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A PROOF OF LOCAL BACKGROUND INDEPENDENCE OF CLASSICAL CLOSED STRING FIELD THEORY

Ashoke Sen[⋆]

Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400005, India

and

Barton Zwiebach[†]

Center for Theoretical Physics
LNS and Department of Physics
MIT, Cambridge, MA 02139, U.S.A.

ABSTRACT

⋆ E-mail address: sen@tifrvax.tifr.res.in, sen@tifrvax.bitnet
† E-mail address: zwiebach@irene.mit.edu, zwiebach@mitlns.bitnet.
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We give a complete proof of local background independence of the classical master action for closed strings by constructing explicitly, for any two nearby conformal theories in a CFT theory space, a symplectic diffeomorphism between their state spaces mapping the corresponding non-polynomial string actions into each other. We uncover a new family of string vertices, the lowest of which is a three string vertex satisfying exact Jacobi identities with respect to the original closed string vertices. The homotopies between the two sets of string vertices determine the diffeomorphism establishing background independence. The linear part of the diffeomorphism is implemented by a CFT theory-space connection determined by the off-shell three closed string vertex, showing how string field theory induces a natural interplay between Riemann surface geometry and CFT theory space geometry.

1. Introduction and Summary

One of the most important open questions in string field theory is that of finding a manifestly background independent formulation. A consistent quantum closed string field theory already exists [1 − 11]. It is written using the Batalin-Vilkovisky (BV) formalism which turned out to be remarkably efficient for string field theory. This string field theory, however, requires for its formulation a choice of a conformal field theory defining a consistent background for string propagation. Such choice would not be necessary in a manifestly background independent formulation, where consistent backgrounds would arise as classical solutions. Since closed string field theory is not *manifestly* background independent, the obvious question is whether it is background independent *at all.* In this paper we prove that closed string field theory is indeed independent of the background in which it is formulated, as long as the backgrounds are related by marginal deformations. Since our proof is geometrical, we believe that it may provide crucial insight for the construction of a manifestly background independent closed string field theory.

A string field theory, in the BV formulation, is defined by a master action S, which is a function on a subspace $\widehat{\mathcal{H}}$ of the state space of the chosen CFT, and a symplectic structure ω, or BV antibracket, on $\widehat{\mathcal{H}}$. The string field is just an arbitrary element of $\widehat{\mathcal{H}}$. In writing down closed string field theory one has to make two types of choices. The first one, as mentioned above, consists of choosing a conformal theory from the space of two dimensional theories. The second one, apparently on a totally different footing, is a choice of string vertices for the string field action. This choice of vertices determines how the Feynman diagrams of the resulting string field theory decompose the moduli spaces of Riemann surfaces. A canonical choice of string vertices arises from minimal area metrics, but other choices are possible. The choice of an n-string vertex for classical closed string theory, is the choice of a collection of n-punctured spheres, each having specific choices of local coordinates (defined up to phases) around each puncture. This amounts to choosing a subspace of $\widehat{\mathcal{P}}_n$, the space of all inequivalent n-punctured spheres with all possible choices of local coordinates on the punctures. Therefore, the choice of vertices is a choice of subspaces from spaces of decorated Riemann surfaces. It was shown recently [12], that string field theories corresponding to different choices of string vertices, are, in fact, related by field transformations canonical with respect to the BV antibracket. This shows that these different string field theories represent the same theory written in terms of different variables.

Given that we only know how to formulate closed string field theory once we choose a conformal field theory, the problem of background independence is formulated as follows. Let x and y denote two different conformal theories, and let $\widehat{\mathcal{H}}_x$ and $\widehat{\mathcal{H}}_y$ be their respective state spaces. Let (S_x, ω_x) and (S_y, ω_y) be their respective master actions and BV structures. Background independence would mean that there is a string field transformation that establishes the physical equivalence of the two theories. More precisely, we have to find a diffeomorphism relating $\widehat{\mathcal{H}}_x$ to $\widehat{\mathcal{H}}_y$, such that under its action the respective master actions and BV structures are taken into each other. The main purpose of the present paper is to construct this diffeomorphism explicitly for the case when we have nearby conformal field theories related by an exactly marginal operator. We call this the problem of *local* background independence. The natural setting for this problem is therefore that of a space of conformal field theories. The state spaces of the conformal field theories then form a vector bundle over the space of conformal theories. We will see in this paper how the choice of string vertices, necessary for writing a string field theory, provides local geometrical structure on this vector bundle. Thus string field theory is seen to induce a natural interplay between Riemann surface theory and theory space geometry. The geometrical structure induced on CFT theory-space is essential to our proof of background independence.

Since the BV master action is not the gauge invariant classical action nor the gauge fixed action (even though both arise from the master action by simple operations), physical background independence may not require background independence of the master action. Our success in proving that the master action is background independent provides further evidence of the deep significance of the

BV formulation of string theory.

The problem of local background independence was addressed earlier in refs.[13 − 15] where it was shown that, up to cubic order in the string field, the classical actions of two string field theories, formulated around two nearby conformal field theories, can be related by a field redefinition. Due to various technical complications, the result could not be extended to higher orders in the string field. Moreover, there was no natural geometric construction of the field redefinition that takes one string field theory action to the other.

Largely stimulated by this work, much progress has been made in understanding deformations of conformal field theories [16 − 21]. It was understood that having a space of conformal theories implies the existence of connections on the vector bundle of state spaces over this theory space. A connection is necessary to formulate precisely (and covariantly!) the intuitive idea that correlation functions vary smoothly as we move in theory space. A connection is also necessary to construct a conformal theory using the state space of another conformal theory. There is no unique connection on this vector bundle, and specific choices must be made for specific purposes. In [20] a unified description of all possible connections was given by generalizing the variational formula of Sonoda [22]. A particularly useful connection $\widehat{\Gamma}_\mu$, was seen to satisfy the following variational formula:

$$D_\mu(\,\widehat{\Gamma}\,)\langle\,\Sigma\,| = -\frac{1}{\pi}\int\limits_{\Sigma-\cup_i D_i^{(1)}} d^2z\ \langle\,\Sigma; z\,|\mathcal{O}_\mu\rangle. \tag{1.1}$$

In here, $\langle\,\Sigma\,|$ are the states of the operator formalism encoding all the correlators of the punctured Riemann surface Σ. This bra is a section on the vector bundle. In the right hand side we have the integral, over the surface minus the *unit* disks around each puncture, of the insertion of the exactly marginal operator $\mathcal{O}_\mu$ (in the operator formalism this insertion is done by contracting a section $\langle\Sigma; z|$, corresponding to the surface Σ with an extra puncture at z, with the state $|\mathcal{O}_\mu\rangle$). Our whole input from the fact that we have a theory space will be that a connection $\widehat{\Gamma}_\mu$

satisfying the above formula must exist. String field theory contact interactions, such as those defining the classical closed string field theory vertices $\mathcal{V}_n$, are specified by punctured spheres Σ whose unit disks around the punctures cover fully and precisely the surfaces. It follows from the above formula that, for such surfaces, there is no region to integrate over, and therefore, the covariant derivative $D_\mu(\,\widehat{\Gamma}\,)$ of the closed string field theory vertices vanishes.

Coming back to the question of background independence, it should be emphasized that, while we are discussing an infinitesimal variation δx^μ in theory space, the diffeomorphism relating the two relevant state spaces (*i.e.,* the redefinition relating the corresponding string fields) is not linear. It is actually nonpolynomial. The field independent part of it is a constant shift corresponding to a perturbation by an exactly marginal operator. The linear part of the map can be interpreted as defining a theory space connection Γ_μ. We find that background independence, to quadratic order in the string field, requires that the symplectic form be a covariantly constant section (in theory space), and that the covariant derivative of the BRST operator must be given by $D_\mu(\,\Gamma\,)Q = \langle V^{(3)}|c\bar{c}\mathcal{O}_\mu\rangle$, where $\langle V^{(3)}|$ is the off-shell three string vertex of closed string field theory (with one of its state spaces turned into a ket). This formula is remarkable in that the Riemann surface data encoding the three string vertex of string field theory determines a particular connection in theory space. The question of background independence to quadratic order is therefore the question whether a connection Γ_μ satisfying the two conditions stated above exists. We find such a connection. This is done by first showing that the canonical connection $\widehat{\Gamma}_\mu$ satisfies an equation of the type $D_\mu(\,\widehat{\Gamma}\,)Q = \langle V'^{(3)}|c\bar{c}\mathcal{O}_\mu\rangle$ where $\langle V'^{(3)}|$ is a new three string field vertex. This vertex has an asymmetric puncture, where $c\bar{c}\mathcal{O}_\mu$ is inserted, but is symmetric under the exchange of the other two punctures (see Fig.1, in §3.3). It is then simple to show that $\Gamma_\mu - \widehat{\Gamma}_\mu$ can be expressed in terms of an interpolating three string vertex $\mathcal{B}_3$ representing a deformation from $\mathcal{V}_3$ to $\mathcal{V}'_3$ (the surfaces, or points in $\widehat{\mathcal{P}}_3$, corresponding to the string field vertices $\langle V^{(3)}|$ and $\langle V'^{(3)}|$ respectively).

In proceeding to higher orders in the redefinition rather interesting properties of the new vertex $\langle V'^{(3)}|$ come into light. If we denote by $[\,,\,]$ the standard star product arising from the three string vertex, and by $[\,,\,]'$ the star product arising from the new string vertex, we then find that

$$\big[A_1,\, [A_2, A_3]\,\big]' \pm \text{cyclic} = 0\,. \tag{1.2}$$

The new three string vertex, together with the standard three string vertex, satisfies a strict Jacobi like identity. An on-shell version of this identity is sufficient to guarantee that the quadratic part of the diffeomorphism exists. The new product $[\,,\,]'$ also satisfies consistency conditions with respect to the higher products of closed string field theory. We find that

$$\big[A_1,\, [A_2, A_3, \cdots, A_N]\,\big]' \pm \text{cyclic} = 0\,, \quad N \geq 3\,. \tag{1.3}$$

An on-shell version of these identities is sufficient to guarantee the existence of the desired diffeomorphism to all orders. We believe the fact that these identities hold off-shell could prove necessary for a general analysis of background independence where we must consider shifts of the string field corresponding to arbitrary operators. This new product could also be a useful tool in the understanding of homotopy Lie Algebras [23]. Moreover, as we will discuss in §9, it opens up the possibility of constructing string field actions without using the BRST operator, or perhaps involving more than one string field. Such versions of string field theory could represent progress towards a manifestly background independent formulation.

The pattern that emerges is as follows. If we denote the closed string vertices by $\mathcal{V}_3, \mathcal{V}_4, \cdots$, and the new string vertex associated to $\langle V'^{(3)}|$ is denoted by $\mathcal{V}'_3$, we first find a homotopy $\mathcal{B}_3$ between $\mathcal{V}'_3$ and $\mathcal{V}_3$. A new vertex $\mathcal{V}'_4$ is then constructed by twist-sewing (sewing and integrating over twist) $\mathcal{V}_3$ to all the surfaces in $\mathcal{B}_3$. As a consequence of the above mentioned consistency conditions, the boundaries of $\mathcal{V}_4$ and $\mathcal{V}'_4$ turn out to coincide. This is essential to be able to define a satisfactory interpolating vertex $\mathcal{B}_4$ between them. This process is continued recursively. At every stage, a new vertex $\mathcal{V}'_N$ is obtained by twist-sewing one lower dimensional string vertex $\mathcal{V}$, with one interpolating vertex $\mathcal{B}$, in all possible ways. The consistency conditions guarantee that the boundaries of $\mathcal{V}'_N$ and $\mathcal{V}_N$ coincide, and therefore, one can define the new interpolating vertex $\mathcal{B}_N$. The end result is an infinite family of vertices $\mathcal{V}'_3, \mathcal{V}'_4 \cdots$, and an infinite family of interpolating vertices $\mathcal{B}_3, \mathcal{B}_4 \cdots$. These interpolating vertices define the full diffeomorphism implementing background independence.

In a manifestly background independent formulation of string theory a change of background should be implemented by a simple shift of the string field. A possible way to achieve this would have the string field be the coordinates labelling the space of two dimensional field theories. For open string field theories such a manifestly background independent approach has been proposed in Refs.[24,25], again based on the BV formalism. Other issues involving background independence have been discussed in Ref.[26]. We feel intuitively, that a measure of the degree of background independence of a formulation is provided by the simplicity of the field redefinition that takes us from string field theory in one background to another. As we have sketched above, the relevant diffeomorphism has a clear geometric interpretation. This leads us to believe that the present formulation of closed string field theory may not be far from a manifestly background independent formalism. For the case of the standard covariant open string field theory [27], we show that the redefinition is given by a shift plus a linear transformation.

The plan of the paper is as follows. In §2 we give all the background material that is necessary for our analysis. In §3 we develop some preliminary results essential for our proof. These include a discussion of the canonical connection $\widehat{\Gamma}_\mu$ in the presence of a ghost conformal theory, a computation of the covariant derivative of the BRST operator, and a study of the connectivity property of the spaces of symmetric string vertices. In §4 we set up the general conditions for background independence of closed string field theory, and explore their explicit forms for the case of nearby conformal theories. Since we work in the Batalin-Vilkovisky formalism, background independence of the theory requires existence of a field redefinition which maps not only the action, but also the symplectic structure of the theory formulated around one background to those in the theory formulated around a different background. In §5 we prove background independence to quadratic and cubic order in the string field. This section develops most of the intuition necessary for the later generalization to all orders. In §6 we discuss in detail the new three string vertex $\mathcal{V}_3'$, and prove (1.2), and (1.3), in particular, we explain why they hold off-shell. We also find that $\mathcal{V}_3'$ can be used in conjuction with the standard closed string vertices to find a new way to construct the moduli spaces $\mathcal{M}_n$ with the use of fewer than usual Feynman diagrams. §7 gives the construction of the symplectic diffeomorphism to all orders in the string field. In §8 we turn to the question of existence of field redefinitions that relate string field theories formulated around backgrounds which are finite distance away, but are related to each other by a set of marginal deformations. We show that the field redefinitions required in this case satisfy a set of differential equations and prove that their integrability conditions are always satisfied. Therefore the question of existence is reduced to proving that in the process of integrating the diffeomorphism one does not encounter infinities. We argue, but do not prove, that this should be the case.

We conclude this paper in §9. There we present a proof of local background independence for open string field theory. We explore the possibility of extending our analysis to string field theories based on general (*i.e.* non-overlap) vertices, and, propose a setup for a proof of *quantum* background independence of closed string field theory. We speculate, on the basis of our results, on formulations of closed string theory with a higher degree of background independence.

2. Review

In this section we begin by reviewing the basics of closed string field theory. We will describe the various moduli spaces of surfaces relevant to off-shell amplitudes, and the properties of differential forms in these moduli spaces, with particular attention to the action of the BRST operator and to sewing properties. We then give the precise definition of the symplectic structure relevant to closed string field theory, as it arises from the symplectic structure on the state space of a CFT including the reparametrization ghosts. We explain why, in the Batalin-Vilkovisky (BV) formalism, the symplectic structure is necessary, in addition to the master action, to specify the theory. Finally, we review the earlier work in background independence of closed string field theory and discuss its relation to the present work.

2.1 Basics in String Field Theory

The main geometrical input to the construction of the classical closed string field theory is the set of string vertices $\mathcal{V}_n$. The string vertices are properly thought as subspaces of the space $\widehat{\mathcal{P}}_n = \mathcal{P}_n / \sim$, where $\mathcal{P}_n$ is the space of n-punctured Riemann spheres equipped with local coordinates around each of the n punctures, and $/ \sim$ indicates that two identical punctured spheres, with local coordinate systems that differ by a constant phase around each puncture, should be identified. Local coordinates up to phases are defined by "coordinate curves", Jordan closed curves homotopic to the punctures that correspond to the locus $|z| = 1$ of the local coordinate z, with $z = 0$ the puncture.

Given a point in $\mathcal{P}_n$, there is an obvious projection to $\widehat{\mathcal{P}}_n$, which consists in forgetting about the phase of the local coordinate. There is another projection π from $\widehat{\mathcal{P}}_n$ to $\mathcal{M}_n$, the moduli space of n-punctured spheres, consisting of forgetting about the local coordinates. This allows us to regard $\widehat{\mathcal{P}}_n$ as a fiber bundle, with $\mathcal{M}_n$ as the base space, and the space of the local coordinate systems at the punctures modulo phases, as the fiber. We denote by σ a section of this fiber bundle. This can be summarized in the diagram

$$\begin{array}{c} \mathcal{P}_n \\ \Big\downarrow \\ \widehat{\mathcal{P}}_n \\ \sigma \Big\uparrow \Big\downarrow \pi \\ \mathcal{M}_n \end{array} \tag{2.1}$$

In classical closed string field theory, points in the subspace $\mathcal{V}_n$ correspond to "restricted polyhedra" [1,2]. They represent contact interactions, which means that for each n-punctured sphere (with local coordinates) corresponding to a point in $\mathcal{V}_n$, the disks determined by the coordinate curves cover fully the surface. Moreover the subspaces $\mathcal{V}_n$ satisfy the recursion relation

$$\partial\,\mathcal{V}_n \;=\; -\,\frac{1}{2} \sum_{\substack{n_1,n_2\geq 3 \\ n_1+n_2=n+2}} \mathbf{S}\big(\,\mathcal{V}_{n_1}\,\times\,\mathcal{V}_{n_2}\,\big)\,, \tag{2.2}$$

where $\times$ indicates the subspace obtained by sewing each surface of the subspace $\mathcal{V}_{n_1}$ with each surface of the subspace $\mathcal{V}_{n_2}$, with sewing parameter $t = \exp i\theta$, and $\theta \in [0, 2\pi]$. $\mathbf{S}$ denotes the sum over different splittings of the n labelled punctures into two (unordered) sets, one with $n_1 - 1$ punctures, to be attached to the free puntures of of $\mathcal{V}_{n_1}$, and the other with $n_2 - 1$ punctures, to be attached to the free punctures of $\mathcal{V}_{n_2}$. Each inequivalent contribution is counted twice in the sum on the right hand side, due to symmetry under the exchange of the two vertices. This is compensated for by the explicit factor of 1/2. In the left hand side $\partial\mathcal{V}_n$ denotes the boundary of $\mathcal{V}_n$. The spaces $\mathcal{V}_n$ are actually subspaces of a globally defined section σ over $\mathcal{M}_n$. This is the section determined by minimal area metrics.

2.1.1 Reflectors.

In the operator formulation of conformal field theory, to every n-punctured surface Σ with local coordinates, one assigns a state $\langle\Sigma| \in \mathcal{H}^{*\otimes n}$, where $\mathcal{H}$ denotes the Hilbert space of states in the conformal field theory. The basic requirement is that the states must give a representation of the algebra of sewing of Riemann surfaces. A particularly useful state is the "reflector" state $\langle R_{12}| \in \mathcal{H}^* \otimes \mathcal{H}^*$, representing a two punctured sphere with local coordinates $z_1 = z$, and $z_2 = 1/z$, about $z = 0$, and $z = \infty$, respectively. Since there is a globally defined conformal map exchanging the punctures and the local coordinates (the map $z \to 1/z$), this state is symmetric under the exchange of the state space labels: $\langle R_{12}| = \langle R_{21}|$. This state defines a bilinear form (a metric) on $\mathcal{H}$. Choose a basis $|\Phi_i\rangle$ of states in $\mathcal{H}$, and let $\epsilon(\Phi_i) \equiv i$, denote the grassmanality of the operator Φ_i. We define the metric components

$$g_{ij} \equiv \langle R_{12}|\Phi_i\rangle_1 |\Phi_j\rangle_2 \,. \tag{2.3}$$

Here, g_{ij} are numbers, and are non-vanishing only when $i + j$ is even. This metric satisfies

$$g_{ij} = (-)^{ij} g_{ji} \,, \quad \epsilon(g_{ij}) = i + j = \text{even} \,, \tag{2.4}$$

One defines the inverse metric g^{ij} satisfying $g^{ik} g_{kj} = g_{jk} g^{ki} = \delta^i_j$, and, as a consequence

$$g^{ij} = -(-)^{(i+1)(j+1)} g^{ji} \,, \quad \epsilon(g^{ij}) = i + j = \text{even} \,. \tag{2.5}$$

We define states with upper indices as $|\Phi^i\rangle \equiv g^{ij}|\Phi_j\rangle$. The reflector state is used to introduce a dual basis. One defines

$$\langle\Phi^i| \equiv \langle R\,|\Phi^i\rangle \,. \tag{2.6}$$

It then follows that $\langle\Phi^i|\Phi_j\rangle = \delta^i_j$, and

$$\langle R_{12}| = g_{ij}\ {}_2\langle\Phi^j|_1\langle\Phi^i| \,. \tag{2.7}$$

One can also introduce the ket reflector $|R_{12}\rangle$

$$|R_{12}\rangle = |\Phi_i\rangle_1\, g^{ij}\, |\Phi_j\rangle_2\,, \tag{2.8}$$

also symmetric under the exchange of its state spaces. The contraction

$$\langle R_{12}|R_{23}\rangle = {}_3\mathbf{1}_1\,, \tag{2.9}$$

gives the relabeling operator. The bra $\langle R_{12}|$ satisfies the following properties

$$\langle R_{12}|(c_0^{(1)} + c_0^{(2)}) = \langle R_{12}|(b_0^{(1)} - b_0^{(2)}) = \langle R_{12}|(Q^{(1)} + Q^{(2)}\,) = 0. \tag{2.10}$$

and similar properties with b, c replaced by $\bar{b}$, $\bar{c}$. The ket reflector satisfies analogous properties

$$(c_0^{(1)} + c_0^{(2)})|R_{12}\rangle = (b_0^{(1)} - b_0^{(2)})|R_{12}\rangle = (Q^{(1)} + Q^{(2)}\,)|R_{12}\rangle = 0. \tag{2.11}$$

The dynamical closed string field $|\Psi\rangle$ corresponds to a state in the subspace $\widehat{\mathcal{H}}$ of $\mathcal{H}$, spanned by the elements of $\mathcal{H}$ that are annihilated by $L_0 - \bar{L}_0$, and $b_0 - \bar{b}_0 \equiv b_0^-$,

$$\widehat{\mathcal{H}} = \Big\{ |A\rangle \;\Big|\; |A\rangle \in \mathcal{H}, \text{ and, } (L_0 - \bar{L}_0)|A\rangle = (b_0 - \bar{b}_0)|A\rangle = 0 \Big\}. \tag{2.12}$$

This makes it convenient to introduce the bra

$$\langle R'_{12}| = \langle R_{12}|\mathcal{P}_1\mathcal{P}_2\,, \tag{2.13}$$

where

$$\mathcal{P} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(L_0 - \bar{L}_0)}\,, \tag{2.14}$$

is the projection to rotationally invariant ($L_0 = \bar{L}_0$) states.

2.1.2 Differential Forms.

The basic objects for the construction of the string field interactions arise from differential forms defined on the tangent space $T_{\Sigma}\mathcal{P}_n$ based at the surface Σ. We let $\Omega_{\Sigma}^{(k)n}$ denote a $(2n-6+k)$-form labelled by n arbitrary states $|\Psi_1\rangle, \ldots |\Psi_n\rangle$ in $\mathcal{H}$. We will generally omit from the form the label Σ corresponding to the surface. These forms are explicitly given by [11,28,29,3]

$$\Omega^{(k)n}_{\Psi_1\cdots\Psi_n}(V_1,\cdots,V_{2n-6+k}) = (2\pi i)^{(3-n)}\langle\Sigma|\,\mathbf{b}(\mathbf{v}_1)\cdots\mathbf{b}(\mathbf{v}_{2n-6+k})|\Psi_1\rangle\cdots|\Psi_n\rangle. \tag{2.15}$$

The Schiffer vector $\mathbf{v}_r = (v_r^{(1)}(z),\cdots v_r^{(n)}(z))$ creates the deformation of the surface Σ specified by the tangent $V_r \in T_{\Sigma}\mathcal{P}_n$, and the antighost insertions are defined by

$$\mathbf{b}(\mathbf{v}) = \sum_{i=1}^{n}\Bigg(\oint b^{(i)}(z_i)v^{(i)}(z_i)\frac{dz_i}{2\pi i} + \oint \overline{b}^{(i)}(\overline{z}_i)\overline{v}^{(i)}(\overline{z}_i)\frac{d\overline{z}_i}{2\pi i}\Bigg). \tag{2.16}$$

Here $\oint$ is defined such that $\oint dz/z = \oint d\bar{z}/\bar{z} = 2\pi i$. Since there are no global sections in $\mathcal{P}_n$ we must work on $\widehat{\mathcal{P}}_n$ where there are global sections. It can be shown that for $|\Psi_i\rangle \in \widehat{\mathcal{H}}$ the above differential forms descend to well-defined forms on $T_{\Sigma}\widehat{\mathcal{P}}_n$ [29,11].

