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ELECTRIC MAGNETIC DUALITY IN STRING THEORY

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ABSTRACT

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The electric-magnetic duality transformation in four dimensional heterotic string theory discussed by Shapere, Trivedi and Wilczek is shown to be an exact symmetry of the equations of motion of low energy effective field theory even after including the scalar and the vector fields, arising due to compactification, in the effective field theory. Using this duality transformation we construct rotating black hole solutions in the effective field theory carrying both, electric and magnetic charges. The spectrum of extremal magnetically charged black holes turns out to be similar to that of electrically charged elementary string excitations. We also discuss the possibility that the duality symmetry is an exact symmetry of the full string theory under which electrically charged elementary string excitations get exchanged with magnetically charged soliton like solutions. This proposal might be made concrete following the suggestion of Dabholkar et. al. that fundamental strings may be regarded as soliton like classical solutions in the effective field theory.

1. Introduction

In a recent paper, Shapere et. al.[1] showed that the equations of motion of the coupled Einstein-Maxwell-axion-dilaton system, that occur in the low energy effective action in string theory, is invariant under an electric-magnetic duality transformation [2] [3] that also interchanges the strong and weak coupling limits of string theory.⋆ In this paper we shall show that this symmetry is valid even when we include the extra massless scalar and vector fields that arise from the (toroidal) compactification of ten dimensional heterotic string theory to four dimensions.

As was discussed in ref.[1], the duality transformation, together with the invariance under the shift of the axion field, form an SL(2,**R**) group. One immediate consequence of this symmetry is that one can use this transformation on known electrically charged classical solutions to generate solutions carrying both electric and magnetic charges [1].† We apply these transformations on the known rotating charged black hole solution in string theory [5] to construct rotating dyonic black hole solutions in string theory. We also find that when the string coupling constant in equal to unity (with suitable normalization), the spectrum of purely magnetically charged extremal black holes has a remarkable similarity with the spectrum of electrically charged elementary string excitations.

Next we investigate the possibility that the duality symmetry is an exact symmetry of string theory under which the electrically charged elementary string excitations get exchanged with magnetically charged solitons in the theory [6] [7]. Since instanton corrections break the symmetry involving translation of the axion field to the discrete group **Z**, this would imply that string theory has SL(2,**Z**) as its symmetry group. We show that if this is the case, then points in the configuration

⋆ Although the authors of ref.[1] claim that the duality symmetry is valid for a restricted class of backgrounds for which $2F_{\mu\rho}\tilde{F}_{\nu}{}^{\rho}+2F_{\nu\rho}\tilde{F}_{\mu}{}^{\rho}-G_{\mu\nu}F_{\rho\sigma}\tilde{F}^{\rho\sigma}$ vanishes, it is easy to see that this term vanishes identically in four dimensions, and does not impose any restriction on the backgrounds.

† The duality symmetry of the effective field theory in the absence of the axion field has been used even before for this purpose [4].

space of string theory related by the SL(2,$\mathbf{Z}$) transformation must be identified. We explore some of the consequences of this identification.

This proposal is made more concrete by following the suggestion of Dabholkar et. al.[8] that strings themselves can be regarded as (possibly singular) classical solutions of the effective field theory. In that case, the quantization of the zero modes of this classical solution should reproduce the full spectrum of the string theory. Thus if the effective field theory has SL(2,$\mathbf{Z}$) symmetry, then the full string theory should also have this symmetry. In this context we show that in the duality invariant effective field theory that includes the scalar and vector fields arising due to compactification, the bosonic zero modes of the four dimensional string like solution of ref.[8] are in one to one correspondance to the bosonic degrees of freedom of heterotic string moving in four dimensions. The fermionic degrees of freedom of the fundamental string, on the other hand, are expected to arise from the fermionic zero modes generated by supersymmetry transformation of the original solution. Thus in order to establish SL(2,$\mathbf{Z}$) invariance of the full (effective) string theory, we need to show the duality invariance of the effective action even after including the fermionic variables, higher derivative terms, and quantum corrections in the theory. We do not address this problem in this paper.

2. Duality Symmetry of Low Energy Effective Action

The first part of the section will contain a review of the results derived in ref.[1]. In the second part we shall generalise the results to a background more general than the one considered in ref.[1] and show that duality symmetry holds even for this more general background. We consider critical heterotic string theory in four dimensions with the extra six dimensions compactified, and, to start with, consider a restricted set of background field configurations provided by the metric $G_{S\mu\nu}$, the antisymmetric tensor field $B_{\mu\nu}$, the dilaton field Φ, and a single $U(1)$

gauge field A_μ. The effective action at low energy is given by,

$$S = -\int d^4x \sqrt{-\det G_S} e^{-\Phi} (-R_S - G_S^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{12} G_S^{\mu\mu'} G_S^{\nu\nu'} G_S^{\tau\tau'} H_{\mu\nu\tau} H_{\mu'\nu'\tau'}$$
$$+ \frac{1}{8} G_S^{\mu\mu'} G_S^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'}) \tag{2.1}$$

where,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.2}$$

$$H_{\mu\nu\rho} = (\partial_\mu B_{\nu\rho} + \text{ cyclic permutations}) - (\Omega_3(A))_{\mu\nu\rho} \tag{2.3}$$

$$(\Omega_3(A))_{\mu\nu\rho} = \frac{1}{4}(A_\mu F_{\nu\rho} + \text{ cyclic permutations}) \tag{2.4}$$

