

F-theory and Orientifolds

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Abstract

By analyzing *F*-theory on *K3* near the orbifold limit of *K3* we establish the equivalence between *F*-theory on *K3* and an orientifold of type IIB on T^2 , which in turn, is related by a T-duality transformation to type I theory on T^2 . By analyzing the *F*-theory background away from the orbifold limit, we show that non-perturbative effects in the orientifold theory splits an orientifold plane into two planes, with non-trivial $SL(2, \mathbb{Z})$ monodromy around each of them. The mathematical description of this phenomenon is identical to the Seiberg-Witten result for $N=2$ supersymmetric $SU(2)$ gauge theory with four quark flavors. Points of enhanced gauge symmetry in the orientifold / *F*-theory are in one to one correspondence with the points of enhanced global symmetry in the Seiberg-Witten theory.

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1 Introduction and Summary

Besides establishing connections between apparently different string theories, recent developments in string theory have also provided us with some novel ways of compactifying string theory. One such procedure, now known as F -theory[1, 2, 3, 4, 5, 6, 7], involves type IIB string compactification where the dilaton and the scalar from the Ramond-Ramond (RR) sector (which we shall refer to as the axion) are not constant, but vary on the internal manifold. Given a manifold M that admits elliptic fibration, *i.e.* has the structure of a fiber bundle whose fiber is a two dimensional torus and base is some manifold B , one defines F -theory on M as the type IIB theory on B , with the axion-dilaton modulus of the type IIB theory being set equal to the complex structure modulus of the fiber. Since in general this modulus varies as we move on B , the axion-dilaton modulus of the type IIB theory will vary as we move on B . In particular as we travel along non-trivial closed cycles on B the fiber can undergo non-trivial $SL(2,Z)$ transformation. This would imply that the axion-dilaton modulus of the type IIB theory will undergo non-trivial $SL(2,Z)$ transformation as we move along closed cycles of B .

Many of the F -theory compactifications have been conjectured to be equivalent to more conventional compactifications of heterotic string theory. In particular, F -theory on an elliptically fibered $K3$ surface has been conjectured to be dual to heterotic string theory on a two dimensional torus[1]. One of the purposes of this paper is to establish this equivalence. Like many of the dualities in string theory and M-theory which can be established by working at the orbifold limit of the compact manifolds[8], the duality between F -theory on $K3$ and heterotic string theory on T^2 is established by going to a special point in the $K3$ moduli space where it can be identified to a Z_2 orbifold of a four torus. We show that in this limit the F -theory background reduces to a conventional background where the axion-dilaton modulus remains constant as we move in the internal space. More specifically this background can be identified to that of an orientifold[9, 10, 11] of type IIB theory which can be analyzed by the conventional conformal field theory techniques and is in fact related by T -duality to type I string theory on T^2 .³ Thus at this special point in the moduli space F -theory on $K3$ is identical to type I theory on T^2 , which, in turn has been conjectured to be equivalent to heterotic string theory on T^2 [12, 13, 14, 15]. Once we have established the equivalence between the heterotic string

³We have been informed by C. Vafa that he has independently made a similar observation.

theory on T^2 and F -theory on $K3$ at a special point in the moduli space, we can argue that the equivalence must hold at all points in the moduli space since we can deform both theories away from this specific point by switching on appropriate background fields.

We also explicitly study deformations of the F -theory, as well as the orientifold theory, away from this special point in the moduli space. The moduli space of the orientifold theory is characterized by the vacuum expectation value of the Higgs field in the adjoint representation of the gauge group, or equivalently, locations of the sixteen seven-branes on the internal two dimensional manifold. On the other hand the F -theory moduli space is characterized by the moduli of elliptically fibered $K3$ surfaces. Both theories are described by background field configurations which consist of dilaton-axion modulus with non-trivial dependence on the coordinates of the internal two dimensional manifold. Explicit comparison of the two sets of field configuration reveals that they are identical in the two theories at weak coupling, but differ for finite coupling. This difference is non-perturbative in the coupling constant of the orientifold theory. If we focus on the physics near one of the four orientifold planes, then the mathematical description of the field configuration turns out to be identical to the Seiberg-Witten solution of the $N=2$ supersymmetric $SU(2)$ gauge theory with four quark flavors[16]. In the analysis of ref.[16] the moduli space of $N=2$ supersymmetric $SU(2)$ gauge theory was characterized by a gauge invariant quantity u , and the complex ‘coupling constant’ τ varies as we move in the u plane. The F -theory background is identical to this configuration, with u labelling the coordinate of the base B and τ denoting the axion-dilaton modulus. On the other hand the orientifold background corresponds to the classical limit of this configuration. As is well known from the analysis of ref.[16], the classical limit of this background is singular as $Im(\tau)$ becomes negative in some regions. Thus we expect the F -theory background to describe the quantum corrected version of the orientifold background.

The $SU(2)$ gauge theory with four quark flavors is characterized by five parameters, – the asymptotic value of τ , and the four quark masses in the Yang-Mills theory. In the orientifold theory, these four mass parameters denote the locations of the four seven branes around an orientifold plane. (These in turn can be related to the vacuum expectation value of a scalar field belonging to the adjoint representation of the gauge group $SO(8)$.) As was discussed in ref.[16], at special points in the space of the parameters m_i the $N=2$ supersymmetric Yang-Mills theory has enhanced global symmetry group G . It turns out that precisely at these special points the orientifold / F -theory develops enhanced gauge

symmetry G .