The above forms satisfy the basic identity (Ref.[11], Eqn.(7.49))

$$\Omega^{(k+1)n}_{(\sum Q)\Psi_1\cdots\Psi_n} = (-)^{k+1}\,\mathrm{d}\Omega^{(k)n}_{\Psi_1\cdots\Psi_n}, \tag{2.17}$$

which holds both for forms in $\mathcal{P}$ or $\widehat{\mathcal{P}}$. Therefore, the BRST operator Q acts as an exterior derivative on the extended moduli spaces. We will drop the off-shell states from the formulas by writing the forms as bras in $(\mathcal{H}^*)^{\otimes n}$:

$$\Omega^{(k)n}_{\Psi_1\cdots\Psi_n} = \langle\Omega^{(k)n}|\,\Psi_1\rangle\cdots|\Psi_n\rangle. \tag{2.18}$$

Then, Eqn.(2.17) reads

$$\big\langle\Omega^{(k+1)n}\,\big|\sum_{i=1}^{n} Q^{(i)} = (-)^{k+1}\,\mathrm{d}\,\big\langle\Omega^{(k)n}\big|\,, \tag{2.19}$$

The form $\langle\Omega^{(0)N}|$ (in $T_{\Sigma}\widehat{\mathcal{P}}_N$), integrated over the subspace $\mathcal{V}_N$ of $\widehat{\mathcal{P}}_N$, defines the

N-string interaction vertex*

$$\langle V^{(N)}| = \int_{\mathcal{V}_N} \langle \Omega^{(0)N}| . \tag{2.20}$$

More precisely, this equation should include the kets (in $\widehat{\mathcal{H}}$) that, upon contraction, give a number. In terms of these vertices, the closed string field theory master action is given by

$$S = \frac{1}{2}\langle R'_{12}|\, c_0^{-(2)}\, Q^{(2)}|\Psi\rangle|\Psi\rangle + \sum_{N=3}^{\infty}\frac{1}{N!}\langle\, V^{(N)}|\Psi\rangle_1\cdots|\Psi\rangle_N \equiv \sum_{N=2}^{\infty}\frac{1}{N!}\langle\, V^{(N)}|\Psi\rangle_1\cdots|\Psi\rangle_N \tag{2.21}$$

with $c_0^- = (c_0 - \bar{c}_0)/2$, and, with the master field $|\Psi\rangle$ an element of $\widehat{\mathcal{H}}$. Here, for convenience, we have set the string coupling constant g to 1. The appropriate factors of g can be easily recovered by a rescaling $|\Psi\rangle \to g|\Psi\rangle$ in the final result. The statistics of the expansion coefficients of $|\Psi\rangle$, along the basis vectors of $\widehat{\mathcal{H}}$, are chosen in such a way that $|\Psi\rangle$ is always even.

2.1.3 Sewing Property.

The forms Ω (in $T_\Sigma\mathcal{P}$) also satisfy a sewing property. We introduce a ket

$$|\widetilde{\mathcal{S}}(\theta)_{12}\rangle = \frac{1}{2\pi} b_0^{-(1)} e^{i\theta(L_0^{(1)} - \overline{L}_0^{(1)})}|R_{12}\rangle\,, \tag{2.22}$$

which, apart from the b_0^- insertion, has the geometrical meaning of sewing with a twist angle θ. One can then prove (following the methods of Ref.[11], §8) that

$$\big\langle \Omega^{(1)n_1+1}_{\Sigma_1}\,\big|\,\big\langle \Omega^{(0)n_2+1}_{\Sigma_2}\,\big|\,\widetilde{\mathcal{S}}(\theta)\big\rangle \;=\; \big\langle \Omega^{(0)n_1+n_2}_{\Sigma_1\cup_\theta\Sigma_2}\,\big|\,, \tag{2.23}$$

where the proper interpretation of this equation is that the left hand side acts on $2n_1 - 3$ tangent vectors of $T_{\Sigma_1}\mathcal{P}_{n_1+1}$, and $2n_2 - 4$ tangent vectors of $T_{\Sigma_2}\mathcal{P}_{n_2+1}$,

* Although the bosonic closed string field theory action can be constructed in terms of the vertices $\langle\Omega^{(0)n}|$ only, the $\langle\Omega^{(k)n}|$ for $k < 0$ are essential for construction of fermionic string field theory[30].

while the right hand side acts on $2n_1+2n_2-6=(2n_1-3)+(2n_2-4)+1$ vectors. Of those, the first $(2n_1-3)$ are vectors in $T_{\Sigma_1\cup_\theta\Sigma_2}\mathcal{P}_{n_1+n_2}$, each of which creates the deformation of the sewn surface that would be produced by deforming Σ_1 with the corresponding vector in $T_{\Sigma_1}\mathcal{P}_{n_1+1}$, and then sewing to Σ_2. The next $(2n_2-4)$ vectors arise in a completely analogous fashion. The last vector is $\partial/\partial\theta$, and is the generator of twist. It arises from the b_0^- insertion in the ket $|\widetilde{\mathcal{S}}\rangle$. We also define

$$|\mathcal{S}_{12}\rangle \equiv \int_0^{2\pi} d\theta |\widetilde{\mathcal{S}}(\theta)_{12}\rangle = b_0^{-(1)}|R'_{12}\rangle \,. \tag{2.24}$$

The sewing ket $|\mathcal{S}_{12}\rangle$ is the familiar ket relevant to "twist-sewing", that is, sewing with integration over the twist angle. Moreover, it follows from (2.11) that the sewing ket $|\mathcal{S}\rangle$ is also symmetric. The integrated version of (2.23) in $\widehat{\mathcal{P}}$ will be useful for us. Let $\mathcal{B}$ denote a $(2n_1-3)$ subspace of $\widehat{\mathcal{P}}_{n_1+1}$ and let $\mathcal{V}$ denote a $(2n_1-4)$ subspace of $\widehat{\mathcal{P}}_{n_2+1}$. We then find

$$\int_{\mathcal{B}} \langle \Omega^{(1)n_1+1} \,|\, \int_{\mathcal{V}} \langle \Omega^{(0)n_2+1} \,|\, \mathcal{S}\rangle \;=\; \int_{\mathcal{B}\times\mathcal{V}} \langle \Omega^{(0)n_1+n_2} \,| \,. \tag{2.25}$$

Here in the right hand side $\mathcal{B}\times\mathcal{V}$ is the (oriented) subspace of $\widehat{\mathcal{P}}_{n_1+n_2}$ obtained by twist-sewing every element of $\mathcal{B}$ to every element of $\mathcal{V}$.⋆

2.2 Batalin-Vilkovisky Structures

We would like to review the BV structure that exists in a supermanifold, and the BV structure that exists in the vector space $\widehat{\mathcal{H}}$. The BV structure is nothing else than a symplectic structure. For a vector space we need a bilinear odd-nondegenerate form. For the case of a manifold we need, in addition, that the form be closed.

⋆ The orientation of $\mathcal{B}\times\mathcal{V}$ is fixed by an ordered basis $[\cdots]$ of tangent vectors at each point. The induced orientation of $\mathcal{B}\times\mathcal{V}$ is $[[\mathcal{B}],[\mathcal{V}],\partial/\partial\theta]$, where the vectors $[\mathcal{B}]$ and $[\mathcal{V}]$ were defined below eqn.(2.23).

2.2.1 Symplectic form on a Supermanifold

We follow the conventions of Ref.[12]. On a manifold the symplectic form ω reads

$$\omega = -dz^i\, \omega_{ij}(z)\, dz^j\,. \tag{2.26}$$

The form ω is odd, nondegenerate and closed. Nondegeneracy means that the matrix ω_{ij} is invertible, and the inverse matrix is denoted by ω^{ij}. One has

$$\omega^{ik}\, \omega_{kj} = \omega_{jk}\, \omega^{ki} = \delta^i_j\,. \tag{2.27}$$

The following properties hold

$$\begin{aligned} &\epsilon(\omega_{ij}) = \epsilon(\omega^{ij}) = i + j + 1\ , \\ &\omega_{ij} = -(-)^{ij}\omega_{ji}\ , \\ &\omega^{ij} = -(-)^{(i+1)(j+1)}\omega^{ji}\ . \end{aligned} \tag{2.28}$$

Using (2.26) we derive the following transformation laws

$$\begin{aligned} \omega_{pq}(\xi) &= \frac{\partial_l z^i}{\partial \xi^p}\, \omega_{ij}(z)\, \frac{\partial_r z^j}{\partial \xi^q}\,, \\ \omega^{pq}(\xi) &= \frac{\partial_r \xi^p}{\partial z^i}\, \omega^{ij}(z)\, \frac{\partial_l \xi^q}{\partial z^j}\,, \end{aligned} \tag{2.29}$$

where ∂_l and ∂_r denote left and right derivatives respectively.

2.2.2 Symplectic form on $\widehat{\mathcal{H}}$

Let $A, B \in \widehat{\mathcal{H}}$, be vectors in the even-dimensional supervector space $\widehat{\mathcal{H}}$. The symplectic form $\omega(\cdot\,,\cdot)$ must have the following exchange property:

$$\omega\,(A, B) = -(-)^{AB}\,\omega\,(B, A)\,. \tag{2.30}$$

In closed string field theory, $\widehat{\mathcal{H}}$ is the vector space defined in (2.12), and the physically relevant choice of symplectic form reads

$$\omega\,(A,B) = \langle R'_{12}\,|c_0^{-(2)}\,|A\rangle_1\,|B\rangle_2 \equiv \langle\omega_{12}\,|\,A\rangle_1\,|B\rangle_2\,. \tag{2.31}$$

The property (2.30) is easily verified using (2.10). We now introduce component notation as follows

$$\begin{aligned} \langle\omega_{12}| &= \langle R'_{12}|c_0^{-(2)} \equiv -\ {}_1\langle\Phi^i|\ \omega_{ij}(x)\ \ {}_2\langle\Phi^j|, \\ |\mathcal{S}_{12}\rangle &= b_0^{-(1)}|R'_{12}\rangle \equiv\ |\Phi_i\rangle_1\ (-)^{j+1}\omega^{ij}(x)\ \ |\Phi_j\rangle_2\,, \end{aligned} \tag{2.32}$$

where the sewing ket $|\mathcal{S}_{12}\rangle$ was introduced in Eqn.(2.24). It follows from Eqn.(2.32) that

$$\omega_{ij} = (-)^{i+1}\langle\Phi_i|c_0^-|\Phi_j\rangle \quad \text{and} \quad \omega^{ij} = -\langle\Phi^i|b_0^-|\Phi^j\rangle\,, \tag{2.33}$$

are real numbers which are non-vanishing only if the ghost numbers of the states $|\Phi_i\rangle$ and $|\Phi_j\rangle$ add up to five. Thus ω_{ij} defined this way automatically satisfies the first of eqs.(2.28). Moreover, it is clear from the reflector properties that

$$\langle\omega_{12}| = -\langle\omega_{21}| \quad \text{and} \quad |\mathcal{S}_{12}\rangle = |\mathcal{S}_{21}\rangle\,. \tag{2.34}$$

These equations together with our definitions in (2.32) imply the expected exchange properties

$$\omega_{ij} = -(-)^{ij}\omega_{ji}, \quad \text{and} \quad \omega^{ij} = -(-)^{(i+1)(j+1)}\,\omega^{ji}, \tag{2.35}$$

It follows from (2.9) that $\langle R'_{12}|R'_{23}\rangle = {}_3\mathcal{P}_1$, where the operator on the right is an operator that changes the state space label of states from one to three, and, at the same time projects into the $L_0 = \bar{L}_0$ subspace. We use this to evaluate the contraction of $\langle\omega|$ with $|\mathcal{S}\rangle$. One readily finds

$$\langle\omega_{12}|\mathcal{S}_{23}\rangle = \langle R'_{12}|c_0^{-(2)}b_0^{-(2)}|R'_{23}\rangle = \langle R'_{12}|b_0^{-(3)}c_0^{-(3)}|R'_{23}\rangle = b_0^{-(3)}c_0^{-(3)}\,{}_3\mathcal{P}_1\ =\ {}_3\mathcal{P}_1\,, \tag{2.36}$$

where the last equality holds in the restricted space we work on (where states are annihilated by b_0^-). Equation (2.36) implies that our definitions in (2.32) give, as expected,

$$\omega_{ik}\,\omega^{kj} \;=\; \delta_i^j\,. \tag{2.37}$$

2.2.3 Antibracket

This structure arises from the symplectic structure in $\widehat{\mathcal{H}}$ as follows [11]. Consider a set of basis vectors $|\widetilde{\Phi}^i\rangle$ such that

$$\omega\,(\,|\widetilde{\Phi}^i\rangle, |\Phi_j\rangle) = -\delta^i_j. \tag{2.38}$$

We can construct the tilde states as follows

$$|\widetilde{\Phi}^i\rangle \equiv (-)^i \langle\Phi^i\,|\mathcal{S}\rangle\,. \tag{2.39}$$

It is straightforward to verify that (2.38) is satisfied by this definition. It is also simple to see that $|\widetilde{\Phi}^i\rangle = b_0^-\,|\Phi^i\rangle$. The string field is then expanded as

$$|\Psi\rangle = \sum_i |\Phi_i\rangle\,\psi^i = \sum_{g(\Phi_s)\leq 2} \big(\,|\Phi_s\rangle\,\psi^s + |\widetilde{\Phi}^s\rangle\,\psi^*_s\,\big)\,, \tag{2.40}$$

where ψ^s are fields, and ψ^*_s are antifields. The second sum is only over states of ghost number less than or equal to two. Since the ghost number of $|\widetilde{\Phi}^s\rangle$ is five minus the ghost number of $|\Phi_s\rangle$, the sum actually runs over a complete basis (in fact, a symplectic basis of $\widehat{\mathcal{H}}$). The antibracket of two functions A and B, of the string field is defined as

$$\big\{A\,,B\,\big\} = \frac{\partial_r A}{\partial\psi^s}\,\frac{\partial_l B}{\partial\psi^*_s} \;-\; \frac{\partial_r A}{\partial\psi^*_s}\,\frac{\partial_l B}{\partial\psi^s}\,. \tag{2.41}$$

It is a straightforward calculation to prove that

$$\big\{A\,,B\,\big\} = \frac{\partial_r A}{\partial\psi^i}\omega^{ij}\frac{\partial_l B}{\partial\psi^j} \;=\; (-)^{B+1}\frac{\partial\,A}{\partial\,|\Psi\rangle}\,\frac{\partial\,B}{\partial\,|\Psi\rangle}\,|\mathcal{S}\rangle\,, \tag{2.42}$$

where the sewing ket $|\mathcal{S}\rangle$ is gluing the two state spaces left open by the differentiation with respect to the string field. There is no need to specify left or right derivatives because the string field is even.

2.3 Equivalence of Master Actions

In a conventional field theory, two actions give classically equivalent physics if they are related to each other via field redefinitions. In this case the tree level S matrices calculated from these two actions are identical. In the Batalin-Vilkovisky formalism, however, specifying the action does not completely specify the theory, even at the tree level. One also needs the symplectic structure ω_{ij} to specify the theory completely at the tree level. It enters the theory in three different ways:

1. The master action S must satisfy

$$\{S, S\} = 0\,, \tag{2.43}$$

   where $\{\ ,\ \}$ is the antibracket defined with respect to the symplectic structure ω (see (2.42)). Thus, for a generic change of ω, the master action S will not even remain a solution of the master equation.

2. The physical observables $\mathcal{O}$ must satisfy

$$\{S, \mathcal{O}\} = 0\,. \tag{2.44}$$

   Thus, even if we change ω in such a way that Eqn.(2.43) is still satisfied, the observables in the original theory do not remain observables in the new theory.

3. Finally, given a set of observables $\mathcal{O}_i$, their correlation function is calculated as,

$$\langle \prod_i \mathcal{O}_i \rangle = \int_L d\psi e^{-S} \prod_i \mathcal{O}_i\,, \tag{2.45}$$

   where L is a lagrangian submanifold of the full manifold M of configurations of the master fields. It has dimension equal to half of that of M, and satisfies the property that, for any two tangent vectors $t^i \frac{\partial}{\partial \psi^i}$, $\tilde{t}^i \frac{\partial}{\partial \psi^i}$ in the tangent space $T(L)$ of L,

$$\tilde{t}^i \, \omega_{ij} \, t^j = 0 \, . \tag{2.46}$$

Thus the definition of a lagrangian submanifold depends on the choice of ω.* This shows that even if we choose a set of observables and change ω in such a way that both eqs.(2.43) and (2.44) still hold with the new ω, their is no *a priori* guarantee that the correlation functions of these observables will be the same in the two theories.

This shows that if we want to prove the equivalence of two theories, it is not enough to find a field redefinition which maps one master action to the other. If such field redefinition, in addition, maps the symplectic structure of one theory to the other, the theories are clearly equivalent.† In the next section we shall show how this can be done for string field master actions and symplectic structures arising from two different conformal field theories.

2.4 Review of Earlier Work

Background independence of closed string field theory has been analyzed earlier in refs.[13,14,15]. In this subsection we shall briefly review these results, and discuss their relationship with present analysis.

We begin with a review of ref.[13], which analyzes background independence of the quadratic part of the string field theory action. Let us consider two conformal field theories CFT and CFT$'$ related by an infinitesimal marginal deformation $(\delta\lambda/\pi) \int d^2z \, \mathcal{O}$. Let Q and Q' denote the BRST charges of the two string theories formulated around these two conformal field theories, and $\langle \, | \, \rangle$ and $\langle \, | \, \rangle'$ be the

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⋆ For a given choice of ω, the right hand side of Eqn.(2.45) is independent of the choice of lagrangian submanifold. This is the main result of BV theory. For extensions, see [31].

† It seems likely that, under reasonable assumptions, this mapping of both action and symplectic structure, is not only sufficient, but is also necessary to prove the equivalence of two theories.

BPZ inner products in these two theories. Then, in a certain choice of basis, one finds that

$$
\begin{aligned}
Q' - Q =& \frac{\delta\lambda}{2\pi i} \oint\limits_{|z|=\epsilon} \left(dz \bar{c}(\bar{z}) \mathcal{O}(z,\bar{z}) + d\bar{z} c(z) \mathcal{O}(z,\bar{z}) \right) , \\
\langle A|B\rangle' - \langle A|B\rangle =& - \frac{\delta\lambda}{\pi} \int\limits_{\epsilon \le |z| \le 1/\epsilon} d^2 z \langle A|\mathcal{O}(z,\bar{z})|B\rangle .
\end{aligned}
\tag{2.47}
$$

for some number ϵ. Also let $\delta\lambda|\widehat{\mathcal{O}}\rangle = \delta\lambda|c\bar{c}\mathcal{O}\rangle$ denote the classical solution in string field theory formulated around CFT that represents CFT$'$, and

$$
\hat{Q} = Q + \delta\lambda\, [\widehat{O},\ ] \,, \tag{2.48}
$$

be the nilpotent operator [32] that appears in the kinetic term of the string field theory action formulated around CFT after we shift fields by an amount $\delta\lambda|\widehat{\mathcal{O}}\rangle$. It was shown that there is a transformation $S \equiv e^{\delta\lambda K}$ such that acting on states in $\widehat{\mathcal{H}}$

$$
\hat{Q} = S Q' S^{-1}, \quad \langle S\Phi_1|c_0^-|S\Phi_2\rangle = \langle \Phi_1|c_0^-|\Phi_2\rangle', \quad |\Phi_i\rangle \in \widehat{\mathcal{H}} \tag{2.49}
$$

to order $\delta\lambda$.

To compare this result to the results of the present paper we have to express Eqn.(2.49) in a different language. Writing $Q' = Q + \delta\lambda\partial_\lambda Q$, and noting that,

$$
\begin{aligned}
\langle \Phi_1|c_0^-|\Phi_2\rangle' - \langle \Phi_1|c_0^-|\Phi_2\rangle =& (-)^{\Phi_1} \ ({}'\langle\omega_{12}| - \langle\omega_{12}|)|\Phi_1\rangle_1|\Phi_2\rangle_2 \\
=& (-)^{\Phi_1} \delta\lambda (\partial_\lambda \langle\omega_{12}|)|\Phi_1\rangle_1|\Phi_2\rangle_2 \ ,
\end{aligned}
\tag{2.50}
$$

we can rewrite eqs.(2.49), to order $\delta\lambda$, as

$$
\partial_\lambda Q + [K, Q] = [\widehat{\mathcal{O}}\,,\ ] \,, \quad \partial_\lambda\langle\omega_{12}| - \langle\omega_{12}|(K^{(1)} + K^{(2)}) = 0 \,. \tag{2.51}
$$

As we shall see in §4, by demanding the background independence of the quadratic terms of the master action, we arrive precisely at equations of the form (2.51), which are conditions on the covariant derivatives of Q and $\langle\omega_{12}|$ with respect to the connection K. Thus the proof of existence of K given in ref.[13] may be taken as the proof of background independence of the quadratic part of the master action.‡ The proof given in [13] was based on showing how to construct matrix elements of K satisfying Eqn.(2.51). However, no closed expression for K as an operator was obtained. In this paper we shall obtain a closed expression for this operator, which we shall call Γ_μ, by expressing it in terms of geometric objects in the moduli space $\widehat{\mathcal{P}}_3$ of three punctured spheres (with local coordinates at the punctures).

In refs. [14,15] an attempt was made to prove the background independence of cubic and higher order terms of the string field theory. In particular, it was shown that there is an explicit field redefinition which transforms the *classical* string field theory action formalated around CFT, to the *classical* string field theory action formulated around CFT$'$, up to cubic terms. Again, the proof involved explicit construction of the different coefficients that appear in the field redefinition. In this paper we find a field redefinition that relates the *master* actions formulated around these two conformal field theories *to all orders* in the string field. Moreover, the objects which describe the the field redefinition are expressed in terms of geometric objects in the moduli spaces $\widehat{\mathcal{P}}_n$ of n-punctured Riemann spheres (with local coordinates at the punctures).

‡ The analysis of ref.[13] was for the classical action, and not the master action. In this case only a combination of the two equations in (2.51), specifying the covariant derivative of $\langle\omega_{12}|(Q^{(1)}+Q^{(2)})$, is necessary for background independence. However, a proof of existence of K satisfying both equations separately, was given in ref.[13]. As we shall see in §4, covariant constancy of $\langle\omega_{12}|$ is necessary for the symplectic structure of the theory formulated around CFT to get mapped to the symplectic structure of the theory formulated around CFT$'$.

3. Connections and Symmetric Vertices

In this section we develop some of the basic results that will be necessary to carry out our proof of background independence. After defining covariant derivatives on (super) vector bundles, we discuss how the connection $\widehat{\Gamma}_\mu$ [18,9,17,20] is defined in the presence of ghosts. We then compute the covariant derivative of the string field vertices with respect to this connection. This includes the covariant derivative of the BRST operator, which appears together with the symplectic form in the definition of the two string vertex. It is here that the asymmetric three string vertex $\mathcal{V}_3'$ makes its appearance. Finally, we prove that the space of symmetric closed string vertices is connected.