In writing down the action (2.1), we have set the string coupling constant to unity. We can represent a background for which the string coupling constant is not unity by having an asymptotic value of Φ different from zero. The subscript S denotes that we are using the σ-model metric, which is related to the Einstein metric $G_{\mu\nu}$ through the relation

$$G_{S\mu\nu} = e^{\Phi} G_{\mu\nu} \tag{2.5}$$

From now on we shall express all the quantities in terms of the Einstein metric instead of the σ-model metric and all the indices will be raised or lowered with the Einstein metric. The equations of motion derived from the action (2.1) take the form:

$$R_{\mu\nu} - \frac{1}{2} D_\mu \Phi D_\nu \Phi - \frac{1}{2} G_{\mu\nu} D^\rho D_\rho \Phi - \frac{1}{4} e^{-2\Phi} H_{\mu\rho\tau} H_\nu{}^{\rho\tau} - \frac{1}{4} e^{-\Phi} F_{\mu\rho} F_\nu{}^\rho = 0 \tag{2.6}$$

$$D_\rho (e^{-2\Phi} H^{\mu\nu\rho}) = 0 \tag{2.7}$$

$$D_\mu (e^{-\Phi} F^{\mu\nu}) + \frac{1}{2} e^{-2\Phi} H_{\rho\mu}{}^{\nu} F^{\rho\mu} = 0 \tag{2.8}$$

$$D^\mu D_\mu \Phi + \frac{e^{-2\Phi}}{6} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{e^{-\Phi}}{8} F_{\mu\nu} F^{\mu\nu} = 0 \tag{2.9}$$

where D_μ denotes covariant derivative. The Bianchi identity for $H_{\mu\nu\rho}$ is given by,

$$(\sqrt{-\det G})^{-1} \epsilon^{\mu\nu\rho\sigma} \partial_\mu H_{\nu\rho\sigma} = -\frac{3}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \tag{2.10}$$

where,

$$\tilde{F}^{\mu\nu} = \frac{1}{2} (\sqrt{-\det G})^{-1} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{2.11}$$

Using eq.(2.7) we can define a scalar field Ψ such that,

$$H^{\mu\nu\rho} = -(\sqrt{-\det G})^{-1} e^{2\Phi} \epsilon^{\mu\nu\rho\sigma} \partial_\sigma \Psi \tag{2.12}$$

Eq.(2.10) then gives,

$$D^\mu (e^{2\Phi} D_\mu \Psi) = \frac{1}{8} F_{\mu\nu} \tilde{F}^{\mu\nu} \tag{2.13}$$

Let us now define a complex field λ as,

$$\lambda = \Psi + i e^{-\Phi} \equiv \lambda_1 + i \lambda_2 \tag{2.14}$$

Eqs.(2.6), (2.9), (2.13) and (2.8) then gives,

$$R_{\mu\nu} = \frac{\partial_\mu \bar{\lambda} \partial_\nu \lambda + \partial_\nu \bar{\lambda} \partial_\mu \lambda}{4(\lambda_2)^2} + \frac{1}{4} \lambda_2 F_{\mu\rho} F_\nu^{\ \rho} - \frac{1}{16} \lambda_2 G_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{2.15}$$

$$\frac{D^\mu D_\mu \lambda}{(\lambda_2)^2} + i \frac{D_\mu \lambda D^\mu \lambda}{(\lambda_2)^3} - \frac{i}{16} F_{-\mu\nu} F_-^{\ \mu\nu} = 0 \tag{2.16}$$

$$D_\mu (\lambda F_+^{\ \mu\nu} - \bar{\lambda} F_-^{\ \mu\nu}) = 0 \tag{2.17}$$

where,

$$F_\pm = F \pm i \tilde{F} \tag{2.18}$$

In terms of the fields $F_\pm$, the Bianchi identity for $F_{\mu\nu}$ takes the form:

$$D_\mu (F_+^{\ \mu\nu} - F_-^{\ \mu\nu}) = 0 \tag{2.19}$$

We now note that eqs.(2.15)-(2.17), and (2.19) are invariant under the following transformations:

$$\lambda \to \lambda + c \tag{2.20}$$

where c is a real number, and,

$$\lambda \to -\frac{1}{\lambda}, \qquad F_+ \to -\lambda F_+, \qquad F_- \to -\bar{\lambda} F_- \tag{2.21}$$

Invariance of all the equations under (2.20) is manifest. Under (2.21), eq.(2.16) is invariant, eqs.(2.17) and (2.19) get interchanged, and eq.(2.15) transforms to itself plus an extra term, given by,

$$-\frac{\lambda_1(\lambda_2)^2}{|\lambda|^2}(2F_{\mu\rho}\tilde{F}_\nu{}^\rho + 2F_{\nu\rho}\tilde{F}_\mu{}^\rho - g_{\mu\nu}F_{\rho\tau}\tilde{F}^{\rho\tau}) \tag{2.22}$$

The term given in eq.(2.22), however, vanishes identically in four dimensions, showing that (2.21) is a genuine symmetry of the equations of motion. The two transformations together generate the full SL(2,**R**) group under which $\lambda \to (a\lambda+b)/(c\lambda+d)$ with $ad - bc = 1$, and $F_+ \to -(c\lambda + d)F_+$.