It was also noted in ref.[16] that the $SL(2,Z)$ action on τ has to be accompanied by a triality action on the representations of the global symmetry group $SO(8)$ which transform the parameters m_i in a non-trivial manner. Thus we would expect that in the orientifold / F -theory, the $SL(2,Z)$ action on the axion-dilaton modulus will have to be accompanied by triality action on the representations of the gauge group $SO(8)$, and in particular on the Higgs vacuum expectation values (locations of seven branes) represented by the parameters m_i . We explicitly verify this triality action of $SL(2,Z)$ in the dual heterotic string theory on T^2 where $SL(2,Z)$ is part of the T-duality group of the theory.

Finally, by comparing the masses of BPS states in the F -theory and in the orientifold theory, we show that in F -theory the masses of BPS states can be expressed in terms of period integrals of the holomorphic two form on the (complex) surface on which F -theory is compactified. Whenever one or more of the period integrals vanish, the corresponding BPS state(s) become massless, and at the same time the surface becomes singular. This is a reflection of the relationship between singularities of the surface and the appearance of enhanced gauge symmetries in the corresponding F -theory compactification.

The paper is organized as follows. Section 2 is devoted to studying the F -theory on $K3$ at the orbifold limit of $K3$ and comparing the background field configuration describing this F -theory compactification with the background of an orientifold of type IIB theory on T^2 . In section 3 we study deformation of both, the F -theory and the orientifold backgrounds away from this special point in the moduli space.

2 F -theory on $K3$ in the Orbifold Limit

Let us begin with the following elliptically fibered $K3$ surface

$$y^2 = x^3 + f(z)x + g(z) \tag{2.1}$$

where $x, y, z \in CP^1$, $f(z)$ is a polynomial of degree eight, and $g(z)$ is a polynomial of degree twelve in z . This describes a torus for each point on CP^1 labelled by the coordinate z . The modular parameter $\tau(z)$ of the torus is determined in terms of the ratio f^3/g^2 through the relation

$$j(\tau(z)) = \frac{4 \cdot (24f)^3}{27g^2 + 4f^3}, \tag{2.2}$$

where

$$j(\tau) = \frac{(\theta_1^8(\tau) + \theta_2^8(\tau) + \theta_3^8(\tau))^3}{\eta(\tau)^{24}}. \quad (2.3)$$

By definition, compactification of F -theory on this particular $K3$ corresponds to compactification of type IIB theory on CP^1 labelled by z , with

$$a(z) + ie^{-\Phi(z)/2} = \tau(z). \quad (2.4)$$

Here a denotes the RR scalar field and Φ denotes the dilaton field. Physically such a background corresponds to a configuration of twenty four seven branes of the type IIB theory transverse to CP^1 and situated at the zeroes of

$$\Delta \equiv 4f^3 + 27g^2. \quad (2.5)$$

In the generic case, the twenty four zeroes of Δ are distinct from each other and neither f nor g vanishes at the zeroes of Δ . If z_i denotes such a zero of Δ then from eq.(2.2) and (2.5) we see that near such a point

$$j(\tau(z)) \sim \frac{1}{z - z_i}. \quad (2.6)$$

This gives

$$\tau(z) \simeq \frac{1}{2\pi i} \ln(z - z_i), \quad (2.7)$$

up to $SL(2, Z)$ transformation.

We shall consider a special point in the moduli space of this compactification where $\tau(z)$ is independent of z . From eq.(2.2) we see that this requires

$$f^3/g^2 = \text{constant}. \quad (2.8)$$

Since g and f are polynomials in z of order twelve and eight respectively, the solution to eq.(2.8) is given by

$$g = \phi^3, \quad f = \alpha\phi^2, \quad (2.9)$$

where α is a constant and ϕ is a polynomial in z of degree four. By a rescaling of y and x we can set the coefficient of z^4 in ϕ to be one. Thus ϕ has the form⁴

$$\phi = \prod_{i=1}^4 (z - z_i) \quad (2.10)$$

⁴A special case of this where $z_1 = z_2 = 0$ and $z_3 = z_4 = \infty$ has been discussed in [3].

where z_i are constants. From eqs.(2.2), (2.5), (2.9) and (2.10) we get

$$\Delta = (4\alpha^3 + 27) \prod_{i=1}^4 (z - z_i)^6, \quad (2.11)$$

and

$$j(\tau) = \frac{4 \cdot (24\alpha)^3}{27 + 4\alpha^3}. \quad (2.12)$$

Thus this particular compactification corresponds to a configuration where the twenty four seven-branes are grouped into four sets of six coincident seven-branes, situated at the points z_1, \dots, z_4 . τ is constant over CP^1 ; however there is an $SL(2, Z)$ monodromy

$$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \quad (2.13)$$

around each of the points z_i . This can be seen by noting that as z moves once around the point z_i , y changes sign. This corresponds to the hyperelliptic involution of the torus, represented by the $SL(2, Z)$ matrix given in eq.(2.13).

The metric on the base can be read out from the formulae derived in ref.[17]. Up to an overall multiplicative constant, it is given by

$$ds^2 = \frac{dzd\bar{z}}{\prod_i (z - z_i)^{1/2} (\bar{z} - \bar{z}_i)^{1/2}}. \quad (2.14)$$

Thus there is a deficit angle of π at each of the points z_i . Thus the base has the geometry of T^2/\mathcal{I}_2 , where \mathcal{I}_2 acts on the torus by inverting the sign of both the coordinates of the torus. The modular parameter λ of the torus is determined in terms of the cross ratio

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}, \quad (2.15)$$

which is invariant under an $SL(2, C)$ transformation of the base CP^1 .

