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An Ambiguity in Fermionic String Perturbation Theory*

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ABSTRACT

Recent investigation by Verlinde and Verlinde has shown that the fermionic string loop amplitudes change by a total derivative term in the moduli space under a change of basis of the supermoduli. This ambiguity is addressed in the context of the heterotic string theory, and shown to be a consequence of an inherent ambiguity in defining integration over the variables of a Grassmann algebra—in this case the Grassmann valued coordinates of the supermoduli space. A resolution of this ambiguity in genus-two within this formalism is also presented.

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1. Introduction

Recently Verlinde and Verlinde[1] have written down an expression for the super (heterotic) string functional integral which can be used to carry out explicit computation of the partition function and various amplitudes in fermionic string theories. Their formulae however seem to suffer from the defect that all amplitudes depend on the particular choice of basis of the super-Beltrami differentials in terms of which one expands the gravitino field. In ref.[1] it was shown that under a change of basis of super-Beltrami differentials the partition function or any other n-point amplitude changes by a total derivative in the moduli space. Normally one would ignore such terms. However, as has become clear recently[2], such terms cannot always be ignored. For example, following the analysis of ref.[1] one can show that in any compactification of heterotic string theory which preserves $N = 1$ space-time supersymmetry at the string tree level, the dilaton tadpole (or, equivalently, the partition function) is a total derivative in the moduli space. This, however, does not necessarily imply the vanishing of the dilaton tadpole. In a special class of theories, one loop string effect may break space-time supersymmetry by generating a Fayet-Iliopoulos D term[3,4]. It is precisely in these theories that the two loop dilaton tadpole receives a non-vanishing contribution from the boundary of the genus two moduli space [2].

Since space-time supersymmetry breaking terms arise as total derivatives in the moduli space, it is imperative at this stage to understand the other type of total derivative terms arising from the change of basis of the super-Beltrami differentials. This will help us gain deeper insight into the structure of fermionic string perturbation theory, and will enable us to isolate unambiguously the space-time supersymmetry breaking effects.

In this paper we shall study the total derivative ambiguity in the fermionic string functional integral. In particular, we shall concentrate our attention on (compactified) heterotic string theory. We shall see that this ambiguity is very generic and is related to an inherent ambiguity in the definition of integrals over elements of a Grassmann algebra (in this case the variables describing the supermoduli space). We also show how to resolve this ambiguity in the case of genus two surfaces through various considerations of modular invariance and BRST invariance. Currently we do not have a solution to this problem beyond genus two surfaces, although we believe that similar considerations could be powerful enough to lead to a resolution of this problem in the case of arbitrary genera.

Sec. 2 of this paper reviews the result of Verlinde and Verlinde[1], and identifies the total derivative terms in the heterotic string partition function arising from a change of basis of the super-Beltrami differentials. We also show in this section that the answer for the heterotic string partition function given in ref.[1] is invariant under a change in the basis of Beltrami and super-Beltrami differentials induced by an ordinary reparametrization. In sec. 3 we discuss an inherent ambiguity in defining integration over the variables of a Grassmann algebra, and show that this is the origin of the total derivative ambiguity in the heterotic string partition function. In sec. 4 we discuss the resolution of this ambiguity for genus two Riemann surfaces. We summarize our results in sec. 5.

2. The total derivative ambiguity in the heterotic string partition function

We shall start our discussion by writing down an expression for the two loop heterotic string partition function following ref.[1],

$$W = \int D[XBC]\Big(\prod_{i=1}^{6} dm_i\Big)\Big(\prod_{a=1}^{2} d\varsigma^a\Big)\Big(\prod_{i=1}^{6}(\eta_i \mid B)\Big) \\ \Big(\prod_{a=1}^{2}\delta\big((\chi_a \mid B)\big)\Big)e^{-S(X,B,C,m_i,\varsigma^a)}, \tag{2.1}$$

where $\{m_i\}$ are the six moduli for the genus two surface and $\{\varsigma^a\}$ are the two supermoduli [5]; B denotes the reparametrization ghosts b_{zz}, $\bar{b}_{\bar{z}\bar{z}}$ and the super-reparametrization ghost $\beta_{z\theta}$. Similarly C stands for the reparametrization ghost fields c^z, $\bar{c}^{\bar{z}}$ as well as the super-reparametrization ghost field γ^θ. X denotes the set of all the matter fields. The inner products $(\eta_i \mid B)$, $\delta((\chi_a \mid B))$ in (2.1), which are there to soak up the various ghost zero modes, are defined as follows:

$$
\begin{aligned}
(\eta_i \mid B) &= \int d^2 z \{\eta_{i\bar{z}}{}^{z} b_{zz} + \bar{\eta}_{iz}{}^{\bar{z}} \bar{b}_{\bar{z}\bar{z}} + \eta_{i\bar{z}}{}^{\theta}\beta_{z\theta}\} \\
(\chi_a \mid B) &= \int d^2 z \chi_{a\bar{z}}{}^{\theta}\beta_{z\theta}
\end{aligned}
\tag{2.2}
$$

where η_i, χ_a form a basis for the super-Beltrami differentials and are defined through the following equations:

$$
\begin{aligned}
\eta_{i\bar{z}}{}^{z} &= g_{z\bar{z}}\frac{\delta g^{zz}}{\delta m_i}, \qquad \bar{\eta}_{iz}{}^{\bar{z}} = g_{z\bar{z}}\frac{\delta g^{\bar{z}\bar{z}}}{\delta m_i}, \\
\eta_{i\bar{z}}{}^{\theta} &= \frac{\delta \chi_{\bar{z}}^{\theta}}{\delta m_i}, \qquad \chi_{a\bar{z}}{}^{\theta} = \frac{\delta \chi_{\bar{z}}^{\theta}}{\delta \varsigma^a}.
\end{aligned}
\tag{2.3}
$$

Here we have assumed that the choice of the metric $g^{\alpha\beta}(m_i)$ is independent of the odd coordinates ς^a.* In writing down eq.(2.1) we have ignored the complex structure on the moduli space.† We have also suppressed the summation over spin structures. This sum may be regarded as included in the integration over m_i if these are viewed as coordinates of the spin moduli space, the moduli space of Riemann surfaces with spin structures. $(z,\bar{z})$ denotes a coordinate system such that $g^{zz} = g^{\bar{z}\bar{z}} = 0$ everywhere on the Riemann surface at the particular point $\{m_i\}$ in the moduli space where we are evaluating the string integrand. Thus the choice of the coordinate system $(z,\bar{z})$ varies with the moduli. In defining the various partial derivatives appearing in eq.(2.3), however, we must keep the coordinate system fixed. In other words, after we choose the specific coordinate system $(z,\bar{z})$ by demanding that $g^{\alpha\beta}(m_i)$ is diagonal everywhere on the Riemann

* As we shall see later, this assumption is related to the total derivative ambiguity.

