first need to verify that the left hand side of eq.(4.7) vanishes when either of $\hat{\Phi}^c_{3,\hat{r}}$, $\hat{\Lambda}^{(\hat{\rho})}$ or $\hat{\Psi}^{(\hat{m})}$ is BRST trivial. To this end, note that under the antisymmetric tensor gauge transformation, the transformation of the fields have only linear term ($\delta B_{\mu\nu} \propto \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$), hence $B^{(3)}_{ij\kappa}$ vanishes in this case. Thus we need to look for contribution to the left hand side of eq.(4.7) from general coordinate transformation. For a general coordinate transformation labelled by the parameter $\epsilon_\mu(x)$, we see from eq.(3.11) that,

$$b_0^-|\hat{\Lambda}^{(\hat{\rho})}\rangle = -i \int d^D k \hat{\epsilon}_\mu(k)(c_1 \alpha^\mu_{-1} - \bar{c}_1 \bar{\alpha}^\mu_{-1})|k\rangle \tag{A.7}$$

It is easy to see that for no $\hat{\epsilon}_\mu$ this can be written as $Q_B|s\rangle$. Thus $b_0^-|\hat{\Lambda}^{(\hat{\rho})}\rangle$ must be a (non-zero) member of the BRST cohomology. A straightforward analysis of the BRST cohomology shows that such states are given by $\hat{\epsilon}_\mu(k) = -(1/\sqrt{2}g)\epsilon_\mu \delta^{(D)}(k)$, corresponding to rigid translation by an amount ϵ_μ (see eqs.(3.9)). Thus in this case,

$$B^{(3)}_{ij\kappa}\hat{\phi}^{(\hat{m})}_j\hat{\eta}^{(\hat{\rho})}_\kappa = i\epsilon_\mu k^\mu \hat{\phi}^{(\hat{m})}_i \tag{A.8}$$

where k is the D-momentum carried by $\hat{\phi}^{(\hat{m})}_j$.⋆ Since $C^{(0)}_{ri}$ is block diagonal in the momentum space (i.e. if $\phi^{(m)}_j$ carries momentum k then $b_0^-|\Psi^{(m)}\rangle \equiv C^{(0)}_{rj}\phi^{(m)}_j|\Phi_{2,r}\rangle$ also carries momentum k), we can express $C^{(0)}_{\hat{r}i}B^{(3)}_{ij\kappa}\hat{\phi}^{(\hat{m})}_j$ as $(i\epsilon_\mu k^\mu)C^{(0)}_{\hat{r}j}\hat{\phi}^{(\hat{m})}_j$. Using eq.(A.1) we now see that,

$$C^{(0)}_{\hat{r}j}\hat{\phi}^{(\hat{m})}_j = \langle\hat{\Phi}^c_{3,\hat{r}}|\hat{\Psi}^{(\hat{m})}\rangle \tag{A.9}$$

Hence if either $\langle\hat{\Phi}^c_{3,\hat{r}}|$ or $b_0^-|\hat{\Psi}^{(\hat{m})}\rangle$ is BRST trivial, $C^{(0)}_{\hat{r}j}\hat{\phi}^{(\hat{m})}_j$ vanishes. This, in turn, shows that the left hand side of eq.(4.7) also vanishes unless $\langle\hat{\Phi}^c_{3,\hat{r}}|$, $b_0^-|\hat{\Psi}^{(\hat{m})}\rangle$ and $b_0^-|\hat{\Lambda}^{(\hat{\rho})}\rangle$ are all (non-zero) elements of the BRST cohomology.

