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ON THE BACKGROUND INDEPENDENCE OF STRING FIELD THEORY

III. Explicit Field Redefinitions

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ABSTRACT

Given two conformal field theories related to each other by a marginal perturbation, and string field theories constructed around such backgrounds, we show how to construct explicit redefinition of string fields which relate these two string field theories. The analysis is carried out completely for quadratic and cubic terms in the action. Although a general proof of existence of field redefinitions which relate higher point vertices is not given, specific examples are discussed. Equivalence of string field theories formulated around two conformal field theories which are not close to each other, but are related to each other by a series of marginal deformations, is also discussed. The analysis can also be applied to study the equivalence of different formulation of string field theories around the same background.

1. INTRODUCTION

A complete formulation of closed bosonic string field theory has been given in the last few years in terms of non-polynomial interactions [1] [2] (see also ref.[3]), and quantization of this string field theory has also been carried out $[4-8]$. Such a field theory can be formulated not only in the background of flat space-time, but also in the background of any arbitrary conformal field theory [9] following the formulation of refs.[10][11]. The natural question to ask is whether the string field theory is background independent, i.e. given the string field theory action around two different conformal field theory backgrounds, whether we can find a redefinition of string fields which relates these two actions.

The question was partially answered in two previous papers [12] [13]. In these papers we studied the relationship between string field theories formulated around two neighbouring conformal field theories, $-$ CFT and CFT$''$, $-$ related by a marginal perturbation. Let Ψ and $\tilde{\Psi}$ denote the string fields corresponding to string field theories formulated around CFT and CFT$''$ respectively, and $S(\Psi)$ and $\tilde{S}(\tilde{\Psi})$ be the corresponding string field theory actions. It was shown in ref.[12] that there is a classical solution Ψ_{cl} of the equations of motion $(\partial S/\partial\Psi)=0$ such that if we define $\hat{\Psi}=\Psi-\Psi_{cl}$ and $\hat{S}(\hat{\Psi})=S(\Psi)-S(\Psi_{cl})$, then the kinetic operator of $\hat{S}(\hat{\Psi})$ is related to the kinetic operator of $\tilde{S}(\tilde{\Psi})$ by a similarity transformation. Furthermore, the linearized form of the gauge transformation of $\hat{\Psi}$ is also related to that of $\tilde{\Psi}$ by the same similarity transformation. The analysis was carried out to the first order in the perturbation which relates CFT and CFT$''$. In other words, ref.[12] established the equivalence between the quadratic terms in the actions $\hat{S}(\hat{\Psi})$ and $\tilde{S}(\tilde{\Psi})$. On the other hand, in ref.[13] we analyzed the physical S-matrix elements in the two theories described by the actions $\hat{S}(\hat{\Psi})$ and $\tilde{S}(\tilde{\Psi})$, and showed that they are the same. The analysis was carried out for all three point amplitudes and all N-point tachyonic amplitudes.

In this paper, we shall show with the help of the results of refs.[12] and [13] that one can construct explicit redefinition of string fields which converts the action

$\tilde{S}(\tilde{\Psi})$ to $\hat{S}(\hat{\Psi})$, and hence to $S(\Psi)$. The analysis is carried out completely for the cubic vertices in the two theories, using a method similar to the one used in ref.[14]. (The analysis for the quadratic terms was already done in ref.[12]). We also discuss some specific features that appear in the analysis of the higher order vertices, and show, in special cases, how explicit field redefinitions may be found which relate the vertices in $\tilde{S}(\tilde{\Psi})$ and $\hat{S}(\hat{\Psi})$. But a general proof of equivalence of these two vertices is not given.

Assuming that a complete set of field redefinitions can be found which relate $S(\Psi)$ and $\tilde{S}(\tilde{\Psi})$ to first order in the perturbation parameter λ, one can ask if the analysis can be extended beyond first order in λ and hence can be used to relate string field theories around backgrounds that are not necessarily close to each other but can be obtained from each other by a series of marginal deformations. Intuitively it is clear that the string field theories constructed around two such backgrounds will also be related by field redefinition, which can be built by combining successive field redefinitions which relate two nearby conformal field theories. We give a general algorithm for finding these finite field redefinitions in terms of the infinitesimal ones.

The plan of the paper is as follows. Sect.2 contains a precise formulation of the problem that we are going to study, as well as its relationship with the work of ref.[12]. In sect.3 we prove the existence of field redefinitions that relate the cubic vertices of these two theories, and also give the general algorithm for constructing these field redefinitions. Sect.4 contains a discussion of the corresponding analysis for higher point vertices. Sect.5 deals with the case where the two conformal field theories are not necessarily close. We also use the result of this section to construct the classical solution in string field theory representing the perturbed conformal field theory to second order in the perturbation parameter. We conclude in sect.6 with a discussion of our results and some comments. The three appendices contain some of the technical results needed in the analysis of sects. 3, 4 and 5.