† The six real moduli may be grouped into three complex moduli m_p, $m_{\bar{p}}$; and the metric may be chosen such that $\eta_{\bar{p}\bar{z}}{}^{z} = \bar{\eta}_{pz}{}^{\bar{z}} = 0$.

surface, we evaluate $g^{zz}(m_i + \delta m_i)$, $g^{\bar{z}\bar{z}}(m_i + \delta m_i)$ and $\chi_{\bar{z}}^{\theta}(m_i + \delta m_i)$ in this coordinate system, and take $\delta m_i \to 0$ limit of appropriate ratios to calculate the (super)-Beltrami differentials using eq.(2.3).

The standard practice for dealing with the super-moduli has been so far to perform the super-moduli integration explicitly using the rules for Grassmann integration. This leads us to,

$$W = \int D[XBC] \prod_{i=1}^{6} dm_i e^{-S_0} \prod_{a=1}^{2} \delta((\chi_a \mid B)) \\ \Big[\prod_{a=1}^{2} \{(\chi_a \mid T) + \frac{\partial}{\partial \varsigma^a}\} \prod_{i=1}^{6} (\eta_i \mid B)\Big]_{\varsigma^a=0}, \tag{2.4}$$

where S_0 is the part of the action S obtained by setting the gravitino field to zero. The first term in the curly bracket arises because of the coupling of the two dimensional gravitino to the world-sheet supersymmetry current $(T_F)_{z\theta}$, while the second term is due to the fact that ${\eta_{i\bar{z}}}^{\theta}$ appearing in $(\eta_i \mid B)$ depends on the supermoduli ς^a.

Expression (2.4) for the heterotic string functional integral may be made more explicit through a particular choice of basis of the super-Beltrami differentials ${\chi_{a\bar{z}}}^{\theta}$ given by,

$${\chi_{a\bar{z}}}^{\theta} = \delta^{(2)}(z - z_a), \qquad (a = 1, 2), \tag{2.5}$$

where $\{z_a\}$ are *a priori* two arbitrary points on the Riemann surface. In this basis it is not difficult to show that expression (2.4) can be written as,

$$W=-\int\prod_{i=1}^{6}dm_iD[XBC]e^{-S_0}\Big[Y(z_1)Y(z_2)\prod_{i=1}^{6}(\eta_i\mid B)_0$$
$$+Y(z_1)\partial\xi(z_2)\sum_{j=1}^{6}(-1)^{j+1}\frac{\partial z_2}{\partial m_j}\prod_{i\neq j}(\eta_i\mid B)_0$$
$$+Y(z_2)\partial\xi(z_1)\sum_{j=1}^{6}(-1)^{j+1}\frac{\partial z_1}{\partial m_j}\prod_{i\neq j}(\eta_i\mid B)_0$$
$$-\partial\xi(z_1)\partial\xi(z_2)\sum_{j=1}^{6}\sum_{k>j}(-1)^{j+k}\prod_{i\neq j,k}(\eta_i\mid B)_0\Big(\frac{\partial z_1}{\partial m_j}\frac{\partial z_2}{\partial m_k}-\frac{\partial z_1}{\partial m_k}\frac{\partial z_2}{\partial m_j}\Big)\Big] \tag{2.6}$$

where we have used the bosonization prescription of ref.[6,1]:

$$\beta=\partial\xi e^{-\phi},\qquad\gamma=\eta e^{\phi}, \tag{2.7}$$

in which case[1]

$$\delta(\beta)=e^{\phi},\qquad\delta(\gamma)=e^{-\phi}. \tag{2.8}$$

We have also defined,

$$Y=:e^{\phi}T_F:=c\partial\xi+e^{\phi}T_F^{matter}-\frac{1}{4}\{\partial\eta e^{2\phi}b+\partial(\eta e^{2\phi}b)\}=\{Q_B,\xi\} \tag{2.9}$$

where Q_B is the BRST charge.* This is nothing but the picture changing operator[6]. Finally $(\eta_i\mid B)_0$ is obtained from $(\eta_i\mid B)$ by setting all the supermoduli to zero inside $(\eta_i\mid B)$. In writing down eq.(2.6) we have utilized the fact that

$$\frac{\partial}{\partial\varsigma^a}(\eta_i\mid B)=\int d^2z\frac{\partial^2\chi_{\bar z}{}^{\theta}}{\partial\varsigma^a\partial m_i}\beta_{z\theta}=\frac{\partial z_a}{\partial m_i}\partial\beta(z_a) \tag{2.10}$$

for the basis in (2.5). We have also suppressed a factor of $\xi(z_0)$ which must be inserted in (2.6) to absorb the ξ zero mode. As in the calculation of the (super)-

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* We may take for Q_B either the BRST charge (Q_B^R) associated with the right moving BRST current,or the sum of the BRST charges ($Q_B^R+Q_B^L$) associated with the right and the left moving BRST currents, since $\{Q_B^L,\xi\}=0$. For later analysis, it will prove to be convenient to take Q_B to be $Q_B^R+Q_B^L$ at this stage.