This background can now be given an orientifold interpretation as follows. By studying the action of various symmetry transformations on the massless fields of the theory it is easy to verify that the $SL(2, Z)$ transformation (2.13) can be identified to the transformation $(-1)^{FL} \cdot \Omega$ of the type IIB theory, where Ω denotes the orientation reversal transformation (exchange of left and right moving modes on the world sheet) and $(-1)^{FL}$ changes the sign of all the Ramond sector states on the left. Thus we have type IIB compactification on T^2/\mathcal{I}_2 such that as we go once around each fixed point on T^2/\mathcal{I}_2 the theory comes back to itself transformed by the symmetry $(-1)^{FL} \cdot \Omega$. In other words,

the theory can be identified to type IIB on T^2 , modded out by the Z_2 transformation $(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$. This is an orientifold[11], and, as we shall see, is related to type I theory on T^2 by a T -duality transformation. We shall denote this theory as type IIB on $T^2/(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$. In this case, each of the four orientifold planes carry -4 units of seven-brane charge, which need to be neutralized by putting sixteen seven branes transverse to T^2/\mathcal{I}_2 [11]. At a generic point in the moduli space, where the seven-branes are located at arbitrary positions on T^2/\mathcal{I}_2 , the seven brane charges are not neutralized locally and as a result $\tau(z)$ varies on T^2/\mathcal{I}_2 . On the other hand, in order that the field configuration matches with the one obtained from F -theory, $\tau(z)$ must be constant on T^2/\mathcal{I}_2 . Thus the seven-brane charges must be neutralized pointwise on T^2/\mathcal{I}_2 . This happens if the sixteen seven branes are grouped into four sets of four coincident seven-branes, and these four sets are placed at the four orientifold planes. This would give a field configuration identical to the one obtained from the F -theory configuration.

This establishes the equivalence between F -theory compactification on $K3$ and an orientifold of type IIB theory at a special point of the moduli space. By making an $R \rightarrow (1/R)$ duality transformation on both the circles of T^2 we can map the Z_2 transformation $(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$ to the transformation Ω .⁵ Since modding out the type IIB theory by Ω produces type I theory, we see that the orientifold is related by T -duality to type I theory on T^2 . This in turn has been conjectured to be equivalent to heterotic string theory on T^2 [12, 13, 14, 15]. Thus through this chain of arguments we have been able to establish the conjectured duality between F -theory compactification on $K3$ and heterotic string theory on T^2 . Although this duality was established only at one point in the moduli space, we can deform both theories away from this special point by switching on appropriate background fields, and hence the duality must hold at all points in the moduli space.

We can also study the enhanced non-abelian gauge symmetries at this special point in the moduli space, both from the orientifold view point as well as the F -theory viewpoint. From the orientifold viewpoint we have an $SO(8)$ gauge symmetry associated with each orientifold plane, since four seven-branes and their images meet there. Thus we get an $(SO(8))^4$ non-abelian gauge symmetry at this special point. On the other hand, in order to study the enhancement of gauge symmetry from the F -theory viewpoint, we need to study what kind of singularities appear at the points z_i ($1 \leq i \leq 4$) at this special point in the moduli space. To see this note that after a suitable rescaling of the various

⁵This fact has been independently observed in ref.[18].

coordinates, eq.(2.1) near the point $z = z_1$ takes the form:

$$\tilde{y}^2 \simeq \tilde{x}^3 + \alpha \tilde{x} \tilde{z}^2 + \tilde{z}^3, \quad (2.16)$$

where,

$$\tilde{y} = y \prod_{i=2}^4 (z_1 - z_i)^{3/2}, \quad \tilde{x} = x \prod_{i=2}^4 (z_1 - z_i), \quad \tilde{z} = (z - z_1). \quad (2.17)$$

This corresponds to a D_4 type singularity of the $K3$ surface and hence corresponds to an enhanced $SO(8)$ gauge symmetry. Since there are four such singular points, we get a net non-abelian gauge group $(SO(8))^4$, in agreement with the answer obtained from the orientifold analysis.

3 Deforming away from the Orbifold Limit of $K3$

In this section we shall discuss deforming both the F -theory and the orientifold theory away from the special point in the moduli space considered in the previous section and compare the results. In the F -theory such a deformation would correspond to splitting the six coincident zeroes of Δ away from each other. On the other hand, for the orientifold, this corresponds to moving the four coincident seven-branes away from the orientifold plane. In order to learn the physics of the situation we can focus our attention on one of the four orientifold planes. Equivalently, instead of studying F theory on the orbifold T^4/\mathcal{I}_4 (where \mathcal{I}_4 denotes changing the sign of all the four coordinates on T^4) and the type IIB on $T^2/(-1)^{FL} \cdot \Omega \cdot \mathcal{I}_2$, we consider F -theory on $T^2 \times R^2/\mathcal{I}_4$ and type IIB on $R^2/(-1)^{FL} \cdot \Omega \cdot \mathcal{I}_2$. Let z denote the complex coordinate on R^2/\mathcal{I}_2 , and $z = 0$ denote the fixed point, so that at the special point in the moduli space considered in the previous section the metric takes the form:

$$ds^2 = dzd\bar{z}/z^{1/2}\bar{z}^{1/2}. \quad (3.1)$$

Also let τ_0 be the constant value of τ away from the singular point. We shall study the deformation of this background keeping the asymptotic metric fixed to be of the form (3.1) and asymptotic τ fixed at τ_0 . The collective coordinates of this theory describe an N=1 supersymmetric $SO(8)$ gauge theory in eight dimensions. The moduli space of this theory is characterized by the vacuum expectation value of the complex scalar field

ϕ belonging to the adjoint representation of the gauge group. At a generic point in the moduli space this vacuum expectation value takes the form:

$$\langle \phi \rangle = \begin{pmatrix} i\sigma_2 c_1 & & & \\ & i\sigma_2 c_2 & & \\ & & i\sigma_2 c_3 & \\ & & & i\sigma_2 c_4 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (3.2)$$

where c_i are complex parameters.