We can now restrict our attention to the case where $\langle\hat{\Phi}^c_{3,\hat{r}}|$, $|\hat{\Psi}^{(\hat{m})}\rangle$ and $|\hat{\Lambda}^{(\hat{\rho})}\rangle$ are all (non-zero) elements of the BRST cohomology. First let us consider the case where $|\hat{\Lambda}^{(\hat{\rho})}\rangle$ corresponds to gauge transformation associated with antisymmetric tensor field. As remarked before, in this case the left hand side of eq.(4.7) vanishes. The right hand side is given by eq.(A.6). Standard analysis of BRST cohomology shows that $b_0^-|\hat{\Lambda}^{(\hat{\rho})}\rangle$ must have the form $\xi_\mu(c_1\alpha^\mu_{-1}+\bar{c}_1\bar{\alpha}^\mu_{-1})|0\rangle$, whereas $b_0^-|\hat{\Psi}^{(\hat{m})}\rangle$ and $\langle\hat{\Phi}^c_{3,\hat{r}}|$ can be taken to be of the form $a_{\mu\nu}(k)c_1\bar{c}_1\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}|k\rangle$, and $\langle -k|\alpha^\mu_1\bar{\alpha}^\nu_1\bar{c}_{-1}c_{-1}c_0^+e_{\mu\nu}(k)$ respectively, with $k^\mu a_{\mu\nu} = k^\nu a_{\mu\nu} = k^\mu e_{\mu\nu} = k^\nu e_{\mu\nu} = 0$, $k^2 = 0$. Since each of these are primary states, it is straightforward to compute expression (A.6). It turns out to vanish, showing that eq.(4.7) is satisfied for the case where $\hat{\eta}^{(\hat{\rho})}_\kappa$ is taken to be an antisymmetric tensor gauge transformation.

We now look at the case where $|\hat{\Lambda}^{(\hat{\rho})}\rangle$ (or equivalently $\hat{\eta}^{(\hat{\rho})}_\kappa$) correspond to general coordinate transformation. As remarked before, the only (non-zero) element of BRST cohomology is generated by rigid translation, and using eq.(A.9) and the

⋆ We have chosen a basis $\{\phi^{(m)}_j\}$ such that $\phi^{(m)}_j$ has a fixed momentum.

discussion above it, we can bring the left hand side of eq.(4.7) to the form:

$$i\epsilon_\mu k^\mu \langle \Phi^c_{3,r} | \hat{\Psi}^{(\hat{m})} \rangle \tag{A.10}$$

The right hand side of eq.(4.7) is given by eq.(A.6). From eq.(A.7) and the fact that $\hat{\epsilon}_\mu = (-1/\sqrt{2}g)\epsilon_\mu \delta^{(D)}(k)$ (eq.(3.9)), we get,

$$b_0^- |\hat{\Lambda}^{(\hat{\rho})}\rangle = \frac{i}{\sqrt{2}g}\epsilon_\mu (c_1 \alpha^\mu_{-1} - \bar{c}_1 \bar{\alpha}^\mu_{-1})|0\rangle \tag{A.11}$$

$\langle \hat{\Phi}^c_{3,\hat{r}}|$ and $b_0^-|\hat{\Psi}^{(\hat{m})}\rangle$ have the same form as given in the previous paragraph. Since $\hat{\Phi}^c_{3,\hat{r}}$, $b_0^- \hat{\Lambda}^{(\hat{\rho})}$ and $b_0^- \hat{\Psi}^{(\hat{m})}$ are all dimension $(0,0)$ primary fields, it is straightforward to evaluate eq.(A.6) and we get the answer:

$$i\epsilon_\mu k^\mu \langle \hat{\Phi}^c_{3,\hat{r}} | \hat{\Psi}^{(\hat{m})} \rangle \tag{A.12}$$

which agrees with eq.(A.10).

Finally, note that if $k = 0$, then the set $\{|\hat{\Psi}^{(\hat{m})}\rangle\}$ contains an extra physical state of the form $(c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})|0\rangle$, and the set $\{\langle \hat{\Phi}^c_{3,\hat{r}}|\}$ contains an extra physical state of the form $\langle 0|(c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})c_0^+$. It is easy to see that both for $|\hat{\Lambda}^{(\hat{\rho})}\rangle$ representing a rigid translation or a rigid antisymmetric tensor gauge transformation, the right and the left hand side of eq.(4.7) vanishes.

This completes the proof that eq.(4.7) is satisfied for all values of $\hat{r}$, $\hat{\rho}$ and $\hat{m}$.

We now turn to eq.(4.11). Defining $b_0^-|\Lambda^{(\rho)}\rangle = \eta^{(\rho)}_\kappa D^{(0)}_{\alpha\kappa}|\Phi_{1,\alpha}\rangle$ as before, and noting that,

$$\begin{aligned} C^{(0)}_{sj} B^{(2)}_{j\kappa'} \eta^{(\rho')}_{\kappa'} |\Phi_{2,s}\rangle =& A^{(2)}_{s\alpha} D^{(0)}_{\alpha\kappa'} \eta^{(\rho')}_{\kappa'} |\Phi_{2,s}\rangle \\ =& \langle \Phi^c_{3,s}| c_0^- Q_B |\Phi_{1,\alpha}\rangle D^{(0)}_{\alpha\kappa'} \eta^{(\rho')}_{\kappa'} |\Phi_{2,s}\rangle \\ =& Q_B b_0^- |\Lambda^{(\rho')}\rangle \end{aligned} \tag{A.13}$$