Beltrami differentials, $\frac{\partial z_a}{\partial m_i}$ in eq.(2.6) is calculated by first fixing the coordinate system (z,$\bar{z}$) in which $g^{\alpha\beta}$ becomes diagonal everywhere on the Riemann surface at the specific point $\{m_i\}$ in the moduli space, then calculating $z_a(m_i)$ and $z_a(m_i + \delta m_i)$ in this coordinate system, and finally taking the limit $\delta m_i \to 0$ of the appropriate ratios.

In writing down eq.(2.1) we have fixed the choice of gauge, *i.e.* in the field space of two dimensional metric and gravitino fields, we have chosen a slice orthogonal to the gauge directions. At this point one may like to check if the partition function is invariant under a different choice of slice, related to the original slice by an infinitesimal reparametrization $v^\alpha(m_i)$. Since in order to express the partition function at a point $\{m_i\}$ in the moduli space we use a coordinate system $(z, \bar{z})$ in which $g^{\alpha\beta}$ is diagonal everywhere on the Riemann surface, there is no change of $g^{\alpha\beta}(m_i)$ expressed in the coordinate system $(z, \bar{z})$. As a result, in this coordinate system, $v^z(m_i, z, \bar{z}) = v^{\bar{z}}(m_i, z, \bar{z}) = 0$. However, expressed in the coordinate system $(z, \bar{z})$, $v^z(m_i + \delta m_i, z, \bar{z})$ $(v^{\bar{z}}(m_i + \delta m_i, z, \bar{z}))$ does not necessarily vanish. Consequently, under reparametrization, the various quantities appearing in eq.(2.6) change as,

$$\begin{aligned} \Delta\eta_{i\bar{z}}{}^{z} &= \frac{\partial}{\partial m_i}(\partial_{\bar{z}} v^z), \qquad \Delta\bar{\eta}_{iz}{}^{\bar{z}} = \frac{\partial}{\partial m_i}(\partial_z v^{\bar{z}}) \\ \Delta(\frac{\partial z_a}{\partial m_i}) &= \frac{\partial v^z(z_a)}{\partial m_i} \end{aligned} \tag{2.11}$$

Substituting this in eq.(2.6) one can calculate the net change in W under a reparametrization. The calculation is more or less straightforward, once we note that $(\Delta\eta \mid B)_0$ can be expressed as a total derivative in z, $\bar{z}$, and receives contribution only from those regions of integration in z, $\bar{z}$ plane where $b_{zz}(z)$ developes poles, *i.e.* at the location of the $Y(z_a)$, since $Y(z_a)$ involves a $c(z_a)\partial\xi(z_a)$ term. The contribution from these regions may be readily evaluated from the known operator product $b(z)c(w)$. The final expression for ΔW may be shown to vanish identically point by point in the moduli space. This means that W

is invariant under a change of basis of the (super)-Beltrami differentials induced by reparametrization. Among other things this has the following consequence. Suppose we find a consistent choice of basis of the super-Beltrami differentials in which the location of the points z_1 and z_2 changes as a function of the moduli $(\frac{\partial z_a}{\partial m_i} \neq 0)$. Then, by a reparametrization (which itself is a function of the moduli) we can choose another basis equivalent to the original one, where the location of the points z_1 and z_2 are independent of the moduli, all dependence being transfered to the Beltrami differentials η_i, $\bar{\eta}_i$. It is often convenient to choose such a basis since it simplifies W enormously, only the first term in eq.(2.6) remains. In general, however, after we choose the z_a's in this way, we no longer have the freedom of a further reparametrization which allows us to set $\eta_{\bar{p}\bar{z}}{}^{z} = \bar{\eta}_{pz}{}^{\bar{z}} = 0$, where m_p, $m_{\bar{p}}$ are complex coordinates in the moduli space.

Next we investigate the behavior of W under a change of basis of super-Beltrami differentials. Since the freedom of choosing an arbitrary basis for the super-Beltrami differentials may be traced to supersymmetry invariance of the original system before gauge fixing, this calculation will reflect the change in W under a change in the gauge slice, related to the original slice by a supersymmetry transformation. We choose for simplicity a particular supersymmetry transformation u^θ which changes the point z_1 to $\tilde{z}_1 = z_1 + \Delta z_1$ leaving z_2 fixed.* We shall now represent $Y(z_1) - Y(\tilde{z}_1) \equiv \Delta z_1 \{Q_B, \partial\xi(z_1)\}$ as the contour integral of the BRST current around the point z_1, deform the BRST current away from z_1 and try to shrink it to a point. The only obstruction to this deformation will be poles in the argument of the BRST current at the location of $\partial\xi(z_2)$ and $(\eta_i \mid B)_0$.† The residue at $\partial\xi(z_2)$ is given by $\partial Y(z_2)$, whereas the residue at the location of $(\eta_i \mid B)_0$ is given by $(\eta_i \mid T)_0$, T being the full stress tensor of the system. The insertion of $(\eta_i \mid T)_0$ in a correlator may, in turn, be expressed as $-\frac{\partial S_0}{\partial m_i}$ inserted

* The fact that such a transformation exists will be demonstrated later.

† The BRST contour passes through $Y(z)$ since $Y(z)$ is BRST invariant.