3.1 Covariant Derivatives on the Vector Bundle

Let $\mathcal{F}_n$ denote the vector bundle with the space of conformal field theories, labelled by the coordinates $\{x^\mu\}$, as the base space, and fiber $(\widehat{\mathcal{H}}_x)^{\otimes n}$ for $n \geq 0$ and $(\widehat{\mathcal{H}}_x^*)^{\otimes(-n)}$ for $n < 0$. We begin by defining the connection matrix

$$\Gamma_\mu \equiv |\Phi_j\rangle\, \Gamma_{\mu i}^{\;\; j}\, \langle \Phi^i|\,, \tag{3.1}$$

with $\epsilon(\Gamma_{\mu i}^{\;\; j}) = (-)^{i+j}$. The connections that will be of relevance for us in this paper will all have the property that $\Gamma_{\mu i}^{\;\; j}$ are real numbers and vanish unless $|\Phi_i\rangle$ and $|\Phi_j\rangle$ have the same ghost numbers. If $|A(x)\rangle$ and $\langle B(x)|$ denote sections of $\mathcal{F}_1$ and $\mathcal{F}_{-1}$ respectively, then, the covariant derivatives acting on these sections are defined to be

$$\begin{aligned} D_\mu\,(\,\Gamma\,)\,|A\,\rangle &\equiv \;\; (\partial_\mu + \Gamma_\mu)\,|A\,\rangle \\ D_\mu\,(\,\Gamma\,)\,\langle B\,| &\equiv \;\; \partial_\mu\langle B| \,-\, \langle B\,|\Gamma_\mu \end{aligned} \tag{3.2}$$

It is clear that this definition preserves contraction of state spaces, namely $D_\mu(\Gamma)\langle A|B\rangle = \partial_\mu\langle A|B\rangle$. The covariant derivative of general sections is obtained using the above derivatives and keeping in mind that the derivatives act tensorially. The covariant derivative of the symplectic section is given by

$$\begin{aligned} D_\mu\,(\Gamma\,)\langle\omega\,| &= -\,D_\mu\,(\Gamma\,)\big(\,{}_1\langle\Phi^i|\;\omega_{ij}(x)\;\;{}_2\langle\Phi^j|\big) \\ &= -\,{}_1\langle\Phi^i|\,\Big(\partial_\mu\,\omega_{ij} - (-)^{i(i'+1)}\,\Gamma_{\mu i}^{\;\;i'}\;\omega_{i'j} - \omega_{ij'}\,\Gamma_{\mu j}^{\;\;j'}\Big)\;{}_2\langle\Phi^j|\,. \end{aligned} \tag{3.3}$$

This is sometimes written conventionally as

$$\big(\mathcal{D}_\mu(\Gamma)\,\omega\,\big)_{ij} = \partial_\mu\,\omega_{ij} - (-)^{i(i'+1)}\,\Gamma_{\mu i}^{\;\;i'}\;\omega_{i'j} - \omega_{ij'}\,\Gamma_{\mu j}^{\;\;j'}\,. \tag{3.4}$$

It is useful to introduce another kind of covariant derivative, one relevant to functions on the whole vector bundle $\mathcal{F}_1$. Let $S(\psi^i, x)$ be such a function, with ψ^i denoting the coordinates of $\widehat{\mathcal{H}}$. We then define

$$\mathrm{D}_\mu(\Gamma)\,S \equiv \partial_\mu S - \frac{\partial_r S}{\partial\psi^i}\,\Gamma_{\mu j}^{\;\;i}\psi^j\,. \tag{3.5}$$

This covariant derivative examines the variation of the function as we move in the base along δx, and, on the fiber by parallel transport. For functions that arise naturally from sections, such as

$$F\,(\,|\Psi\rangle\,, x\,) = \langle\,\Sigma\,(x)\,|\Psi\rangle|\Psi\rangle\cdots|\Psi\rangle\,, \tag{3.6}$$

with $\langle\,\Sigma(x)\,|$ a section of $\mathcal{F}_{-n}$, and, $|\Psi\rangle = |\Phi_i\rangle\,\psi^i$ a grassman even ket, one can readily verify that

$$\mathrm{D}_\mu(\Gamma)\,F\,(\,|\Psi\rangle\,, x\,) = \Big(D_\mu(\Gamma)\langle\,\Sigma\,(x)\,|\Big)\,|\Psi\rangle\cdots|\Psi\rangle\,. \tag{3.7}$$

3.2 The connection $\widehat{\Gamma}$ upon inclusion of ghosts

In this subsection we discuss the extension of some of the theory-space geometry results of [20] to the case when the space of CFT's is made of theories each of which is the tensor product of a matter CFT times the standard CFT of reparametrization ghosts. We discuss, in particular, the canonical connection $\widehat{\Gamma}_\mu$. These results will be used in the next subsection for the computation of the covariant derivative, with respect to $\widehat{\Gamma}_\mu$, of the string field vertices.

One of the main results of [20] was that the variational formula of ref.[22] can be generalized to allow a unified description of all possible connections. It was argued that such a variational formula could, in fact, be taken as a definition of what we mean by having a theory space. The formula reads

$$D_\mu(\Gamma\,)\langle\Sigma\,| = -\frac{1}{\pi}\int\limits_{\Sigma-\cup_i D_i} d^2z\;\langle\,\Sigma; z\,|\mathcal{O}_\mu\rangle - \sum_{i=1}^{n}\langle\,\Sigma\,|\,\omega_\mu^{(i)}. \tag{3.8}$$

where the surface sections $\langle\,\Sigma\,|$ of the operator formalism encode all correlators on the punctured surface Σ. The state $|\mathcal{O}_\mu\rangle$ is exactly marginal, and is integrated over the surface minus some disks D_i around the punctures. Finally, the operator one forms ω_μ represent similarity transformations acting on each state space of $\langle\,\Sigma\,|$. Given a domain D, and a one form ω_μ , there must exist a connection Γ_μ such that the above equation holds. In particular, taking $\omega_\mu = 0$, and $D_i = D_i^{(1)}$ to be unit disks, we are guaranteed the existence of the corresponding connection $\widehat{\Gamma}$ satisfying

$$D_\mu(\,\widehat{\Gamma}\,)\langle\,\Sigma\,| = -\frac{1}{\pi}\int\limits_{\Sigma-\cup_i D_i^{(1)}} d^2z\;\langle\,\Sigma; z\,|\mathcal{O}_\mu\rangle. \tag{3.9}$$

For string field theory, the relevant CFT theory-space is made of theories each of which is a matter theory CFT_M, tensored with the reparametrization ghost theory CFT_G. The ghost CFT is never changed, and therefore, the coordinates that parametrize the total space are the coordinates arising from the specification of the matter theory CFT_M. When we consider this theory space, the variational formula (3.9) holds with $|\mathcal{O}_\mu\rangle$ the ghost number zero state created by the action of $\mathcal{O}_\mu(z)\otimes\mathbf{1}$ on the SL(2,C) vacuum, where $\mathcal{O}_\mu(z)$ is constructed purely out of the matter fields. It follows that the covariant derivative of $\langle\Sigma|$ does not change the ghost number of the bra. Indeed, in the convention where both the in and out vacuum have ghost number zero, the bra $\langle\Sigma|$, corresponding to a surface of genus g with n punctures, has ghost number $6g-6+6n$. Upon contraction, one loses six units of ghost number, and therefore, $|\mathcal{O}_\mu\rangle$ must be of ghost number zero for $\langle\Sigma; z|\mathcal{O}_\mu\rangle$ to be of the same ghost number as $\langle\Sigma|$.

Our argument of background independence will assume the existence of $\widehat{\Gamma}_\mu$, as this is conceptually equivalent to the assumption of having a theory space. This connection will be taken to be known, and this will be our basic input from theory space geometry. It is useful, however, to give a construction of $\widehat{\Gamma}_\mu$ in terms of the connection $\widehat{\Gamma}^M_\mu$ relevant to the theory space of CFT_M (without the ghosts). This latter connection satisfies the variational formula

$$D_\mu(\,\widehat{\Gamma}^M\,)\langle\,\Sigma^M\,| = -\frac{1}{\pi}\int\limits_{\Sigma-\cup_i D_i^{(1)}} d^2z\,\langle\,\Sigma^M; z\,|\mathcal{O}^M_\mu\,\rangle\,, \tag{3.10}$$

where we have added the superscripts M to remind ourselves that we are dealing with the matter theory alone. We also denote by $|\phi_i^M\rangle$ a basis in $\mathcal{H}_M$, and the connection coefficients read $\widehat{\Gamma}^{Mj}_{\mu i}$. Now consider the ghost CFT, and choose as a basis of states the Fock space states formed, by acting on the vacuum, with the usual ghost and antighost oscillators $(c_n, \bar{c}_n, b_n, \bar{b}_n)$. Denote such basis states by $|\phi_I^G\rangle$. It then follows that the basis states of $\mathrm{CFT}_M\otimes\mathrm{CFT}_G$ are given by $|\phi_i^M\rangle\otimes|\phi_I^G\rangle$. We now claim that the connection $\widehat{\Gamma}_\mu$ on the full (tensored) theory space is given by

$$\widehat{\Gamma}_{\mu\,(i,I)}^{\;\;(j,J)} = \widehat{\Gamma}^{Mj}_{\mu i}\,\delta_I^J\,. \tag{3.11}$$

The Kronecker delta in the ghost labels implies that the connection essentially ignores the ghosts. More precisely, we have

$$D_\mu(\widehat{\Gamma})\Big(\,|\phi_i^M\rangle\otimes|\phi_I^G\rangle\,\Big) = \Big(\,D_\mu(\widehat{\Gamma}^M)|\phi_i^M\rangle\,\Big)\otimes|\phi_I^G\rangle\,. \tag{3.12}$$

In each tensored theory, the surface states are given by $\langle\Sigma| = \langle\,\Sigma^M\,|\otimes\langle\,\Sigma^G\,|$, with the M and G superscripts denoting the matter theory and the ghost theory respectively. Moreover, $|\mathcal{O}_\mu\rangle = |\mathcal{O}_\mu^M\rangle\otimes|0^G\rangle$. Let us now show that the ansatz (3.11), together with the matter variational formula (3.10), indeed give us the variational formula (3.9) for the tensored theory space. We begin with the left hand side of (3.9)

$$\begin{aligned} D_\mu(\widehat{\Gamma})\langle\,\Sigma\,| = D_\mu(\widehat{\Gamma})\Big(\langle\,\Sigma^M\,|\otimes\langle\,\Sigma^G\,|\Big) &= \Big(\,D_\mu(\widehat{\Gamma}^M)\langle\,\Sigma^M\,|\Big)\otimes\langle\,\Sigma^G\,| \\ &= -\frac{1}{\pi}\int\limits_{\Sigma-\cup_i D_i^{(1)}} d^2z\,\langle\,\Sigma^M;z\,|\mathcal{O}_\mu^M\rangle\otimes\langle\,\Sigma^G\,|\,, \end{aligned} \tag{3.13}$$

where in the first step we used (3.12), and in the second step we used (3.10). Since the vacuum state deletes punctures we can write $\langle\,\Sigma^G\,| = \langle\,\Sigma^G;z|0^G\rangle$, and back in (3.13) we find

$$\begin{aligned} D_\mu(\widehat{\Gamma})\langle\,\Sigma\,| &= -\frac{1}{\pi}\int\limits_{\Sigma-\cup_i D_i^{(1)}} d^2z\,\Big(\,\langle\,\Sigma^M;z\,|\otimes\langle\,\Sigma^G;z|\,\Big)\Big(|\mathcal{O}_\mu^M\rangle\otimes|0^G\,\rangle\,\Big) \\ &= -\frac{1}{\pi}\int\limits_{\Sigma-\cup_i D_i^{(1)}} d^2z\,\langle\,\Sigma;z\,|\mathcal{O}_\mu\rangle\,, \end{aligned} \tag{3.14}$$

which is the desired relation.

Let $\mathcal{A}$ denote an operator constructed purely from the ghost fields. It can then be written explicitly as follows

$$\mathcal{A} \;=\; A_I^J(x)\,\Big[\mathbf{1}^M\otimes\Big(\,|\phi_J^G\rangle\langle\phi^{IG}|\,\Big)\Big]\,. \tag{3.15}$$

The covariant derivative, computed with the help of (3.11), gives

$$D_\mu(\widehat{\Gamma})\mathcal{A} = \;(\partial_\mu A_I^J)(x)\,\Big[\mathbf{1}^M\otimes\Big(\,|\phi_J^G\rangle\langle\phi^{IG}|\,\Big)\Big]\;\equiv\;\partial_\mu\mathcal{A}\,. \tag{3.16}$$

Consider now the ghost operators $\{c_n,\overline{c}_n,b_n,\overline{b}_n\}$. Given that we have defined the basis states of CFT_G in terms of these operators acting on the vacuum, their matrix elements (the analogs of $\mathcal{A}_I^J$) are all constants throughout theory space. It then follows that the covariant derivative of each of these operators vanishes

$$D_\mu(\,\widehat{\Gamma}\,)\{c_n, \bar{c}_n, b_n, \bar{b}_n\} = 0\,. \tag{3.17}$$

3.3 Covariant Derivatives $D_\mu(\,\widehat{\Gamma}\,)$ of String Field Vertices

We shall now compute covariant derivatives of N-string field vertices $\langle V^{(N)}|$ with the connection $\widehat{\Gamma}$. We note that the string field vertices, for $N \geq 3$ all arise from punctured spheres whose local coordinates cover fully the surfaces. Therefore, for each sphere Σ in $\mathcal{V}_N$, $\Sigma - \cup_i D_i^{(1)}$ vanishes, and as a consequence $D_\mu(\,\widehat{\Gamma}\,)\langle\Sigma| = 0$ (see (3.9)). Moreover, recall (§2.1) that the string field vertex takes the form $\langle V^{(N)}| = \int_{\mathcal{V}_N} \langle \Sigma | \mathbf{b} \cdots \mathbf{b}$, where $\mathbf{b}$ are antighost insertions. Such insertions have nothing whatsoever to do with CFT_M, they simply construct a measure on $\widehat{\mathcal{P}}_N$. Therefore $D_\mu(\,\widehat{\Gamma}\,)\,\mathbf{b} = 0$. All this implies that the covariant derivative of the string field vertex $\langle V^{(N)}|$ vanishes

$$D_\mu(\,\widehat{\Gamma}\,)\langle V^{(N)}| = 0 \quad \text{for } N \geq 3. \tag{3.18}$$

We would like to compute now the covariant derivative of the two string vertex $\langle V^{(2)}| = \langle \omega_{12}| Q^{(2)}$. To this end we will first calculate the covariant derivative of the symplectic form, and then the covariant derivative of the BRST operator. The symplectic form is given by $\langle \omega_{12}| = \langle R'_{12}| c_0^{-(2)}$, where, $\langle R'_{12}| = \langle R_{12}|\mathcal{P}_1\mathcal{P}_2$ ((2.13)). Since $\langle R_{12}|$ is an overlap two string vertex, $D_\mu(\,\widehat{\Gamma}\,)\langle R_{12}| = 0$. Furthermore, $D_\mu(\,\widehat{\Gamma}\,)(L_0 - \overline{L}_0) = 0$ (Ref.[20], Eqn.5.5)* implies that $D_\mu(\,\widehat{\Gamma}\,)\mathcal{P} = 0$. These

---

$\star$ This can be easily seen using the definition of $D_\mu(\,\widehat{\Gamma}\,)L_n$, and $D_\mu(\,\widehat{\Gamma}\,)\overline{L}_n$, given below, and, noting that with our definition of $\oint$, $\oint_{|z|=1} z d\bar{z} f = \oint_{|z|=1} \bar{z} dz f$ for any function f of z and $\bar{z}$.

results combine to give

$$D_\mu(\,\widehat{\Gamma}\,)\langle R'_{12}| = 0\,. \qquad (3.19)$$

This equation, expressing the metric compatibility of $\widehat{\Gamma}$, together with $D_\mu(\,\widehat{\Gamma}\,)\,c_0^- = 0$ ((3.17)), implies that the covariant derivative of the symplectic section vanishes

$$D_\mu(\,\widehat{\Gamma}\,)\langle\,\omega\,| = 0. \qquad (3.20)$$

We now turn to the computation of $D_\mu(\,\widehat{\Gamma}\,)\,Q$. It was found in [17,20] that the covariant derivative of the Virasoro operators, with respect to $\widehat{\Gamma}$ is given by

$$D_\mu(\widehat{\Gamma}\,)\,L_n = \oint_{|z|=1} \frac{d\bar{z}}{2\pi i}\; z^{n+1}\,\langle 0, z, \infty^*|\mathcal{O}_\mu\rangle = \oint_{|z|=1} \frac{d\bar{z}}{2\pi i}\; z^{n+1}\,\mathcal{O}_\mu(z,\bar{z}), \qquad (3.21)$$

with $D_\mu(\widehat{\Gamma})\overline{L}_n$ given by a similar expression (recall $\oint dz/z = \oint d\bar{z}/\bar{z} = 2\pi i$). It then follows from $Q = \sum(c_{-n}L_n + \bar{c}_{-n}\overline{L}_n)$, and Eqn.(3.17), that

$$D_\mu(\,\widehat{\Gamma}\,)\,Q = \frac{1}{2\pi i}\oint_{|z|=1} (dz\bar{c}(\bar{z})\mathcal{O}_\mu(z,\bar{z}) + d\bar{z}c(z)\mathcal{O}_\mu(z,\bar{z})). \qquad (3.22)$$

This resembles Eqn.(2.47) for $\epsilon = 1$. We are now set for the computation of $D_\mu(\,\widehat{\Gamma}\,)\,\langle V^{(2)}|$.

3.3.1 Claim:

The covariant derivative of $\langle V^{(2)}|$ is given by

$$D_\mu(\,\widehat{\Gamma}\,)\,\langle V^{(2)}| = D_\mu(\,\widehat{\Gamma}\,)\left(\langle\,\omega_{12}\,|\,Q^{(2)}\right) = \langle\, V'^{(3)}_{123}\,|\,\widehat{\mathcal{O}}_\mu\rangle_3\,, \qquad (3.23)$$

where $|\widehat{\mathcal{O}}_\mu\rangle = |c\bar{c}\mathcal{O}_\mu\rangle$, and the bra $\langle V'^{(3)}_{123}|$ is the surface state corresponding to a three punctured sphere $\mathcal{V}'_3$, shown in Fig. 1, and described as follows. In the uniformizing coordinate z, it is punctured at $z = 0$, with a local coordinate $z_1(z) =$

z, at $z = \infty$, with local coordinate $z_2(z) = 1/z$, and at $z = 1$, with local coordinate $z_3(z) = h(z)$ left arbitrary. The right hand side of Eqn.(3.23) is independent of the choice of z_3, since $|\widehat{\mathcal{O}}_\mu\rangle$, which is inserted at $z = 1$, is primary and of dimension zero. It should be emphazised that this equation holds only upon contraction with states in $\widehat{\mathcal{H}}$, namely, it is a strict identity if we multiply from the right by $b_0^- \mathcal{P}$ for each state space.

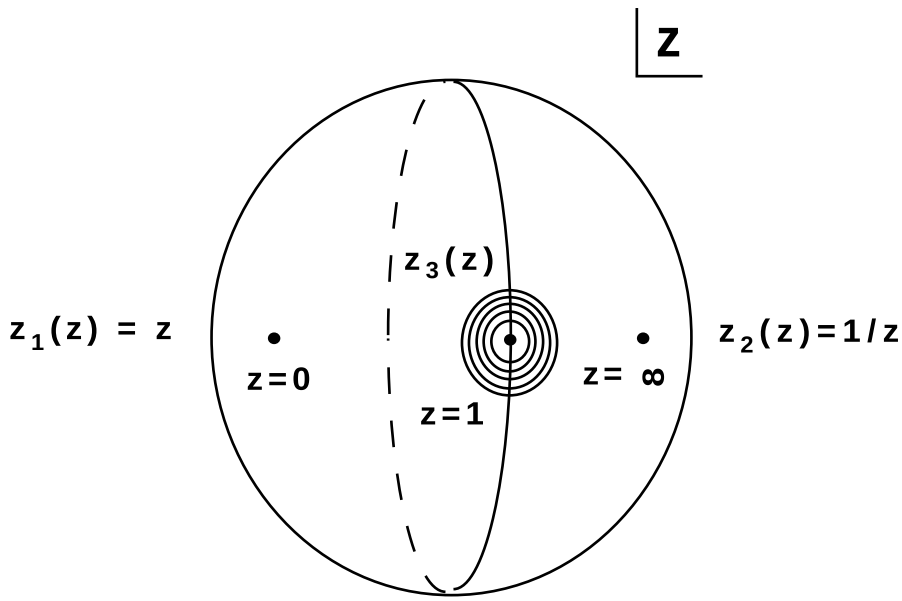


Figure 1. Here we show the asymmetric three punctured sphere $\mathcal{V}_3'$. The local coordinates $z_1(z) = z$, based at $z = 0$, and $z_2(z) = 1/z$, based at $z = \infty$, cover the sphere fully. The coordinate $z_3(z)$, based at $z = 1$, is undetermined.

3.3.2 Proof.

Using (3.20) and (3.22), we get,

$$D_\mu(\hat{\Gamma})\,(\langle\omega_{12}|Q^{(2)}) = \langle\omega_{12}|\frac{1}{2\pi i}\oint\limits_{|z|=1}\Big(dz\bar{c}(\bar{z})\mathcal{O}_\mu(z,\bar{z}) + d\bar{z}c(z)\mathcal{O}_\mu(z,\bar{z})\Big)^{(2)}, \quad (3.24)$$

where the operator inside the contour integral is an operator on the state space (2). We therefore need to show that

$$\langle\omega_{12}|\frac{1}{2\pi i}\oint\limits_{|z|=1}\Big(dz\bar{c}(\bar{z})\mathcal{O}_\mu(z,\bar{z}) + d\bar{z}c(z)\mathcal{O}_\mu(z,\bar{z})\Big)^{(2)} = \langle V'^{(3)}_{123}|\widehat{\mathcal{O}}_\mu\rangle_3 \qquad (3.25)$$

This can be done following a similar analysis in ref.[13]. Consider the left hand side of the equation, and separate out the ghost zero mode from the bra $\langle\omega\,|$

$$\langle R'{}_{12}|\frac{1}{2\pi i}\oint\limits_{|z|=1}\Big(dz\,c_0^-\,\bar{c}(\bar{z})\mathcal{O}_\mu(z,\bar{z}) + d\bar{z}\,c_0^-\,c(z)\mathcal{O}_\mu(z,\bar{z})\Big)^{(2)}. \qquad (3.26)$$

Using Virasoro Ward identities we find

$$\begin{aligned}\bar{c}\mathcal{O}_\mu(z=e^{i\theta}) &= e^{-i\theta}e^{i(L_0-\overline{L}_0)\theta}\;\bar{c}\mathcal{O}_\mu(z=1)\,e^{-i(L_0-\overline{L}_0)\theta}\,,\\ c\mathcal{O}_\mu(z=e^{i\theta}) &= e^{i\theta}\;e^{i(L_0-\overline{L}_0)\theta}\;c\mathcal{O}_\mu(z=1)\,e^{-i(L_0-\overline{L}_0)\theta}.\end{aligned} \qquad (3.27)$$

The operator $e^{i(L_0-\overline{L}_0)\theta}$ on the left commutes with c_0^-, and gives one acting on the primed reflector, and, the operator $e^{-i(L_0-\overline{L}_0)\theta}$ on the right, gives one acting on states in $\widehat{\mathcal{H}}$. Using these relations we can explicitly perform the z, $\bar{z}$ integrals in Eqn.(3.26), and bring it to the form:

$$\langle R'{}_{12}|\,\Big(c_0^-(c(1)+\bar{c}(1))\mathcal{O}_\mu(1)\Big)^{(2)} \qquad (3.28)$$

On the other hand, from the geometrical description of $\langle V'^{(3)}|$ we have that

$$\langle V'^{(3)}_{123}|\widehat{\mathcal{O}}_\mu\rangle_3 = \langle R'{}_{12}|\,\Big(c(1)\bar{c}(1)\mathcal{O}_\mu(1)\Big)^{(2)}, \qquad (3.29)$$

since the third puncture sits at $z_2 = 1$. We now must show that (3.28) and the right hand side of (3.29) agree upon contraction with states annihilated by b_0^-.

The simplest way to verify this is to multiply both expressions from the right by the factor $b_0^{-(2)} b_0^{-(1)}$. One must then move $b_0^{-(2)} b_0^{-(1)}$ all the way to the left and use $\langle R'_{12}| b_0^{-(2)} b_0^{-(1)} = 0$, which follows from the properties of the reflector and $(b_0^-)^2 = 0$. This establishes the desired result. $\square$

3.4 Symmetric Vertices and Their Deformations

Our analysis of symplectic (or antibracket preserving) diffeomophisms requires careful consideration of the meaning of symmetric vertices and their deformations. In this section we will develop the necessary results on symmetric string vertices. The basic result that we need is that given two symmetric string vertices, there is a continuous deformation of one vertex into the other via symmetric string vertices. In other words, the space of symmetric string vertices is connected. We will prove this result by using the methods of Ref. [33]. We discuss explicitly, because of their special features, the cases of two and three string vertices. We then consider all higher string vertices.