Let us now consider a more general class of field configurations where some of the fields originating due to the dimensional reduction from ten to four dimensions are present. More specifically, we shall consider heterotic string theory in ten dimensions, with six of the directions compactified on a torus. Let x^α denote the ten coordinates, and $G^{(10)}_{S\alpha\beta}$, $B^{(10)}_{\alpha\beta}$, $A^{(10)I}_\alpha$, and $\Phi^{(10)}$ denote the ten dimensional fields ($1 \le I \le 16$). (Note that we restrict our background gauge fields to the $U(1)^{16}$ subgroup of the full non-abelian group.) The subscript S of G again denotes that this is the metric that appear directly in the σ model. Let μ, ν denote the four dimensional indices ($0 \le \mu, \nu \le 3$), and m, n denote the six dimensional indices ($4 \le m, n \le 9$). We shall restrict to background field configurations which depend only on the four coordinates x^μ. We now define [9] [10],

$$\hat{G}_{mn} = G^{(10)}_{Smn}, \qquad \hat{B}_{mn} = B^{(10)}_{mn}, \qquad \hat{A}^I_m = A^{(10)I}_m \tag{2.23}$$

$$C_\mu^m = \hat{G}^{mn} G^{(10)}_{Sn\mu}, \qquad A^I_\mu = A^{(10)I}_\mu - \hat{A}^I_m C^m_\mu, \qquad D_{m\mu} = B^{(10)}_{m\mu} - B_{mn} C^n_\mu - \frac{1}{4}\hat{A}^I_m A^I_\mu \tag{2.24}$$

$$G_{S\mu\nu} = G^{(10)}_{S\mu\nu} - G^{(10)}_{Sm\mu} G^{(10)}_{Sn\nu} \hat{G}^{mn}, \quad B_{\mu\nu} = B^{(10)}_{\mu\nu} - B^{(10)}_{mn} C^m_\mu C^n_\nu - \frac{1}{2}(C^m_\mu D_{m\nu} - C^m_\nu D_{m\mu}) \tag{2.25}$$

$$\Phi = \Phi^{(10)} - \frac{1}{2}\ln\det\hat{G} \tag{2.26}$$

$$F^{(A)I}_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu, \qquad F^{(C)m}_{\mu\nu} = \partial_\mu C^m_\nu - \partial_\nu C^m_\mu, \qquad F^{(D)}_{m\mu\nu} = \partial_\mu D_{m\nu} - \partial_\nu D_{m\mu} \tag{2.27}$$

$$H_{\mu\nu\rho} = [(\partial_\mu B_{\nu\rho} + \frac{1}{2} C^m_\mu F^{(D)}_{m\nu\rho} + \frac{1}{2} D_{m\mu} F^{(C)m}_{\nu\rho}) + \text{ cyclic permutations}] - (\Omega_3(A))_{\mu\nu\rho} \tag{2.28}$$

$$F_{\mu\nu} = \begin{pmatrix} F^{(C)m}_{\mu\nu} \\ F^{(D)}_{m\mu\nu} \\ -\frac{1}{\sqrt{2}} F^{(A)I}_{\mu\nu} \end{pmatrix} \tag{2.29}$$

In these equations $\hat{G}^{mn}$ denotes components of inverse of the matrix $\hat{G}_{mn}$. We also define 28×28 matrices M and L as,

$$M = \begin{pmatrix} P & Q & R \\ Q^T & S & U \\ R^T & U^T & V \end{pmatrix} \tag{2.30}$$

$$L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix} \tag{2.31}$$

where I_n denotes $n \times n$ identity matrix, T denotes transpose of a matrix, and,

$$\begin{aligned} P^{mn} =& \hat{G}^{mn}, \qquad Q^m{}_n = \hat{G}^{mp}(\hat{B}_{pn} + \frac{1}{4}\hat{A}^I_p \hat{A}^I_n), \qquad R^{mI} = \frac{1}{\sqrt{2}}\hat{G}^{mp}\hat{A}^I_p \\ S_{mn} =& (\hat{G}_{mp} - \hat{B}_{mp} + \frac{1}{4}\hat{A}^I_m \hat{A}^I_p)\hat{G}^{pq}(\hat{G}_{qn} + \hat{B}_{qn} + \frac{1}{4}\hat{A}^J_q \hat{A}^J_n) \\ U_m{}^I =& \frac{1}{\sqrt{2}}(\hat{G}_{mp} - \hat{B}_{mp} + \frac{1}{4}\hat{A}^J_m \hat{A}^J_p)\hat{G}^{pq}\hat{A}^I_q, \qquad V^{IJ} = \delta^{IJ} + \frac{1}{2}\hat{A}^I_p \hat{G}^{pq}\hat{A}^J_q \end{aligned} \tag{2.32}$$