in the correlator[6,7,1]. Combining all the terms together we find,

$$
\begin{aligned}
\Delta_{SUSY}W = &\int \prod_{i=1}^{6} dm_i \sum_{j=1}^{6} \frac{\partial}{\partial m_j}\Big[\int D[XBC]e^{-S_0}(-1)^j \\
&\Big\{\partial\xi(z_1)\Delta z_1 Y(z_2)\prod_{i\neq j}(\eta_i \mid B)_0 + \partial\xi(z_1)\Delta z_1\partial\xi(z_2)\sum_{k>j}(-1)^k\frac{\partial z_2}{\partial m_k}\prod_{i\neq j,k}(\eta_i \mid B)_0 \\
&- \partial\xi(z_1)\Delta z_1\partial\xi(z_2)\sum_{k<j}(-1)^k\frac{\partial z_2}{\partial m_k}\prod_{i\neq j,k}(\eta_i \mid B)_0\Big\}\Big]
\end{aligned}
\tag{2.12}
$$

which agrees with the general formula given in ref.[1]. We can now explicitly evaluate it on a genus two Riemann surface. In order to soak up all the ghost charges we need a net factor of $e^{2\phi}$ in the correlator. Thus the only non-vanishing answer comes from the first term inside the curly bracket, and only the part of Y containing $e^{2\phi(z_2)}$ will contribute. This part of Y may be formally written as $\frac{1}{2}e^{\phi(z_2)}\gamma(z_2)b(z_2)$. This singular operator product must be defined through a suitable subtraction procedure so as to be consistent with BRST invariance, and is given in eq.(2.9). Finally, by expanding $b(z_2)$ in terms of the zero mode wave-functions, this can be be brought into the following final form,

$$
\begin{aligned}
\Delta_{SUSY}W = -\frac{1}{2}\int \prod_{i=1}^{6} dm_i \sum_{j=1}^{6} \frac{\partial}{\partial m_j}\Big[D[XBC]e^{-S_0}\partial\xi(z_1)\Delta z_1 \\
e^{\phi(z_2)}\gamma(z_2)\prod_{i=1}^{6}(\eta_i \mid B)_0\sum_{p=1}^{3}(A^{-1})_{pj}h^{p}_{zz}(z_2)\Big]
\end{aligned}
\tag{2.13}
$$

where the matrix A_{jp} $(A_{j\bar{p}})$ is defined through the inner product,

$$
A_{ip} = \int d^2z\,\eta_{i\bar{z}}{}^{z}h^{(p)}_{zz}, \qquad A_{i\bar{p}} = \int d^2z\,\bar{\eta}_{iz}{}^{\bar{z}}\bar{h}^{(\bar{p})}_{\bar{z}\bar{z}}, \qquad (1\leq i\leq 6,\ \ 1\leq p\leq 3). \tag{2.14}
$$

where $\{h^{(p)}_{zz}\}$ is a basis for the holomorphic quadratic differentials. A^{-1} is the inverse of the 6×6 matrix constructed from A_{ip}, $A_{i\bar{p}}$. This total derivative term receives a contribution from the boundary where the genus two surface breaks up into two genus one surfaces. The contribution from this boundary for arbitrary choice of the points z_1, z_2 may be evaluated using the method of ref.[2], and may be shown to be non-vanishing at least for theories where Fayet-Iliopoulos D-terms are generated at one string loop order[3,4].

3. The physical origin of the total derivative ambiguity

We would like now to identify the origin of the total derivative terms arising in (2.13) as a result of change of basis of the super-Beltrami differentials. The freedom of choosing an arbitrary basis of super-Beltrami differentials is traceable to local world sheet supersymmetry invariance of the theory. Given any gravitino field $\chi_{\bar{z}}^{\theta}$, we may find a supersymmetry transformation v^{θ} such that,

$$\chi_{\bar{z}}^{\theta} = \partial_{\bar{z}} v^{\theta} + \sum_{a=1}^{2} \varsigma^{a} \delta^{(2)}(z - z_a). \tag{3.1}$$

The proof of (3.1) relies on the fact that for given z_1, z_2 there exists a unique Greens function $G(z,w)$ with poles at z_1, z_2 with unique residues $R_1(w)$, $R_2(w)$ satisfying,

$$\partial_{\bar{z}} G(z,w) = \delta^{(2)}(z-w) + \sum_{a=1}^{2} R_a(w) \delta^{(2)}(z - z_a), \tag{3.2}$$

where $G(z,w)$ is a $-\frac{1}{2}$ differential in the z plane and $\frac{3}{2}$ differential in the w plane. v^{θ} and ς^{a} in (3.1) are then given by,

$$\begin{aligned} v^{\theta} &= \int d^2w G(z,w) \chi_{\bar{w}}^{\theta}(w), \\ \varsigma^{a} &= -\int d^2w R_a(w) \chi_{\bar{w}}^{\theta}(w). \end{aligned} \tag{3.3}$$

Now let us assume that we have a fixed basis of super-Beltrami differentials given by $\delta^{(2)}(z - z_a)$ and that we would like to find a supersymmetry transformation parameter $u^{\theta}(z)$ which changes the basis to $\delta^{(2)}(z - \tilde{z}_a)$. $u^{\theta}(z)$ is then given by the solution to,

$$\sum_a \varsigma^a \delta^{(2)}(z - z_a) = \partial_{\bar{z}} u^\theta + \sum_a \tilde{\varsigma}^a \delta^{(2)}(z - \tilde{z}_a), \tag{3.4}$$

i.e.

$$u^\theta = \int d^2w \tilde{G}(z,w) \sum_a \varsigma^a \delta^{(2)}(w - z_a) = \sum_a \varsigma^a \tilde{G}(z, z_a), \tag{3.5}$$

where $\tilde{G}(z,w)$ is the Greens function satisfying,

$$\partial_{\bar{z}} \tilde{G}(z,w) = \delta^{(2)}(z-w) + \sum_a \tilde{R}_a(w) \delta^{(2)}(z - \tilde{z}_a). \tag{3.6}$$

At this point we may note that $\tilde{G}(z,w)$ could be identified with a β, γ correlation function as follows:

$$\tilde{G}(z,w) = \langle \gamma(z) \beta(w) e^{\phi(\tilde{z}_1)} e^{\phi(\tilde{z}_2)} \rangle, \tag{3.7}$$

which possesses poles at $z = w$, $\tilde{z}_1$ and $\tilde{z}_2$ and zeroes at $w = \tilde{z}_1$ and $\tilde{z}_2$.