First let us describe this deformation in the orientifold theory. In this case the deformed configuration will correspond to an orientifold plane carrying -4 units of seven-brane charge (which we shall take to be fixed at $z = 0$) and four seven branes at arbitrary coordinates z_i . Note that if we use the freedom of rescaling z we can eliminate one of the z_i 's from the set of independent parameters, but we do not use this freedom as this will change the asymptotic form of the metric. These four complex parameters can be related to the parameters c_i introduced earlier by working in the coordinate

$$w = z^{1/2}, \quad (3.3)$$

which is the natural coordinate on R^2 (as opposed to z which is the natural coordinate on R^2/\mathcal{I}_2). In this coordinate system the orientifold plane carries -8 units of seven brane charge, and there are eight seven branes on R^2 distributed in an \mathcal{I}_2 invariant fashion. $\pm c_i$ are the locations of these eight seven branes in R^2 [11]. Thus we have

$$z_i = c_i^2. \quad (3.4)$$

Since the seven-brane charge is no longer neutralized locally, τ is no longer a constant in the z plane. Naively, we would expect it to be of the form:

$$\tau(z) = \tau_0 + \frac{1}{2\pi i} \left(\sum_{i=1}^4 \ln(z - z_i) - 4 \ln z \right), \quad (3.5)$$

since there is $+1$ unit of seven brane charge at each z_i , and -4 unit of seven brane charge at $z = 0$. This solution is characterized by five parameters, $-\tau_0$ and the four z_i 's. A closer examination of the solution reveals however that the solution does not make sense everywhere in the z plane. In particular, close to $z = 0$, $Im(\tau)$ becomes large and negative, violating the bound $Im(\tau) \geq 0$.⁶ Thus we would expect that strong coupling effects will

⁶The situation is similar to the result of ref.[15] in one compact dimension, where the supergravity solution broke down for a sufficiently large separation between the orientifold plane and the eight branes. In the present case the solution breaks down for arbitrarily small but finite separation between the orientifold plane and the seven branes due to stronger short distance divergence in two dimensions.

modify the solution near the $z = 0$ point. We shall now see that the F -theory description of the background provides us with precisely such a modification. Note however that due to the high degree of supersymmetry present in the problem, we expect the moduli space of the theory to remain unmodified by quantum corrections, at least locally. Thus c_i (or equivalently z_i) should continue to label the moduli space of the theory, but the background field configuration describing the theory will no longer be of the form (3.5).

From the F -theory viewpoint, the deformation of $T^2 \times R^2/\mathcal{I}_4$ away from the orbifold limit is described by a surface of the form

$$y^2 = x^3 + \tilde{f}(z)x + \tilde{g}(z), \quad (3.6)$$

where \tilde{f} and \tilde{g} are now polynomials in z of degree two and three respectively. This gives a total of seven complex parameters to begin with. Of this one parameter can be removed by an overall shift of z and another can be removed by a rescaling of x and y . This again leaves us with five complex parameters. In order to relate these five parameters to those obtained in the orientifold description it is convenient to choose these five parameters in a specific manner. In the analysis of N=2 supersymmetric $SU(2)$ gauge theory with four quark flavors[16] Seiberg and Witten were led to a similar surface parametrized by four quark masses m_i ($1 \leq i \leq 4$) and the complex coupling constant $\tau_0 \equiv (\theta/\pi + 8\pi i/g^2)$. The surface was given by

$$y^2 = W_1W_2W_3 + A(W_1T_1(e_2 - e_3) + W_2T_2(e_3 - e_1) + W_3T_3(e_1 - e_2)) - A^2N, \quad (3.7)$$

where

$$W_i = x - e_i\tilde{z} - e_i^2R, \quad A = (e_1 - e_2)(e_2 - e_3)(e_3 - e_1), \quad (3.8)$$

$$e_1 - e_2 = \theta_3^4(\tau_0), \quad e_3 - e_2 = \theta_1^4(\tau_0), \quad e_1 - e_3 = \theta_2^4(\tau_0), \quad e_1 + e_2 + e_3 = 0, \quad (3.9)$$

$$\tilde{z} = z - \frac{1}{2}e_1R, \quad (3.10)$$

$$\begin{aligned} R &= \frac{1}{2} \sum_i m_i^2, \\ T_1 &= \frac{1}{12} \sum_{i>j} m_i^2 m_j^2 - \frac{1}{24} \sum_i m_i^4, \\ T_2 &= -\frac{1}{2} \prod_i m_i - \frac{1}{24} \sum_{i>j} m_i^2 m_j^2 + \frac{1}{48} \sum_i m_i^4, \end{aligned}$$

$$\begin{aligned}
T_3 &= \frac{1}{2} \prod_i m_i - \frac{1}{24} \sum_{i>j} m_i^2 m_j^2 + \frac{1}{48} \sum_i m_i^4, \\
N &= \frac{3}{16} \sum_{i>j>k} m_i^2 m_j^2 m_k^2 - \frac{1}{96} \sum_{i \neq j} m_i^2 m_j^4 + \frac{1}{96} \sum_i m_i^6.
\end{aligned} \tag{3.11}$$