(we have used eq.(3.13)); we can express the right hand side of eq.(4.11) as,

$$g[\langle f_1 \circ \hat{\Phi}^c_{3,\hat{r}}(0) f_2 \circ b_0^- \Lambda^{(\rho)}(0) f_3 \circ Q_B b_0^- \Lambda^{(\rho')}(0)\rangle - \langle f_1 \circ \hat{\Phi}^c_{3,\hat{r}}(0) f_2 \circ \Lambda^{(\rho')}(0) f_3 \circ Q_B b_0^- \Lambda^{(\rho)}(0)\rangle] \tag{A.14}$$

Since the closed string vertex is completely symmetric, we can replace f_2 by f_3 and

f_3 by f_2 in the last term in eq.(A.14). Using the fact that $\hat{\Phi}^c_{3,\hat{r}}$ is BRST invariant, we can deform the Q_B contour in the first term so that it acts on $b_0^-\Lambda^{(\rho)}(0)$ instead of $b_0^-\Lambda^{(\rho')}(0)$. We pick up two minus signs in this process, one from reversing the contour, the other from commuting Q_B through $b_0^-\Lambda^{(\rho)}$. The result is that the first term exactly cancels the second term, showing that the right hand side of eq.(4.11) vanishes identically.

What about the left hand side? Note that $\phi_j \equiv B^{(2)}_{j\kappa}\eta^{(\rho)}_\kappa$ represents a field configuration of the low energy effective field theory that is obtained from the zero field configuration by the gauge transformation generated by $\eta^{(\rho)}_\kappa$. Acting on this field configuration with the gauge transformation $\eta^{(\rho')}_{\kappa'}$ we generate a field configuration,

$$\phi_i + B^{(2)}_{i\kappa'}\eta^{(\rho')}_{\kappa'} + B^{(3)}_{ij\kappa'}\phi_j\eta^{(\rho')}_{\kappa'} = B^{(2)}_{i\kappa}\eta^{(\rho)}_\kappa + B^{(2)}_{i\kappa'}\eta^{(\rho')}_{\kappa'} + B^{(3)}_{ij\kappa'}\eta^{(\rho')}_{\kappa'}B^{(2)}_{j\kappa}\eta^{(\rho)}_\kappa \quad \text{(A.15)}$$

Antisymmetrizing the above expression in ρ and ρ', we get a field configuration generated by the commutator of the two transformations generated by $\eta^{(\rho)}_\kappa$ and $\eta^{(\rho')}_{\kappa'}$. This configuration is given by,

$$B^{(3)}_{ij\kappa}B^{(2)}_{j\kappa'}\eta^{(\rho)}_\kappa\eta^{(\rho')}_{\kappa'} - (\rho \leftrightarrow \rho') \quad \text{(A.16)}$$

Since the gauge algebra of the low energy effective field theory closes off-shell, the commutator of two gauge transformations is another gauge transformation. Let $\eta^{(0)}_\kappa$ be the parameter labelling this new gauge transformation. Then eq.(A.16) may be expressed as,

$$B^{(2)}_{i\kappa}\eta^{(0)}_\kappa \quad \text{(A.17)}$$

Using eq.(3.13), and the fact that $A^{(2)}_{\hat{r}\alpha} = \langle\hat{\Phi}^c_{3,\hat{r}}|c_0^- Q_B|\Phi_{1,\alpha}\rangle = 0$, the left hand side of eq.(4.11) may be written as,

$$C^{(0)}_{\hat{r}i}B^{(2)}_{i\kappa}\eta^{(0)}_\kappa = A^{(2)}_{\hat{r}\alpha}D^{(0)}_{\alpha\kappa}\eta^{(0)}_\kappa = 0 \quad \text{(A.18)}$$

Thus we see that the left hand side of eq.(4.11) also vanishes identically. This proves that eq.(4.11) is satisfied for all values of $\hat{r}$, ρ and ρ'.