Under a supersymmetry transformation the two dimensional metric changes as,

$$\begin{aligned} \Delta g^{zz} &= -\frac{1}{2} g^{z\bar{z}} \chi_{\bar{z}}^\theta u^\theta, \\ \Delta g^{\bar{z}\bar{z}} &= 0. \end{aligned} \tag{3.8}$$

As in the derivation of eq.(2.13), we shall now consider the case where $\tilde{z}_2 = z_2$, so that $\tilde{G}(z, z_2) \equiv \tilde{G}(z, \tilde{z}_2) = 0$. Δg^{zz} given in (3.8) induces a change in the moduli given by,

$$\begin{aligned} \Delta m_i &= \sum_{p=1}^{3} (A^{-1})_{pi} \int d^2z h_{zz}^{(p)} \Delta g^{zz} g_{\bar{z}z} \\ &= \frac{1}{2} \sum_{p=1}^{3} (A^{-1})_{pi} \varsigma^1 \varsigma^2 h_{zz}^{(p)}(z_2) \Delta z_1 \langle e^{\phi(z_2)} \gamma(z_2) \partial \xi(z_1) \rangle. \end{aligned} \tag{3.9}$$

using eqs.(2.3), (2.14), (3.8), (2.5), (3.5), (3.7) and (2.7). This shows that a supersymmetry transformation which changes the super-Beltrami differentials also shifts the moduli by an even nilpotent element of the Grassmann algebra. Normally one could get rid of this shift by a change of variables of integration $m_i \to m_i - \Delta m_i$. This is however not possible if the moduli space has boundaries, since such a change of variables will cause a shift in the limits of integration. This may be illustrated by the following simple example[8]. Let us define the superspace integral,

$$I = \int_a^b dx \int d\theta_1 d\theta_2 f(x, \theta_1, \theta_2), \tag{3.10}$$

where x is an even element of the Grassmann algebra, and θ_1, θ_2 are the odd elements. If we make a change of variables,

$$x = y + g(y)\theta_1\theta_2 \tag{3.11}$$

and neglect to change the limits of integration on y, while carrying out everything else correctly, we see that I shifts by a total derivative term under the above change of variables:

$$I \to I - \int_a^b dy \frac{\partial}{\partial y}\{f(y,0,0)g(y)\}. \tag{3.12}$$

This is precisely the origin of the total derivative ambiguity in the fermionic string functional integral W. Suppose we define W by eq.(2.6) for a specific choice of points z_1, z_2, interpreting the integral over m_i in this equation to be the integral over the moduli space of an ordinary genus two Riemann surface. The analysis presented above shows that if we make another choice for z_1, z_2 (say $\tilde{z}_1$, $\tilde{z}_2$) in (2.6), and still interpret the integral over m_i as an integration over the ordinary moduli space of a genus two Riemann surface, the new W will differ from the old one by a total derivative term. With Δm_i given by (3.9), the shift in W

according to eqs. (2.1) and (3.12) should be,

$$\Delta W = -\frac{1}{2}\int \prod_{i=1}^{6} dm_i \sum_{j=1}^{6} \frac{\partial}{\partial m_j}\Big[\int D[XBC]e^{-S_0}\prod_{i=1}^{6}(\eta_i \mid B)_0 \\ \partial\xi(z_1)\Delta z_1 e^{\phi(z_2)}\gamma(z_2)\sum_{p=1}^{3}(A^{-1})_{pj} h_{\bar{z}}^{(p)z}(z_2)\Big] \tag{3.13}$$

which is precisely what we found in (2.13) through entirely different considerations. On the other hand, if we were careful about shifting the limits of integration of the moduli, we would not have run into this discrepancy.

While the above analysis identifies the origin of the problem, it does not give a resolution. In order to make sense of the expression (2.6) for W where the integration over the supermoduli has already been performed, we must interpret the integration over m_i to be over the space of real numbers labelling the ordinary moduli space of genus two Riemann surfaces. What the above analysis shows is that if for a certain choice of basis of the super-Beltrami differentials we restrict the integration over the moduli in (2.1) to the space of real numbers, then for a different choice of basis, the integration over the moduli should run over the space of (real numbers plus an even nilpotent element of the Grassmann algebra depending on the supermoduli), and we would not be able to perform the integration over the supermoduli before integrating over the moduli, and express the answer in the form of eq.(2.6) for this new choice of basis. However the analysis does not tell us at this stage for what choice of basis (if any) of super-Beltrami differentials we should restrict the integration over the moduli to the space of real numbers. This is what we shall try to answer next. Before that, however, we should mention that our analysis above also makes it possible to compare the results of ref.[1] to recent developments based on the theory of super-Riemann surfaces[9], (super-)algebraic geometry and (super-)holomorphic factorization theorems[10,11] which express the integration measure for the superstring partition function in terms of super-algebraic quantities, independent of any choice of basis. In this case the integration variables are even and odd elements of the Grassmann algebra, and one must interprete the integration over the even elements as a contour integral following ref.[8], where the contour describes an embedding of the (spin)-moduli space of an ordinary genus two Riemann surface into the supermoduli space. The result of integration does not depend on the choice of the contour as long as the boundaries of the contour are kept fixed. Since the moduli space of genus two Riemann surface has boundaries, so does the corresponding contour, and there is an ambiguity in the final answer depending on the choice of the boundary. This is precisely the same ambiguity that appears in the answer of ref.[1]. In other words, the ambiguity does not lie in the determination of the integration measure, but in defining the integration contour.

4. Resolution of the ambiguity for genus two Riemann surfaces

We now turn our attention to the question of resolving the above ambiguity for genus two surfaces. In particular, we shall set up criteria for choosing the appropriate set of super-Beltrami differentials for the formalism of ref.[1] to make sense. We shall use two different approaches, one based on the direct study of the supersymmetry of the functional integral, the other on BRST invariace. As we shall see, both approaches lead to the same answer.