The surface described by eq.(3.7) is not exactly of the form (3.6) since the coefficient of the x^2 term does not vanish, but this can be removed by an overall z independent shift in the coordinate x . We shall use the same set of parameters $\{m_i\}$ and τ_0 to label the F -theory background, and show that we get a consistent map between the orientifold and the F -theory by postulating the following simple relation between the parameters m_i labelling the F -theory and the parameters c_i labelling the orientifold:

$$m_i = c_i, \quad 1 \leq i \leq 4. \tag{3.12}$$

This would imply that up to an $SO(8)$ gauge transformation, the vacuum expectation value of the adjoint scalar ϕ is given by

$$\left(\begin{array}{cccc}
i\sigma_2 m_1 & & & \\
& i\sigma_2 m_2 & & \\
& & i\sigma_2 m_3 & \\
& & & i\sigma_2 m_4
\end{array} \right), \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3.13}$$

In the analysis of Seiberg and Witten the coordinate z on the base represented the gauge invariant modulus representing the square of the Higgs expectation value. There are six singularities in the z plane, which, in ref[16], signalled the appearance of massless charged particles at these points in the moduli space. In the weak coupling (large $Im(\tau_0)$) limit, four of these singularities are located at $z \simeq m_i^2$ representing points where the four quarks become massless. The two other singularities are located close to the origin within a distance of order $\exp(i\pi\tau_0/2)$, and represent points where massless monopoles and dyons appear in the spectrum. Thus in the $\tau_0 \rightarrow i\infty$ limit, these two singularities coalesce, and we recover the semiclassical picture where the $SU(2)$ gauge symmetry is restored at the origin and the W^\pm states become massless. Indeed, for large $Im(\tau_0)$ and $|z| \gg |e^{i\pi\tau_0/2}|$ the solution described in (3.8) can be rewritten as

$$\tau(z) \simeq \tau_0 + \frac{1}{2\pi i} \left(\sum_{i=1}^4 \ln(z - m_i^2) - 4 \ln z \right). \tag{3.14}$$

The coefficient 4 in front of $\ln z$ is due to the fact that the W^\pm bosons carry twice the electric charge of a quark, and the relative $-$ sign between the two terms is due to the

fact that the W^\pm belong to a vector multiplet whose contribution to the β -function has sign opposite to that of a hypermultiplet. This configuration is identical to the one given in eq.(3.5) describing the orientifold configuration, provided we identify z_i with m_i^2 , *i.e.* c_i with m_i . This shows that in the weak coupling limit the background field configuration corresponding to an orientifold is identical to the one described by F -theory even away from the special point in the moduli space described in the last section. However, the background field configuration describing the orientifold breaks down close to the orientifold point where the coupling constant becomes strong. On the other hand the F -theory background makes sense everywhere in the z plane. This leads us to conclude that the F -theory provides the correct description of the background field configuration of this theory, and the orientifold background must be modified by quantum correction so as to coincide with the F -theory background. This would imply in particular that the orientifold plane $z = 0$ is split into two planes due to quantum corrections, in a manner analogous to the splitting of the $z = 0$ point in the moduli space of N=2 supersymmetric $SU(2)$ gauge theories into two points. The splitting, being of order $\exp(i\pi\tau_0/2)$, is non-perturbative in the orientifold coupling constant, and is not visible in the perturbation theory.

The above analysis shows the equality of the parameters m_i and c_i in the weak coupling limit where we can directly compare the orientifold background with the F -theory background. We shall now show that eq.(3.12) gives the correct map between the points of enhanced gauge symmetry in the orientifold and the F -theory descriptions even away from the weak coupling region. In subsection 3.3 we shall trace the origin of this simple relation between m_i and c_i to the fact that $m_i \pm m_j$ are related to the period integrals of the holomorphic two form on the surface described by eq.(3.7).

3.1 Unbroken Gauge Symmetries

For a Higgs vacuum expectation value of the form (3.2) the unbroken gauge symmetry is the subgroup of $SO(8)$ that commutes with this matrix. In particular, if n of the c_i 's are equal and non-zero, then we recover an $SU(n)$ gauge symmetry, whereas if n of them are equal and zero, we recover an $SO(2n)$ gauge symmetry. In the orientifold description these enlarged gauge symmetries are associated with coincident seven branes[19]. We shall now show that with the identification of m_i with c_i , we get the same enhanced gauge symmetries in the F -theory at these special points. In the F -theory background