2. FORMULATION OF THE PROBLEM

We begin this section with a precise formulation of the problem that we want to solve. Let CFT and CFT$''$ be two different conformal field theories, both with central charge 26, and hence both providing consistent background for the formulation of string theory. Let $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$ denote string fields for the string field theories formulated around CFT and CFT$''$ respectively, and $S(\Psi)$ and $\tilde{S}(\tilde{\Psi})$ be the actions of the corresponding string field theories. In order to prove background independence of string field theory, one needs to find a functional relationship of the form $\Psi = f(\tilde{\Psi})$ such that,

$$S(f(\tilde{\Psi})) = \tilde{S}(\tilde{\Psi}) + \text{constant} \tag{2.1}$$

Let us denote by CFTG (CFTG$''$) the combined conformal field theory of the ghost system and CFT (CFT$''$). If $\mathcal{H}$ ($\tilde{\mathcal{H}}$) denote the complete Hilbert space of CFTG (CFTG$''$), and L_n, $\bar{L}_n$ ($\tilde{L}_n$, $\tilde{\bar{L}}_n$) denote the total Virasoro generators of CFTG (CFTG$''$), then $b_0^-|\Psi\rangle$ ($b_0^-|\tilde{\Psi}\rangle$) is an arbitrary state in $\mathcal{H}$ ($\tilde{\mathcal{H}}$) with ghost number 2 and annihilated by b_0^- and $L_0^- \equiv L_0 - \bar{L}_0$ ($\tilde{L}_0 \equiv \tilde{L}_0 - \tilde{\bar{L}}_0$). Then the actions $S(\Psi)$ and $\tilde{S}(\tilde{\Psi})$ are given by,

$$S(\Psi) = \frac{1}{2}\langle\Psi|Q_B b_0^-|\Psi\rangle + \sum_{N=3}^{\infty}\frac{g^{N-2}}{N!}\{\Psi^N\} \tag{2.2}$$

and

$$\tilde{S}(\tilde{\Psi}) = \frac{1}{2}\langle\tilde{\Psi}|\tilde{Q}_B b_0^-|\tilde{\Psi}\rangle'' + \sum_{N=3}^{\infty}\frac{g^{N-2}}{N!}\{\tilde{\Psi}^N\}'' \tag{2.3}$$

with the gauge invariance,

$$b_0^-\delta|\Psi\rangle = Q_B b_0^-|\Lambda\rangle + \sum_{N=3}^{\infty}\frac{g^{N-2}}{(N-2)!}[\Psi^{N-2}\Lambda] \tag{2.4}$$

$$b_0^-\delta|\tilde{\Psi}\rangle = \tilde{Q}_B b_0^-|\tilde{\Lambda}\rangle + \sum_{N=3}^{\infty}\frac{g^{N-2}}{(N-2)!}[\tilde{\Psi}^{N-2}\tilde{\Lambda}]'' \tag{2.5}$$

where $b_0^-|\Lambda\rangle$ ($b_0^-|\tilde{\Lambda}\rangle$) are states of ghost number 1 in $\mathcal{H}$ ($\tilde{\mathcal{H}}$) annihilated by b_0^-

and L_0^- ($\tilde{L}_0^-$). Q_B ($\tilde{Q}_B$) is the nilpotent BRST operator acting in $\mathcal{H}$ ($\tilde{\mathcal{H}}$), and $\langle|\rangle$ ($\langle|\rangle''$) denote the BPZ inner product [15] in CFTG (CFTG$''$). Thus the operators L_n, Q_B ($\tilde{L}_n$, $\tilde{Q}_B$) etc. have appropriate hermiticity properties with respect to the inner product $\langle|\rangle$ ($\langle|\rangle''$) but not with respect to the inner product $\langle|\rangle''$ ($\langle|\rangle$). $[A_1 \ldots A_N]$ and $\{A_1 \ldots A_N\} \equiv (-1)^{n_1+1}\langle A_1|[A_2 \ldots A_N]\rangle$ denote multilinear maps from N-fold tensor product of $\mathcal{H}$ to $\mathcal{H}$ and C respectively, and are constructed in terms of correlation functions in CFTG [10] [1] [2] [9] [12] [13]. Here n_1 denotes the ghost number of the state $|A_1\rangle$. Similarly, $[\tilde{A}_1 \ldots \tilde{A}_N]''$ and $\{\tilde{A}_1 \ldots \tilde{A}_N\}'' \equiv (-1)^{n_1+1}\langle\tilde{A}_1|[\tilde{A}_2 \ldots \tilde{A}_N]''\rangle''$ denote multilinear maps from N-fold tensor product of $\tilde{\mathcal{H}}$ to $\tilde{\mathcal{H}}$ and C respectively, and are constructed in terms of correlation functions in CFTG$''$.

Let us introduce a basis of states $|\Phi_{n,r}\rangle$ in the subspace $\mathcal{H}_n$ of $\mathcal{H}$ of ghost number n, and annihilated by b_0^- and L_0^-. Similarly, let $|\tilde{\Phi}_{n,r}\rangle$ be a basis of states in the subspace $\tilde{\mathcal{H}}_n$ of $\tilde{\mathcal{H}}$ of ghost number n and annihilated by b_0^- and $\tilde{L}_0^-$. Then we may write,

$$b_0^-|\Psi\rangle = \sum_r \psi_r|\Phi_{2,r}\rangle \tag{2.6}$$

$$b_0^-|\tilde{\Psi}\rangle = \sum_r \tilde{\psi}_r|\tilde{\Phi}_{2,r}\rangle \tag{2.7}$$

It was shown in ref.[12] that the operators $\tilde{L}_m$, $\bar{\tilde{L}}_m$ in $\tilde{\mathcal{H}}$ may be represented in the Hilbert space $\mathcal{H}$ and vice versa, that is, there is a natural isomorphism between the Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$.⋆ Let us denote the image of $|\tilde{\Phi}_{n,r}\rangle$ in $\mathcal{H}$ under this isomorphism also by $|\tilde{\Phi}_{n,r}\rangle$. Then $|\Phi_{n,r}\rangle$ and $|\tilde{\Phi}_{n,r}\rangle$ are related by a linear transformation of the form:

$$|\tilde{\Phi}_{n,r}\rangle = \sum_s V_{sr}^{[n]}|\Phi_{n,s}\rangle \tag{2.8}$$

⋆ This was done at finite value of the regulator which controls the short distance singularities. In our analysis we shall keep the regulator finite throughout our analysis, and set it to zero only at the very end. For a coordinate independent description see ref.[16].