To start with we shall discuss the constraints imposed by modular invariance. For that, let us set up a fixed coordinate system $\vec{r} = (x, y)$ on the genus two surface, and assign a metric $g^{\alpha\beta}(\vec{r}, m_i)$ on the surface for each point on the Teichmuller space once and for all.* This leads to a particular choice of Beltrami differentials ${\eta_{i\bar{z}}}^{z}$ and ${\bar{\eta}_{iz}}^{\bar{z}}$. Let us also choose a set of super-Beltrami differentials $\delta^{(2)}(\vec{r} - \vec{r}_a(m_i))$, $(a = 1, 2)$. Let us now consider two points m_i and $\tilde{m}_i$ in the

* By definition, at any point of the moduli space, z, $\bar{z}$ always denote the coordinate system where the metric is diagonal ($g^{zz} = g^{\bar{z}\bar{z}} = 0$). Thus unlike the coordinate system $\vec{r}$, the coordinate system z, $\bar{z}$ varies with the moduli.

Teichmuller space, related to each other by a modular transformation. Then there exists a global diffeomorphism which converts the metric $g^{\alpha\beta}(\vec{r}, m_i)$ to $g^{\alpha\beta}(\vec{r}, \tilde{m}_i)$. The requirement of modular invariance then tells us that under the same global diffeomorphism $\vec{r}_1(m_i)$, $\vec{r}_2(m_i)$ should go to the points $\vec{r}_1(\tilde{m}_i)$, $\vec{r}_2(\tilde{m}_i)$ (or $\vec{r}_2(\tilde{m}_i)$, $\vec{r}_1(\tilde{m}_i)$). Once the choice of the points $\vec{r}_1$, $\vec{r}_2$ satisfy this criterion, the answer for the superstring partition function (after integration over the moduli) is independent of the choice of these points, except at the boundary of the moduli space. Note that this criterion is incompatible with the proposal [12] (see also ref. [13]) that the points z_a should be taken to coincide with some of the ramification points of the genus two surface.

If we choose complex coordinates in the moduli space, and choose the Beltrami differentials appropriately so as to satisfy the constraints $\eta_{\bar{p}\bar{z}}{}^{z} = \bar{\eta}_{pz}{}^{\bar{z}} = 0$, then, in order for the partition function W given in (2.6) to have appropriate holomorphic factorization properties [14],† the complex coordinates z_1 and z_2 appearing in eq.(2.6) must be holomorphic functions of the moduli (i.e. $\frac{\partial z_a}{\partial m_{\bar{p}}}$ must vanish). However, the requirement of modular invariance stated above may not, in general, be compatible with the requirement that the z_as are holomorphic functions of the moduli. Again, this does not contradict the results of refs.[10,11] since the loss of holomorphicity occurs in the choice of the integration contour, and not in the integrand, when expressed as a function of the even and the odd coordinates of the supermoduli space. Another source of the lack of holomorphicity in the fermionic string functional integral has been discussed in ref.[12].

Now let us discuss how to choose these points at the boundary of the moduli space. We have seen that world sheet supersymmetry (and hence BRST invariance) of the theory is lost for an arbitrary choice of basis of the supermoduli due to a non-vanishing contribution to Δg^{zz} under a supersymmetry transformation

† For a compactified heterotic string theory, the partition function associated with the internal variables does not, in general, have any holomorphic factorization property. Thus here holomorphic factorization refers only to the part of the partition function associated with the ghost fields and the non-compact directions.

as given in (3.8). If we can choose the basis for the super-Beltrami differentials in such a way that with $\chi_{\bar{z}}^{\theta}$ given by $\sum_{a=1}^{2}\varsigma^{a}\delta^{(2)}(z-z_a)$, Δg^{zz} given in (3.8) vanishes identically at the boundary of the moduli space, then the invariance of the partition function under infinitesimal supersymmetry transformation is recovered without any need to change the limits of integration of m_i. Fortunately, there exists a simple choice of basis which achieves this. The relevant boundary of a genus two moduli space contains two genus one Riemann surfaces T_1 and T_2 with punctures at P_1 and P_2 respectively. The allowed reparametrization and supersymmetry transformation parameters are those which are globally well defined on T_1 (T_2) and vanish at the points P_1 (P_2). As a result, if we choose z_1 and z_2 to coincide with P_1 and P_2 respectively, then Δg^{zz} in (3.8) will vanish identically. The functional integral defined this way will be invariant under (infinitesimal) supersymmetry transformation, and hence BRST transformation, including boundary terms.*

Note also that this choice of basis is consistent with the requirement of modular invariance at the boundary. The group of modular transformations at the boundary which leave the boundary fixed includes as its subgroup independent modular transformations on the tori T_1 and T_2. Since P_1 and P_2 are the positions of the punctures on the tori T_1 and T_2, the global diffeomorphisms associated with these modular transformations leave fixed the points P_1 and P_2 and hence z_1 and z_2. Another element of the modular group at the boundary is the transformation which interchanges the tori T_1 and T_2. Under this transformation P_1 and P_2 are interchanged. Hence our choice of basis at the boundary remains invariant under these transformations.

We shall now check explicitly that the above choice of basis is consistent with the constraints of BRST invariance and the decoupling of the unphysical states. Let us suppose that we would like to calculate the correlation function of n BRST

* A more direct proof of BRST invariance will be given below.

invariant vertex operators:

$$\prod_{k=1}^{n} \int d^2 y_k \langle V_k(y_k) \rangle \tag{4.1}$$

and let us assume further that V_1 describes a spurious state, *i.e.* $V_1(y_1) = [Q_B, \tilde{V}_1(y_1)]$, for some $\tilde{V}_1$. Then in order to obtain a unitary theory, the corresponding amplitude must vanish. To see how this happens for the specific basis we have chosen above, we express V_1 as a contour integral of the BRST current around $\tilde{V}_1(y_1)$ and deform the BRST contour. As before, the BRST contour picks up residues from the location of $(\eta_i \mid B)$ as well as $\partial\xi(z_i)$, and the final result may be expressed as a total derivative in the moduli space. Hence this receives contributions only from the boundary of the moduli space. The relevant boundary in this case is where the genus two surface breaks up into two tori. A suitable set of coordinates in this region of the moduli space are the moduli τ_1, τ_2 of the two tori, and a complex parameter t such that $t = 0$ describes the boundary. The relevant term near $t = 0$ looks like,