described by eq.(3.7), when n of the m_i 's are equal and non-zero, n of the zeroes of Δ , representing the point where these n quarks become massless in the corresponding $SU(2)$ gauge theory, coincide. Thus Δ has an n th order zero, but typically, neither $f(z)$ nor $g(z)$ vanish there. According to the table in ref.[3] this corresponds to an A_{n-1} type singularity and hence an enhanced $SU(n)$ gauge symmetry. The case where n of the m_i 's vanish is somewhat more complicated, and we need to carry out the analysis separately for each n . The relevant values of n are 2, 3 and 4. For $n = 4$ all m_i 's vanish, and the equation of the surface reduces to the form (2.16). This corresponds to a D_4 type singularity and hence the F -theory on such a surface has an enhanced $SO(8)$ gauge symmetry. For $n = 3$ the singularity structure near the origin is that of N=2 supersymmetric Yang-Mills theory with three massless quarks. According to the analysis of Seiberg and Witten, there are two singularities near the origin. Using the results of ref.[16] one can easily verify that near one of the singular points Δ has a fourth order zero with f and g being finite at that point. According to ref.[3] this corresponds to an A_3 type singularity, and hence an enhanced $SU(4) \equiv SO(6)$ gauge symmetry. Finally, for $n = 2$, the singularity structure near the origin is that of an N=2 supersymmetric $SU(2)$ gauge theory with two flavors of massless quarks. According to the analysis of Seiberg and Witten this theory again has two singular points near the origin, and at each of these singular points Δ has a second order zero where f and g remain finite. Using the table of ref.[3] we see that each of these singularities is of A_1 type, and hence we expect an enhanced $SU(2) \times SU(2) \equiv SO(4)$ gauge symmetry at this point. This shows that the identification of the parameters m_i with c_i gives us enhanced gauge symmetries in the F -theory at correct points in the moduli space.

In the parameter space of N=2 supersymmetric $SU(2)$ gauge theories with four flavors of quarks, there are special points of enhanced *global symmetry*. In particular, when n of the quark masses are equal but non-zero, we have a global $SU(n)$ symmetry, whereas if n of the quarks are massless, we have an enhanced global $SO(2n)$ symmetry. The analysis of the previous paragraph shows that whenever the N=2 supersymmetric $SU(2)$ gauge theory develops an enhanced global symmetry, the corresponding F -theory develops the same enhanced gauge symmetry. This has a simple interpretation in view of the identification of m_i with the parameters c_i . In the N=2 supersymmetric Yang-Mills theory, the quark mass matrix is exactly of the form given in (3.13), and the unbroken global symmetry group at any point in the parameter space is the subgroup of $SO(8)$ that commutes with

this matrix. But this is precisely the subgroup of $SO(8)$ that remains unbroken in the corresponding orientifold / F -theory compactification, since (3.13) represents the vacuum expectation value of the adjoint Higgs field in this theory. Thus we see that there is a one to one correspondence between the points of enhanced global symmetries arising in the supersymmetric Yang-Mills theory and the points of enhanced gauge symmetries arising in the orientifold / F -theory.

3.2 Triality

We shall now discuss the observation of ref.[16] that $SL(2,Z)$ transformations have a triality action on the representations of the global symmetry group $SO(8)$. In particular, the transformation $\tau \rightarrow -1/\tau$ acts by exchanging the vector and the spinor representations of $SO(8)$ and acts on the mass parameters m_i as

$$\begin{aligned}
m_1 &\rightarrow \frac{1}{2}(m_1 + m_2 + m_3 + m_4), \\
m_2 &\rightarrow \frac{1}{2}(m_1 + m_2 - m_3 - m_4), \\
m_3 &\rightarrow \frac{1}{2}(m_1 - m_2 + m_3 - m_4), \\
m_4 &\rightarrow \frac{1}{2}(m_1 - m_2 - m_3 + m_4).
\end{aligned} \tag{3.15}$$

This would imply, in particular, that in the specific orientifold compactification that we are considering, the action $\tau_0 \rightarrow -1/\tau_0$ on the asymptotic τ by itself is not a symmetry of the theory, but it must also act on the locations c_i (m_i) of the seven branes (or, equivalently, on the vacuum expectation values of the adjoint Higgs field) according to eq.(3.15).

Is there a way to verify this independently? From the point of view of the type IIB orientifold, the symmetry $\tau \rightarrow -1/\tau$ is a non-perturbative symmetry, and hence the action of this transformation on the representation of the gauge group will be difficult to study explicitly. However, by using the type I – heterotic equivalence in ten dimensions, or equivalently, the F -theory – heterotic duality in eight dimensions, one can map the non-perturbative S-duality transformation of the orientifold theory to a perturbative T-duality transformation in the heterotic string theory, where the action of this transformation on the representations of the gauge group can be studied explicitly. To do this we need to first patch together four copies of the solution we have been discussing so as to describe type IIB compactification on $T^2/(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$. This can easily be done when the distance

between the fixed points is large compared to the distance between any given fixed point and the four seven-branes around it. For this we rewrite eq.(3.7) (after a constant shift of x and a suitable rescaling of x and y) as

$$y^2 = x^3 + x\alpha(\tau_0)\tilde{f}(z, \{m_i\}, \tau_0) + \tilde{g}(z, \{m_i\}, \tau_0), \quad (3.16)$$

where $\alpha(\tau_0)$ is given by eq.(2.12), and \tilde{f} and \tilde{g} are polynomials in z of degree two and three respectively, with the coefficients of the leading power in z set equal to one in both of them. Then the equation of the full $K3$ surface, obtained by patching together four of these solutions will be given by

$$y^2 = x^3 + x\alpha(\tau_0) \prod_{s=1}^4 \tilde{f}(z - z_s, \{m_i^{(s)}\}, \tau_0) + \prod_{s=1}^4 \tilde{g}(z - z_s, \{m_i^{(s)}\}, \tau_0), \quad (3.17)$$

where \tilde{f} and \tilde{g} are the same functions that appear in eq.(3.16). With the help of an $SL(2, \mathbb{C})$ transformation on z , we can set $(z_1, z_2, z_3) = (0, 1, -1)$. The solution is then characterized by the set of sixteen $\{m_i^{(s)}\}$ ($1 \leq i \leq 4, 1 \leq s \leq 4$), τ_0 and z_4 . When all the m_i 's associated with all the four orbifold points vanish, this $K3$ surface is described by eq.(2.1) with g and f given by eqs.(2.9), (2.10). The unbroken symmetry group in this theory is $(SO(8))^4$. Since the same τ_0 represents the value of τ away from each of the four fixed points on T^2/\mathcal{I}_2 , the $\tau_0 \rightarrow -1/\tau_0$ transformation must exchange the vector and the spinor representation of each $SO(8)$.