We can now expand $b_0^-|\Psi\rangle$ as,

$$b_0^-|\Psi\rangle = \sum_r \psi'_r |\tilde{\Phi}_{2,r}\rangle \tag{2.9}$$

where ψ_r and ψ'_r are related by,

$$\psi_r = \sum_s V^{[2]}_{rs} \psi'_s \tag{2.10}$$

Using eqs.(2.2), (2.3), (2.7) and (2.9) we may express $S(\Psi)$ and $\tilde{S}(\tilde{\Psi})$ as,

$$S(\Psi) = \sum_{N=2}^{\infty} \frac{1}{N!} A^{(N)}_{r_1 \ldots r_N} \psi'_{r_1} \ldots \psi'_{r_N} \tag{2.11}$$

$$\tilde{S}(\tilde{\Psi}) = \sum_{N=2}^{\infty} \frac{1}{N!} \tilde{A}^{(N)}_{r_1 \ldots r_N} \tilde{\psi}_{r_1} \ldots \tilde{\psi}_{r_N} \tag{2.12}$$

where,

$$A^{(2)}_{r_1 r_2} = \langle \tilde{\Phi}_{2,r_1} | Q_B c_0^- | \tilde{\Phi}_{2,r_2} \rangle \tag{2.13}$$

$$A^{(N)}_{r_1 \ldots r_N} = g^{N-2} \{ (c_0^- \tilde{\Phi}_{2,r_1}) \ldots (c_0^- \tilde{\Phi}_{2,r_N}) \} \quad \text{for } N \geq 3 \tag{2.14}$$

$$\tilde{A}^{(2)}_{r_1 r_2} = \langle \tilde{\Phi}_{2,r_1} | \tilde{Q}_B c_0^- | \tilde{\Phi}_{2,r_2} \rangle'' \tag{2.15}$$

$$\tilde{A}^{(N)}_{r_1 \ldots r_N} = g^{N-2} \{ (c_0^- \tilde{\Phi}_{2,r_1}) \ldots (c_0^- \tilde{\Phi}_{2,r_N}) \}'' \quad \text{for } N \geq 3 \tag{2.16}$$

We now seek the following form of functional relationship between ψ'_r and $\tilde{\psi}_r$:

$$\psi'_r \equiv (V^{[2]})^{-1}_{rs} \psi_s = \psi^{(0)}_r + \sum_{N=1}^{\infty} \frac{1}{N!} S^{(N-1)}_{r s_1 \ldots s_N} \tilde{\psi}_{s_1} \ldots \tilde{\psi}_{s_N} \tag{2.17}$$

which satisfies eq.(2.1). Here $\psi^{(0)}_r$ and $S^{(N-1)}_{r s_1 \ldots s_N}$ are some constants. Without any loss of generality we can choose $S^{(N-1)}$ to be symmetric in its last N indices.

Proof of background independence of string field theory now reduces to showing the existence of appropriate $\psi_r^{(0)}$ and $S^{(N-1)}_{rs_1\ldots s_N}$ satisfying eq.(2.1). In order that the term linear in $\tilde\psi$ on the left hand side of eq.(2.1) vanishes, $b_0^-|\Psi^{(0)}\rangle = \sum_r \psi_r^{(0)}|\tilde\Phi_{2,r}\rangle$ must be a solution of the classical equations of motion derived from the action $S(\Psi)$.

Let us define,

$$\hat\psi_r = \psi'_r - \psi_r^{(0)}, \qquad b_0^-|\hat\Psi\rangle = \sum_r \hat\psi_r|\tilde\Phi_{2,r}\rangle \tag{2.18}$$

Using this, we may express eq.(2.11) as,

$$\begin{aligned} S(\Psi) &= S(\Psi^{(0)}+\hat\Psi) = S(\Psi^{(0)}) + \sum_{N=2}^\infty \frac{1}{N!}\hat A^{(N)}_{r_1\ldots r_N}\hat\psi_{r_1}\ldots\hat\psi_{r_N} \\ &\equiv S(\Psi^{(0)}) + \hat S(\hat\Psi) \end{aligned} \tag{2.19}$$

where,

$$\begin{aligned} \hat A^{(N)}_{r_1\ldots r_N} &= \sum_{M=2}^\infty \frac{g^{M+N-2}}{(M-2)!}\{(\Psi^{(0)})^{M-2}(c_0^-\tilde\Phi_{2,r_1})\ldots(c_0^-\tilde\Phi_{2,r_N})\} \\ &= g^N\{(c_0^-\tilde\Phi_{2,r_1})\ldots(c_0^-\tilde\Phi_{2,r_N})\}' \qquad \text{for } N\ge 3 \\ \hat A^{(2)}_{r_1r_2} &= \langle\tilde\Phi_{2,r_1}|Q_Bc_0^-|\tilde\Phi_{2,r_2}\rangle + \sum_{M=3}^\infty \frac{g^{M-2}}{(M-2)!}\{(\Psi^{(0)})^{M-2}(c_0^-\tilde\Phi_{2,r_1})(c_0^-\tilde\Phi_{2,r_2})\} \\ &= \langle\tilde\Phi_{2,r_1}|\hat Q_Bc_0^-|\tilde\Phi_{2,r_2}\rangle \end{aligned} \tag{2.20}$$

where,

$$\{A_1\ldots A_N\}' = \sum_{M=2}^\infty \frac{g^{M-2}}{(M-2)!}\{(\Psi^{(0)})^{M-2}A_1\ldots A_N\} \tag{2.21}$$

$$\hat Q_Bb_0^-|A\rangle = Q_Bb_0^-|A\rangle + \sum_{M=3}^\infty \frac{g^{M-2}}{(M-2)!}[(\Psi^{(0)})^{M-2}A] \tag{2.22}$$

and,

$$[A_1\ldots A_N]' = \sum_{M=2}^\infty \frac{g^{M-2}}{(M-2)!}[(\Psi^{(0)})^{M-2}A_1\ldots A_N] \tag{2.23}$$