3.4.1 Two String Vertices.

A two string vertex is a two punctured sphere with a coordinate curve $\mathcal{C}$ (§2.1) surrounding each puncture. If we consider the punctures to be at $z = 0$, and at $z = \infty$, with z the uniformizing coordinate, there is a one complex parameter family of conformal maps taking the punctured sphere into itself, namely, the maps $z \to az$, with a constant. Two two-string vertices are identical if their corresponding coordinate curves $(\mathcal{C}_1, \mathcal{C}_2)$, and $(\mathcal{C}'_1, \mathcal{C}'_2)$ are mapped into each other by the map $T_a : z \to az$, for some a:

$$(T_a\, \mathcal{C}_1, T_a\, \mathcal{C}_2) = (\mathcal{C}'_1, \mathcal{C}'_2)\,. \tag{3.30}$$

A two string vertex is defined to be *symmetric* if any well defined map on the sphere exchanging the two punctures exchanges the coordinate curves up to the above equivalence. The map $I_b : z \to b/z$ is the most general map exchanging the punctures (at zero and infinity). Thus a vertex is symmetric if

$$(I_b\, \mathcal{C}_1, I_b\, \mathcal{C}_2) = (T_a\, \mathcal{C}_2, T_a\, \mathcal{C}_1)\,. \tag{3.31}$$

It follows from the above that

$$(I_1\, \mathcal{C}_1, I_1\, \mathcal{C}_2) = (T_c\, \mathcal{C}_2, T_c\, \mathcal{C}_1)\,. \tag{3.32}$$

for $c = a/b$. Thus, a two string vertex is symmetric if there is a constant c such that the above relation holds. It follows from (3.32) that a symmetric two string vertex is always of the form

$$(\mathcal{C}_1\,,\ I_1\, T_c\, \mathcal{C}_1\,)\ ,\ \ (c \neq 0). \tag{3.33}$$

We now want to show that given two symmetric vertices $(\mathcal{C}_1, I_1\, T_c\, \mathcal{C}_1\,)$, and $(\mathcal{C}_1', I_1\, T_{c'}\, \mathcal{C}_1')$, with c and c' two constants different from zero, there is a continuous deformation taking one into the other, such that, at every stage we have a symmetric two string vertex. To this end, we introduce a homotopy $c(t)$ interpolating between the two constants: $c(0) = c$, $c(1) = c'$, and $c(t) \neq 0$ for all $t \in [0,1]$. Since the coordinate curves $\mathcal{C}_1$ and $\mathcal{C}_1'$ are Jordan closed curves surrounding $z = 0$, they are homotopic, and therefore, there is a homotopy $\mathcal{C}_1(t)$ such that $\mathcal{C}_1(0) = \mathcal{C}_1$, and $\mathcal{C}_1(1) = \mathcal{C}_1'$. It is then clear that $\big(\, \mathcal{C}_1(t)\ ,\ I_1 \circ T_{c(t)} \mathcal{C}_1(t)\ \big)$ provides the desired homotopy between the two string vertices. What we did was elementary, we deformed arbitrarily one of the coordinate curves, and defined the other coordinate curve to be such that we obtain a symmetric two string vertex at every stage of the deformation.

3.4.2 Three string vertices.

The maps taking a three punctured sphere into itself arise from a map T that cycles the three punctures, and a map E that exchanges two punctures leaving the other fixed. A three string vertex is said to be symmetric if T cycles the coordinate curves, and, E exchanges two coordinate curves leaving the other fixed. The requirement of invariance of this coordinate curve under E implies that given an arbitrary coordinate curve around one puncture, it is not always possible to obtain a symmetric vertex. It was shown in [33], however, that given a coordinate curve $\mathcal{C}_1$ satisfying $E\mathcal{C}_1 = \mathcal{C}_1$, the vertex $(\mathcal{C}_1, T\mathcal{C}_1, T^2\mathcal{C}_1)$ is symmetric. Moreover, all symmetric three-string vertices can be written in this way. Thus, given two symmetric vertices $(\mathcal{C}_1, T\mathcal{C}_1, T^2\mathcal{C}_1)$ and $(\mathcal{C}_1', T\mathcal{C}_1', T^2\mathcal{C}_1')$, we must find a homotopy $\mathcal{C}_1(t)$ between $\mathcal{C}_1$ and $\mathcal{C}_1'$ with $E\mathcal{C}_1(t) = \mathcal{C}_1(t)$. This requirement is easily visualized if $\mathcal{C}_1(t)$ is chosen to surround the puncture at $z = 0$, and the other two punctures are placed at $z = 1$ and $z = -1$. Then, the map E takes the form $E : z \to -z$, and it acts on $\mathcal{C}_1(t)$ by reflection around the origin. This means that $\mathcal{C}_1(t)$ can be broken into two open curves $\mathcal{C}^u(t)$ and $\mathcal{C}^l(t)$ (for upper and lower) whose endpoints, one on the positive real axis, and the other on the negative real axis, coincide, with the map E exchanging the open curves. The curves $\mathcal{C}^u(t)$ and $\mathcal{C}^l(t)$ are homotopic to open curves lying fully on the upper and lower half plane respectively. Thus, the open curves $\mathcal{C}_1^u$ and $\mathcal{C'}_1^u$ are homotopic, and any arbitrary homotopy between them can be extended by reflection to a consistent homotopy of the curves $\mathcal{C}_1$ and $\mathcal{C}_1'$. The vertex $(\mathcal{C}_1(t), T\mathcal{C}_1(t), T^2\mathcal{C}_1(t))$ then defines a homotopy, via symmetric closed string vertices, between the two symmetric vertices.

In our analysis we shall also need to construct homotopies between three string vertices which are not fully symmetric, but symmetric under the exchange of two legs. Such vertices are characterized by the coordinate curves $(\mathcal{C}_1, \mathcal{C}_2, E\mathcal{C}_2)$ with $\mathcal{C}_1$ satisfying $\mathcal{C}_1 = E\mathcal{C}_1$, but $\mathcal{C}_2$ arbitrary. The homotopy between two such vertices maintaining the exchange symmetry $2 \leftrightarrow 3$ is given by $(\mathcal{C}_1(t), \mathcal{C}_2(t), E\mathcal{C}_2(t))$, where $\mathcal{C}_1(t)$ is the homotopy between $\mathcal{C}_1$ and $\mathcal{C}_1'$ satisfying $\mathcal{C}_1(t) = E\mathcal{C}_1(t)$, and $\mathcal{C}_2(t)$ is any arbitrary homotopy between $\mathcal{C}_2$ and $\mathcal{C}_2'$.

3.4.3 Higher String Vertices

A vertex $\mathcal{V}_n\,(n \geq 4)$ is a subspace of $\widehat{\mathcal{P}}_n$, typically, of dimensionality $2n-6$. It is said to be symmetric if $\mathcal{V}_n$ includes, for each punctured surface with local coordinates, all the copies of this surface that differ only by the assignment of the labels $\{1, 2\cdots, n\}$ to the underlying punctures. Two string vertices $\mathcal{V}_n$ and $\mathcal{V}_n'$ will be said to be in the same *class* if their boundaries coincide as punctured Riemann surfaces without local coordinates. We claim that given two symmetric string vertices $\mathcal{V}_n$ and $\mathcal{V}_n'$ in the same class, there is a homotopy $\mathcal{V}_n(t)$, such that $\mathcal{V}_n(0) = \mathcal{V}_n$, $\mathcal{V}_n(1) = \mathcal{V}_n'$, and for all $t \in [0,1]$, $\mathcal{V}_n(t)$ is a symmetric vertex in the same class.

This homotopy is simple to build when both $\mathcal{V}_n$, and $\mathcal{V}_n'$, define sections σ, and σ' respectively, over $\mathcal{M}_n$. Then both vertices determine a common space $D_n = \pi(\mathcal{V}_n) = \pi(\mathcal{V}_n') \subset \mathcal{M}_n$, of labeled punctured surfaces (without local coordinates), and $\mathcal{V}_n = \sigma(D_n)$ and $\mathcal{V}_n' = \sigma'(D_n)$. Our aim is to define a homotopy $\sigma(t)\big(\mathcal{D}_n\big) \in \widehat{\mathcal{P}}_n$, such that $\sigma(0) = \sigma$, $\sigma(1) = \sigma'$, and $\sigma(t)\big(\mathcal{D}_n\big)$is symmetric and in the same class for all $t \in [0,1]$. First consider a homotopy taking each local coordinate z_i for each surface on $\sigma(\mathcal{D}_n)$ into the coordinate kz_i, with k a sufficiently large constant so that for each surface $\sigma(P)$ $(P \in \mathcal{D}_n)$, the new coordinate curves lie completely within the corresponding coordinate curves of the surface $\sigma'(P)$. (The constant $k < \infty$ is guaranteed to exist because D_n is compact). Since this deformation is independent of the labelling of the punctures it defines a symmetric deformation manifestly preserving the class of the vertex. We can now imagine each surface $\sigma'(P)$ as equipped with two coordinate curves around each puncture; the one arising from $\sigma(P)$ by the above deformation, completely inside the one defined by the section σ'. Let $(\mathcal{C}_1, \cdots \mathcal{C}_n)$ denote the small curves and $(\mathcal{C}_1', \cdots \mathcal{C}_n')$ denote the big curves. We now define the homotopy $(\mathcal{C}_1(t), \cdots \mathcal{C}_n(t))$ as follows. Let m be the map taking the annulus determined by the curves $\mathcal{C}_i$ and $\mathcal{C}_i'$ to the standard annulus $1 \leq |z| \leq 2$. The homotopy is provided by inverse images of the circles $|z| = 1+t$, for $t \in [0,1]$, namely $\mathcal{C}_i(t) = m^{-1}\{|z| = 1 + t\}$. This clearly gives us a continuous path on the fiber over P from $\sigma(P)$ up to $\sigma'(P)$. We now define $\sigma(t)$ to act in precisely this way for any surface on D_n. We claim that $\sigma(t)(D_n)$ is a section, that is, a continuous map from D_n to $\widehat{\mathcal{P}}_n$. This should be clear, since nearby surfaces $P, P' \in D_n$ must be mapped to nearby surfaces $\sigma(t)(P)\,, \sigma(t)(P') \in \widehat{\mathcal{P}}_n$ for any fixed value of t. The sections $\sigma(t)(D_n)$ must be symmetric since the deformations are done without reference to the labels of the punctures, it only involves the punctured surfaces and their local coordinates. It is also clear that for exceptional surfaces with automorphisms, that is, conformal maps that exchange punctures, there is no complication.[⋆] Any automorphism of a surface must correspond to conformal maps that take the annuli considered above into themselves or into each other. This is because the maps must take the inner circles into each other and the outer circles into each other. But any such map must take, in the standard picture of the annulus as $1 \leq |z| \leq 2$, the constant $|z|$ circles into each other. This shows that our homotopy must respect the automorphisms.

A final point concerns the case when the vertices are not sections. This is the case, for example, when the projection from (some subspace of) $\mathcal{V}_n$ to $\mathcal{M}_n$ is many to one. It is enough to discuss the case when one of the vertices is a section and the other is not, since once we know how to construct such symmetric homotopy, we can find a homotopy between each non-section vertex and a common (section) vertex of the same class, and by composition we find the desired homotopy between the non-section vertices. The idea, for taking a non-section vertex into a section one goes as follows. We extend the section vertex arbitrarily (not even keeping symmetry) so that the projection of the non-section vertex into $\mathcal{M}_n$ is inside the projection of the extended section into $\mathcal{M}_n$. We then do exactly as we did above for every surface in the nonsection vertex, we produce the two curves around each puncture and construct the homotopy. This homotopy flattens the non-section vertex over the section vertex. In this process we obtain folds due to the surfaces

⋆ This implies that the present argument also applies to two-punctured and three-punctured spheres, as particular cases.

that are contained more than once in the non-section. The folds now cancel out, since they represent identical spaces with opposite orientation. This gives us the desired homotopy.

During our analysis, we shall encounter vertices which are symmetric in all but one of the external legs. Such vertices can also be deformed into each other maintaining their symmetry. This is done by following exactly the same procedure as above of deforming the coordinate curves around all the punctures, including the special one.

4. Formulating the Problem of Background Independence of CSFT

In this section we shall discuss the issues involved in proving background independence of the string field theory action. We denote by x^μ the set of coordinates labelling the moduli space of the conformal field theories. For each point x^μ in the moduli space, the state space $\widehat{\mathcal{H}}_x$ contains the states of all ghost numbers of CFT_x annihilated by b_0^- and L_0^-. The state space $\widehat{\mathcal{H}}_x$ is a symplectic vector space, namely it is endowed with an odd nondegenerate two-form ω_x. Acting on two vectors $A, B \in \widehat{\mathcal{H}}_x$, we have $\omega_x(A, B) \sim \omega_{x\,ji} A^i B^j$, with $\omega_{x\,ij}$ constants. The string field $|\Psi_x\rangle$ is an element of $\widehat{\mathcal{H}}_x$, and the string field master action $S_x\left(|\Psi_x\rangle\right)$ is a map from $\widehat{\mathcal{H}}_x$ to the space of real numbers*. We have included an extra explicit dependence on the conformal theory at x^μ as a subscript of the action. This takes into account the fact that the construction of the action makes use of ingredients of the conformal theory in question, such as the BRST operator and correlators. As has already been emphasized, in the BV formalism, both, the master action S_x and the symplectic form ω_x are crucial in specifying the theory.

An issue that will play a role at various points of our discussion is whether the

$\star$ More precisely, since we are dealing with the master action, it is a map to the space of even elements of a grassmann algebra, which we will continue to denote as R.

state space $\widehat{\mathcal{H}}_x$ should be thought of as a vector space or as a manifold[†]. The string field, by definition, is a vector in the state space $\widehat{\mathcal{H}}_x$. Any point in this state space represents a configuration of the string field. In conventional field theory, however, the set of field configurations naturally define a manifold, for example, the space of metrics in gravity. Therefore it is sometimes convenient to think of $\widehat{\mathcal{H}}_x$ as a manifold, with the string field defining coordinates on it. When a vector space is viewed as a manifold, the tangent space at any point p on the manifold is naturally identified with the vector space itself. This allows us to define a symplectic form on the *manifold* $\widehat{\mathcal{H}}_x$, from the symplectic form ω_x on the *vector space* $\widehat{\mathcal{H}}_x$. This symplectic form on the manifold is necessary to be able to define the antibracket of two functions on the manifold. Using the natural coordinates induced by the basis vectors of $\widehat{\mathcal{H}}_x$, we see that the components of the symplectic form that we have obtained on the manifold $\widehat{\mathcal{H}}_x$ are constants. The action S_x may now be regarded as a map from the manifold $\widehat{\mathcal{H}}_x$ to R. A general invertible string field redefinition is then naturally thought of as a diffeomorphism of the manifold $\widehat{\mathcal{H}}_x$ into itself. A particularly relevant subclass of diffeomorphisms are those that preserve the symplectic structure on the manifold $\widehat{\mathcal{H}}_x$. Such diffeomorphisms have featured in the proof that two string field theories formulated on $\widehat{\mathcal{H}}_x$ but using different string field vertices are physically equivalent [12]. We will sometimes separate out linear maps arising from the general diffeomorphisms and then it will be useful to use the picture of $\widehat{\mathcal{H}}_x$ as a linear vector space.[‡]

† Throughout this paper all vector spaces and manifolds, are actually supervector spaces and supermanifolds, respectively.

‡ On general grounds, we can expect that the correspondance between the vector space $\widehat{\mathcal{H}}$ and the manifold of field configurations holds only locally. It may happen that some points in $\widehat{\mathcal{H}}$ far away from the origin do not represent allowed configurations, for example, a fluctuation $h_{\mu\nu}$ of a background metric $\hat{g}_{\mu\nu}$ making the total metric negative. Or it could be that $\widehat{\mathcal{H}}$ actually represents only a patch in the space of all allowed field configurations.

4.1 The General Conditions for Background Independence

The question of background independence of string field theory may now be formulated as follows. Given two string field actions $S_x : \widehat{\mathcal{H}}_x \to R$ and $S_y : \widehat{\mathcal{H}}_y \to R$, formulated around two different conformal theories x and y, we demand the existence of a diffeomorphism

$$F_{y,x} : \widehat{\mathcal{H}}_x \to \widehat{\mathcal{H}}_y \,, \tag{4.1}$$

between the corresponding spaces $\widehat{\mathcal{H}}_x$ and $\widehat{\mathcal{H}}_y$ such that

$$\begin{aligned} \omega_x &= {F_{y,x}}^* \, \omega_y \ , \\ S_x &= {F_{y,x}}^* \, S_y \ , \end{aligned} \tag{4.2}$$

with ${F_{y,x}}^*$ denoting the pullback performed using the diffeomorphism $F_{y,x}$. These equations imply that the diffeomorphism maps *both* the symplectic structure and master action on $\widehat{\mathcal{H}}_y$ to those in $\widehat{\mathcal{H}}_x$. The question of background independence is simply the question whether such symplectic, or antibracket preserving, diffeomorphism exists.

In order to make our discussion more concrete, let us choose a complete set of basis states $|\Phi_i\rangle$ in $\widehat{\mathcal{H}}_x$ for all values of x. The target space fields ψ_x^i are defined to be the components of the string field along the basis vectors

$$|\Psi_x\rangle = \sum_i |\Phi_i\,\rangle \, \psi_x^i \,. \tag{4.3}$$

The string field action $S(\psi_x, x)$ is a function of the string field coordinates ψ_x^i and the coordinate x labelling the space of conformal field theories. Eqs.(4.1) and (4.2) may now be rewritten as

$$\psi_y^i = F^i \left(\psi_x, x, y\right), \tag{4.4}$$

together with the background independence conditions

$$
\begin{aligned}
S\,(\,\psi_x\,,x\,) &= S\,(\,\psi_y\,,y\,)\,, \\
\omega_{i'j'}(x) \;&= \frac{\partial_l F^i}{\partial\psi_x^{i'}}\;\omega_{ij}(y)\;\frac{\partial_r F^j}{\partial\psi_x^{j'}}\;,
\end{aligned}
\tag{4.5}
$$

where ∂_r and ∂_l, as usual, denote derivatives from the right and from the left respectively. We have not included a ψ dependence in the ω_{ij}'s, since, as argued above, they are constants in this coordinate system.

4.2 Background Independence for Nearby Backgrounds

Let us now consider the case of nearby conformal field theories corresponding to the points x and $x+\delta x$. Since we have a vector bundle there is a notion of smoothness in the choice of basis vectors throughout theory space. Therefore, an infinitesimal shift in theory space must require an infinitesimal transformation F^i

$$
\psi^i_{x+\delta x} = F^i\,(\psi_x\,,\,x,x+\delta x) = \;\psi^i_x\;+\;\delta x^\mu\;\cdot f^i_\mu\,(\psi_x\,,x) + \mathcal{O}(\delta x^2)\;, \tag{4.6}
$$

for some function f^i_μ. For $y=x+\delta x$, equations (4.5) and (4.6) demand that

$$
\begin{aligned}
&\frac{\partial\omega_{i'j'}(x)}{\partial x^\mu} + \frac{\partial_l f^i_\mu}{\partial\psi_x^{i'}}\,\omega_{ij'}(x) + \omega_{i'j}(x)\frac{\partial_r f^j_\mu}{\partial\psi_x^{j'}} = 0\,. \\
&\frac{\partial S(\psi_x,x)}{\partial x^\mu} + \frac{\partial_r S(\psi_x,x)}{\partial\psi^i_x} f^i_\mu = 0,
\end{aligned}
\tag{4.7}
$$

The question of background independence of string field theory under infinitesimal change of background reduces to the question of existence of $f^i_\mu(\psi_x\,,x)$ satisfying equations (4.7).

Let us now define objects $\Gamma_\mu^{\;i}$, $\Gamma_{\mu j}^{\;i}$, and h^i_μ by separating out of f^i_μ the ψ independent part $-\Gamma_\mu^{\;i}$, and the part linear in ψ:

$$
f^i_\mu\,(\psi_x\,,x) \;= -\Gamma_\mu^{\;i}\,(x) - \Gamma_{\mu j}^{\;i}\,(x)\,\psi^j_x \;-\; h^i_\mu\,(\psi_x\,,x)\,, \tag{4.8}
$$

where h^i_μ contains quadratic and higher order terms in ψ_x. As the notation suggests, the linear part of f^i_μ defines a connection $\Gamma_{\mu i}^{\;j}(x)$ on the vector bundle of state spaces over CFT theory space. We shall restrict to field redefinitions that preserve ghost number.[⋆] This will imply, among other things, that $\Gamma_{\mu j}^{\;\,i}$ is non-vanishing only if the states $|\Phi_i\rangle$ and $|\Phi_j\rangle$ carry the same ghost number. This also shows that $\epsilon(\Gamma_{\mu j}^{\;\,i}) = (-1)^{i+j} = 1$, which is consistent with the fact the $\Gamma_{\mu j}^{\;\,i}$ are real numbers. Needless to say, covariant derivatives involving the connection $\Gamma_{\mu j}^{\;\,i}$ appear when we analyze the content of our background independence equations. Upon partial expansion, equations (4.7) become the equations

$$\big(\mathcal{D}_\mu(\Gamma)\,\omega\big)_{i'j'} - \frac{\partial_l h^i_\mu}{\partial \psi^{i'}_x}\,\omega_{ij'}(x) \;-\; \omega_{i'j}(x)\,\frac{\partial_r h^j_\mu}{\partial \psi^{j'}_x} = 0\,, \tag{4.9}$$

$$\mathrm{D}_\mu\, S - \frac{\partial_r S}{\partial \psi^i}\,(\,\Gamma_\mu^{\;\,i} + h^i_\mu\,) \;= 0, \tag{4.10}$$

The covariant derivative D_μ was defined in (3.5), and the covariant derivative of ω, is the standard covariant derivative of sections on a vector bundle defined in Eqn.(3.4). Indeed, being ψ independent, the symplectic form can be viewed as a section on the vector bundle.

We shall first analyze the consequences of Eqn.(4.9). Since h^i_μ is quadratic and higher orders in ψ, the ψ^i_x independent part of this equation gives

$$D_\mu(\Gamma)\langle\omega| = 0, \tag{4.11}$$

whereas the ψ^i_x dependent part of this equation gives

$$\frac{\partial_l h^i_\mu}{\partial \psi^{i'}_x}\,\omega_{ij'}(x) \;+\; \omega_{i'j}(x)\,\frac{\partial_r h^j_\mu}{\partial \psi^{j'}_x} = 0\,. \tag{4.12}$$

Since ω is ψ independent, we can write the above equation in the following form

$$(-)^{i'j'}\,\frac{\partial_r(h^i_\mu\,\omega_{ij'})}{\partial \psi^{i'}_x} \;-\; \frac{\partial_r(h^j_\mu\,\omega_{ji'})}{\partial \psi^{j'}_x} = 0 \tag{4.13}$$

It is convenient to use index free notation to appreciate the meaning of the

⋆ Here ghost number refers to the ghost number in the string field theory in the BV formalism. This is equal to the ghost number of the state in the first quantised theory minus 2.

above equations. We take

$$|h_\mu\rangle = |\Phi_i\rangle\, h^i_\mu = \sum_{N=2}^{\infty} \frac{1}{N!}\ {}_{(01\cdots N)}\langle\, \Gamma^{(N+1)}_\mu | \mathcal{S}_{0e}\rangle\, |\Psi\rangle_1 \cdots |\Psi\rangle_N\,, \tag{4.14}$$

where the object $\langle\, \Gamma^{(N+1)}_\mu|$ introduced in the above expansion is a bra in $(\widehat{\mathcal{H}}^*)^{\otimes(N+1)}$. Since $|h_\mu\rangle$ and $|\Psi\rangle$ are even, and $|\mathcal{S}\rangle$ is odd, $\langle\, \Gamma^{(N+1)}_\mu|$ must be odd. Its first state space, denoted as '0', has been contracted with $|\mathcal{S}_{0e}\rangle$, where e, for external, is the label of the resulting state in the left hand side of the equation. By definition, $\langle\, \Gamma^{(N+1)}_\mu|$ is symmetric on the state space labels 1 to N. It is a simple calculation using (2.32) and (2.36) to show that

$$h^i_\mu\, \omega_{ij} = -\sum_{N=2}^{\infty} \frac{1}{N!}\ {}_{(01\cdots N)}\langle\, \Gamma^{(N+1)}_\mu |\Psi\rangle_1 \cdots |\Psi\rangle_N |\Phi_j\rangle_0\,. \tag{4.15}$$

Back in (4.13) we find, that for each value of $N \geq 2$

$$(-)^{i'j'}\, \frac{\partial_r}{\partial\psi^{i'}} \Big(\langle\, \Gamma^{(N+1)}_\mu |\Psi\rangle_1 \cdots |\Psi\rangle_N |\Phi_{j'}\rangle_0 \Big) - \frac{\partial_r}{\partial\psi^{j'}} \Big(\langle\, \Gamma^{(N+1)}_\mu |\Psi\rangle_1 \cdots |\Psi\rangle_N |\Phi_{i'}\rangle_0 \Big) = 0\,, \tag{4.16}$$

which gives

$$\langle\, \Gamma^{(N+1)}_\mu|\, \Psi\rangle_1 \cdots |\Psi\rangle_{N-1} \Big(|\Phi_{i'}\rangle_N |\Phi_{j'}\rangle_0 - |\Phi_{i'}\rangle_0 |\Phi_{j'}\rangle_N \Big) = 0\,. \tag{4.17}$$

This equation implies that the bra $\langle\, \Gamma^{(N+1)}_\mu|$ must be symmetric between its zeroth state space, which was, a priori, on a different footing, and any other state space. Therefore $\langle\, \Gamma^{(N+1)}_\mu|$ must be a *totally* symmetric vertex.