We shall now verify this explicitly by mapping this to the dual heterotic description. For this we regard this as an orientifold, and make $R \rightarrow (1/R)$ T-duality transformation in each of the compact directions (which we shall denote by the coordinates x^8 and x^9) to map this into type I theory on T^2 . In this process the RR scalar, that forms the real part of τ , gets mapped to B'_{89} where B' denotes the rank two anti-symmetric tensor in the RR sector. Under the type I - heterotic duality, this gets mapped to B_{89} where B denotes the rank two anti-symmetric tensor in the heterotic string theory. Thus τ gets mapped to the Kahler modulus of the heterotic string theory on T^2 , and hence the S-duality group $SL(2, \mathbb{Z})$ of the orientifold theory gets mapped to the $SL(2, \mathbb{Z})$ T-duality symmetry of the $SO(32)$ heterotic string theory compactified on T^2 . In particular the transformation $\tau \rightarrow -1/\tau$ will correspond to $R \rightarrow 1/R$ duality on both the circles in the heterotic string theory, together with an exchange of the coordinates x^8 and x^9 . We shall denote this transformation by σ and show that it induces a triality transformation on the representations on $(SO(8))^4 \subset SO(32)$.

We need to study the $SO(32)$ heterotic string theory near a point in the moduli space where the $SO(32)$ gauge group has been broken down to $(SO(8))^4$. This corresponds to introducing $SO(32)$ Wilson lines U_8 and U_9 along x^8 and x^9 given by

$$U_8 = \begin{pmatrix} -I_8 & & & \\ & -I_8 & & \\ & & I_8 & \\ & & & I_8 \end{pmatrix}, \quad U_9 = \begin{pmatrix} -I_8 & & & \\ & I_8 & & \\ & & -I_8 & \\ & & & I_8 \end{pmatrix}, \quad (3.18)$$

where I_n denotes $n \times n$ identity matrix. We shall analyze this theory using the fermionic description of the heterotic string theory. Without the Wilson lines, the $SO(32)$ heterotic string contains a conformal field theory of 32 free left-moving Majorana fermions, and this conformal field theory is modded out by a Z_2 transformation that changes the sign of all the thirty two fermions[20]. This Z_2 modding out is responsible for the GSO projection on the left, and the existence of twisted sector states belonging to the spinor representation of $SO(32)$. We shall group the 32 fermions into four groups of eight each and denote this Z_2 transformation as $(- - - -)$ in order to denote that it acts as $-I_8$ on all four groups of fermions. Introduction of the Wilson lines (3.18) corresponds to further modding out the theory by a $Z_2 \times Z_2$ transformation, generated by

$$(- - ++)(x^8 \rightarrow x^8 + \pi), \quad (3.19)$$

and

$$(- + - +)(x^9 \rightarrow x^9 + \pi). \quad (3.20)$$

We shall use the convention in which the transformation $-I_8$ of $SO(8)$ changes the sign of the vector (v) and the conjugate spinor (c) representations of $SO(8)$ but leaves the spinor (s) representations of $SO(8)$ invariant. Let us now consider an untwisted sector state that transforms in the vector representations of the first two $SO(8)$. We shall denote such a state by $(v_1 v_2)$. This state is even under $(- - ++)$ and odd under $(- + - +)$. Thus invariance under (3.19) and (3.20) requires that the state carries even unit of momentum along x^8 and odd unit of momentum along x^9 . The duality transformation σ converts this to a state carrying odd unit of winding along x^8 and even unit of winding along x^9 . In other words, this would correspond to a state in the twisted sector of (3.19). Such a state belongs to the conjugacy class $(s_1 s_2)$. Similar analysis shows that the transformation σ takes a state in the conjugacy class $(v_i v_j)$ to the conjugacy class $(s_i s_j)$ for all i, j . (Due to the twisting by the $(- - - -)$ transformation that commutes with σ , we do not

distinguish between conjugacy classes $(s_i s_j)$ and $(s_k s_l)$ if (i, j, k, l) is a permutation of the set $(1, 2, 3, 4)$.) This shows that σ does exchange the vector and spinor representations of each of the four $SO(8)$'s as expected from the F -theory description.

From the analysis of ref.[16] we also know that the transformation $\tau \rightarrow \tau + 1$ induces an $SO(8)$ parity transformation. Thus we would expect that in type IIB on $T^2/(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$, $\tau \rightarrow \tau + 1$ must be accompanied by a parity transformation in all four $SO(8)$'s. This however corresponds to the $SO(32)$ gauge transformation

$$\begin{pmatrix} I_7 & & & & & & & \\ & -1 & & & & & & \\ & & I_7 & & & & & \\ & & & -1 & & & & \\ & & & & I_7 & & & \\ & & & & & -1 & & \\ & & & & & & I_7 & \\ & & & & & & & -1 \end{pmatrix}. \quad (3.21)$$

Since (3.21) by itself is a symmetry of the theory, in this theory $\tau \rightarrow \tau + 1$ is also a symmetry by itself without being accompanied by any action on the Higgs field. As a result we do not expect to see any non-trivial action of this transformation on the representations of $(SO(8))^4$ in the dual heterotic string theory.