Note that these matrices M and L are related to similar matrices defined in ref.[11]

by a similarity transformation. In terms of these variables, the action of ref.[11] (with $D = 10$, $p = 16$) may be written as [10],

$$S = \int d^4x \sqrt{-\det G_S} e^{-\Phi} \Big(R_S + G_S^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{12} G_S^{\mu\mu'} G_S^{\nu\nu'} G_S^{\tau\tau'} H_{\mu\nu\rho} H_{\mu'\nu'\tau'} \\ - \frac{1}{4} G_S^{\mu\mu'} G_S^{\nu\nu'} F^T_{\mu\nu} LMLF_{\mu'\nu'} + \frac{1}{8} G_S^{\mu\nu} Tr(\partial_\mu ML \partial_\nu ML) \Big) \tag{2.33}$$

Let us now define $G_{\mu\nu}$, Ψ and λ through eqs.(2.5), (2.12), and (2.14). We also define $\tilde{F}$ through eq.(2.11) and,

$$F_{\pm\mu\nu} = -MLF_{\mu\nu} \pm i\tilde{F}_{\mu\nu} \tag{2.34}$$

keeping in mind that F now is a 28 dimensional column vector. Using the relations,

$$M^T = M, \qquad M^T LM = L \tag{2.35}$$

the equations of motion derived from the action (2.33) and the Bianchi identities can be shown to be invariant under the duality transformation:

$$G_{\alpha\beta} \to G_{\alpha\beta}, \qquad \lambda \to -\frac{1}{\lambda}, \qquad F_+ \to -\lambda F_+, \qquad F_- \to -\bar{\lambda} F_- \tag{2.36}$$

Proof of invariance of the equations of motion of the metric, gauge fields, and the dilaton-axion field follows exactly as before. The only new feature appears in the proof of the equation of motion of G_{mn}, B_{mn} and A^I_m. Let $\{\phi_i\}$ denote these set of fields. The dependence of the action on these fields is through the matrix M appearing in the last two terms in the action. Of these, the last term is manifestly invariant under the duality transformation, hence the contribution of this term to the equation of motion $\delta S/\delta\phi_i = 0$ is duality invariant. On the other hand, the contribution to the left hand side of the equation of motion from the last but one term is given by,

$$-\frac{\lambda_2}{4}\sqrt{-\det G}G^{\mu\mu'}G^{\nu\nu'}F^T_{\mu\nu}L\frac{\delta M}{\delta\phi_i}LF_{\mu'\nu'} \tag{2.37}$$

From eq.(2.36) we see that under the duality transformation,

$$F \to -\lambda_1 F - \lambda_2 ML\tilde{F} \tag{2.38}$$

Transforming eq.(2.37) by this transformation, and using the equation,

$$\frac{\delta M}{\delta\phi_i}LM + ML\frac{\delta M}{\delta\phi_i} = 0 \tag{2.39}$$

we see that eq.(2.37) is invariant under the duality transformation.

3. Rotating Dyonic Black Holes

In ref.[5] a solution of the equations of motion derived from the effective action given in eq.(2.1) was found which represents rotating charged black hole. The solution was given by,

$$\begin{aligned} ds^2 =& -\frac{\rho^2 + a^2\cos^2\theta - 2m\rho}{\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2}}dt^2 + \frac{\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2}}{\rho^2 + a^2 - 2m\rho}d\rho^2 \\ &+ (\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2})d\theta^2 - \frac{4m\rho a\cosh^2\frac{\alpha}{2}\sin^2\theta}{\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2}}dtd\phi \\ &+ \{(\rho^2 + a^2)(\rho^2 + a^2\cos^2\theta) + 2m\rho a^2\sin^2\theta + 4m\rho(\rho^2 + a^2)\sinh^2\frac{\alpha}{2} + 4m^2\rho^2\sinh^4\frac{\alpha}{2}\} \\ &\times \frac{\sin^2\theta}{\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2}}d\phi^2 \end{aligned} \tag{3.1}$$

$$\Phi = -\ln\frac{\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2}}{\rho^2 + a^2\cos^2\theta} \tag{3.2}$$

$$A_\phi = -\frac{2m\rho a\sinh\alpha\sin^2\theta}{\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2}} \tag{3.3}$$

$$A_t = \frac{2m\rho \sinh\alpha}{\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2}} \tag{3.4}$$

$$B_{t\phi} = \frac{2m\rho a \sinh^2\frac{\alpha}{2}\sin^2\theta}{\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2}} \tag{3.5}$$