$$\begin{aligned} I = \int dt d\bar{t} d^2\tau_1 d^2\tau_2 \frac{\partial}{\partial t} \Big[\int D[XBC] e^{-S_0} (\eta_1 \mid B)_0 (\eta_2 \mid B)_0 (\eta_{\bar{1}} \mid B)_0 (\eta_{\bar{2}} \mid B)_0 \\ (\eta_{\bar{t}} \mid B)_0 Y(z_1) Y(z_2) \int d^2 y_1 \tilde{V}_1(y_1) \prod_{k=2}^{n} \int d^2 y_k V_k(y_k) \Big] \\ \equiv \int dt d\bar{t} d^2\tau_1 d^2\tau_2 \frac{\partial}{\partial t} M \end{aligned} \tag{4.2}$$

where η_i $(\eta_{\bar{i}})$ (i=1,2) are the Beltrami differentials dual to τ_i $(\bar{\tau}_i)$ and η_t $(\eta_{\bar{t}})$ is the Beltrami differential dual to t $(\bar{t})$. In order for the above boundary contribution to be non-zero, M should behave as $\frac{1}{t}$ as $t \to 0$. It is not difficult to show that $(\eta_{\bar{t}} \mid B)$ contributes a factor of $\frac{1}{\bar{t}}$ in this limit. The asymptotic behavior of the rest of the terms may be inferred using the factorization theorem[15]. The potentially dangerous contribution comes from the configuration where all the vertex operators are on one torus so that there is no momentum flow through the pinch.* Without loss of generality we may take them on the torus T_1. In this case M in eq.(4.2) above may be expressed as,

$$M \sim \sum_{\{\Phi\}} \bar{t}^{-1} t^{h_\Phi} \bar{t}^{\bar{h}_\Phi - 1} \Big\langle Y(z_1) \prod_{k=2}^{n} V_k(y_k) \tilde{V}_1(y_1) (\eta_1 \mid B)_0 (\eta_{\bar{1}} \mid B)_0 \bar{c}(P_1) \Phi^\dagger(P_1) \Big\rangle_{T_1}$$
$$\langle \Phi(P_2) \bar{c}(P_2) Y(z_2) (\eta_2 \mid B)_0 (\eta_{\bar{2}} \mid B)_0 \rangle_{T_2}, \tag{4.3}$$

where Φ is an arbitrary operator that does not contain any ξ zero mode, and $(h_\Phi, \bar{h}_\Phi)$ denote the conformal dimensions of Φ. Also we have displayed the $\bar{c}$ content of the operators inserted at P_1, P_2 explicitly in writing down eq.(4.3). We want (4.3) to vanish identically. This will happen if,

$$\langle \Phi(P_2) \bar{c}(P_2) Y(z_2) (\eta_2 \mid B) (\eta_{\bar{2}} \mid B) \rangle_{T_2} \tag{4.4}$$

vanishes identically for all possible operators Φ of conformal dimension (0,1).

In order to proceed further, we shall restrict ourselves to the specific theories where the boundary terms are known to become important. These are the theories where Fayet-Iliopoulos terms are generated at one loop order in compactified heterotic string theory. First let us consider the part of $Y(z_2)$ proportional to $e^{2\phi(z_2)}$ (eq.(2.9)). Using various ghost charge conservations one can see that the only dimension (0,1) operator contributing to (4.4) is of the form $c(P_2)\partial c(P_2) e^{-2\phi(P_2)} U^{(a)}(P_2) \partial\xi(P_2)$, where $U^{(a)}(P_2)$ is a dimension (0,1) current

* If there is momentum flow ℓ through the pinch, the integrand has a factor of $| t |^{2\ell^2}$ in the $t \to 0$ limit. Hence the boundary terms may be made to vanish by analytically continuing ℓ^2 to large positive values, except when all the vertex operators are on one torus, so that ℓ vanishes identically, or when all but one vertex operators are on one torus, so that ℓ^2 is fixed by on-shell constraints on the external momenta. Possible BRST anomalies associated with the second type of boundaries may be removed by carrying out appropriate mass renormalization of the external states[16]. Part of the BRST anomaly associated with the first type of boundaries are associated with the presence of massless tadpoles in the theory, and should be removed by shifting various fields[17]. Here we shall concentrate on theories which do not have a tadpole at one loop, and hence any BRST anomaly from the first type of boundaries will reflect a sickness of the theory.

associated with a gauge symmetry. One can explicitly compute the relevant correlator by combining the formalisms of ref.[1] and [4], and show that it vanishes in the $z_2 \to P_2$ limit after the sum over spin structures is performed. If we consider the $c\partial\xi$ term in Y, then it can be seen by using various ghost charge conservation that there is no dimension (0,1) operator Φ that contributes to (4.4). This leaves us with the term $e^{\phi(z_2)}T_F^{matter}(z_2)$ in Y. Ghost charge conservation demands that the operator Φ is of the form:

$$\Phi(P_2) = c(P_2)e^{-\phi(P_2)}f(P_2) \tag{4.5}$$

where f is an operator of conformal dimension $(\frac{1}{2},1)$. As was shown in ref.[4], f must either have the form $\psi^\mu\bar{\partial}X^\mu$, or it must be constructed totally from the internal fields. If it is of the form $\psi^\mu\bar{\partial}X^\mu$, then, as $z_2 \to P_2$,

$$\begin{aligned}\lim_{z_2\to P_2} & c(P_2)e^{-\phi(P_2)}\psi^\mu(P_2)\bar{\partial}X^\mu(P_2)\bar{c}(P_2)e^{\phi(z_2)}T_F^{matter}(z_2)\\ &\sim c(P_2)\bar{c}(P_2)\bar{\partial}X^\mu(P_2)\partial X^\mu(P_2)\end{aligned} \tag{4.6}$$

and the corresponding correlator vanishes due to the vanishing of the one loop dilaton tadpole[18]. (Otherwise one has to cancel this BRST anomaly by shifting appropriate fields[17]). On the other hand, if f is constructed out of the internal fields,