Detailed properties of $\hat Q_B$, $[A_1\ldots A_N]'$ and $\{A_1\ldots A_N\}'$ have been analyzed in ref.[9]. Identifying the constant in eq.(2.1) with $S(\Psi^{(0)})$ we can now express this equation as,

$$\hat{S}(\hat{\Psi}) = \tilde{S}(\tilde{\Psi}) \tag{2.24}$$

with eq.(2.17) taking the form:

$$\hat{\psi}_r = \sum_{N=1}^{\infty} \frac{1}{N!} S^{(N-1)}_{rs_1\dots s_N} \tilde{\psi}_{s_1} \dots \tilde{\psi}_{s_N} \tag{2.25}$$

Using eqs.(2.12), (2.19), (2.24) and (2.25) we get,

$$\tilde{A}^{(2)}_{r_1 r_2} = S^{(0)}_{s_1 r_1} S^{(0)}_{s_2 r_2} \hat{A}^{(2)}_{s_1 s_2} \tag{2.26}$$

$$\begin{aligned} \tilde{A}^{(3)}_{r_1 r_2 r_3} =& \hat{A}^{(2)}_{s_1 s_2} (S^{(0)}_{s_1 r_1} S^{(1)}_{s_2 r_2 r_3} + S^{(0)}_{s_1 r_2} S^{(1)}_{s_2 r_1 r_3} + S^{(0)}_{s_1 r_3} S^{(1)}_{s_2 r_1 r_2}) \\ &+ S^{(0)}_{s_1 r_1} S^{(0)}_{s_2 r_2} S^{(0)}_{s_3 r_3} \hat{A}^{(3)}_{s_1 s_2 s_3} \end{aligned} \tag{2.27}$$

and more generally,

$$\begin{aligned} \tilde{A}^{(N)}_{r_1\dots r_N} = & \sum_{\substack{\{n, N_1, \dots, N_n\} \\ n\ge 2, N_i \ge 1, \sum N_i = N}} \frac{1}{n! \prod_i N_i!} \Big(\hat{A}^{(n)}_{s_1\dots s_n} S^{(N_1-1)}_{s_1 r_1 \dots r_{N_1}} S^{(N_2-1)}_{s_2 r_{(N_1+1)} \dots r_{(N_1+N_2)}} \dots S^{(N_n-1)}_{s_n r_{(N-N_n+1)} \dots r_N} \\ & + \text{All permutations of } r_1 \dots r_N \Big) \end{aligned} \tag{2.28}$$

We shall consider the case where CFT and CFT$''$ are related to each other by an exactly marginal perturbation. Also, in this section, we shall work to first order in the perturbation parameter λ. For definiteness, we take CFT to be a direct sum of a theory of some free scalar fields (one of which is time-like) and an internal conformal field theory; and take the marginal operator that relates CFT to CFT$''$ to be a dimension $(1,1)$ operator φ in this internal conformal field theory. It was shown in ref.[12] that if the two dimensional action for CFT$''$ is obtained by adding a term of the form $-\lambda \int d^2z\varphi(z,\bar{z})$ to the two dimensional action of CFT, then $\Psi^{(0)}$ is given by,

$$b_0^-|\Psi^{(0)}\rangle = \frac{\sqrt{2}\lambda}{g} c_1\bar{c}_1|\varphi\rangle \tag{2.29}$$

It was also shown that to order λ, there is an operator $\mathcal{S}$ (called S in ref.[12]) which acts on states annihilated by b_0^- and $L_0^- = \tilde{L}_0^-$, and relates the BRST charge $\tilde{Q}_B$ of CFTG$''$ to $\hat{Q}_B$ defined in eq.(2.22) through the relation:

$$\hat{Q}_B = \mathcal{S}\tilde{Q}_B\mathcal{S}^{-1} \tag{2.30}$$

Furthermore, $\mathcal{S}$ preserves the inner product between states:

$$\langle \mathcal{S}\Phi_1|c_0^-|\mathcal{S}\Phi_2\rangle = \langle \Phi_1|c_0^-|\Phi_2\rangle'' \tag{2.31}$$

where $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are any two states annihilated by b_0^- and L_0^-. Taking the matrix elements of both sides of eq.(2.30) between the states $\mathcal{S}|\tilde{\Phi}_{2,r_1}\rangle$ and $c_0^-\mathcal{S}|\tilde{\Phi}_{2,r_2}\rangle$, and using eq.(2.31), we get,

$$\langle \mathcal{S}\tilde{\Phi}_{2,r_1}|c_0^-\hat{Q}_B|\mathcal{S}\tilde{\Phi}_{2,r_2}\rangle = \langle \tilde{\Phi}_{2,r_1}|c_0^-\tilde{Q}_B|\tilde{\Phi}_{2,r_2}\rangle'' \tag{2.32}$$

Thus, if we define $S_{rs}^{(0)}$ through the relation,

$$\mathcal{S}|\tilde{\Phi}_{2,r}\rangle = \sum_r S_{sr}^{(0)}|\tilde{\Phi}_{2,s}\rangle \tag{2.33}$$

then, using eqs.(2.15) and (2.20), eq.(2.32) may be written in the form:

$$\hat{A}^{(2)}_{s_1s_2}S^{(0)}_{s_1r_1}S^{(0)}_{s_2r_2} = \tilde{A}^{(2)}_{r_1r_2} \tag{2.34}$$

This is precisely eq.(2.26). Thus we see that $S_{sr}^{(0)}$ defined in eq.(2.33) provides a solution to eq.(2.26).