To summarize, the conditions that the symplectic form is preserved are simply

$$D_\mu\, (\Gamma\,)\, \langle\, \omega\, | = 0\,, \quad \text{and,} \quad \langle\, \Gamma^{(N+1)}_\mu| \in \mathbf{S}\, \big(\, \widehat{\mathcal{H}}^{*\otimes(N+1)}\big)\,. \tag{4.18}$$

We now analyze the consequences of Eqn.(4.10). The ψ independent part of the diffeomorphism is given by

$$|\Phi_i\rangle\,\Gamma^{\,i}_{\mu}\;=|\,\widehat{\mathcal{O}}_\mu\,\rangle\equiv\;{}_0\langle\Gamma^{(1)}_\mu|\mathcal{S}_{0e}\rangle\,,\tag{4.19}$$

where we have introduced, in analogy to (4.14) a bra $\langle\Gamma^{(1)}_\mu|\in\widehat{\mathcal{H}}^*$. This equation indicates that the ψ independent part of the diffeomorphism is the classical solution representing the theory at $x+\delta x$, in the string field theory formulated around the background x. We can now rewrite Eqn.(4.10) more clearly as

$$\mathrm{D}_\mu\,S-\,\frac{\partial_r\,S}{\partial\,|\Psi\rangle}\,\big(\,|\widehat{\mathcal{O}}_\mu\rangle+|h_\mu\,\rangle\big)\;=0,\tag{4.20}$$

Let us now derive the explicit conditions arising from the above equation. Making use of (3.7) and (2.21) it follows that

$$\mathrm{D}_\mu S=\sum_{N=2}^{\infty}\frac{1}{N!}(D_\mu\langle V^{(N)}|)|\Psi\rangle_1\cdots|\Psi\rangle_N\,.\tag{4.21}$$

We then find that the terms quadratic in $|\Psi\rangle$ in Eqn.(4.20) give,

$$D_\mu(\Gamma)\,\big(\langle\omega_{12}|\,Q^{(2)}\,\big)\;=\;\langle V^{(3)}_{123}\,|\,\widehat{\mathcal{O}}_\mu\rangle_3\,.\tag{4.22}$$

and terms involving higher powers of $|\Psi\rangle$ give,

$$\langle\Gamma^{(N)}_\mu|\sum_{i=1}^{N}Q^{(i)}\;=\;-\sum_{m=3}^{N-1}\mathbf{S}\Big(\langle\Gamma^{(N-m+2)}_\mu|\,\langle V^{(m)}|\mathcal{S}\rangle\Big)+D_\mu(\Gamma)\langle V^{(N)}|-\langle V^{(N+1)}|\widehat{O}_\mu\rangle\,.\tag{4.23}$$

In the first term on the right hand side of the above equation $|\mathcal{S}\rangle$ sews any one of the $(N-m+2)$ legs of $\langle\Gamma^{(N-m+2)}|$ to any one of the m legs of $\langle V^{(m)}|$. This gives a bra in $(\widehat{\mathcal{H}}^*)^{\otimes N}$ which is symmetric in its first $N-m+1$ legs and also in the last $m-1$ legs. As it was the case below Eqn.(2.2), $\mathbf{S}$ denotes complete symmetrization of this bra. The total number of terms in $\mathbf{S}(\cdots)$ is $\binom{N}{m-1}$, which is the number of ways of splitting the N external labels into two sets, one to be attached to $\langle V|$, and one to be attached to $\langle\Gamma_\mu|$.

In the next three sections we shall see how to obtain the connection Γ_μ and symmetric $\langle\Gamma^{(N)}_\mu|$'s for $N \geq 3$ satisfying the conditions of background independence expressed in equations (4.18), (4.22) and (4.23). The diffeomorphism implementing background independence will be given by

$$|\Psi\rangle_{x+\delta x} = {}_{x+\delta x}\mathcal{I}_x \Big[\, |\Psi\rangle - \delta x^\mu \Big(\Gamma_\mu|\Psi\rangle + \sum_{\substack{N\geq 0 \\ N\neq 1}} \frac{1}{N!}\; {}_{(01\cdots N)}\langle\, \Gamma^{(N+1)}_\mu|\mathcal{S}_{0e}\rangle\, |\Psi\rangle_1 \cdots |\Psi\rangle_N \Big) \Big]\,. \tag{4.24}$$

This equation follows from our definitions (4.8), (4.14), and (4.19). Note that we have incorporated the ψ independent shift into the sum (the $N = 0$ term) but not the linear term. As we will see later, part of the connection Γ_μ will be incorporated into the sum. Since the left hand side is a string field at $x + \delta x$, while the input $|\Psi\rangle$ in the right hand side is a string field at x, we have included the "copying" operator ${}_x\mathcal{I}_y = \sum_i |\Phi_i(x)\rangle\langle\Phi^i(y)|$, which is the operator that copies a state from one state space to another one. The copying operator, acting on a vector in one state space, gives a vector in another state space, with the same value for all of its components.

5. Background Independence to Quadratic and Cubic Order

We have derived in the previous section the explicit conditions for background independence. The diffeomorphism relating the two theories must be symplectic, that is, it should preserve the BV structure, and we have observed that the linear part of the diffeomorphism has the index structure of a connection. In the present section we will study the first three conditions for background independence, namely

$$D_\mu\,(\Gamma\,)\,\langle\,\omega\,| = \;0, \tag{5.1}$$

$$D_\mu(\Gamma)\,\big(\langle\omega_{12}|\,Q^{(2)}\big) \;=\; \langle V^{(3)}_{123}\,|\,\widehat{\mathcal{O}}_\mu\rangle_3\,, \tag{5.2}$$

$$\langle\Gamma^{(3)}_\mu|\sum_{i=1}^{3} Q^{(i)} \;-\; D_\mu(\Gamma)\langle V^{(3)}| \;=-\langle V^{(4)}|\widehat{\mathcal{O}}_\mu\rangle\,. \tag{5.3}$$

We will show how to solve for the connection Γ_μ, and for the three string vertex $\langle\Gamma^{(3)}_\mu|$. It will then be a simple matter to extend the discussion to all orders. This will be done in the next two sections.

5.1 Finding the String Field Theory Connection Γ

We now show how to obtain a connection Γ_μ satisfying Eqns.(5.1)–(5.3). The middle equation, Eqn.(5.2), that fixes the covariant derivative of the BRST operator in terms of the three string vertex contracted with the marginal operator, will be our main input. As we will see, this equation does not determine the connection completely. Nevertheless, we will write a geometrical expression that solves equation (5.2). It will then be straightforward to see that equation (5.1) is satisfied. The third equation, which in fact, determines part of the connection could give rise to an inconsistency. We explain why this does not happen, and how this last equation can be used to solve for $\langle\Gamma^{(3)}_\mu|$.

The main observation that leads to the solution for Γ is that the canonical connection $\widehat{\Gamma}$ satisfies rather similar equations. We have (see (3.20), (3.23) and (3.18))

$$D_\mu\,(\,\widehat{\Gamma}\,)\,\langle\,\omega\,| = \;0, \tag{5.4}$$

$$D_\mu(\,\widehat{\Gamma}\,)\,\langle\omega_{12}|\,Q^{(2)} \;=\; \langle V'^{(3)}_{123}\,|\,\widehat{\mathcal{O}}_\mu\rangle_3\,, \tag{5.5}$$

$$D_\mu(\,\widehat{\Gamma}\,)\langle V^{(3)}| \;= 0. \tag{5.6}$$

These equations indicate that it should be simpler to try to find the difference between the two connections. We therefore introduce the operator one form $\Delta\Gamma_\mu$ as the difference between the connection Γ_μ and the canonical connection $\widehat{\Gamma}_\mu$

$$\Delta\Gamma_\mu = \Gamma_\mu - \widehat{\Gamma}_\mu. \tag{5.7}$$

Associated to the operator $\Delta\Gamma_\mu$ it is convenient to introduce the bra $\langle\Delta\Gamma_\mu| \in$

$\widehat{\mathcal{H}}^* \otimes \widehat{\mathcal{H}}^*$ as

$$\langle \Delta\Gamma_\mu \,(1,2)| = \langle\, \omega_{12}\,|\,(\Delta\Gamma_\mu)^{(1)} \,. \tag{5.8}$$

The difference between Eqns.(5.1) and (5.4) gives

$$\langle\omega_{12}|\Big(\Delta\Gamma_\mu^{(1)} + \Delta\Gamma_\mu^{(2)}\Big) = 0 \quad \rightarrow \quad \langle\Delta\Gamma_\mu\,(1,2)| - \langle\Delta\Gamma_\mu\,(2,1)| = 0\,, \tag{5.9}$$

which is simply the condition that $\langle\Delta\Gamma_\mu|$ is symmetric. The covariant constancy of $\langle\omega|$ implies that equation (5.2) can be written as

$$\langle\omega_{12}|\,\Big(\partial_\mu Q + [\Gamma_\mu\,, Q]\Big)^{(2)} \;=\; \langle V_{123}^{(3)}\,|\,\widehat{\mathcal{O}}_\mu\rangle_3\,, \tag{5.10}$$

Subtraction of the similar equation following from (5.5) gives

$$\langle\omega_{12}|\,\big[\Delta\Gamma_\mu\,, Q\big]^{(2)} \;=\; -\,\big(\,\langle V_{123}'^{(3)}| - \langle V_{123}^{(3)}|\,\big)\;|\widehat{\mathcal{O}}_\mu\rangle_3\;. \tag{5.11}$$

Making use of $\langle\omega_{12}|(Q^{(2)} + Q^{(1)}) = 0$, and (5.8) we find

$$\langle\,\Delta\Gamma_\mu\,|\,(\,Q^{(1)} + Q^{(2)}\,) = \big(\,\langle V_{123}'^{(3)}| - \langle V_{123}^{(3)}|\big)\;|\widehat{\mathcal{O}}_\mu\rangle_3\;. \tag{5.12}$$

We shall now show how to find a $\langle\Delta\Gamma_\mu|$ satisfying eqs.(5.9) and (5.12).

The right hand side of (5.12) shows the difference between two three string vertices. The two three-string-vertices represent two different points $\mathcal{V}_3$ and $\mathcal{V}_3'$ in the space $\widehat{\mathcal{P}}_3$. Since the vertices are contracted with the marginal operator, they effectively behave as two-string vertices, *i.e.* the right hand side of (5.12) belongs to $\widehat{\mathcal{H}}^* \otimes \widehat{\mathcal{H}}^*$. Both, $\langle V_{123}'^{(3)}|\widehat{\mathcal{O}}_\mu\rangle_3$ and $\langle V_{123}^{(3)}|\widehat{\mathcal{O}}_\mu\rangle_3$ are symmetric two string vertices. Let $\mathcal{B}_3$ be a path in $\widehat{\mathcal{P}}_3$ representing a symmetric homotopy between $\mathcal{V}_3$ and $\mathcal{V}_3'$, namely, every point of $\mathcal{B}_3$ is a three string vertex with a special puncture, and symmetric under the exchange of the other two punctures. This homotopy can be constructed as explained in §3.4. We therefore have

$$\partial \mathcal{B}_3 = \mathcal{V}_3' - \mathcal{V}_3. \tag{5.13}$$

We now claim that

$$\langle \Delta\Gamma_\mu | = -\int_{\mathcal{B}_3} \langle\, \Omega^{(1)3}\, | \widehat{\mathcal{O}}_\mu \rangle_3\,, \tag{5.14}$$

is the solution to Eqs.(5.9) and (5.12). Here $\langle \Omega^{(1)3}|$ (see §2.1) is a one form in $\widehat{\mathcal{P}}_3$. Since the interpolation path is a symmetric homotopy, $\langle \Delta\Gamma_\mu|$ satisfies (5.9). We now verify (5.12) is also satisfied

$$\begin{aligned}
\langle \Delta\Gamma_\mu|\,(Q^{(1)} + Q^{(2)}\,) &= -\int_{\mathcal{B}_3} \langle\, \Omega^{(1)3}\,| \sum_{i=1}^{3} Q^{(i)} | \widehat{\mathcal{O}}_\mu\rangle_3 = \int_{\mathcal{B}_3} \mathrm{d}\, \langle \Omega^{(0)3} |\, \widehat{\mathcal{O}}_\mu\, \rangle_3 \\
&= \int_{\partial\mathcal{B}_3} \langle \Omega^{(0)3} |\, \widehat{\mathcal{O}}_\mu\, \rangle_3 \;=\; \int_{\mathcal{V}_3'} \langle \Omega^{(0)3} |\, \widehat{\mathcal{O}}_\mu\, \rangle_3 \;-\; \int_{\mathcal{V}_3} \langle \Omega^{(0)3} |\, \widehat{\mathcal{O}}_\mu\, \rangle_3 \\
&= \langle\, V'^{(3)}_{123}\, | \widehat{\mathcal{O}}_\mu \rangle_3 - \langle\, V^{(3)}_{123}\, | \widehat{\mathcal{O}}_\mu \rangle_3\,.
\end{aligned} \tag{5.15}$$

where use was made of (2.19) and of (5.13). This proves that (5.12) is satisfied. (Since $\mathcal{V}_3$ ($\mathcal{V}_3'$) refers to a single point in $\widehat{\mathcal{P}}_3$, $\int_{\mathcal{V}_3}$ ($\int_{\mathcal{V}_3'}$) in the second line of the equation simply denotes that the integrand needs to be evaluated at that point.) We shall call $\mathcal{B}_3$ (and the corresponding state $\langle B^{(3)}| \equiv \int_{\mathcal{B}_3} \langle \Omega^{(1)3}|$) the interpolating three-string vertex.

5.1.1 Ambiguities in Γ_μ

The result in (5.14) implies an obvious ambiguity in the connection Γ_μ arising from the possibility of using two different homotopies $\mathcal{B}_3$ and $\mathcal{B}_3'$ between the initial and final three string vertices. Given two such homotopies we can find a two dimensional region $\mathcal{D}_3$ in $\widehat{\mathcal{P}}_3$ such that $\partial\mathcal{D}_3 = \mathcal{B}_3' - \mathcal{B}_3$. Therefore, if we let

$$\langle \Delta\Gamma_\mu | = -\int\limits_{\mathcal{B}_3} \langle\, \Omega^{(1)3}\, | \widehat{\mathcal{O}}_\mu \rangle_3\,, \quad \text{and} \quad \langle \Delta'\Gamma_\mu | = -\int\limits_{\mathcal{B}_3'} \langle\, \Omega^{(1)3}\, | \widehat{\mathcal{O}}_\mu \rangle_3\,, \tag{5.16}$$

we then have that

$$\begin{aligned} \langle \Delta'\Gamma_\mu | - \langle \Delta\Gamma_\mu | &= \Big(-\int\limits_{\mathcal{B}_3'} + \int\limits_{\mathcal{B}_3} \Big) \langle\, \Omega^{(1)3}\, | \widehat{\mathcal{O}}_\mu \rangle_3 = -\int\limits_{\mathcal{D}_3} \mathrm{d}\, \langle\, \Omega^{(1)3}\, | \widehat{\mathcal{O}}_\mu \rangle_3 \\ &= -\int\limits_{\mathcal{D}_3} \langle\, \Omega^{(2)3}\, | \widehat{\mathcal{O}}_\mu \rangle_3\, (Q^{(1)} + Q^{(2)})\,, \end{aligned} \tag{5.17}$$

where use was made of (2.19) in the last step. This ambiguity, of the form $\langle \eta\, |(Q^{(1)} + Q^{(2)})$, could be expected from (5.12) and reflects the fact that the field redefinition that maps S_x to $S_{x+\delta x}$ is determined only up to a gauge transformation.⋆

5.2 Constraints on Γ_μ and solving for $\langle \Gamma_\mu^{(3)} |$

Let us now turn to equation (5.3). Consider the second term in the left hand side and use (5.6) to write it as

$$D_\mu(\,\Gamma\,)\langle V^{(3)}| = D_\mu(\,\widehat{\Gamma}\, + \Delta\Gamma_\mu\,)\langle V^{(3)}| = -\sum_{i=1}^{3} \langle V^{(3)}|\Delta\Gamma_\mu^{(i)}\,. \tag{5.18}$$

Consider any one term in the final right hand side, for example, $\langle V_{123}^{(3)}\, |\Delta\Gamma_\mu^{(1)}$. Using (5.8) and (2.36), this term can be rewritten as $\langle V_{1'23}^{(3)}\, |\langle \Delta\Gamma_\mu(1,0)|\mathcal{S}_{01'}\rangle$, where we get the geometrical picture of twist-sewing the three string vertex to the vertex $\langle \Delta\Gamma_\mu|$. Back in the (5.18) we obtain

$$D_\mu(\,\Gamma\,)\langle V^{(3)}| = -\mathbf{S}\Big(\langle \Delta\Gamma_\mu | \langle V^{(3)} | \mathcal{S}\rangle\Big)\,, \tag{5.19}$$

where, as usual, $\mathbf{S}$ denotes symmetrization on the external legs (one out of $\langle \Delta\Gamma_\mu|$ and two out of $\langle V^{(3)}|$) requiring a total of three terms. Using this result in (5.3)

⋆ The ambiguity is not itself ambiguous, the right hand side of (5.17), by virtue of $Q^2 = 0$, does not depend on the chosen homotopy $\mathcal{D}_3$.

we obtain

$$\langle\Gamma_{\mu}^{(3)}|\sum_{i=1}^{3}Q^{(i)} = -\mathbf{S}\Big(\langle\Delta\Gamma_{\mu}|\langle V^{(3)}|\mathcal{S}\rangle\Big) - \langle V^{(4)}|\widehat{\mathcal{O}}_{\mu}\rangle\,. \tag{5.20}$$

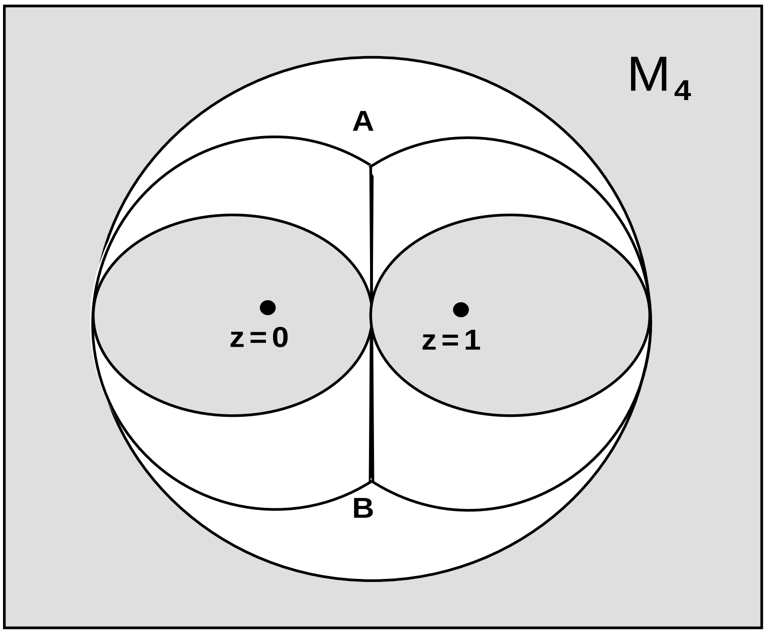
We now notice that, if we contract both sides of the above equation with three BRST invariant states, the left hand side of the equation vanishes identically. The equation then imposes a constraint on $\langle\Delta\Gamma_{\mu}|$. We must show that this constraint is satisfied identically by the expression for $\langle\Delta\Gamma_{\mu}|$ given in Eqn.(5.14). In fact, understanding why this constraint is satisfied, holds the key to solving the equation for the unknown $\langle\Gamma_{\mu}^{(3)}|$. Using the expression (5.14) we may rewrite Eqn.(5.20) as

$$\begin{aligned}\langle\Gamma_{\mu}^{(3)}|\sum_{i=1}^{3}Q^{(i)} &= \mathbf{S}\Big(\int_{\mathcal{B}_3}\langle\Omega^{(1)3}|\int_{\mathcal{V}_3}\langle\Omega^{(0)3}|\mathcal{S}\rangle\,|\widehat{\mathcal{O}}_{\mu}\rangle\Big) - \langle V^{(4)}|\widehat{\mathcal{O}}_{\mu}\rangle \\ &= \int_{\mathcal{V}_4'}\langle\Omega^{(0)4}|\widehat{\mathcal{O}}_{\mu}\rangle_4 - \int_{\mathcal{V}_4}\langle\Omega^{(0)4}|\widehat{\mathcal{O}}_{\mu}\rangle\,,\end{aligned} \tag{5.21}$$

where the region $\mathcal{V}_4'$ is given by

$$\mathcal{V}_4' = \mathbf{S}(\mathcal{B}_3\times\mathcal{V}_3)\,. \tag{5.22}$$

In deriving the second line of Eqn.(5.21) we have used the sewing property (2.25). The first term in the right hand side of (5.21) denotes an integral identical to the second one, but over a different region $\mathcal{V}_4'$. Note that in (5.22) the special puncture, where $\widehat{\mathcal{O}}_{\mu}$ is inserted, is always on $\mathcal{B}_3$ (which we define to be the fourth leg of the vertex $\mathcal{V}_4'$); the symmetrization involves only the other legs (yielding three terms). The set of surfaces in $\mathcal{V}_4'$ corresponds to three Feynman diagrams built by twist-sewing the interpolating three string vertex $\mathcal{B}_3$ and the three string vertex $\mathcal{V}_3$. We shall show that, remarkably, apart from the local coordinates, the set of surfaces $\mathcal{V}_4'$ coincides exactly with the set of surfaces in $\mathcal{V}_4$. This guarantees that, when we contract (5.21) with three physical states, represented by dimension zero primaries, the right hand side vanishes, as desired. When the representatives for the physical states are arbitrary, differing from dimension zero primaries by BRST exact states, the right hand side vanishes because of the additional feature that the boundaries of $\mathcal{V}_4'$ and $\mathcal{V}_4$ agree precisely (except for the local coordinate at the special puncture, – more on this later). These facts will allow us to calculate $\langle\Gamma_\mu^{(3)}|$. Let us therefore explain how $\mathcal{V}_4'$ turns out to be so special.



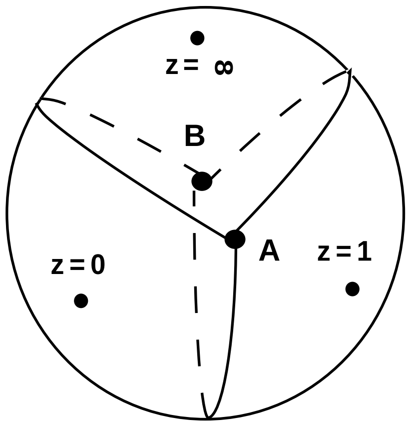


Figure 2. To the left we show the moduli space $\mathcal{M}_4$ of four punctured spheres. The three shaded regions correspond to the surfaces generated by the three Feynman graphs contributing to this amplitude. The three closed string vertex $\mathcal{V}_3$ is shown to the right. Since it is a three punctured sphere it can be identified with $\mathcal{M}_4$.

5.2.1 Restoring Jacobi Identity

It is useful to recall how the standard vertex $\mathcal{V}_4$ arises. If we take two three-string-vertices $\mathcal{V}_3$, and form the three standard Feynman graphs suitable for four-string amplitudes, we do not cover $\mathcal{M}_4$. With $\mathcal{M}_4$ thought itself as the complex plane (compactified at infinity) and punctured at $z = 0, 1$, and ∞, the three Feynman diagrams cover three nonoverlapping disks around $0, 1$, and ∞, as shown in Fig. 2. In this representation, any point z in the plane represents a four-punctured sphere, punctured at $0, 1, z$, and ∞. The region missed, not shaded in the figure, represents the surfaces in $\mathcal{V}_4$. This diagram does not tell, however, how to choose local coordinates on the punctures of the missing surfaces *inside* the region $\mathcal{V}_4$, although it does tell us how to choose them on the boundary of $\mathcal{V}_4$. The boundaries of the three disks, which coincide with the boundary of $\mathcal{V}_4$, correspond to twist-sewing of two three string vertices.