Let us now consider eq.(4.14). If we define $b_0^-|\Psi^{(m')}\rangle = C^{(0)}_{t'j'}\phi^{(m')}_{j'}|\Phi_{2,t'}\rangle$ and $b_0^-|\hat{\Lambda}^{(\hat{\rho})}\rangle = \hat{\eta}^{(\hat{\rho})}_{\kappa} D^{(0)}_{\alpha\kappa}|\Phi_{1,\alpha}\rangle$, then, by standard manipulations, the right hand side of eq.(4.14) may be shown to be proportional to,

$$[\langle f_1 \circ Q_B b_0^- \Psi^{(m')}(0) f_2 \circ b_0^- \hat{\Lambda}^{(\hat{\rho})}(0) f_3 \circ b_0^- \Psi^{(m)}(0)\rangle + (m \leftrightarrow m')] \tag{A.19}$$

Since $Q_B b_0^-|\hat{\Lambda}^{(\hat{\rho})}\rangle = 0$, we can again deform the BRST contour in the first term so that Q_B acts on $b_0^-\Psi^{(m)}(0)$. Using the symmetry of the vertex we can also interchange f_1 and f_3 in the second term, and make appropriate rearrangement of the operators inside the correlator, picking up appropriate signs in the process. The final result is that the two terms in eq.(A.19) exactly cancel each other, thereby showing that the right hand side of eq.(4.14) vanishes identically.

It thus remains to show that the left hand side of eq.(4.14) also vanishes identically. To see this, let us write the action for the low energy effective field theory in the following form:

$$S_{eff}(\phi) = \sum_{N=2}^{\infty} \frac{1}{N} \tilde{B}^{(N)}_{i_1\ldots i_N} \phi_{i_1} \ldots \phi_{i_N} \tag{A.20}$$

Invariance of this action under the gauge transformation given in eq.(2.7) then gives,

$$\begin{aligned} \tilde{B}^{(2)}_{ij} B^{(2)}_{j\kappa} &= 0 \\ \tilde{B}^{(3)}_{ijl} B^{(2)}_{i\kappa} + \tilde{B}^{(2)}_{ij} B^{(3)}_{il\kappa} + (j \leftrightarrow l) &= 0 \end{aligned} \tag{A.21}$$

Multiplying the second of eq.(A.21) by $\hat{\eta}^{(\hat{\rho})}_{\kappa}$ and using eq.(4.4) we get,

$$\tilde{B}^{(2)}_{ij} B^{(3)}_{il\kappa} \hat{\eta}^{(\hat{\rho})}_{\kappa} + (j \leftrightarrow l) = 0 \tag{A.22}$$

Using eqs.(2.1), (2.8), and (A.20) we get,

$$\tilde{B}^{(2)}_{ij} = \tilde{A}^{(2)}_{rt} C^{(0)}_{ri} C^{(0)}_{tj} \tag{A.23}$$

(Note that this relation has already been verified in sect. 3, where the quadratic part of the action obtained from string field theory was shown to agree with that of low energy effective field theory.) Eq.(A.22) then takes the form,

$$\tilde{A}^{(2)}_{rt} C^{(0)}_{ri} C^{(0)}_{tj} B^{(3)}_{il\kappa} \hat{\eta}^{(\hat{\rho})}_{\kappa} + (j \leftrightarrow l) = 0 \tag{A.24}$$

Vanishing of the left hand side of eq.(4.14) is an immediate consequence of this equation.

Hence both sides of eq.(4.14) vanish, and are, therefore equal to each other. This completes the verification of eqs.(4.7), (4.11) and (4.14).

Finally we give an example to show that eq.(4.36) breaks down for specific choices of $\hat{\Phi}^c_{3,\hat{r}}$, $\hat{\eta}^{(\hat{\rho})}_{\kappa}$ and $\phi^{(m)}_j$. We choose,

$$b_0^- |\hat{\Lambda}^{(\hat{\rho})}\rangle \equiv D^{(0)}_{\alpha\kappa} \hat{\eta}^{(\hat{\rho})}_{\kappa} |\Phi_{1,\alpha}\rangle = \frac{i}{\sqrt{2}g} \epsilon_\mu (c_1 \alpha^\mu_{-1} - \bar{c}_1 \bar{\alpha}^\mu_{-1}) |0\rangle \tag{A.25}$$

$$\langle \hat{\Phi}^c_{3,\hat{r}} | = a_{\mu\nu} \langle -k | c_{-1} \bar{c}_{-1} \alpha^\mu_1 \bar{\alpha}^\nu_1 Q_B \tag{A.26}$$