In the next sections we shall show how to obtain solutions of eqs.(2.27) and also discuss some general features of the other equations (2.28). For our analysis it will be more convenient to define $\bar{\psi}_r$ as,

$$\bar{\psi}_r = (S^{(0)})^{-1}_{rs}\hat{\psi}_s \tag{2.35}$$

Then the action $\hat{S}(\hat{\Psi})$ given in eq.(2.19) may be expressed as,

$$\hat{S}(\hat{\Psi}) \equiv \bar{S}(\bar{\Psi}) = \frac{1}{2}\tilde{A}^{(2)}_{rs}\bar{\psi}_r\bar{\psi}_s + \sum_{N=3}^{\infty}\frac{1}{N!}\bar{A}^{(N)}_{r_1\ldots r_N}\bar{\psi}_{r_1}\ldots\bar{\psi}_{r_N} \tag{2.36}$$

where,

$$\bar{A}^{(N)}_{r_1\ldots r_N} = S^{(0)}_{s_1r_1}\ldots S^{(0)}_{s_Nr_N}\hat{A}^{(N)}_{s_1\ldots s_N}, \quad \text{for } N\geq 3 \tag{2.37}$$

The relation (2.25) now takes the form:

$$\bar{\psi}_r = \tilde{\psi}_r + \sum_{N=2}^{\infty}\frac{1}{N!}\bar{S}^{(N-1)}_{rs_1\ldots s_N}\tilde{\psi}_{s_1}\ldots\tilde{\psi}_{s_N} \tag{2.38}$$

where,

$$\bar{S}^{(N-1)}_{rr_1\ldots r_N} = S^{(N-1)}_{sr_1\ldots r_N}(S^{(0)})^{-1}_{rs}, \quad \text{for } N\geq 2 \tag{2.39}$$

Eqs.(2.27) and (2.28) may now be writtten as,

$$\tilde{A}^{(3)}_{r_1r_2r_3} = \bar{A}^{(3)}_{r_1r_2r_3} + (\tilde{A}^{(2)}_{r_1s_1}\bar{S}^{(1)}_{s_1r_2r_3} + \tilde{A}^{(2)}_{r_2s_1}\bar{S}^{(1)}_{s_1r_1r_3} + \tilde{A}^{(2)}_{r_3s_1}\bar{S}^{(1)}_{s_1r_1r_2}) \tag{2.40}$$

and,

$$\begin{aligned}\tilde{A}^{(N)}_{r_1\ldots r_N} =&\bar{A}^{(N)}_{r_1\ldots r_N} + \sum_{\substack{n,M,N_1,\ldots N_n\\ 2\leq M<N, 1\leq n\leq M, N_i\geq 2, M-n+\sum N_i=N}}\frac{1}{n!(M-n)!\prod_i N_i!}\Big(\bar{A}^{(M)}_{r_1\ldots r_{M-n}s_1\ldots s_n}\\ &\times\bar{S}^{(N_1-1)}_{s_1r_{(M-n+1)}\ldots r_{(M-n+N_1)}}\bar{S}^{(N_2-1)}_{s_2r_{(M-n+N_1+1)}\ldots r_{(M-n+N_1+N_2)}}\ldots\bar{S}^{(N_n-1)}_{s_nr_{(N-N_n+1)}\ldots r_N}\\ &+\ \ \text{All permutations of } r_1\ldots r_N\Big)\end{aligned} \tag{2.41}$$

Before we conclude this section, we shall write down the gauge symmetries of the action $\bar{S}(\bar{\Psi})$. The action $\hat{S}(\hat{\Psi})$ is known to have a gauge symmetry of the form [9]

$$b_0^- \delta(|\hat{\Psi}\rangle) = \hat{Q}_B b_0^- |\Lambda\rangle + \sum_{N=3}^{\infty} \frac{g^{N-2}}{(N-2)!} [\hat{\Psi}^{N-2}\Lambda]' \tag{2.42}$$

If we define,

$$b_0^- |\bar{\Psi}\rangle = \sum_r \bar{\psi}_r |\tilde{\Phi}_{2,r}\rangle = \mathcal{S}^{-1} b_0^- |\hat{\Psi}\rangle \tag{2.43}$$

then eqs.(2.42) and (2.30) gives,

$$b_0^- \delta |\bar{\Psi}\rangle = \tilde{Q}_B b_0^- |\bar{\Lambda}\rangle + \sum_{N=3}^{\infty} \frac{g^{N-2}}{(N-2)!} \mathcal{S}^{-1} [\hat{\Psi}^{N-2}\Lambda]' \tag{2.44}$$

where,

$$b_0^- |\bar{\Lambda}\rangle = \mathcal{S}^{-1} b_0^- |\Lambda\rangle \equiv \sum_r \bar{\lambda}_r |\tilde{\Phi}_{2,r}\rangle \tag{2.45}$$

Eq.(2.44), together with the relations (2.43) and (2.45) give the gauge symmetries of the action $\bar{S}(\bar{\Psi})$ given in eq.(2.36). Note that the kinetic terms and the linearized gauge transformations have the same form for $\tilde{S}(\tilde{\Psi})$ and $\bar{S}(\bar{\Psi})$. This fact will be useful to us in the later analysis.