The other components of A_μ and $B_{\mu\nu}$ vanish. (Note that ds^2 here denotes the Einstein metric which was called ds'^2_E in ref.[5].) m, a and α are three parameters which are related to the mass, angular momentum, and charge of the black hole. We shall now perform an SL(2,**R**) transformation on this solution to construct a new solution, and show that this new solution represents a black hole carrying mass, angular momentum, and electric and magnetic type charges. To do this we first need to calculate the various field strengths associated with the solution given in eqs.(3.1)-(3.5). They are given by,

$$F_{\rho\phi} = \frac{2ma\sinh\alpha\sin^2\theta(\rho^2 - a^2\cos^2\theta)}{(\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2})^2} \tag{3.6}$$

$$F_{\theta\phi} = -\frac{4m\rho a\sinh\alpha(\rho^2 + a^2 + 2m\rho\sinh^2\frac{\alpha}{2})\sin\theta\cos\theta}{(\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2})^2} \tag{3.7}$$

$$F_{\rho t} = -\frac{2m\sinh\alpha(\rho^2 - a^2\cos^2\theta)}{(\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2})^2} \tag{3.8}$$

$$F_{\theta t} = \frac{4m\rho a^2\sinh\alpha\sin\theta\cos\theta}{(\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2})^2} \tag{3.9}$$

$$e^{-2\Phi}\sqrt{-G}H^{\rho t\phi} = \frac{2ma\sinh^2\frac{\alpha}{2}(\rho^2 - a^2\cos^2\theta)\sin\theta}{(\rho^2 + a^2\cos^2\theta)^2} \tag{3.10}$$

$$e^{-2\Phi}\sqrt{-G}H^{\theta\phi t} = \frac{4m\rho a\sinh^2\frac{\alpha}{2}\cos\theta}{(\rho^2 + a^2\cos^2\theta)^2} \tag{3.11}$$

From eqs.(2.12), (3.10) and (3.11) we get,

$$\Psi = \frac{2ma\sinh^2\frac{\alpha}{2}\cos\theta}{\rho^2 + a^2\cos^2\theta} \tag{3.12}$$

Also, using the definition of $\tilde{F}$ given in eq.(2.11), we get,

$$\tilde{F}_{\rho\phi} = \frac{4m\rho a^2\sinh\alpha\sin^2\theta\cos\theta}{(\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2})^2} \tag{3.13}$$

$$\tilde{F}_{\theta t} = -\frac{2ma\sinh\alpha(\rho^2 - a^2\cos^2\theta)\sin\theta}{(\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2})^2} \tag{3.14}$$

$$\tilde{F}_{\theta\phi} = \frac{2m\sinh\alpha\sin\theta(\rho^2 - a^2\cos^2\theta)(\rho^2 + a^2 + 2m\rho\sinh^2\frac{\alpha}{2})}{(\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2})^2} \tag{3.15}$$

$$\tilde{F}_{\rho t} = -\frac{4m\rho a\sinh\alpha\cos\theta}{(\rho^2 + a^2\cos^2\theta + 2m\rho\sinh^2\frac{\alpha}{2})^2} \tag{3.16}$$

We can now generate a new solution by performing an SL(2,**R**) transformation on the above solution in the same manner as in ref.[1]. We consider the SL(2,**R**) transformation $\lambda \to -(1+c^2)/(\lambda+c)$, $F_+ \to -(\lambda+c)F_+/\sqrt{1+c^2}$. The transformed solution is given by,

$$\lambda' = -\frac{1+c^2}{\lambda+c}, \qquad ds'^2 = ds^2, \qquad F'_{\mu\nu} = -\frac{\Psi + c}{\sqrt{1+c^2}}F_{\mu\nu} + \frac{e^{-\Phi}}{\sqrt{1+c^2}}\tilde{F}_{\mu\nu} \tag{3.17}$$

We shall not write out the solution in detail, but only give the asymptotic behavior of the solution in order to identify its electric and magnetic charges. The leading components of the electromagnetic field are given by,

$$F'_{\rho t} \simeq \frac{2mc\sinh\alpha}{\sqrt{1+c^2}\rho^2}, \qquad F'_{\theta\phi} \simeq \frac{2m\sinh\alpha\sin\theta}{\sqrt{1+c^2}} \tag{3.18}$$

With appropriate normalization (which sets the coefficient of F^2 term in the action to unity), the electric and magnetic charges carried by the solution may be identified to,

$$Q_{el} = \frac{1}{\sqrt{2}}\frac{mc\sinh\alpha}{\sqrt{1+c^2}}, \qquad Q_{mag} = \frac{1}{\sqrt{2}}\frac{m\sinh\alpha}{\sqrt{1+c^2}} \tag{3.19}$$

Since the metric remains the same, the expressions for the mass M and angular momentum J of the black hole in terms of the parameters m, a and α remain the same as in ref.[5] and are given by,

$$M = \frac{m}{2}(1+\cosh\alpha), \qquad J = \frac{ma}{2}(1+\cosh\alpha) \tag{3.20}$$

As in ref.[5], the coordinate singularities of the metric occur on the surfaces,

$$\rho^2 - 2m\rho + a^2 = 0 \tag{3.21}$$

and give rise to the horizons which shield the genuine singularities of the solution. The extremal limit is approached as $|a| \to m$ from below, when the horizon disappears leaving behind naked singularities. Using eqs.(3.19) and (3.20) we see that in terms of the physical parameters the extremal limit corresponds to

$$M^2 \to |J| + \frac{(Q_{el})^2 + (Q_{mag})^2}{2} \tag{3.22}$$

The expressions for the various fields around a rotating dyonic black hole was found in ref.[12] to linear order in Q_{el} and Q_{mag}.