3.3 BPS States

At a generic point in the moduli space, type IIB on $R^2/(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$ has BPS states representing the $SO(8)$ gauge bosons and their superpartners that have become massive due to Higgs vacuum expectation value of the form (3.2). In the orientifold description these correspond to elementary strings starting on one of the four seven branes and ending on another seven brane. Due to the presence of the orientifold plane there are two topologically distinct ways an open string can stretch between two seven branes. We shall call one of them the direct route, and the other one, differing from the direct route by one unit of winding around the orientifold plane, the indirect route. In the natural coordinate system w on R^2 defined in (3.3) the direct route is a straight line joining the points m_i and m_j and the indirect route is a straight line joining the points m_i and $-m_j$. With suitable convention for the sign of $\{m_i\}$, the mass of a BPS state represented by an open string stretched between the i th seven brane and the j th seven-brane along the direct route is given by $m_i - m_j$. On the other hand the BPS state corresponding to an open

string stretched between the i th and the j th seven brane along the indirect route has a mass given by $m_i + m_j$.

From our previous discussion about the relationship between the orientifold and the F -theory background, it is clear how to represent these BPS states in the corresponding F -theory description. The description is in fact identical to the one given in the orientifold theory, except that winding around the orientifold plane will correspond to winding around the two seven-branes (which we shall label as the fifth and the sixth seven brane) into which the orientifold plane splits. We would now like to ask if the masses of these BPS states can be expressed in terms of some natural objects in the F -theory. This is important since the parameters m_i were introduced via eq.(3.7)-(3.11) in an *ad hoc* fashion in order to parametrize the F -theory background and hence one would like to know if there is any reason why the masses of BPS states should have simple expression in terms of these parameters. We shall now see that these masses can in fact be expressed in terms of natural objects in the F -theory.

For a given open string BPS state starting at the i th seven brane and ending at the j -th seven brane, let us introduce a closed curve C in the z plane that travels around the i th seven brane in anti-clockwise direction, goes to the j th seven brane following the contour of the open string, travels it in the clockwise direction, and comes back to the i th seven-brane by following the contour of the open string in the opposite direction. Then the mass of the open string state can be written as

$$\oint_C \partial_z a_D, \quad (3.22)$$

with a_D as defined in ref.[16]. (Note that we have renamed the variable u in ref.[16] as z). To test the validity of eq.(3.22) we simply use the fact[16] that as we move around the i th seven brane in the anti-clockwise direction

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \rightarrow \begin{pmatrix} a_D + a + m_i \\ a \end{pmatrix}. \quad (3.23)$$

With the help of this equation, (3.22) reproduces the mass formula $(m_i - m_j)$ for the open string state stretched between the i th and the j th string along the direct route. On the other hand, along a closed curve that winds once around the fifth and the sixth seven branes, the vector $\begin{pmatrix} a_D \\ a \end{pmatrix}$ suffers a monodromy represented by the matrix[16]

$$\begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}. \quad (3.24)$$

Thus the shift $a_D \rightarrow a_D - a - m_j$, when transported along such a curve, corresponds to a shift $a_D \rightarrow a_D - a + m_j$. As a result, (3.22) evaluated along the curve C associated with the indirect route between the i th and the j th seven brane gives an answer $m_i + m_j$. This agrees with the mass formula obtained from the orientifold theory.

We shall now try to reexpress (3.22) in terms of integral of the holomorphic two form on the surface (3.7) along a closed two cycle. For this we note that[16]

$$a_D = \oint_b \lambda, \quad (3.25)$$

where λ is the one form introduced in section 17 of ref.[16], and b denotes the b cycle of the torus represented by eq.(3.7) for fixed z . We can then reexpress (3.22) as

$$\oint_C \partial_z \oint_b \lambda = \oint_S \omega, \quad (3.26)$$

where

$$\omega \equiv d\lambda \propto \frac{dx \wedge dz}{y}, \quad (3.27)$$

is the holomorphic two form on the surface described by eq.(3.7) and S is the two (real) dimensional surface swept out by the b cycle of the torus as we move along the closed curve C in the z plane. Since a_D comes back to a_D up to a constant shift on being transported around the curve C , the b -cycle of the torus comes back to the b cycle on being transported around C , and hence S is a closed surface. Note that in defining the surface S we had to choose a specific cycle on the torus which we defined as the b -cycle, and hence broke manifest $SL(2, Z)$ symmetry of the mass formula. But this is expected since we are analyzing states that arise from *elementary strings* stretched between two seven-branes, and elementary strings are not invariant under $SL(2, Z)$ transformation[21]. In particular, the mass formula given in eq.(3.26) is to be multiplied by the square root of the string tension of the elementary string.

Eq.(3.26) gives an expression for the masses of BPS states in F -theory on a (complex) surface in terms of period integrals of the holomorphic two form on this surface. When one or more of these period integrals vanish, the surface becomes singular, and at the same time the corresponding BPS states become massless, signalling the appearance of enhanced gauge symmetries in the theory.

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