3. ANALYSIS OF CUBIC TERMS IN THE ACTION

In this section we shall show that it is possible to choose suitable $S^{(1)}_{sr_1r_2}$ (or, equivalently, $\bar{S}^{(1)}_{sr_1r_2}$) so as to satisfy eq.(2.27) (or, equivalently, eq.(2.40)). We shall choose to work with the quantities $\bar{S}^{(1)}_{sr_1r_2}$ and eq.(2.40), since the various equations we shall encounter during this analysis take a more compact form in terms of these variables. To begin with, we divide the basis states $\{|\tilde{\Phi}_{n,r}\rangle\}$ into three sets: physical states $\{\tilde{\Phi}_{n,k_n}\rangle\}$, unphysical states $\{|\tilde{\Phi}_{n,\alpha_n}\rangle\}$, and pure gauge states $\{\tilde{\Phi}_{n,\alpha_{n-1}}\rangle\}$, satisfying the relations [13]:

$$\begin{aligned} \tilde{Q}_B|\tilde{\Phi}_{n,k_n}\rangle =&0 \\ \sum_{k_n} a_{k_n}|\tilde{\Phi}_{n,k_n}\rangle \neq& \tilde{Q}_B b_0^-|s\rangle \text{ for any } \{a_{k_n}\} \text{ and any } |s\rangle \end{aligned} \tag{3.1}$$

$$\tilde{Q}_B \sum_{\alpha_n} a_{\alpha_n}|\tilde{\Phi}_{n,\alpha_n}\rangle \neq 0 \text{ for any } \{a_{\alpha_n}\} \tag{3.2}$$

$$|\tilde{\Phi}_{n,\alpha_{n-1}}\rangle = \tilde{Q}_B|\tilde{\Phi}_{n-1,\alpha_{n-1}}\rangle \tag{3.3}$$

We shall further group the BRST invariant basis states $\{|\tilde{\Phi}_{n,k_n}\rangle\}$ and $\{|\tilde{\Phi}_{n,\alpha_{n-1}}\rangle\}$ into a single group, and denote this by $\{|\tilde{\Phi}_{n,\mu_n}\rangle\}$. We shall now study eq.(2.40) for different choices of the indices r_1, r_2 and r_3, and show that it is possible to choose $\bar{S}^{(1)}_{srr'}$ such that eq.(2.40) is satisfied for all possible values of the indices r_1, r_2 and r_3.

Case I. All three indices correspond to unphysical states

In this case, let us choose,

$$|\tilde{\Phi}_{2,r_1}\rangle = |\tilde{\Phi}_{2,\alpha_2}\rangle, \quad |\tilde{\Phi}_{2,r_2}\rangle = |\tilde{\Phi}_{2,\beta_2}\rangle, \quad |\tilde{\Phi}_{2,r_3}\rangle = |\tilde{\Phi}_{2,\gamma_2}\rangle \tag{3.4}$$

Since $\tilde{A}^{(2)}_{rs} = \langle\tilde{\Phi}_{2,r}|\tilde{Q}_B c_0^-|\tilde{\Phi}_{2,s}\rangle$ is non-vanishing only when $|\tilde{\Phi}_{2,r}\rangle$ and $|\tilde{\Phi}_{2,s}\rangle$ are unphysical states, the only non-vanishing components of $\tilde{A}^{(2)}$ are of the form $\tilde{A}^{(2)}_{\alpha_2\beta_2}$. We may now express eq.(2.40) as,

$$\tilde{A}^{(3)}_{\alpha_2\beta_2\gamma_2} = \bar{A}^{(3)}_{\alpha_2\beta_2\gamma_2} + (\tilde{A}^{(2)}_{\alpha_2\delta_2}\bar{S}^{(1)}_{\delta_2\beta_2\gamma_2} + \tilde{A}^{(2)}_{\beta_2\delta_2}\bar{S}^{(1)}_{\delta_2\alpha_2\gamma_2} + \tilde{A}^{(2)}_{\gamma_2\delta_2}\bar{S}^{(1)}_{\delta_2\alpha_2\beta_2}) \tag{3.5}$$

From eq.(3.2) and the definition of $\tilde{A}^{(2)}$ it is clear that the matrix $\tilde{A}^{(2)}_{\alpha_2\beta_2}$ does not have any left or right eigenvector with zero eigenvalue. Hence it is an invertible matrix.[⋆] Let $M_{\alpha_2\beta_2}$ be this inverse. Eq.(3.5) may then be satisfied by choosing,

$$\bar{S}^{(1)}_{\alpha_2\beta_2\gamma_2} = \frac{1}{3} M_{\alpha_2\delta_2} (\tilde{A}^{(3)}_{\delta_2\beta_2\gamma_2} - \bar{A}^{(3)}_{\delta_2\beta_2\gamma_2} + L^{(1)}_{\delta_2\beta_2\gamma_2}) \tag{3.6}$$

where $L^{(1)}_{\delta_2\beta_2\gamma_2}$ is an arbitrary tensor which is symmetric in β_2 and γ_2 and satisfies:

$$L^{(1)}_{\alpha_2\beta_2\gamma_2} + L^{(1)}_{\beta_2\gamma_2\alpha_2} + L^{(1)}_{\gamma_2\alpha_2\beta_2} = 0 \tag{3.7}$$

In particular, we may take $L^{(1)}$ to be 0, but eq.(3.7) represents the most general solution of eq.(3.5). As we shall see later in this section, $L^{(1)}$ needs to be adjusted appropriately in order to ensure that the coefficients $\bar{S}^{(1)}$ do not have any discontinuity as a function of the momenta of various external states.