and,

$$\begin{aligned} b_0^- |\Psi^{(m)}\rangle \equiv & C^{(0)}_{rj} \phi^{(m)}_j |\Phi_{2,r}\rangle \\ = & [h_{\mu\nu} c_1 \bar{c}_1 \alpha^\mu_{-1} \bar{\alpha}^\nu_{-1} - \frac{1}{\sqrt{2}} (k^\nu h_{\nu\mu} - \frac{1}{2} k_\mu h^\nu_\nu) c_0^+ (c_1 \alpha^\mu_{-1} - \bar{c}_1 \bar{\alpha}^\mu_{-1}) \\ & + \frac{1}{2} h^\mu_\mu (c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})] |k\rangle \end{aligned} \tag{A.27}$$

with $h_{\mu\nu} = h_{\nu\mu}$. In other words, $\hat{\eta}^{(\hat{\rho})}_{\kappa}$ denotes a rigid translation with parameter ϵ_μ, $\langle \hat{\Phi}^c_{3,\hat{r}}|$ denotes a pure gauge (BRST-exact) state and $\phi^{(m)}_j$ corresponds to an off-shell graviton background with momentum k (see eq.(3.12)). In this case the left hand side of the equation is proportional to,

$$i\epsilon \cdot k C^{(0)}_{\hat{r}j} \phi^{(m)}_j = i\epsilon . k \langle \hat{\Phi}^c_{3,\hat{r}} | \Psi^{(m)} \rangle \tag{A.28}$$

On the other hand, the right hand side of the equation is given by,

$$g \langle f_1 \circ \hat{\Phi}^c_{3,\hat{r}}(0) f_2 \circ b_0^- \hat{\Lambda}^{(\hat{\rho})}(0) f_3 \circ b_0^- \Psi^{(m)}(0) \rangle \tag{A.29}$$

which can be shown to be equal to

$$i\epsilon.k\langle\hat{\Phi}^c_{3,\hat{r}}|\Psi^{(m)}\rangle + \text{ extra terms proportional to the equations of motion.} \quad \text{(A.30)}$$

The expressions (A.28) and (A.29) therefore differ unless the background fields $\phi_j^{(m)}$ are solutions of the equations of motion. This, in turn, shows that we cannot solve the set of equations (4.1) with $K^{(1)} = 0$.

REFERENCES

N1 J. Scherk and J. Schwarz, Nucl. Phys. **B81** (1974) 118, Phys. Lett. **B52** (1974) 347. N2 T. Yoneya, Nuovo Cimento Lett. **8** (1973) 951; Prog. Theor. Phys. **51** (1974) 1907; **56** (1976) 1310. N3 M. Saadi and B. Zwiebach, Ann. Phys. **192** (1989) 213; T. Kugo, H. Kunitomo, and K. Suehiro, Phys. Lett. **226B** (1989) 48; N4 T. Kugo and K. Suehiro, Nucl. Phys. **B337** (1990) 434. N5 M. Kaku and J. Lykken, Phys. Rev. **D38** (1988) 3067; M. Kaku, preprints CCNY-HEP-89-6, Osaka-OU-HET 121. N6 C. Schubert, MIT preprint CTP 1977. N7 H. Hata, Phys. Lett. **217B** (1989) 438, 445; Nucl. Phys. **B329** (1990) 698; **B339** (1990) 663. N8 H. Sonoda and B. Zwiebach, Nucl. Phys. **B331** (1990) 592. N9 M. Saadi, Mod. Phys. Lett. **A5** (551) 1990; Int. J. Mod. Phys. **A6** (1991) 1003. N10 B. Zwiebach, Mod. Phys. Lett. **A5** (1990) 2753; Phys. Lett. **B241** (1990) 343; Comm. Math. Phys. **136** (1991) 83. N11 L. Hua and M. Kaku, Phys. Lett. **B250** (1990) 56; M. Kaku, Phys. Lett. **B250** (1990) 64. N12 H. Hata and M. Maeno, preprint KUNS-1037 (1990). N13 W. Siegel, Introduction to String Field Theory, World Scientific, 1988. N14 A. Sen, Phys. Lett. **B241** (1990) 350, Nucl. Phys. **B345** (1990) 551. N15 S. Mukherji and A. Sen, Nucl. Phys. **B363** (1991) 639. N16 B. Zwiebach, Phys. Lett. **B156** (1985) 315 N17 B. Zwiebach, as quoted in ref. [6]. N18 A. LeClair, M. Peskin and C. Preitschopf, Nucl. Phys. **B317** (1989) 411,464. N19 A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333.