Let us now figure out what region of moduli space is obtained with the three diagrams of $\mathbf{S}(\mathcal{B}_3 \times \mathcal{V}_3)$. To this end it is easiest to examine the boundaries of the regions. Since twist sewing does not introduce boundaries, and $\mathcal{V}_3$ is a point, the boundaries arise from

$$\partial\big(\,\mathbf{S}(\mathcal{B}_3 \times \mathcal{V}_3)\big) = \mathbf{S}\left((\partial\mathcal{B}_3) \times \mathcal{V}_3\right) = \mathbf{S}\left(\mathcal{V}_3' \times \mathcal{V}_3\right) - \mathbf{S}(\mathcal{V}_3 \times \mathcal{V}_3)\,, \tag{5.23}$$

where use was made of (5.13). The boundary $-\mathbf{S}(\mathcal{V}_3 \times \mathcal{V}_3)$ corresponds to the configurations arising from twist-sewing of two three-string vertices. For a given Feynman diagram, they coincide with the boundary of a shaded disk in the figure. This boundary coincides precisely with $\partial\mathcal{V}_4$ as the recursion relation (2.2) indicates (the factor of one-half is absent because the special puncture breaks the symmetry leading to double counting in (2.2)). How about the boundary $\mathbf{S}(\mathcal{V}_3' \times \mathcal{V}_3)$? In each of the three Feynman diagrams contributing to this boundary, the special puncture of $\mathcal{V}_3'$ is not sewn. The sewn configuration can be viewed as a new copy of $\mathcal{V}_3$ coupling the two free legs of $\mathcal{V}_3$, and the free leg of $\mathcal{V}_3'$, with the special puncture of $\mathcal{V}_3'$ landing on the coordinate curve of the puncture that comes from

$\mathcal{V}_3'$. The twist makes the special puncture travel around that coordinate curve. For the other two Feynman graphs, the special puncture will land on the coordinate curves of the other two punctures of the final vertex $\mathcal{V}_3$. Since the three string vertex $\mathcal{V}_3$ is an overlap (the coordinate curves coincide two at a time), the three boundaries cancel out as the special puncture travels each piece of the coordinate curves in opposite directions. It is fun to see where in $\mathcal{M}_4$ this cancellation is taking place. For this purpose, we identify the three punctured sphere $\mathcal{V}_3$ with $\mathcal{M}_4$, by thinking of $z = 0, 1$, and ∞, as the three punctures of $\mathcal{V}_3$ (see Fig. 2). The coordinate curves are then nothing else but the familiar lines joining points A, and B of the figure. Therefore, each Feynman diagram of $\mathbf{S}(\mathcal{B}_3 \times \mathcal{V}_3)$ covers the region interpolating from the boundary of a shaded disk up to the coordinate curve. The three diagrams together cover the missing region $\mathcal{V}_4$. The cancellation of the boundaries $\mathbf{S}(\mathcal{V}_3' \times \mathcal{V}_3)$ is due to the tight fit of the three regions. Clearly the vertex $\mathcal{V}_3'$ has nice properties with respect to the vertex $\mathcal{V}_3$, the different channels agree. We refer to this as $\mathcal{V}_3'$ restoring a Jacobi identity to $\mathcal{V}_3$.

The above cancellation of boundaries, if it is to happen in $\widehat{\mathcal{P}}_4$, requires careful consideration of the local coordinate on the special puncture. The above argument proves that the boundaries cancel if we ignore the local coordinate on the special puncture. This is really all we need for the present application, since $\widehat{\mathcal{O}}_\mu$ is inserted there. Thus $\mathcal{V}_4'$ and $\mathcal{V}_4$ have the same boundaries, if we ignore the local coordinate at the special puncture. This shows that their projections to $\mathcal{M}_4$ have the same boundaries, and are therefore identical.* We will show in §6 that remarkably, it is simple to choose the coordinate on the special puncture in $\mathcal{V}_3'$ such that the above cancellation takes place fully off-shell (on $\widehat{\mathcal{P}}_4$). We will therefore have that the boundaries of $\mathcal{V}_4'$ and $\mathcal{V}_4$ agree strictly.

$\star$ Since $\mathcal{V}_4'$ need not be a section, when projecting down to $\mathcal{M}_4$ one should remember that surfaces produced more than once cancel out in pairs.

5.2.2 Determination of $\langle\Gamma^{(3)}_\mu|$

We shall now show how to construct $\langle\Gamma^{(3)}_\mu|$ satisfying (5.21). We give the construction for the case where local coordinates on the special puncture have been chosen so that, as a result, $\partial\mathcal{V}'_4 = \partial\mathcal{V}_4$ strictly (see §6). The case when we do not choose coordinates on the special puncture is treated exactly analogously.

Since $\mathcal{V}'_4$ and $\mathcal{V}_4$ are symmetric (in three legs) closed string vertices with common boundary, there is a symmetric homotopy between the two vertices keeping the boundary fixed (see §3.4). Let $\mathcal{B}_4$ denote the subspace generated by the homotopy. It then follows that

$$\partial\mathcal{B}_4 = \mathcal{V}'_4 - \mathcal{V}_4, \tag{5.24}$$

and then

$$\langle\Gamma^{(3)}_\mu| = -\int_{\mathcal{B}_4} \langle\Omega^{(1)4}|\widehat{\mathcal{O}}_\mu\rangle\,, \tag{5.25}$$

provides the desired solution of (5.21). Since $\mathcal{B}_4$ is a symmetric homotopy the bra $\langle\Gamma^{(3)}_\mu|$ is symmetric in its three state spaces, as required to have a symplectic diffeomorphism. We refer to $\mathcal{B}_4$ (and the corresponding state $\langle B^{(4)}| \equiv \int_{\mathcal{B}_4}\langle\Omega^{(1)4}|$) as the interpolating four-string vertex.

If no coordinate is chosen at the special puncture, one must introduce a projection $\pi_f : \widehat{\mathcal{P}}_n \to \widehat{\mathcal{P}}'_n$, with $\widehat{\mathcal{P}}'_n$ the space where the local coordinate around the special puncture is forgotten. As long as we integrate objects contracted with $|\widehat{\mathcal{O}}_\mu\rangle$, our integrals can be thought as integrals on $\widehat{\mathcal{P}}'$. Since the π_f projections of $\mathcal{V}'_4$ and $\mathcal{V}_4$ have the same boundary, and define symmetric vertices, there is a symmetric homotopy $\mathcal{W}_4 \in \widehat{\mathcal{P}}'_4$ interpolating between them, and $\langle\Gamma^{(3)}_\mu| = \int_{\mathcal{W}_4}\langle\Omega^{(1)4}|\widehat{\mathcal{O}}_\mu\rangle$ provides the desired solution.

6. New Vertices and Identities

In this section we will study the new geometrical structures relevant to the problem of background independence. We will develop the results beyond what is strictly necessary for the problem of background independence, as studied in the present paper. Our results suggest that the new family of vertices are relevant for studying shifts of the string field that are completely general, namely, do not correspond to dimension zero primary fields. We begin by a detailed analysis of the asymmetric three punctured sphere $\mathcal{V}'_3$. We show that, the condition of off-shell exchange symmetry between the two symmetric punctures of $\mathcal{V}'_3$, constrains the local coordinate on the asymmetric puncture enough, to guarantee *off-shell* consistency properties for $\mathcal{V}'_3$, with respect to the polyhedral closed string vertices. We then explain how $\mathcal{V}'_3$ can be used, in conjuction with the standard closed string vertices, to produce a good cover of the moduli spaces $\mathcal{M}_n$ of punctured spheres.

6.1 The Three Punctured Sphere $\mathcal{V}'_3$

In the previous section we made use of a special three punctured sphere $\mathcal{V}'_3$. This sphere, using the uniformizing coordinate z, is punctured at $z = 0$, with a local coordinate $z_1(z) = z$, and punctured at $z = \infty$, with local coordinate $z_2 = 1/z$. The asymmetric puncture is located at $z = 1$. This is the puncture where the marginal field is inserted, making the local coordinate z_3 at this puncture irrelevant. We now want to fix this coordinate in order to achieve the strongest possible identities.

In the same way as the three punctured sphere $\mathcal{V}_3$ is used to define the string field product $[\ ,\ ]$, we can use the yet-to-be fully specified sphere $\mathcal{V}'_3$ to define a new product $[\ ,\ ]'$ as follows

$$[A, B]' \equiv \langle {V'}^{(3)}_{123} |\, \mathcal{S}_{3e}\rangle \,|A\rangle_1 \,|B\rangle_2 \,. \tag{6.1}$$

Here the states A, and B, are inserted on the punctures at zero, and at infinity, respectively, and the product comes out of the asymmetric (third) puncture. The state space of the product is labelled by e, for external. It is natural to demand that this product be symmetric, or graded commutative,

$$[A, B]' = (-)^{AB}\,[B, A]'. \tag{6.2}$$

This condition will restrict the choice of coordinate at the third puncture.

We discussed in §5 the nice interplay between the three string vertex $\mathcal{V}_3$ of closed string field theory and $\mathcal{V}_3'$. That interplay can be summarized in the following relation

$$\Big[A_1,\, [A_2, A_3\,] \Big]'(-)^{A_1(1+A_3)} \;+\; \text{cyclic} \;=\; 0, \tag{6.3}$$

which was guaranteed by the pictures to hold on shell, that is, the left hand side vanishes when contracted with dimension zero primary states. It is clearly of interest to know if the coordinate z_3 of the sphere $\mathcal{V}_3'$ can be chosen so that the above identity holds strictly. The surprising result we will prove is that, the condition on z_3 required by graded commutativity ((6.2)), is actually sufficient to guarantee that (6.3) holds strictly. Equation (6.3) is a curious variation on the usual Jacobi identity of a Lie bracket. In a homotopy Lie algebra we have a bracket that satisfies a Jacobi identity weakly; here we have found another bracket with a curious compatibility with the original bracket. Furthermore, the product $[\;,\;]'$ also satisfies remarkable properties with respect to the higher string vertices. Again, if (6.2) holds, we can show that

$$\Big[A_1, [\,A_2, \cdots, A_n\,] \Big]' \pm \text{cyclic} = 0. \tag{6.4}$$

This identity (in fact, only its on-shell version) will be necessary to complete our proof of background independence to higher orders. The sign factors can be written out explicitly using Eqn.(4.14) of Ref.[11]. If all A's are even, all terms in (6.4) have a plus sign.

The geometrical version of the above identities is summarized by the expression

$$\mathbf{S}\big(\mathcal{V}_3' \times \mathcal{V}_M\big) = 0\,, \quad M \geq 3\,, \tag{6.5}$$

which indicates that the various subspaces of $\widehat{\mathcal{P}}_n$ arising from twist-sewing $\mathcal{V}_3'$ to the surfaces in $\mathcal{V}_M$ add up to zero. As usual, $\mathbf{S}$ symmetrizes over the free puncture of $\mathcal{V}_3'$ and the free punctures of $\mathcal{V}_M$ (the special puncture of $\mathcal{V}_3'$ is not symmetrized over).

As an aside, we observe that the $\mathcal{V}_3'$ sphere could be used to define another product

$$A \circ B = \langle V'^{(3)}_{123}|\, \mathcal{S}_{1e}\rangle\, |A\rangle_2\, |B\rangle_3\,, \tag{6.6}$$

where the first state in the product is inserted on the second puncture, the second state is inserted on the third puncture (the asymmetric puncture), and the product comes out of the first puncture. It is clear that this product is not symmetric

$$A \circ B \neq (-)^{AB} B \circ A. \tag{6.7}$$

This product may be eventually useful to write some new interactions for string fields (see §9), but will not be explored systematically here.

6.2 The Local Coordinate in the Asymmetric Puncture of $\mathcal{V}_3'$

The sphere $\mathcal{V}_3'$, using the uniformizing coordinate z, is punctured at $z = 0$, with a local coordinate $z_1(z) = z$, and punctured at $z = \infty$, with local coordinate $z_2 = 1/z$. The asymmetric puncture, whose local coordinate z_3 has not been fixed, is located at $z = 1$. It is natural to demand that the manifest exchange symmetry between the local coordinates at zero and infinity be respected by the asymmetric puncture. The map $z \to 1/z$, which exchanges z_1 and z_2, indeed leaves the asymmetric puncture at $z = 1$ fixed, but we need more. Points near this puncture must be mapped to points having the same z_3 coordinate, up to a common constant phase. Since the exchange map squared is the identity, the phase must be simply (± 1). Moreover, since the map, near $z = 1$, acts as a reflection, the local coordinate cannot remain unchanged and we must choose the minus sign. We therefore require

$$z_3(z) = -z_3(1/z). \tag{6.8}$$

It is not hard to parametrize the most general solution of the above equation. Since z_3 must vanish at $z = 1$, we can, without loss of generality, write

$$z_3(z) = f_O\left(\frac{z-1}{z+1}\right) \equiv h(z)\,, \tag{6.9}$$

where f_O is a function that vanishes at zero, and, is one to one inside a disk around the origin. Thus, for small y, $f_O(y) \propto y$. Eqn.(6.8) implies that the function f_O must be odd. We see that the condition of off-shell symmetry, under the exchange of the two symmetric punctures, leaves the coordinate at the asymmetric puncture fairly unconstrained.

Nevertheless, it is not possible to make a choice of f_O such that the vertex $\mathcal{V}_3'$ becomes cyclic, as it is simple to prove that the cyclic map $T(z) = 1/(1-z)$ cannot map the local coordinate z_1 into the local coordinate z_3. It should also be no surprise that there is no choice of f_O that makes this vertex fully symmetric on the three punctures. The simplest possible choice of f_O, namely, the identity function, makes $\mathcal{V}_3'$ a projective vertex. It should be clear from our analysis of three punctured spheres in §3.4, that any coordinate around the asymmetric puncture that could, in principle, be extended to a fully symmetric vertex by choosing related coordinates at zero and at infinity, is a consistent choice for the coordinate in the asymmetric puncture of $\mathcal{V}_3'$. This means, for example, that the asymmetric puncture could simply keep the local coordinate of the closed string vertex $\mathcal{V}_3$.

We now claim that off-shell exchange symmetry of $\mathcal{V}_3'$ (condition (6.8)), is all we need to get equations (6.3), and (6.4) to hold off-shell. As discussed in §5.2 the overlap nature of the three string vertex is responsible for the on-shell version

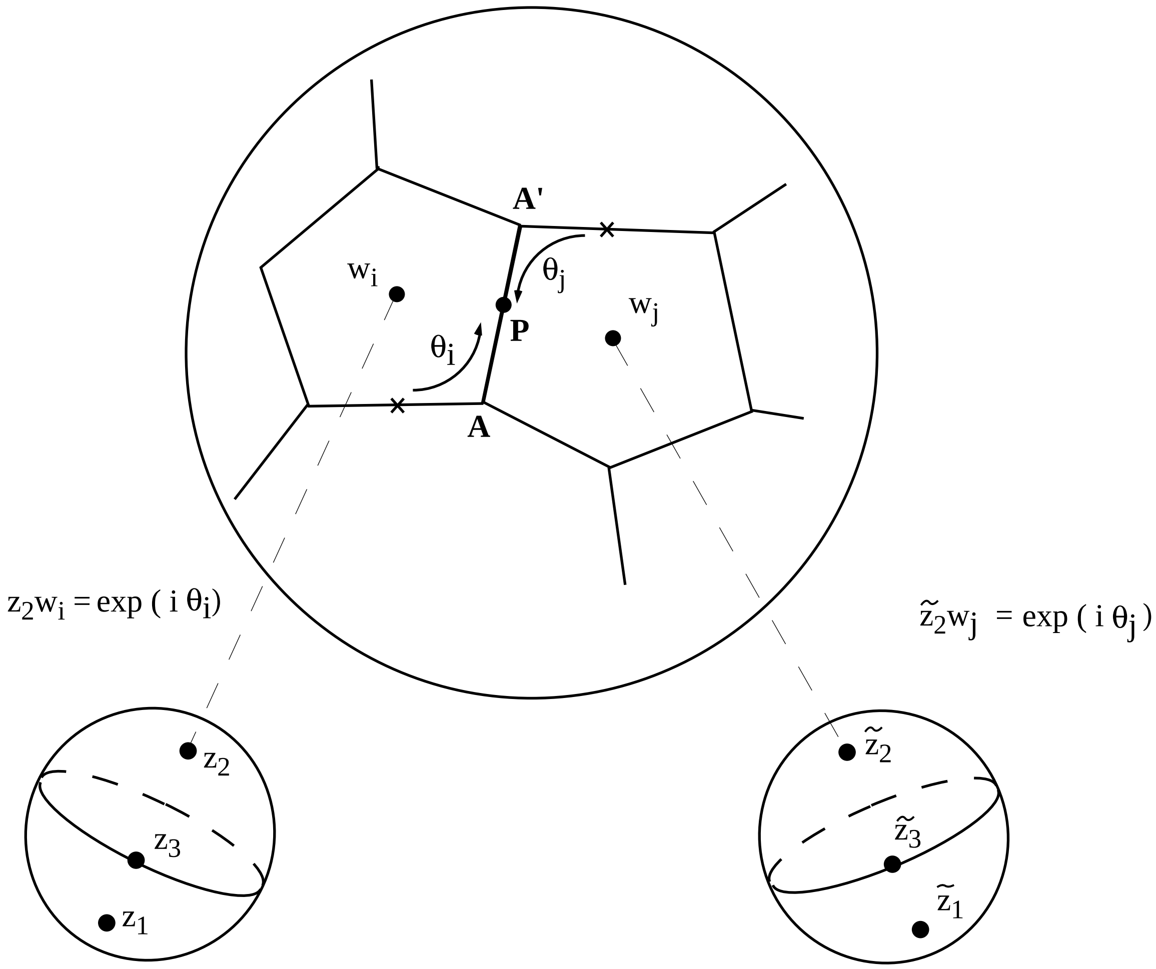


Figure 3. This figure is used to derive the conditions on the local coordinate at the asymmetric puncture of $\mathcal{V}_3'$ in order for equations (6.3), and (6.4) to hold off-shell. We explore what happens when the asymmetric puncture lands on the edge AA' of the polyhedron in two different ways.

of (6.3). Since all the higher string vertices, the restricted polyhedra, are also contact interactions, we will treat the general situation. We first need to make a preliminary observation. Consider an arbitrary polyhedron, as shown in Fig. 3 , and let w_i and w_j be the local coordinates in two adjacent faces which share the edge AA' of the polyhedron. Let us now show that, in a neighborhood of this edge, the coordinates have the transition function $w_i\, w_j = \exp(i\phi)$, for some phase ϕ. This need not be true for a generic contact interaction, but happens here because the closed string theory contact interactions arise from Jenkins-Strebel quadratic differentials [1]. We can think of the interaction represented by the polyhedron, as having semiinfinite cylinders (with metrics) of circumference 2π, whose boundaries are glued *isometrically* following the instructions of the polyhedron. This means that every edge of the polyhedron is parametrized by length, consistently, from the viewpoint of the two cylinders attaching to the edge. Consider a point $P \in AA'$ with local coordinate $w_i(P) = \exp(i\theta_i)$ and $w_j(P) = \exp(i\theta_j)$. From the point of view of the cylinder attached to the ith face, θ_i measures the distance of P from some fixed point along the boundary of the cylinder, whereas from the point of view of the cylinder attached to the jth face, θ_j measures the distance of P from some other fixed point along the boundary of the cylinder (see figure). Thus

$$\theta_i + \theta_j \equiv \phi\,, \tag{6.10}$$

is a constant, independent of the choice of P. It then follows that the transition function must be of the form

$$w_i(P')\, w_j(P') = \exp[i\phi]\,, \tag{6.11}$$

since it clearly works for $P' = P$; it works for $P' \in AA'$ by our arguments about the parametrization of the edge, and therefore by analyticity must work in a neighborhood of the edge.⋆ This proves our statement about transition functions.

Let us now address the issue of duality. We must therefore consider two configurations. In the first one, a sphere $\mathcal{V}_3'$, with coordinates (z_1, z_2, z_3), is sewn to the polyhedron via the relation

$$z_2\, w_i = \exp(i\theta_i)\,, \tag{6.12}$$

so that the special puncture of $\mathcal{V}_3'$ lands at P (since $z_2 = 1$, for the special puncture,

⋆ This is consistent with the fact that the quadratic differentials $(dw_1)^2/w_1^2$ and $(dw_2)^2/w_2^2$ defined for each disk, must agree on the edge.

and $w_i = \exp(i\theta_i)$, for P). In the second configuration, another sphere $\mathcal{V}_3'$, with coordinates $(\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3)$, is sewn to the polyhedron via the relation

$$\widetilde{z}_2\, w_j = \exp(i\theta_j)\,, \tag{6.13}$$

so that the special puncture of $\mathcal{V}_3'$ again lands at P (since $\widetilde{z}_2 = 1$, for the special puncture, and $w_j = \exp(i\theta_j)$, for P). In doing this we have made sure that the two configurations, corresponding to two polyhedra with an extra puncture on an edge, are conformally equivalent surfaces. We must now see if the local coordinates agree. The local coordinates corresponding to the faces of the polyhedra agree, as we verify next. In the first configuration, the coordinate z_1 ends up as the coordinate of the i-th face, while in the second configuration, w_i remains as the local coordinate. It follows from $z_2 z_1 = 1$, and Eqn.(6.12), that

$$w_i = z_1 \exp(i\theta_i)\,, \tag{6.14}$$

which shows that the two coordinates simply differ by a phase. Conversely, in the second configuration, the coordinate $\widetilde{z}_1$ ends up as the coordinate of the j-th face, while in the first configuration, w_j remains as the local coordinate. It follows from $\widetilde{z}_2 \widetilde{z}_1 = 1$, and Eqn.(6.13), that

$$w_j = \widetilde{z}_1 \exp(i\theta_j)\,, \tag{6.15}$$

again showing just a phase difference. The less clear point is that the local coordinates arising in the first configuration from z_3 (located at P) agrees with the local coordinate arising in the second configuration from $\widetilde{z}_3$. Consider a point P' near P and let $z_1(P')$ and $\widetilde{z}_1(P')$ be its coordinate in the first and second configurations respectively. It follows by multiplication of the last two equations that

$$w_i(P')\, w_j(P') = z_1(P')\, \widetilde{z}_1(P') \exp[i(\theta_i + \theta_j)] = z_1(P')\, \widetilde{z}_1(P') \exp[i\phi]\,, \tag{6.16}$$

and, using (6.11), it follows that

$$z_1(P')\,\widetilde{z}_1(P') = 1\,. \tag{6.17}$$

If we use the uniformizing coordinate z on $\mathcal{V}_3'$ (such that $z_1 = z$), we can write $z_3(p) = h(z(p)) = h(z_1(p))$. Using this language, it follows that the duality condition, namely, the requirement that the z_3 and $\widetilde{z}_3$ coordinates of P' agree, gives

$$z_3(P') = \pm\widetilde{z}_3(P') \quad \longrightarrow \quad h\,(z_1(P')) = \pm\, h\,(\widetilde{z}_1(P')\,. \tag{6.18}$$

It therefore follows, using (6.17), that we must have

$$h\,(z_1(P')) = \pm\, h(\,1/z_1(P')\,)\,, \tag{6.19}$$

which we recognize immediately as the condition (6.8) for off-shell symmetry of $\mathcal{V}_3'$ under the exchange of the first and second punctures (the plus sign cannot be realized). This completes our argument for off-shell duality. In this way we have established that (6.3) holds off-shell. Eqn.(6.4) also holds off-shell, since off-shell duality holds for any n-polyhedron, and therefore, for the full $\mathcal{V}_n$ space defining the product of $(n-1)$ string fields.

6.3 A Surprising Covering Result

It is a simple consequence of our previous discussion that we can produce a smooth covering of the moduli space $\mathcal{M}_4$ of four punctured spheres, with three string diagrams built by sewing, with the standard propagator, a vertex $\mathcal{V}_3$, and a vertex $\mathcal{V}_3'$. In the three string diagrams, the special puncture retains its label, and the labels of the other punctures are exchanged, as usual, to get a symmetric combination. The reason this covers smoothly moduli space is that, when the propagators collapse, the three string diagrams, by our duality argument, match precisely, and when the propagators are infinitely long, we get the proper degenerations.

We claim the following generalization of this result: A complete (smooth) cover of the moduli space of n-punctured Riemann surfaces is generated with all the standard tree-level Feynman diagrams that can be built using the vertices $\{\mathcal{V}_3 \cdots \mathcal{V}_{n-1}\}$, and the vertex $\mathcal{V}_3'$, with the condition that in each diagram, $\mathcal{V}_3'$ appears once. The label of the special puncture is always the same.

A proof by induction would only require to show that all (non-degenerate) boundaries of the Feynman graphs match, since, the good cover of the lower dimensional moduli spaces guarantees we cannot miss any degeneration. We will only sketch an argument for this matching of boundaries. In building the moduli space of n-punctured spheres, the Feynman diagrams with lowest number of propagators are $\mathbf{S}(\mathcal{V}_{n-1} \cdot \mathcal{V}_3')$ which have one propagator, indicated by the dot between the vertices. The (non-degenerate) boundaries of this graph arise from the collapsed propagator, yielding $\mathbf{S}(\mathcal{V}_{n-1} \times \mathcal{V}_3') = 0$ (Eqn.(6.5)), and, from the standard boundary of the vertex $\mathcal{V}_{n-1}$, yielding terms of the form $\mathcal{V}_p \times \mathcal{V}_q \cdot \mathcal{V}_3'$. These boundaries, cancel with the boundaries of the Feynman diagrams $\mathcal{V}_p \cdot \mathcal{V}_q \cdot \mathcal{V}_3'$ arising from the collapse of the first propagator. When the second propagator in this graph collapses, it yields the boundaries $\mathcal{V}_p \cdot \mathcal{V}_q \times \mathcal{V}_3'$, which do not cancel out by themselves. They would, if the term was of the form $\mathcal{V}_p \cdot (\mathcal{V}_q \times \mathcal{V}_3')$, where the parenthesis indicates that the propagator connects to either a leg in $\mathcal{V}_q$, or a leg in $\mathcal{V}_3'$. We therefore need to introduce the Feynman graphs of the type $\mathcal{V}_p \cdot \mathcal{V}_3' \cdot \mathcal{V}_q$, which provide the missing boundaries, when a propagator collapses. The graphs $\mathcal{V}_p \cdot \mathcal{V}_q \cdot \mathcal{V}_3'$, together with $\mathcal{V}_p \cdot \mathcal{V}_3' \cdot \mathcal{V}_q$, amount to all graphs with two propagators, and one $\mathcal{V}_3'$ vertex. In these graphs, the boundaries arising from the vertices, would then be matched with boundaries from all graphs with three propagators, involving one $\mathcal{V}_3'$ vertex. This goes on until all boundaries are accounted for.