Case II. One of the indices corresponds to a BRST invariant state, others correspond to unphysical states

Since eq.(2.40) is completely symmetric in the indices r_1, r_2 and r_3, we may, without any loss of generality, choose:

$$|\tilde{\Phi}_{2,r_1}\rangle = |\tilde{\Phi}_{2,\mu_2}\rangle, \quad |\tilde{\Phi}_{2,r_2}\rangle = |\tilde{\Phi}_{2,\alpha_2}\rangle, \quad |\tilde{\Phi}_{2,r_3}\rangle = |\tilde{\Phi}_{2,\beta_2}\rangle \tag{3.8}$$

Eq.(2.40) then takes the form:

$$\tilde{A}^{(3)}_{\mu_2\alpha_2\beta_2} = \bar{A}^{(3)}_{\mu_2\alpha_2\beta_2} + (\tilde{A}^{(2)}_{\alpha_2\delta_2}\bar{S}^{(1)}_{\delta_2\mu_2\beta_2} + \tilde{A}^{(2)}_{\beta_2\delta_2}\bar{S}^{(1)}_{\delta_2\mu_2\alpha_2}) \tag{3.9}$$

This equation may be solved by choosing,

$$\bar{S}^{(1)}_{\delta_2\mu_2\beta_2} = \bar{S}^{(1)}_{\delta_2\beta_2\mu_2} = \frac{1}{2} M_{\delta_2\alpha_2} (\tilde{A}^{(3)}_{\alpha_2\mu_2\beta_2} - \bar{A}^{(3)}_{\alpha_2\mu_2\beta_2} + L'^{(1)}_{\alpha_2\beta_2\mu_2}) \tag{3.10}$$

where $L'^{(1)}$ is an arbitrary tensor, satisfying,

$$L'^{(1)}_{\alpha_2\beta_2\mu_2} + L'^{(1)}_{\beta_2\alpha_2\mu_2} = 0 \tag{3.11}$$

Case III. Two of the indices correspond to BRST invariant states, the

⋆ Although $\tilde{A}^{(2)}_{\alpha_2\beta_2}$ is an infinite dimensional matrix, it is block diagonal in the basis which are chosen to be $\tilde{L}_0^+$ and the momentum eigenstates, with each block a finite dimensional matrix. Thus the results for finite dimensional matrices can be applied here.

third one corresponds to an unphysical state

In this case we choose,

$$|\tilde{\Phi}_{2,r_1}\rangle = |\tilde{\Phi}_{2,\mu_2}\rangle, \quad |\tilde{\Phi}_{2,r_2}\rangle = |\tilde{\Phi}_{2,\nu_2}\rangle, \quad |\tilde{\Phi}_{2,r_3}\rangle = |\tilde{\Phi}_{2,\alpha_2}\rangle \tag{3.12}$$

Eq.(2.40) now takes the form:

$$\tilde{A}^{(3)}_{\mu_2\nu_2\alpha_2} = \bar{A}^{(3)}_{\mu_2\nu_2\alpha_2} + \tilde{A}^{(2)}_{\alpha_2\delta_2}\bar{S}^{(1)}_{\delta_2\mu_2\nu_2} \tag{3.13}$$

which can be satisfied by choosing,

$$\bar{S}^{(1)}_{\delta_2\mu_2\nu_2} = M_{\delta_2\alpha_2}(\tilde{A}^{(3)}_{\alpha_2\mu_2\nu_2} - \bar{A}^{(3)}_{\alpha_2\mu_2\nu_2}) \tag{3.14}$$

Case IV. All three indices correspond to BRST invariant states

In this case, we choose,

$$|\tilde{\Phi}_{2,r_1}\rangle = |\tilde{\Phi}_{2,\mu_2}\rangle, \quad |\tilde{\Phi}_{2,r_2}\rangle = |\tilde{\Phi}_{2,\nu_2}\rangle, \quad |\tilde{\Phi}_{2,r_3}\rangle = |\tilde{\Phi}_{2,\tau_2}\rangle \tag{3.15}$$

and eq.(2.40) takes the form:

$$\tilde{A}^{(3)}_{\mu_2\nu_2\tau_2} = \bar{A}^{(3)}_{\mu_2\nu_2\tau_2} \tag{3.16}$$

Note that the coefficients $\bar{S}^{(1)}$ have dropped out of this equation. Hence this equation cannot be satisfied by adjusting $\bar{S}^{(1)}$, it must be satisfied identically. We shall now show that this is indeed the case. First let us consider the case where at least one of the states $|\tilde{\Phi}_{2,\mu_2}\rangle$, $|\tilde{\Phi}_{2,\nu_2}\rangle$ and $|\tilde{\Phi}_{2,\tau_2}\rangle$ correspond to a pure gauge state. Without any loss of generality we can take this to be the state $|\tilde{\Phi}_{2,\mu_2}\rangle$. Thus we take,

$$|\tilde{\Phi}_{2,\mu_2}\rangle = |\tilde{\Phi}_{2,\alpha_1}\rangle = \tilde{Q}_B|\tilde{\Phi}_{1,\alpha_1}\rangle \tag{3.17}$$