If we start from the action (2.33), then we can construct more general dyonic solutions carrying electric and magnetic charges associated with the gauge fields A^I_μ, C^m_μ and $D_{m\mu}$ as follows. We start from the charge neutral Kerr solution as in ref.[5], and then perform the $O(6,1)\times O(22,1)$ transformation [3] [13 − 15] [11] (as in ref.[5]) that mixes the time coordinate with the six left moving and twenty two right moving coordinates. This gives rise to a solution carrying electrical type A^I_μ, C^m_μ and $D_{m\mu}$ charges. The explicit construction may be carried out in two stages. In the first stage, we perform an $O(22)$ transformation on the solution given by eqs.(3.1)-(3.5). This generates a solution with 21 extra parameters n^I, p^m ($1 \le I \le 16$, $1 \le m \le 6$, $\sum_I n^I n^I + \sum_m p^m p^m = 1$), in terms of which various components of the gauge fields A^I_μ, C^m_μ and $D_{m\mu}$ are given by,

$$A^I_\mu = n^I A_\mu, \qquad C^m_\mu = -D_{m\mu} = \frac{1}{2} p^m A_\mu \tag{3.23}$$

where A_μ are given by eqs.(3.3) and (3.4). The metric, the dilaton, and the anti-symmetric tensor fields remain unchanged.

In the next stage, we reexpress the solution as a solution in ten dimensions using the relations given in eqs.(2.23)-(2.26), and transform it by a Lorentz boost that mixes the time coordinate with the six internal coordinates. The independent parameters may be characterized by the magnitude of the boost and a five dimensional unit vector in the six dimensional internal space denoting the direction of the boost. Since the results are somewhat messy to write down, we shall not write them down here. In particular, note that now, besides the gauge fields, the metric and the axion-dilaton fields, the scalar fields characterizing the matrix M also acquire non-trivial configuration. In order to show that the singularities of the resulting four dimensional metric at $\rho^2 - 2m\rho + a^2 = 0$ are coordinate singularities, we need to show the existence of explicit gauge and coordinate transformations which make the field configuration non-singular on this surface. This is relatively straightforward to show for non-rotating solutions ($a = 0$), and we shall restrict ourselves to this case from now on.

The asymptotic behaviour of the transformed solution is easier to calculate, since the $O(6,1) \times O(22,1)$ transformation laws of various fields are simple if we keep terms upto linear order in the fields [14] [11]. For this we start from the charge neutral Schwarzchild solution and perform an $O(6,1) \times O(22,1)$ transformation on it directly. If we define the 6 and 22 dimensional vectors $\vec{Q}^{(el)}_L$ and $\vec{Q}^{(el)}_R$ respectively through the asymptotic relations:

$$\frac{1}{2\sqrt{2}}(F^{(C)m}_{\rho t} + F^{(D)}_{m\rho t}) \simeq \frac{(Q^{(el)}_L)_m}{r^2} \tag{3.24}$$

$$\begin{pmatrix} \frac{1}{2\sqrt{2}}(F^{(C)m}_{\rho t} - F^{(D)}_{m\rho t}) \\ \frac{1}{2\sqrt{2}}F^{(A)I}_{\rho t} \end{pmatrix} \simeq \frac{1}{r^2} \begin{pmatrix} (Q^{(el)}_R)_m \\ (Q^{(el)}_R)^I \end{pmatrix} \tag{3.25}$$

then the mass M and the charges $\vec{Q}_{eL}$ and $\vec{Q}_{eR}$ of the final solution may be expressed in terms of the parameters of the original solution as,

$$\begin{aligned} M =& \frac{1}{2}m(1 + \cosh\alpha\cosh\beta) \\ \vec{Q}^{(el)}_R =& \frac{m}{\sqrt{2}}\cosh\beta\sinh\alpha\ \vec{u} \\ \vec{Q}^{(el)}_L =& \frac{m}{\sqrt{2}}\cosh\alpha\sinh\beta\ \vec{v} \end{aligned} \tag{3.26}$$

where $\vec{u}$ and $\vec{v}$ are 22 and 6 dimensional unit vectors respectively, and α and β are two boost angles, characterizing the $O(22,1)$ and $O(6,1)$ rotation matrices.

The solution carrying magnetic type charges may be obtained from this solution by the SL(2,**R**) transformation discussed above. Again, the steps involved in this construction are purely algebraic, and hence we shall not carry out the explicit construction of the solution here. The mass and angular momenta of the solution are given by the same expressions as in eq.(3.26); the electric and magnetic charge vectors of the transformed solution are related to those defined in eq.(3.26) by the relations:

$$\vec{Q}^{(el)\prime}_{\substack{R\\L}} = \frac{c}{\sqrt{1+c^2}}\vec{Q}^{(el)}_{\substack{R\\L}}; \vec{Q}^{(mag)\prime}_{\substack{R\\L}} = \frac{1}{\sqrt{1+c^2}}\vec{Q}^{(el)}_{\substack{R\\L}} \tag{3.27}$$