In retrospect, we can argue that this result is a consequence of the background independence of string field theory. If $S(|\Psi\rangle)$ denotes the string field theory action at the point x, then,

$$\widehat{S}(\,|\Psi\rangle\,) \;\equiv\; S(\,|\Psi\rangle\,) + \delta x^{\mu} \langle \frac{\partial S}{\partial |\Psi\rangle} | \widehat{\mathcal{O}}_{\mu} \rangle \ , \tag{6.20}$$

denotes the string field theory action obtained when the string field is shifted by an amount $\delta x^\mu|\widehat{\mathcal{O}}_\mu\rangle$. In this action we use the state space $\widehat{\mathcal{H}}_x$. On the other, we can consider the string field action based on $\widehat{\mathcal{H}}_{x+\delta x}$, which is of the form $\widetilde{S}(\,|\Psi\rangle) = \sum_{N=2}\frac{1}{N!}\langle V^{(N)}|\Psi\rangle\cdots|\Psi\rangle$, where both the bras, and the kets, refer to objects in $\widehat{\mathcal{H}}_{x+\delta x}$. Since contraction is invariant under parallel transport, we can parallel transport both the bras, and the kets, from $\widehat{\mathcal{H}}_{x+\delta x}$ to $\widehat{\mathcal{H}}_x$ without changing the value of the action. This can be done with any connection. Using the connection $\widehat{\Gamma}_\mu$, we have

$$\widetilde{S}(|\Psi\rangle) = \sum_{N=2}\frac{1}{N!}\Big(\langle V^{(N)}| + \delta x^\mu D_\mu(\,\widehat{\Gamma}\,)\langle V^{(N)}|\Big)\big(\,|\Psi\rangle + \delta x^\mu \widehat{\Gamma}_\mu|\Psi\rangle\,\big)^N\,, \qquad (6.21)$$

where we made use of standard parallel transport formulas (see [20], §2.4). All objects in this right hand side refer to $\widehat{\mathcal{H}}_x$. We now do the field redefinition $|\widetilde{\Psi}\rangle \equiv |\Psi\rangle + \delta x^\mu\widehat{\Gamma}_\mu|\Psi\rangle$, and, use (3.18) and (3.23), to find that $\widetilde{S}$ takes the form

$$\widetilde{S}(|\Psi\rangle) = S(\,|\widetilde{\Psi}\rangle\,) + \tfrac{1}{2}\,\delta x^\mu\,\langle V'^{(3)}_{123}|\widehat{\mathcal{O}}_\mu\rangle_3|\widetilde{\Psi}\rangle_2|\widetilde{\Psi}\rangle_1\,. \qquad (6.22)$$

While formulated on $\widehat{\mathcal{H}}_x$, it still represents string field theory at the background $x+\delta x$. If string field theory is background independent, the actions in (6.20) and (6.22) must be related via a field redefinition involving linear and higher order terms. As a result, the on-shell S-matrix elements calculated from the two theories must be the same. The order δx^μ terms of the S-matrix elements in the theory described by the first action can be computed by treating the order δx^μ term in the action as perturbation, and are given (including the combinatoric factors) by the ordinary Feynman diagrams of the unperturbed theory with one of the legs being $|\widehat{\mathcal{O}}_\mu\rangle$.* This is known to cover the moduli space of punctured spheres. On

$\star$ The calculation is more complicated if there are divergences associated with the insertion of $\widehat{\mathcal{O}}_\mu$ on the external leg; in this case, one has to take into account wave-function renormalization of the external state. For the purpose of this argument we can choose $\widehat{\mathcal{O}}_\mu$ and the external states in such a way that no such divergence is present.

the other hand, using the same reasoning, we see that the order δx^μ terms of the S-matrix elements calculated from the second action are given by Feynman diagrams constructed from the ordinary string vertices with one and only one insertion of the vertex $\langle V'^{(3)}|$, with the state $|\widehat{\mathcal{O}}_\mu\rangle$ inserted at the asymmetric puncture. Thus, these diagrams must also cover the moduli space fully. This establishes the covering result.

7. Constructing The Complete Diffeomorphism

We have constructed so far the first few pieces of the diffeomorphism establishing background independence. Indeed, in §5, by defining $\Gamma_\mu = \Delta\Gamma_\mu + \widehat{\Gamma}_\mu$, with $\widehat{\Gamma}_\mu$ the canonical connection, we were able to solve for the one-form $\Delta\Gamma_\mu$, and hence for Γ_μ. We also solved for the one-form $\langle\Gamma^{(3)}_\mu|$. The purpose of the present section is to find the solution for all the higher bras $\langle\Gamma^{(N)}_\mu|$, with $N \geq 4$. It is useful to define

$$\langle\Gamma^{(2)}_\mu| \equiv \langle\Delta\Gamma_\mu|\,, \tag{7.1}$$

since the one-form $\Delta\Gamma_\mu$ is really on the same footing as any bra $\langle\Gamma^{(N)}_\mu|$. With this definition, the diffeomorphism implementing background independence, as written in (4.24), becomes

$$|\Psi\rangle_{x+\delta x} = {}_{x+\delta x}\mathcal{I}_x\Big[\,|\Psi\rangle - \delta x^\mu\Big(\,\widehat{\Gamma}_\mu|\Psi\rangle + \sum_{N\geq 0}\frac{1}{N!}\,{}_{(01\cdots N)}\big\langle\,\Gamma^{(N+1)}_\mu|\mathcal{S}_{0e}\big\rangle\,|\Psi\rangle_1\cdots|\Psi\rangle_N\Big)\Big]\,. \tag{7.2}$$

It is instructive to interpret the above diffeomorphism as the result of a canonical transformation, followed by parallel transport. It is clear from (2.42), and the symmetry of $\langle\Gamma^{(N)}_\mu|$ that

$$\sum_{N\geq 0}\frac{1}{N!}\,{}_{(01\cdots N)}\big\langle\,\Gamma^{(N+1)}_\mu|\mathcal{S}_{0e}\big\rangle\,|\Psi\rangle_1\cdots|\Psi\rangle_N \;=\; -\Big\{\,\mathbf{U}_\mu\,,\,|\Psi\rangle\,\Big\}\;, \tag{7.3}$$

where the generator $\mathbf{U}_\mu$ of the canonical transformation is given by

$$\mathbf{U}_\mu = \sum_{N\geq 1} \frac{1}{N!}\ {}_{(1\cdots N)}\langle\, \Gamma_\mu^{(N)}|\Psi\rangle_1 \cdots |\Psi\rangle_N\,. \tag{7.4}$$

It then follows that

$$|\Psi\rangle_{x+\delta x} = {}_{x+\delta x}\mathcal{I}_x \Big[\ |\Psi\rangle\ +\ \delta x^\mu \Big\{ \mathbf{U}_\mu\,,\, |\Psi\rangle \Big\} - \delta x^\mu \cdot \widehat{\Gamma}_\mu\, |\Psi\rangle\ \ \Big]\,, \tag{7.5}$$

which shows that the string field at $x + \delta x$ is obtained from the string field at x by first performing a canonical transformation with generator $\mathbf{U}_\mu$, and then performing parallel transport with the canonical flat connection $\widehat{\Gamma}$.

Back to our central topic in this section, the background independence conditions can also be written more clearly with the help of (7.1). Starting from $D_\mu(\widehat{\Gamma})\langle V^{(N)}| = 0$, and following the same steps we performed at the beginning of §5.2, we find

$$D_\mu(\,\Gamma\,)\langle V^{(N)}| = -\mathbf{S}\Big(\langle\Gamma_\mu^{(2)}|\langle V^{(N)}|\mathcal{S}\rangle\Big)\,. \tag{7.6}$$

This relation, back in the background independence condition (4.23), gives us

$$\langle\Gamma_\mu^{(N)}|\sum_{i=1}^{N} Q^{(i)}\ =\ -\sum_{m=3}^{N}\mathbf{S}\Big(\langle\Gamma_\mu^{(N-m+2)}|\,\langle V^{(m)}|\mathcal{S}\rangle\Big)\ -\langle V^{(N+1)}|\widehat{O}_\mu\rangle\,. \tag{7.7}$$

The solutions we have found so far can be written in the form

$$\langle\Gamma_\mu^{(2)}| = -\int_{\mathcal{B}_3}\langle\,\Omega^{(1)3}\,|\widehat{\mathcal{O}}_\mu\rangle_3, \quad \partial\mathcal{B}_3 = \mathcal{V}_3' - \mathcal{V}_3. \tag{7.8}$$

$$\langle\Gamma_\mu^{(3)}| = -\int_{\mathcal{B}_4}\langle\Omega^{(1)4}|\widehat{\mathcal{O}}_\mu\rangle_4\,, \quad \partial\mathcal{B}_4 = \mathcal{V}_4' - \mathcal{V}_4, \tag{7.9}$$

where $\mathcal{V}_4' = \mathbf{S}(\mathcal{B}_3 \times \mathcal{V}_3)$ satisfies $\partial\mathcal{V}_4' = \partial\mathcal{V}_4$. This suggests that the higher order solutions take the form

$$\langle \Gamma_\mu^{(N)} | = - \int\limits_{\mathcal{B}_{N+1}} \langle \Omega^{(1)N+1} | \mathcal{O}_\mu \rangle_{N+1}, \quad \partial \mathcal{B}_{N+1} = \mathcal{V}'_{N+1} - \mathcal{V}_{N+1}\,, \tag{7.10}$$

where $\mathcal{B}_{N+1}$ must be a symmetric (in first N legs) homotopy between $\mathcal{V}_{N+1}$ and some vertex $\mathcal{V}'_{N+1}$. The symmetric homotopy is required in order to to have a symplectic diffeomorphism. We can derive what the vertex $\mathcal{V}'_{N+1}$ should be, by considering condition (7.7), together with our ansatz (7.10). We find

$$- \int\limits_{\mathcal{B}_{N+1}} \langle \Omega^{(1)N+1} | \sum_{i=1}^{N+1} Q^{(i)} \;=\; \sum_{m=3}^{N} \mathbf{S} \int\limits_{\mathcal{B}_{N-m+3}} \langle \Omega^{(1)N-m+3} | \int\limits_{\mathcal{V}_m} \langle \Omega^{(0)m} | \mathcal{S} \rangle \;-\; \int\limits_{\mathcal{V}_{N+1}} \langle \Omega^{(0)N+1} |\,, \tag{7.11}$$

where we peeled off the common state $|\mathcal{O}_\mu\rangle$.* Here $\mathbf{S}$ denotes symmetrization in all the free legs of $\mathcal{B}_{N-m+3}$, except for the $(N-m+3)$-th leg (where $\mathcal{O}_\mu$ is to be attached), and all the free legs of $\mathcal{V}_m$. $|\mathcal{S}\rangle$ sews one of the first $N-m+2$ legs of $\mathcal{B}_{N-m+3}$, with one of the legs of $\mathcal{V}_m$. Making use of (2.19), and (2.25), we rewrite the above equation as

$$\int\limits_{\partial \mathcal{B}_{N+1}} \langle \Omega^{(0)N+1} | = \sum_{m=3}^{N} \int\limits_{\mathbf{S}(\mathcal{B}_{N-m+3} \times \mathcal{V}_m)} \langle \Omega^{(0)N+1} | \;-\; \int\limits_{\mathcal{V}_{N+1}} \langle \Omega^{(0)N+1} |\,, \tag{7.12}$$

where the three integrals have a common integrand. It follows from this equation that

$$\partial \mathcal{B}_{N+1} = \sum_{m=3}^{N} \mathbf{S}(\mathcal{B}_{N-m+3} \times \mathcal{V}_m) - \mathcal{V}_{N+1}\,. \tag{7.13}$$

Upon comparison with (7.10), we conclude that $\mathcal{V}'_{N+1}$ must be given by

$$\mathcal{V}'_{N+1} = \sum_{m=3}^{N} \mathbf{S}\Big(\mathcal{B}_{N-m+3} \times \mathcal{V}_m \Big) \;=\; \mathbf{S}\Big(\mathcal{B}_N \times \mathcal{V}_3 + \cdots + \mathcal{B}_3 \times \mathcal{V}_N \Big). \tag{7.14}$$

This is a simple expression. It says that the new vertex $\mathcal{V}'$, to any order, is obtained by twist-sewing an interpolating vertex $\mathcal{B}$ of lower order, with an old vertex $\mathcal{V}$, in

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⋆ Note that (7.11) implies (7.7), but is a stronger constraint than (7.7).

all possible ways. While this definition always makes sense, Eqn.(7.10) implies a strong constraint. Since $\partial^2 = 0$, we must have that

$$\partial \mathcal{V}'_{N+1} = \partial \mathcal{V}_{N+1} \,. \tag{7.15}$$

Indeed, if this property holds, we can always find a symmetric interpolating vertex $\mathcal{B}_{N+1}$. Therefore, our problem is to show that $\mathcal{V}'$, as defined in (7.14), satisfies (7.15).

7.0.1 Proving the Coincidence of Boundaries.

We shall carry out the proof via induction. Let us assume that we have found new vertices $\mathcal{V}'_3, \cdots, \mathcal{V}'_M$ and the corresponding interpolating vertices $\mathcal{B}_3, \cdots, \mathcal{B}_M$, such that

$$\begin{aligned} \partial \mathcal{B}_n &= \mathcal{V}'_n - \mathcal{V}_n \,, \\ \partial \mathcal{V}'_n &= \partial \mathcal{V}_n \,, \\ \mathcal{V}'_n &= \sum_{m=3}^{n-1} \mathbf{S}(\mathcal{B}_{n-m+2} \times \mathcal{V}_m) \,, \end{aligned} \tag{7.16}$$

for all n in the interval $3 \leq n \leq M$. We then want to show that $\mathcal{V}'_{M+1}$, defined as

$$\mathcal{V}'_{M+1} = \sum_{m=3}^{M} \mathbf{S}(\mathcal{B}_{M-m+3} \times \mathcal{V}_m) \,, \tag{7.17}$$

satisfies $\partial \mathcal{V}'_{M+1} = \partial \mathcal{V}_{M+1}$. This would allow us to define $\mathcal{B}_{M+1}$, and continue the recursion procedure.

We must simply compute the boundary of $\mathcal{V}'_{M+1}$. From eqn.(7.17) we get,

$$\partial \mathcal{V}'_{M+1} = \sum_{m=3}^{M} \mathbf{S}(\partial \mathcal{B}_{M-m+3} \times \mathcal{V}_m) + \sum_{m=4}^{M} \mathbf{S}(\mathcal{B}_{M-m+3} \times \partial \mathcal{V}_m) \tag{7.18}$$

using $\partial \mathcal{V}_3 = 0$. With the help of the first equation in (7.16), and the expression

(2.2) for $\partial\mathcal{V}_m$ we rewrite the above equation as

$$\begin{aligned}\partial\mathcal{V}'_{M+1} = &-\sum_{m=3}^{M}\mathbf{S}(\mathcal{V}_{M-m+3}\times\mathcal{V}_m) + \sum_{m=3}^{M}\mathbf{S}(\mathcal{V}'_{M-m+3}\times\mathcal{V}_m)\\ &-\sum_{m=4}^{M}\sum_{p=3}^{m-1}\mathbf{S}(\mathcal{B}_{M-m+3}\times\mathcal{V}_{m-p+2}\times\mathcal{V}_p),\end{aligned}\tag{7.19}$$

where we have adopted the convention that $\mathcal{A}\times\mathcal{B}\times\mathcal{C}$ denotes set of surfaces obtained by twist-sewing one puncture of $\mathcal{A}$ with one puncture of $\mathcal{B}$, and another puncture of $\mathcal{B}$ with a puncture of $\mathcal{C}$. According to this convention, the last term of the above equation only contains terms where $\mathcal{B}_{M-m+3}$ is sewed to $\mathcal{V}_{m-p+2}$, but not to $\mathcal{V}_p$. This is responsible for the absence of a factor of (1/2) present in Eqn.(2.2). The first term of the right hand side of Eqn.(7.19) is recognized to be $\partial\mathcal{V}_{M+1}$, keeping in mind that $\mathbf{S}$ symmetrizes all but the $(M-m+3)$-th leg of $\mathcal{V}_{M-m+3}$ in this term, thereby accounting for the missing factor of (1/2). We therefore have

$$\partial\mathcal{V}'_{M+1} - \partial\mathcal{V}_{M+1} = \sum_{p=3}^{M}\mathbf{S}(\mathcal{V}'_{M-p+3}\times\mathcal{V}_p) - \sum_{m=4}^{M}\sum_{p=3}^{m-1}\mathbf{S}(\mathcal{B}_{M-m+3}\times\mathcal{V}_{m-p+2}\times\mathcal{V}_p).\tag{7.20}$$

We must now show that the right hand side of the above equation vanishes. The $p=M$ term in the first sum on the right hand side gives $\mathbf{S}(\mathcal{V}'_3\times\mathcal{V}_M)$, and vanishes by Eqn.(6.5). The other terms in this sum can be rewritten using the third of Eqn.(7.16), for $n\leq M$,

$$\sum_{p=3}^{M-1}\sum_{m=p+1}^{M}\mathbf{S}\Big(\,(\mathcal{B}_{M-m+3}\times\mathcal{V}_{m-p+2}\,)\times\mathcal{V}_p\Big) = \sum_{m=4}^{M}\sum_{p=3}^{m-1}\mathbf{S}\Big(\,(\mathcal{B}_{M-m+3}\times\mathcal{V}_{m-p+2}\,)\times\mathcal{V}_p\Big)\,,\tag{7.21}$$

where the extra parenthesis indicate that $\mathcal{V}_p$ is sewn to both legs that come out of $\mathcal{B}_{M-m+3}$, and legs that come out of $\mathcal{V}_{m-p+2}$. Thus $\partial\mathcal{V}'_{M+1}-\partial\mathcal{V}_{M+1}$ is given by the difference between this term and the last term of Eqn.(7.20). This gives

$$\partial\mathcal{V}'_{M+1} - \partial\mathcal{V}_{M+1} = \sum_{m=4}^{M}\sum_{p=3}^{m-1} \mathbf{S}\big(\mathcal{V}_{m-p+2} \times \mathcal{B}_{M-m+3} \times \mathcal{V}_p\big)\,, \tag{7.22}$$

We claim this term vanishes, in fact, each term corresponding to a fixed value of m vanishes:

$$\sum_{p=3}^{m-1} \mathbf{S}\big(\mathcal{V}_{m-p+2} \times \mathcal{B}_{M-m+3} \times \mathcal{V}_p\big) = 0\,. \tag{7.23}$$

Note that all terms in this equation involve the same vertex $\mathcal{B}$. This vertex is symmetric under the exchange of any of its state spaces (except the one where $\widehat{\mathcal{O}}_\mu$ is to be inserted, which cannot be used to sew into the $\mathcal{V}$ vertices). The above relation can be rewritten as

$$\frac{1}{2}\sum_{p=3}^{m-1} \mathbf{S}\Big(\,\mathcal{V}_{m-p+2} \times \mathcal{B}_{M-m+3} \times \mathcal{V}_p \;\;+\;\; \mathcal{V}_p \times \mathcal{B}_{M-m+3} \times \mathcal{V}_{m-p+2}\Big) = 0\,. \tag{7.24}$$

This equation holds because, for each value of p, each of the two terms in the above expression produces the same subspace of $\mathcal{P}_{M+1}$, (this is manifest due to the symmetry of $\mathcal{B}$), but with opposite orientation. A way to show the orientations are opposite is to write the expression for the corresponding string amplitudes and to check they cancel. The amplitude is written as follows

$$\begin{aligned}\langle B^{(M-m+3)}(1,2,\cdots)|\,\Big(\;&\langle V^{(m-p+2)}(1',\cdots)|\langle V^{(p)}(2',\cdots)|\mathcal{S}_{11'}\rangle|\mathcal{S}_{22'}\rangle\\ &+\langle V^{(p)}(1',\cdots)|\langle V^{(m-p+2)}(2',\cdots)|\mathcal{S}_{11'}\rangle|\mathcal{S}_{22'}\rangle\Big)\end{aligned} \tag{7.25}$$

where

$$\langle B^{(N)}| = \int_{\mathcal{B}_N} \langle\Omega^{(1)N}|\,. \tag{7.26}$$

Eqn.(7.25) can be rewritten as

$$\begin{aligned}\langle B^{(M-m+3)}(1,2,\cdots)| \Big(&\langle V^{(m-p+2)}(1',\cdots)|\langle V^{(p)}(2',\cdots)| \\ &+ \langle V^{(m-p+2)}(2',\cdots)|\langle V^{(p)}(1',\cdots)| \Big) |\mathcal{S}_{11'}\rangle|\mathcal{S}_{22'}\rangle \,,\end{aligned} \tag{7.27}$$

which vanishes identically since the product of sewing kets is antisymmetric under the exchanges $1 \leftrightarrow 2$, $1' \leftrightarrow 2'$, while the rest of the expression is manifestly symmetric (recall that all external legs, with the exception of the last one in $\mathcal{B}$, are symmetrized). This proves the desired result, and verifies the consistency of our construction of the full nonlinear diffeomorphism implementing background independence to all orders in the string coupling constant.

8. Backgrounds a Finite Distance Apart

We have proven in §5–§7 the existence of a fully nonlinear infinitesimal diffeomorphism relating string field theories formulated around infinitesimally close conformal field theories. This diffeomorphism established local background independence of closed string field theory. If we have a CFT theory space, it is natural to ask if this proof of local background independence can be extended to the case when the two conformal theories are a finite distance apart in theory space.* The finite distance diffeomorphism would be obtained by integrating the infinitesimal diffeomorphism along a path in theory space joining the two conformal theories [15,18]. There are two aspects to the question of existence of a finite distance diffeomorphism. The first is a formal one. Are there local integrability conditions that must be satisfied in order for the diffeomorphism to be path independent? We show here that there are no such integrability conditions. The second hinges on the fact that we are dealing with an infinite dimensional vector bundle. Is it possible to integrate the diffeomorphism without getting infinities? We will not deal with this question in detail, but will argue that finite distance diffeomorphisms are expected to exist.

$\star$ Distance can be defined using the Zamolodchikov metric.

8.1 Composition Properties

We begin by noting that the diffeomorphism $F_{y,x} : \widehat{\mathcal{H}}_x \to \widehat{\mathcal{H}}_y$ that relates the string master actions S_x and S_y, and the symplectic forms ω_x and ω_y, arising from two different conformal field theories, is ambiguous due to the presence of gauge (and possibly other global) symmetries of the action. A symmetry $\mathbf{g}_x : \widehat{\mathcal{H}}_x \to \widehat{\mathcal{H}}_x$ of the string field theory at x, is a transformation of the string field leaving the action, and the symplectic form invariant, namely $S_x = \mathbf{g}_x^* \, S_x$ and $\omega_x = \mathbf{g}_x^* \, \omega_x$.† It follows that whenever $F_{x,y}$ is a diffeomorphism that relates string theory at x and at y, so is $\mathbf{g}_y \circ F_{y,x} \circ \mathbf{g}_x$. This gives us the equivalence relation:

$$F_{y,x} \approx \mathbf{g}_y \circ \, F_{y,x} \, \circ \, \mathbf{g}_x \, . \tag{8.1}$$

In fact, it is sufficient to consider gauge transformations on the left, since

$$F_{y,x} \, \circ \, \mathbf{g}_x = \bigl(F_{y,x} \, \circ \, \mathbf{g}_x \circ \, F_{x,y} \bigr) \circ \, F_{y,x} \, . \tag{8.2}$$

In the above equation the map in parenthesis is a symmetry transformation at y since it is a diffeomorphism from $\widehat{\mathcal{H}}_y$ preserving S_y and ω_y. This implies that any symmetry transformation applied before performing the diffeomorphism can be written as a symmetry transformation applied after performing the diffeomorphism.

By definition, the diffeomorphisms establishing the equivalence of string field theories at different points in CFT theory space must satisfy a composition law. Given three points x, y, and z, we must have

$$F_{z,x} \approx F_{z,y} \, \circ \, F_{y,x} \, . \tag{8.3}$$

The right hand side is a diffeomorphism from $\widehat{\mathcal{H}}_x$ to $\widehat{\mathcal{H}}_z$ establishing the equivalence of the corresponding string field theories, therefore uniqueness (up to symmetry

† Note that in the Batalin-Vilkovisky formalism only those transformations that preserve the symplectic structure together with the action are genuine symmetries of the (tree level) theory.

transformations) of the diffeomorphism relating two state spaces implies the equality. Explicitly, in the notation of (4.4) this equation reads

$$F\,(\,\psi_x\,,x,z\,) \approx F\,\big(\,F\,(\,\psi_x\,,x,y\,)\,,y\,,z\,\big), \tag{8.4}$$

where we have supressed, for clarity, the vector indices on the string field components and on F.