$$\lim_{z_2\to P_2} c(P_2)e^{-\phi(P_2)}f(P_2)\bar{c}(P_2)e^{\phi(z_2)}T_F^{matter}(z_2) \sim c(P_2)\bar{c}(P_2)g(P_2) \tag{4.7}$$

where,

$$g(P_2) = \lim_{z_2\to P_2}\{T_F^{matter}(z_2)f(P_2)(z_2 - P_2)\} \tag{4.8}$$

This shows that g is the highest component of a superfield whose lowest component is f, and that g has conformal dimension (1,1). Let us denote by J_0 the $U(1)$ charge of the (2,0) superconformal algebra which is always present in the models of the kind considered here. If g carries a non-zero J_0 charge then $\langle c\bar{c}g\rangle_T$ may be shown to vanish using the J_0 charge conservation. On the other hand, if g carries no J_0 charge, then it survives the GSO projection and represents the vertex operator of a physical state. The expectation value of $c\bar{c}g$ on a torus can then be shown to vanish due to the known result about the absence of one loop tadpoles in the theory [18]. (Otherwise one has to cancel this BRST anomaly by shifting fields). Thus we see that with the choice of basis we have made, the null states decouple, and we get a theory consistent with BRST invariance and unitarity.

5. Conclusion

In this paper we have shown that there is an inherent ambiguity in defining the fermionic string functional integral due to an ambiguity in defining integration over the variables of a Grassmann algebra. This ambiguity is the same as the one found by Verlinde and Verlinde [1], namely, the dependence of the physical amplitudes on the choice of the basis of the super-Beltrami differentials in the form of total derivative terms in the moduli space. We also show how to resolve this ambiguity in the (compactified) heterotic string theory in the case of genus two Riemann surfaces using the constraint of BRST invariance. The correct choice of the super-Beltrami differentials obtained through this analysis turns out to be precisely the one used in ref.[2] to calculate two loop dilaton tadpole in compactified heterotic string theory.⋆ However, we do not yet have a resolution of this ambiguity for an arbitrary genus surface. For that we feel a better understanding of the structure of supermoduli space may be needed.

Note added: After this work was completed, we were informed by G. Moore that some related results have been obtained in ref.[19].

⋆ In that paper this choice was arrived at through considerations of modular invariance.

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REFERENCES

1. E. Verlinde and H. Verlinde, Phys. Lett. **B192** (1987) 95.
2. J. J. Atick and A. Sen, SLAC-PUB-4292, to appear in Nucl. Phys. B.
3. M. Dine, N. Seiberg and E. Witten, Nucl. Phys. **B289** (1987) 589; M. Dine, I. Ichinose and N. Seiberg, Nucl. Phys. **B293** (1987) 253.
4. J. J. Atick, L. Dixon and A. Sen, Nucl. Phys. **B292** (1987) 109.
5. G. Moore, P. Nelson and J. Polchinski, Phys. Lett. **B169** (1986) 47; E. D'Hoker and D. Phong, **B278** (1986) 225; G. Moore and P. Nelson, Nucl. Phys. **B274** (1986) 509; S. Giddings and P. Nelson, Harvard preprint HUTP-87/A062 (1987).
6. D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. **B271** (1986) 93.
7. E. Martinec, Nucl. Phys. **B281** (1987) 157; H. Sonoda, Nucl. Phys. **B281** (1987) 546; Phys. Lett. **184B** (1987) 336; T. Eguchi and H. Ooguri, Nucl. Phys. **B282** (1987) 308; E. Verlinde and H. Verlinde, Nucl. Phys. **B288** (1987) 357; M. Dugan and H. Sonoda, Nucl. Phys. **B289** (1987) 227.
8. B. de Witt, Supermanifolds, Cambridge Univ. Press (1984); A. Rogers, J. Math. Phys. **26** (1985) 385, **27** (1986) 710; J. Rabin, Physica **15D** (1985) 65 (Proceedings of the Workshop on Supersymmetry in Physics, held at Los Alamos, N. Mex., Dec., 1983); M. Rothstein, Trans. AMS **299** (1987) 387.
9. D. Friedan, in Unified String Theories, eds. M. Green and D. Gross (World Scientific, 1986); L. Crane and J. Rabin, Commun. Math. Phys., to appear; P. Freund and J. Rabin, EFI-87-22-CHICAGO.

10. E. D'Hoker and D. Phong, PUPT-1029.

11. M. Baranov and A. Shvarts, Pisma ZETF **42** (1985) 340 (JETP Lett. **42** (1986) 419); M. Baranov, Yu Manin, I. Frolov and A. Shvarts, Yad. Phys. **43** (1986) 1053 (Sov. J. Nucl. Phys. **43** (1986) 670); M. Baranov and A. Shvarts, Niels Bohr Inst. preprint (1987).

12. A. Morozov and A. Perelomov, ITEP preprint (1987).

13. J. J. Atick and A. Sen, Phys. Lett. **186B** (1987) 339.

14. A. A. Belavin and V. G. Knizhnik, Phys. Lett. **168B** (1986) 201.

15. D. Friedan and S. Shenker, Nucl. Phys. **B281** (1987) 509; Phys. Lett. **175B** (1986) 287.

16. A. Sen, SLAC-PUB-4383.

17. W. Fischler and L. Susskind, Phys. Lett. **B171** (1986) 383; **B173** (1986) 262. S. R. Das and S. J. Rey, Phys. Lett. **186B** (1987) 328; C. G. Callan, C. Lovelace, C. R. Nappi and S. A. Yost, Princeton preprint PUPT-1027 (1986); PUPT-1045 (1987); J. Polchinski and Y. Cai, Texas preprint UTTG-09-87; W. Fischler, I. Klebanov and L. Susskind, preprint SLAC-PUB-4298; H. Ooguri and N. Sakai, preprint TIT/HEP-117 (UT-512); R. Brustein, D. Nemeschansky and S. Yankielowicz, USC-87/004.

18. E. Martinec, Phys. Lett. **B171** (1986) 189.

19. G. Moore and A. Morozov, IAS preprint.