Let us discuss the purely magnetically charged solutions in some detail, since these solutions, if present, must be regarded as new states in the spectrum of string theory. These solutions are obtained by setting $c = 0$. (For convenience, we shall drop the primes from now on.) In this case, the extremal limit corresponds to $m \to 0$ keeping the physical parameters M, $\vec{Q}^{(mag)}_L$ and $\vec{Q}^{(mag)}_R$ fixed. This, in turn, requires that at least one of the angles α or β approach ∞. By analyzing eqs.(3.26), (3.27) it is easy to see that the condition for extremality may be expressed in terms of the physical parameters as,

$$(M^2 - \frac{(\vec{Q}_R^{(mag)})^2}{2})(M^2 - \frac{(\vec{Q}_L^{(mag)})^2}{2}) = 0 \qquad (3.28)$$

We now note the following amusing coincidence. The spectrum of states in heterotic string theory which do not have any oscillator excitations, is given by (in the normalization convention that we have adopted),

$$M^2 \simeq \frac{(\vec{Q}_R^{(el)})^2}{2} \simeq \frac{(\vec{Q}_L^{(el)})^2}{2} \qquad (3.29)$$

for $M >> 1$. (Note that this is also the limit of the black hole mass in which neglecting the higher derivative terms in the effective action is justified). The angular momentum carried by such states is zero. The two relations (3.28) and (3.29) look identical* if we restrict to states with $(\vec{Q}_L^{(mag)})^2 = (\vec{Q}_R^{(mag)})^2$, except for the interchange of $(\vec{Q}_R^{(el)}, \vec{Q}_L^{(el)})$ with $(\vec{Q}_R^{(mag)}, \vec{Q}_L^{(mag)})$. $(\vec{Q}_R^{(el)}, \vec{Q}_L^{(el)})$ belongs to an even self dual Lorentzian lattice [17] [18], and, if the magnetic charge is quantised according to the rule $\vec{Q}_R^{(mag)}.\vec{Q}_R^{(el)} + \vec{Q}_L^{(mag)}.\vec{Q}_L^{(el)} = integer$, then $(-\vec{Q}_R^{mag}, \vec{Q}_L^{(mag)})$ belongs to the same lattice. This seems to suggest that by including the magnetically charged black holes as new elementary particles in the theory† we may be able to establish the electric magnetic duality as an exact symmetry of string theory.

In principle the similarity in the spectrum could be destroyed by string loop corrections, but in this case we shall expect that space-time supersymmetry will prevent the mass charge relation of extremal black holes from being modified by

⋆ This also indicates that the electrically charged elementary string excitations become extremal charged black holes. Relationship between string matter and extremal charged black holes has also been observed for a more restricted class of solutions in ref.[16].

† The possibility of treating black holes as elementary particles has been discussed in ref.[19]. Even if the magnetically charged black holes of the type discussed here are not of the type that behave as elementary particles according to the analysis of ref.[19], we must still include them as new states in the spectrum since they cannot be formed as composites of ordinary electrically charged matter.

radiative corrections [20]. The extremalilty condition for black holes carrying finite angular momentum do not seem to have such protection against radiative corrections, and hence the analysis becomes more complicated in this case. Also we do not yet have the magnetically charged particles which are dual to the elementary string states that have arbitrary oscillator excitations. In the next section we shall explore the possibility of exact electric-magnetic duality in string theory in some more detail.

4. Can Duality be an Exact Symmetry of String Theory?

In this section we shall explore the possibility that duality is an exact symmetry of string theory under which electrically charged elementary string excitations get exchanged with magnetically charged solitons [6] [7]. We begin by discussing the consequences of duality being an exact symmetry, later we shall come back to the question of how the magnetically charged states that are required for duality to be an exact symmetry of the theory might be constructed. We note the following points:

1. The symmetry $\lambda \to \lambda + c$ is broken down to $\lambda \to \lambda + 1$ by instanton corrections. Hence the complete symmetry group of the theory can at most be SL(2,**Z**) and not SL(2,**R**) [1].

2. Since the fundamental string is an axionic string [21], the axion field changes by 1 as we go around the fundamental string. Hence in the configuration space we must identify the points related by the transformation $T : \lambda \to \lambda + 1$. If the duality transformation $S : \lambda \to -1/\lambda$ together with appropriate transformation on the other fields is to be a genuine symmetry of the theory, then the points in the configuration space related by the transformation STS^{-1} must also be identified. Since T and STS^{-1} can be shown to generate the full SL(2,**Z**) group, this shows that points in the configuration space related by any SL(2,**Z**) transformation must be idetified.

In order to understand the physical significance of this identification, we note that the weak coupling perturbation theory is around a vacuum in which the expectation value of $Im\lambda$ is large, hence the SL(2,**Z**) symmetry acting on a field configuration around this vacuum will take it to another field configuration far removed from it (before the identification under SL(2,**Z**) is made). As a result, perturbation theory around such a vacuum is insensitive to the identification of fields under the SL(2,**Z**) transformation. As an analogy, let us consider a field theory of a scalar field ϕ with potential $(\phi^2 - a^2)^2$, with the identification $\phi \equiv -\phi$. The vacuum configuration corresponds to $\phi = a$ which is not invariant under the $\mathbf{Z}_2$ symmetry, and perturbative quantization of the theory around this point is totally insensitive to the fact that in the configuration space the points ϕ and $-\phi$ are identified. There could, however, be important non-perturbative effects. For example, without the identification in the field space, there will be stable domain walls in the system since there is more than one vacuum configurations. On the other hand, with the identification, there is only one vacuum configuration, and hence there is no stable domain wall.