8.2 Differential Equation for F and its Integrability

In order to find a differential equation for the diffeomorphism F, we apply equation (8.4) for the case when $z = y + \delta y$, to find

$$\begin{aligned} F^i\,(\,\psi_x\,,x,\;y+\delta y\,) &= F^i\,\big(\,F\,(\,\psi_x\,,x,y\,)\,,y\,,y+\delta y\,\big) \\ &= F^i\,(\,\psi_x\,,x,y\,) + \delta y^\mu\;\cdot f^i_\mu\,\big(\,F\,(\,\psi_x\,,x,y\,)\,,y\,\big) + \mathcal{O}(\delta y^2), \end{aligned} \tag{8.5}$$

where use was made of Eqn.(4.6). For the convenience of writing we have replaced the $\approx$ symbol by $=$ in the above equation, but we should always keep in mind that the equality in the above equation is true only in the sense of equivalence defined in Eqn.(8.1). In particular, we are allowed to add any infinitesimal symmetry transformation to the right hand side of the above equation. Eqn.(8.5) then gives

$$\frac{\partial F^i(\psi_x,x,y)}{\partial y^\mu} = f^i_\mu\,\big(\,F(\,\psi_x,x,y),y\,\big)\,. \tag{8.6}$$

Since the existence of f^i_μ has already been proved, the proof of existence of F reduces to showing the integrability of the set of partial differential equations (8.6) with the boundary condition

$$F^i\,(\,\psi_x,x,x) = \psi^i_x. \tag{8.7}$$

Since the infinitesimal diffeomorphism f^i_μ preserves the symplectic structure, it is guaranteed that the finite diffeomorphism $F(\psi_x,x,y)$ obtained by integration, will also map the symplectic structure at the point x to the symplectic structure at the point y.

The integrability conditions for (8.6) arise by taking a further derivative of the equation and antisymmetrizing:

$$\Delta_{\mu\nu} F^i \equiv \Big(\frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu} - \frac{\partial}{\partial y^\nu} \frac{\partial}{\partial y^\mu} \Big) F^i \left(\psi_x \, , x , y \, \right) \, = \, 0 \, . \tag{8.8}$$

Making use of (8.6) to evaluate the second derivatives we find

$$\Delta_{\mu\nu} F^i = \bigg(\frac{\partial f^i_\mu(\psi_y, y)}{\partial y^\nu} + \frac{\partial_r f^i_\mu(\psi_y, y)}{\partial \psi^j_y} f^j_\nu(\psi_y \, , y) \bigg) - (\mu \leftrightarrow \nu) \, . \tag{8.9}$$

If we can show that our solution for f^i_μ, satisfying the local background independence conditions (4.7), implies that $\Delta_{\mu\nu} F^i = 0$, then we would have proved (formal) background independence of string field theory for finite deformations of the background.

Actually the condition $\Delta_{\mu\nu} F^i = 0$ is too strong. This is due to the fact that we are interested in obtaining a solution $F^i(\psi_x, x, y)$ which is single valued *only* when it is regarded as a point in the space of all diffeomorphisms modulo the set of gauge transformations at y. In other words, it is acceptable if integration of Eqn.(8.6) along two different paths gives different $F^i(\psi_x, x, y)$'s which are related by a symmetry transformation of S_y. Indeed, $(\delta_1 x^\mu)(\delta_2 x^\nu)\Delta_{\mu\nu} F^i$ gives the difference between the diffeomorphisms obtained when going from x, to $x{+}\delta_1 x{+}\delta_2 x = y$, along the two obvious paths. Thus all we need is that $\Delta_{\mu\nu} F^i$ be a symmetry at y, namely $(\partial_r S/\partial \psi^i)\Delta_{\mu\nu} F^i = 0$. This gives

$$\frac{\partial_r S\left(\psi_y \, , y\right)}{\partial \psi^i_y} \Big[\Big(\frac{\partial f^i_\mu(\psi_y, y)}{\partial y^\nu} + \frac{\partial_r f^i_\mu(\psi_y, y)}{\partial \psi^j_y} f^j_\nu(\psi_y, y) \Big) - \Big(\mu \leftrightarrow \nu \Big) \Big] = 0 \tag{8.10}$$

We shall now show that this equation is automatically satisfied by the solution of the local background independence conditions. We start with the second equation of (4.7) and differentiate it with respect to x^ν:

$$\begin{aligned}\frac{\partial^2 S}{\partial x^\nu \partial x^\mu} &= -\frac{\partial_r}{\partial \psi_x^i}\Big(\frac{\partial S}{\partial x^\nu}\Big) f_\mu^i - \frac{\partial_r S}{\partial \psi_x^i}\frac{\partial f_\mu^i}{\partial x^\nu} \\ &= -\frac{\partial_r S}{\partial \psi_x^i}\left[\frac{\partial f_\mu^i}{\partial x^\nu} - \frac{\partial_r f_\nu^i}{\partial \psi_x^j} f_\mu^j\right] + \Big(\frac{\partial_r}{\partial \psi_x^i}\frac{\partial_r}{\partial \psi_x^j} S\Big) f_\mu^i f_\nu^j.\end{aligned} \tag{8.11}$$

Upon antisymmetrization in μ and ν, the last term in the right hand side drops out, and the remaining terms are seen to coincide with the desired expression in Eqn. (8.10) upon replacement of x by y. We thus see that the integrability conditions required for obtaining the finite field redefinitions F^i from the infinitesimal field redefinitions given by f_μ^i are automatically satisfied. This is not surprising, however. Since f_μ^i satisfies Eqn.(4.7), we are guaranteed that by integrating Eqn.(8.6) from x to y along any path we must get a transformation $F(\psi_x, x, y)$ that maps S_x to S_y. Thus if we obtain different F's by integrating along different paths, they must differ by a symmetry transformation of S_y.

8.3 Integrability without Divergences ?

We have shown in the above paragraphs that there are no local integrability conditions that ought to be satisfied, *i.e.* our infinitesimal diffeomorphisms can be integrated and we are guaranteed not to run into trouble unless we find infinite quantities. In order to avoid infinities to first approximation, products of the connection Γ_μ must be finite, as is the case for the connection c_μ (or $\bar{c}_\mu$) of Refs.[22,20]. This, of course, cannot be the complete story since the diffeomorphism involves higher order bras $\langle \Gamma_\mu^{(N)}|$, and they must also enter in a full discussion. In fact, a finite-distance field redefinition will involve the products of all the $\langle \Gamma^{(N)}|$'s, and the question of existence of divergence free field redefinitions connecting two string field theories reduces to the question of finiteness of these products.

While a complete analysis ought to be done, it seems plausible that no infinities will arise. The products of $\langle \Gamma^{(N)}|$'s are obtained by sewing punctured spheres, and the only possible divergence in this procedure comes from the configurations in $\langle\Gamma^{(N)}|$ representing surfaces where the special puncture lies on the coordinate curve of some free puncture. When two such configurations are sewn, we may get divergences due to the collision of the special punctures. These dangerous configurations in $\langle\Gamma^{(N)}|$, can all be traced back to the sewing of the special vertex $\mathcal{V}_3'$ to an ordinary string vertex. Thus the only possible sources of divergences may be traced to the introduction of $\langle V'^{(3)}|$ in our analysis. This, in turn, came from the connection $\widehat{\Gamma}_\mu$. The explicit presence of $\widehat{\Gamma}_\mu$ in Eqn.(7.2) will also give rise to divergences during the process of integrating the equations for finite field redefinitions, since, as was shown in ref.[20], the product of two $\widehat{\Gamma}_\mu$'s is divergent, and hence the connection $\widehat{\Gamma}_\mu$ cannot be used to parallel transport over a finite distance. This divergence also appears due to the collision of $\widehat{\mathcal{O}}_\mu$'s, and must be related to the divergence that arises in the process of sewing two $\langle V'^{(3)}|$ vertices due to the collision of the special punctures. (No such divergences occur in the sewing of ordinary string vertices.)

It is clear from the above discussion that all sources of divergence can finally be traced to the introduction of the connection $\widehat{\Gamma}_\mu$ in our analysis. But this appearance is purely fictitious, and is due to the fact that we have chosen to express the connection Γ_μ as a sum of $\widehat{\Gamma}_\mu$, and the difference $\Gamma_\mu - \widehat{\Gamma}_\mu$. This indicates that the integrability analysis of Eqn.(7.2) may be more transparent if we express the connection Γ_μ as the sum of $\widetilde{\Gamma}_\mu$, and $\Gamma_\mu - \widetilde{\Gamma}_\mu$, where $\widetilde{\Gamma}_\mu$ is a connection with finite products (such as c_μ or $\bar{c}_\mu$). Indirect evidence for the finiteness of the field redefinition is provided by the perturbative finiteness of finite classical solutions of string field theory [34] which form the constant shift part of the field redefinition.

9. Discussion

In this paper we have shown that given two nearby conformal field theories CFT and CFT$'$, related to each other via a marginal deformation, and BV string field theories formulated around each of these conformal field theories, there is a field redefinition which relates the two master actions and the antibrackets. The constant shift involved in the field redefinition is given by the classical solution in the string field theory around CFT that represents the background given by CFT$'$. The linear part of the field redefinition can be interpreted as a connection in the space of conformal field theories, which differs from a canonical connection by a term that can be expressed as an integral of a string vertex over a certain region of the extended moduli space of punctured Riemann surfaces. Finally, the non-linear part of the field redefinitions can also be expressed as integrals of appropriate string vertices over regions in the extended moduli space of punctured Riemann surfaces.

9.0.1 Open String Field Theory.

We have carried out our analysis for closed string field theory, but the extension of our analysis to open string theories is straightforward. In fact, a simpler field redefinition, one involving a shift and a linear transformation, suffices to prove open string background independence. Recall that for open string theory the interaction vertices $\langle V^{(N)}|$ vanish for $N \geq 4$. Therefore, if $\langle\Gamma^{(3)}_\mu|$ can be shown to vanish, a consistent solution of eqs.(4.23) is obtained by setting all the higher $\langle\Gamma^{(N)}_\mu|$'s to zero. Proving that the redefinition need not be nonlinear thus reduces to showing that $\langle\Gamma^{(3)}_\mu|$ vanishes. On the other hand, $\langle\Gamma^{(3)}_\mu|$ is to be determined from an equation analogous to Eqn.(5.20), with no $\langle V^{(4)}|$ term. Thus all we need to show is that it is possible to choose $\langle\Delta\Gamma_\mu|$ in such a way that $\mathbf{S}(\langle\Delta\Gamma_\mu|\langle V^{(3)}|\mathcal{S}\rangle$ vanishes. Note that for open strings $|\mathcal{S}_{12}\rangle = |R_{12}\rangle$. Also, since the vertex $\langle V^{(3)}|$ has cyclic symmetry, but no exchange symmetry, $\mathbf{S}$ must explicitly symmetrize in the two external legs of $\langle V^{(3)}|$.

Let us now analyze the quantity $\mathbf{S}(\langle\Delta\Gamma_\mu|\langle V^{(3)}|\mathcal{S}\rangle$. In this expression $\langle V^{(3)}|$ denotes the Witten vertex where half of the first string overlaps with half of the second string, half of the second string overlaps with half of the third string, and half of the third string overlaps with half of the first string, with the strings 1, 2, 3 appearing in an anticlockwise cyclic order. What about $\langle\Delta\Gamma_\mu|$? It satisfies an equation similar to (5.12), except that the right hand side of this equation must involve explicit symmetrization in the state spaces 1 and 2 due to the lack of explicit exchange symmetry of the vertices. A description of the vertex $\mathcal{V}_3'$ is given as follows; the first and the second strings have a complete overlap, and the third string is located at one of the common string endpoints, with the strings 1, 2 and 3 being in anticlockwise cyclic order. The on-shell state $|\widehat{\mathcal{O}}_\mu\rangle \equiv |c\mathcal{O}_\mu\rangle$ is inserted at the third puncture, hence the final result is insensitive to the choice of the coordinate system at the third puncture.

We now consider an interpolating vertex where a length $(1+t)/2$ of the first string coincides with the length $(1+t)/2$ of the second string, a length $(1-t)/2$ of the second string coincides with a length $(1-t)/2$ of the third string, and a length $(1-t)/2$ of the third string coincides with the length $(1-t)/2$ of the first string. The first and the second strings are each of length one, whereas the third string is taken to be of length $(1-t)$, so that there is complete overlap of the three strings.* Again, the strings 1, 2 and 3, are in anticlockwise cyclic order. At $t=0$ this describes Witten vertex, whereas at $t=1$ this describes $\mathcal{V}_3'$. Note that the coordinate system on the third string becomes singular as $t\to 1$, since its length vanishes, but the result is insensitive to the choice of the coordinate system on the third string. In fact, the above description of the interpolating vertex can be taken as a specification of the location of the punctures of the three strings and the coordinate systems of the first and the second string, but not that of the third string. If $\mathcal{B}_3$ denotes the region in $\mathcal{P}_3$† corresponding to the interpolating vertex,

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$\star$ This, of course, is actually the description of a Jenkins-Strebel quadratic differential.

$\dagger$ Now $\mathcal{P}_n$ denotes moduli space of a disk with n punctures at the boundary of the disk.

then we may write‡

$$\langle(\Delta\Gamma_\mu)_{12}| = -\int\limits_{\mathcal{B}_3} (\langle\,\Omega^{(1)3}_{123}\,| - \langle\,\Omega^{(1)3}_{213}\,|\,)\,|\widehat{\mathcal{O}}_\mu\rangle_3\,, \tag{9.1}$$

This is an equation analogous to (5.14), except that it has two terms due to the lack of explicit symmetry of $\langle V^{(3)}|$. The relative $-$ sign between these two terms is due to the fact that the open string master field is anticommuting (unlike the closed string master field which is commuting).

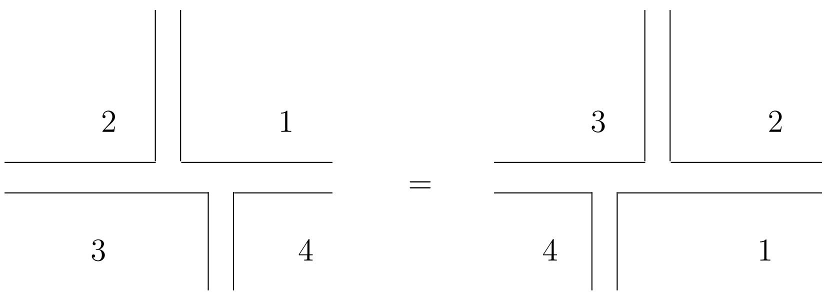
We can now use this expression for $\langle\Delta\Gamma_\mu|$ to compute $\mathbf{S}(\langle\Delta\Gamma_\mu|\langle V^{(3)}|\mathcal{S}\rangle$. $\int_{\mathcal{B}_3}$ simply denotes an integral over t. We now see that for every value of t the contributions to $\mathbf{S}(\langle\Delta\Gamma_\mu|\langle V^{(3)}|\mathcal{S}\rangle$ cancel pairwise. For example, the following pair of terms,

$$(I) \;=\; \langle\,\Omega^{(1)3}_{3'34}(t)\,|\,\langle\,\Omega^{(0)3}_{123''}(t=1/2)\,|\mathcal{S}_{3'3''}\rangle|\widehat{\mathcal{O}}_\mu\rangle_4\,, \tag{9.2}$$

and,

$$(II) \;=\; \langle\,\Omega^{(1)3}_{13'4}(t)\,|\,\langle\,\Omega^{(0)3}_{233''}(t=1/2)|\mathcal{S}_{3'3''}\rangle|\widehat{\mathcal{O}}_\mu\rangle_4\,, \tag{9.3}$$

yielding four strings with the same cyclic order, cancel out since they correspond to identical configurations. This can be seen diagramatically ($(I) = (II)$)



‡ Note that the forms $\langle\Omega^{(k)N}|$ are now given in terms of correlation functions of boundary operators in conformal field theory on the half plane, and as a result has symmetry only under cyclic permutation of the state space labels, but not under their exchange.

This shows that it is possible to construct $\langle \Delta\Gamma_\mu|$ in such a way that a consistent field redefinition is obtained by setting $\langle \Gamma_\mu^{(N)}| = 0$ for $N \geq 3$. In plain english, this means that it is possible to relate the actions of open string field theories formulated around neighboring conformal field theories via a field redefinition which only includes a shift and a linear transformation.

One can compare our result with what one expects in the purely cubic open string field theory [35]. In this formalism, the string field theory action is given by the purely cubic term $\frac{1}{3!}\langle V^{(3)}|\Psi\rangle|\Psi\rangle|\Psi\rangle$. A given background, characterized by a BRST operator Q corresponds to a specific classical solution $Q_L|\mathcal{I}\rangle$, where $|\mathcal{I}\rangle$ is the identity operator of the star product, satisfying $\langle V_{123}^{(3)}|\mathcal{I}\rangle_3 = \langle R_{12}|$. In this case, a shift in the background amounts to a change ΔQ in the BRST charge, and hence, a simple shift $\Delta Q_L|\mathcal{I}\rangle$ in the string field, without any further linear field redefinition. This shift, however, is singular*, since the state $|\mathcal{I}\rangle$ is a singular state. We expect that the field redefinition that we have found is related to the one induced by this simple shift by a (singular) gauge transformation.

9.0.2 Other Directions for Closed String Field Theory

The above discussion naturally raises the question as to whether it is possible to find a formulation of closed string field theory analogous to the purely cubic open string field theory. We do not have a definite answer to this question. We note, however, a surprising fact. The purely cubic closed string field theory action

$$S = \frac{1}{3!}\langle V^{(3)}|\Psi\rangle|\Psi\rangle|\Psi\rangle\,, \tag{9.4}$$

is actually invariant under a gauge transformation

$$\delta_\Lambda\,|\Psi\rangle = \langle V_{123}^{\prime(3)}|\mathcal{S}_{1e}\rangle|\Psi\rangle_2|\Lambda\rangle_3 = \Psi\circ\Lambda\,, \tag{9.5}$$

where $|\Psi\rangle$ is a classical string field (a ghost number two state in $\widehat{\mathcal{H}}$), and $|\Lambda\rangle$ is the gauge transformation parameter (a ghost number one state in $\widehat{\mathcal{H}}$). The product

$\star$ In the sense that string field products involving more than one such field are typically ill-defined.

used in the above transfomation is the asymmetric product where Ψ is inserted on the second puncture of $\mathcal{V}_3'$ and Λ is inserted in the asymmetric puncture. Off-shell gauge invariance follows from Eqn.(6.3). The above action was never thought of as a possible candidate for a closed string field action because it seemed to have no gauge invariance. Is the above gauge invariance an indication that it could, in fact, be a consistent classical action? This is not clear to us. While on-shell, this gauge invariance is equivalent to a gauge invariance generated by the standard $\mathcal{V}_3$, the following properties seem to suggest complications. First, in this case we would not have the standard BV relation between the action and the gauge transformations. Second, the algebra of this gauge transformations may need regularization; when performing two successive gauge transformations the gauge parameters would collide. Third, thanks to Eqn.(6.4), any action of the form

$$S = \sum_{N=3}^{\infty} a_N \langle V^{(N)}|\Psi\rangle_1 \cdots |\Psi\rangle_N \tag{9.6}$$

where the a_N's are arbitrary coefficients, is invariant under the above gauge transformation.

Another possibility for writing new closed string field theory actions could involve the use of two string fields. Given that we now have a vertex $\mathcal{V}_3'$ that naturally distinguishes one puncture from the other two, it is tempting to couple a new string field through this puncture. This brings us to the admittedly speculative possibility that the theory-space connection, or some string field encoding such data, could actually represent a dynamical string variable. In such formulation, elimination of this connection through its field equations would leave a 'background' connection, along with a fully nonlinear action for the string field. This 'background connection' could play the role of fixing the Riemann surface geometry that defines the string interactions. Solving for the connection would amount to fixing the way string theory would cut moduli space.

9.0.3 Extending the Proof of Background Independence.

Our proof of background independence made important use of the fact that the interactions of the standard classical closed string field theory are overlaps. Associated to such vertices, the canonical connection $\widehat{\Gamma}_\mu$ played a prominent role in the analysis. It is clearly possible to construct other string field theories based on non-overlap type vertices (the simplest example being a theory with stubs). Although our analysis has not been done for such theories, it was shown in a recent paper[12] that these different string field theories are related to the standard one by canonical field redefinitions, and hence this field redefinition, combined with the field redefinition we have found in our paper, makes the result of the present paper valid even for string field theories with non overlap vertices.

It would be more instructive, however, to apply our methods directly to these theories. The main difference in this case is that $D_\mu(\widehat{\Gamma})\langle V^{(N)}|$ is no longer zero, but can be expressed as an integral of $\langle\Omega^{(0)N+1}|\widehat{\mathcal{O}}_\mu\rangle$ over a certain region of $\widehat{\mathcal{P}}_{N+1}$. This will give rise to a new term on the right hand side of Eqn.(7.7) (and (5.20)) which has the same structure as the other terms, and hence these equations can be solved in the same way. Although we have not given a direct proof of existence of the solutions of these modified equations, it is guaranteed by the results of ref.[12]. On the other hand, it is possible that a connection different from $\widehat{\Gamma}_\mu$ could be more appropriate to deal with such non-overlap theories.

A related question is whether we could have carried out our analysis of the standard closed string field theory using a reference connection $\widetilde{\Gamma}_\mu$ different from $\widehat{\Gamma}_\mu$. This is not a merely academic question. As argued at the end of last section, a connection different from $\widehat{\Gamma}_\mu$ may be useful to construct a manifestly finite field redefinition relating two distant theories. Again the main difference is that $D_\mu(\widetilde{\Gamma})\langle V^{(N)}|$ would not be zero. If we choose $\widetilde{\Gamma}$ to be the any of the connections Γ_D defined in ref.[20] we can again express $D_\mu(\widetilde{\Gamma})\langle V^{(N)}|$ as an integral of $\langle\Omega^{(0)N+1}|\widehat{\mathcal{O}}_\mu\rangle$ over a certain region of $\widehat{\mathcal{P}}_{N+1}$, and the effect is to modify the right hand side of Eqn.(7.7) (and (5.20)) by the addition of this term. In this case $D_\mu(\widetilde{\Gamma})Q$ also has a form different from $D_\mu(\widehat{\Gamma})Q$, and the result is to modify the right hand side of Eqn.(5.11) by the replacement of $\langle V'^{(3)}|$ by a different three string vertex. Again, all the equations have the same structure as the ones we have analyzed, and can be solved in an identical manner. The existence of the solution of these equations is guaranteed by our result, together with the fact that changing the connection amounts to a linear redefinition of the string fields.

9.0.4 Quantum Background Independence ?

All of our analysis has been done in the context of classical master action. How about the quantum theory? Given that in the BV formulation of closed string field theory one has a well defined quantum master action, the question of quantum background independence is likely to be well defined. A quantum theory, however, is defined by a BV supermanifold $(M, \omega, d\mu)$, where M is the supermanifold, ω the symplectic form, and $d\mu$ a consistent volume element (leading to a nilpotent Δ operator), together with the master action S. It was found in ref.[12], that the symplectic diffeomorphisms relating theories using different string vertices do not preserve the volume element $d\mu$ and the action S separately, but do preserve $d\mu e^{2S}$. This indicates that the symplectic diffeomorphism implementing the physical requirement of background independence also cannot preserve both the volume element and the master action separately. Suppose we are comparing string field theories formulated on $\widehat{\mathcal{H}}_x$ and $\widehat{\mathcal{H}}_y$. Moreover, we have volume elements $d\mu_x$ and $d\mu_y$, respectively. Let L_y be an arbitrary lagrangian submanifold of $\widehat{\mathcal{H}}_y$, and let $d\lambda_y$ be the measure induced on that submanifold by the measure $d\mu_y$ on $\widehat{\mathcal{H}}_y$. The physical requirement of background independence is that the symplectic diffeomorphism from $\widehat{\mathcal{H}}_x$ to $\widehat{\mathcal{H}}_y$ should map $[d\lambda_y\, e^{S_y}]$ to $[d\lambda_x\, e^{S_x}]$ where $d\lambda_x$ is the measure induced from $d\mu_x$, on the lagrangian submanifold obtained as the (inverse) image of L_y under the diffeomorphism. This is all that is required physically. Actually such a result would follow from the possibly stronger condition that the diffeomorphism takes $[d\mu_y\, e^{2S_y}]$ to $[d\mu_x\, e^{2S_x}]$, and we believe it is likely that this stronger condition holds. As a technical point, we note that the presence of higher loop tadpoles in the master action will probably force us to modify even the constant part $\langle \Gamma^{(1)}|$ of the field redefinition from its tree level value.

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