We should also point out that with this identification, at string coupling constant equal to unity, the magnetically charged solutions that we have discussed in the last section are not new states. Instead, a physical state is to be constructed as a linear superposition of the electrically charged state and its duality transform.

The proposal that the SL(2,**Z**) invariance is an exact symmetry of the full string theory can be made more concrete by following the suggestion of Dabholkar et. al.[8] that the elementary string may be regarded as a classical solution of the effective field theory, and that the collective excitations associated with the zero modes of the classical solution correspond to the dynamical degrees of freedom of the elementary string. Quantization of these zero modes would then give rise to the spectrum of the full string theory. In that case, SL(2,**Z**) invariance of the effective field theory (after including all the higher derivative terms and quantum corrections) would automatically imply the SL(2,**Z**) invariance of the full string theory, since, given any classical string configuration, we can construct its dual by

SL(2,**Z**) transformation on the corresponding solution in the effective field theory.

In order to implement this idea, we need to ensure that we include enough degrees of freedom in the effective field theory so that the classical solution describing the string has enough number of deformations (zero modes) which are in one to one correspondence with the degrees of freedom of the fundamental string. At the same time, we need to ensure that this effective field theory is invariant under the duality transformation, and hence should not, for example, contain any charged field. We shall now show that the solution of ref.[8], regarded as a solution of the (duality invariant) equations of motion in the effective field theory described by the action (2.33), has all the degrees of freedom required to describe the bosonic dynamical degrees of freedom of the fundamental string. As was shown in ref.[8], the fermionic degrees of freedom come from the supersymmetry transformation of the original solution. Thus in order to incorporate these degrees of freedom, we must work with the full ($N = 4$) supersymmetric effective field theory by including also the fermionic fields in the theory, and prove that this theory is invariant under duality transformation. We do not address this problem in this paper. Nor do we address the question as to whether the SL(2,**Z**) symmetry of the equations of motion survive when we include higher derivative terms and quantum corrections in the effective action.

In order to see how the bosonic degrees of freedom of the fundamental string are related to the zero modes of the classical solution, let us note that the four bosonic zero modes corrsponding to translation of the solution in the four dimensional space correspond to the variables X^μ of the fundamental string. The remaining degrees of freedom of the fundamental string are the six internal bosonic coordinates X^m, and the sixteen right moving internal coordinates Y^I. Alternatively we can regard them as six left moving and twenty two right moving world sheet currents. These degrees of freedom may be identified to the deformation of the original solution of ref.[8] by the $O(6,1) \times O(22,1)$ transformation discussed in ref.[11], which mixes the left and right components of the time coordinate with the left and right components of the internal coordinates respectively. Such deformations are possible since the original solution is independent of time, as well as internal coordinates. A particular example was already discussed in ref.[22], where by using an $O(1,1)$ transformation of the solution of ref.[8], a charged string solution was constructed.[⋆] To see how the counting of the degrees of freedom work, note that the $O(6) \times O(22)$ transformation that mixes the internal coordinates do not modify the original solution. Hence the number of independent deformations generated by the $O(6,1) \times O(22,1)$ transformation is equal to the dimension of the coset space $(O(6,1)/O(6)) \times (O(22,1)/O(22))$, which is equal to $6 + 22$.[†] The details of the construction are identical to the corresponding construction of the general electrically charged black holes discussed in the last section. This way we can identify the bosonic dynamical degrees of freedom of the fundamental string to the bosonic deformations of the classical solution in the effective field theory. Construction of the general solution deformed by all the bosonic zero modes involves purely algebraic manipulations, and will not be given here.

5. Summary

In this paper we have shown that the electric-magnetic duality transformation is an exact symmetry of the equations of motion of the low energy effective field theory arising in string theory. Using this symmetry we have constructed rotating black hole solution in string theory carrying both, electric and magnetic charge. We have also analyzed the possibility that the duality transformation is an exact symmetry of string theory, by regarding the fundamental string as a classical solution in the effective field theory. We found that in order to make this proposal more concrete, we need to establish that the full supersymmetric effective field theory is invariant under duality transformation. We hope to return to this question in the near future.

⋆ I wish to thank A. Strominger for pointing out the relation between this deformation and the degrees of freedom of the fundamental string.

† Actually in this case there is also a set of deformations belonging to the coset space $O(6,22)/O(6) \times O(22)$ [17] which correspond to changing the lattice of compactification and putting constant background $A_m^{(I)}$ and B_{mn} fields in the theory [18].

We conclude by mentioning that if the duality symmetry indeed turns out to be an exact symmetry of string theory, it will undoubtedly give us information about non-perturbative features of string theory, since the duality transformation interchanges the strong and weak coupling limits of string theory.

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