We shall now show that for this choice of $|\tilde{\Phi}_{2,\mu_2}\rangle$, both, the left and the right hand side of eq.(3.16) vanish as a consequence of gauge invariance. This is intuitively obvious, since it involves on-shell three point amplitudes involving external pure gauge states; but we shall go through the proof in some detail, since it will be useful to derive a similar result in the next section. Let $\{\langle\tilde{\Phi}^c_{n,r}|\}$ denote a basis of states conjugate to $|\tilde{\Phi}_{n,r}\rangle$, satisfying,

$$\langle\tilde{\Phi}^c_{n,r}|\tilde{\Phi}_{n,s}\rangle'' = \delta_{rs} \tag{3.18}$$

Expanding the gauge transformation parameter $|\tilde{\Lambda}\rangle$ given in eq.(2.5) as,

$$b_0^-|\tilde{\Lambda}\rangle = \sum_{\alpha_1}\tilde{\lambda}_{\alpha_1}|\tilde{\Phi}_{1,\alpha_1}\rangle \tag{3.19}$$

we may express eq.(2.5) as,

$$\delta\tilde{\psi}_r = \sum_{N=0}^{\infty}\frac{1}{N!}\tilde{B}^{(N)}_{r\alpha_1 s_1\ldots s_N}\tilde{\psi}_{s_1}\ldots\tilde{\psi}_{s_N}\tilde{\lambda}_{\alpha_1} \tag{3.20}$$

where,

$$\tilde{B}^{(0)}_{r\alpha_1} = \langle\tilde{\Phi}^c_{2,r}|\tilde{Q}_B|\tilde{\Phi}_{1,\alpha_1}\rangle'' = \langle\tilde{\Phi}^c_{2,r}|\tilde{\Phi}_{2,\alpha_1}\rangle'' = \delta_{r\alpha_1} \tag{3.21}$$

and,

$$\tilde{B}^{(N)}_{r\alpha_1 s_1\ldots s_N} = g^N\langle\tilde{\Phi}^c_{2,r}|[(c_0^-\tilde{\Phi}_{2,s_1})\ldots(c_0^-\tilde{\Phi}_{2,s_N})(c_0^-\tilde{\Phi}_{1,\alpha_1})]''\rangle'' \quad \text{for } N\geq 1 \tag{3.22}$$

Invariance of the action (2.12) under the infinitesimal transformation given in eq.(3.20) gives,

$$\tilde{A}^{(2)}_{r_1r_2}\tilde{B}^{(0)}_{r_2\alpha_1} = 0 \tag{3.23}$$

$$(\tilde{A}^{(2)}_{r_1s}\tilde{B}^{(1)}_{s\alpha_1r_2} + \tilde{A}^{(2)}_{r_2s}\tilde{B}^{(1)}_{s\alpha_1r_1}) + \tilde{A}^{(3)}_{r_1r_2s}\tilde{B}^{(0)}_{s\alpha_1} = 0 \tag{3.24}$$

$$\begin{gathered}\Big(\tilde{A}^{(2)}_{r_1s}\tilde{B}^{(2)}_{s\alpha_1r_2r_3} + \tilde{A}^{(2)}_{r_2s}\tilde{B}^{(2)}_{s\alpha_1r_3r_1} + \tilde{A}^{(2)}_{r_3s}\tilde{B}^{(2)}_{s\alpha_1r_1r_2}) + \tilde{A}^{(4)}_{sr_1r_2r_3}\tilde{B}^{(0)}_{s\alpha_1} \\ + (\tilde{A}^{(3)}_{sr_1r_2}\tilde{B}^{(1)}_{s\alpha_1r_3} + \tilde{A}^{(3)}_{sr_1r_3}\tilde{B}^{(1)}_{s\alpha_1r_2} + \tilde{A}^{(3)}_{sr_2r_3}\tilde{B}^{(1)}_{s\alpha_1r_1}) = 0\end{gathered} \tag{3.25}$$

etc. Let us now choose the indices r_1 and r_2 in eq.(3.24) such that,

$$|\tilde{\Phi}_{2,r_1}\rangle = |\tilde{\Phi}_{2,\nu_2}\rangle, \quad |\tilde{\Phi}_{2,r_2}\rangle = |\tilde{\Phi}_{2,\tau_2}\rangle \tag{3.26}$$

In this case $\tilde{A}^{(2)}_{\nu_2s}$ and $\tilde{A}^{(2)}_{\tau_2s}$ vanish since $\langle\tilde{\Phi}_{2,\nu_2}|\tilde{Q}_B = 0 = \langle\tilde{\Phi}_{2,\tau_2}|\tilde{Q}_B$. Using eq.(3.21) we may now bring eq.(3.24) into the form:

$$\tilde{A}^{(3)}_{\nu_2\tau_2\alpha_1} = 0 \tag{3.27}$$

In an exactly similar way the gauge transformation (2.44) may be expressed in component form as,

$$\delta\bar{\psi}_r = \sum_{N=0}^{\infty} \frac{1}{N!} \bar{B}^{(N)}_{r\alpha_1 s_1\ldots s_N} \bar{\psi}_{s_1}\ldots\bar{\psi}_{s_N}\bar{\lambda}_{\alpha_1} \tag{3.28}$$

where,

$$\begin{aligned}
\bar{B}^{(0)}_{r\alpha_1} =& \langle\tilde{\Phi}^c_{2,r}|\tilde{Q}_B|\tilde{\Phi}_{1,\alpha_1}\rangle'' = \tilde{B}^{(0)}_{r\alpha_1} \\
\bar{B}^{(N)}_{r\alpha_1 s_1\ldots s_N} =& g^N\langle\tilde{\Phi}^c_{2,r}|\mathcal{S}^{-1}[(c_0^-\mathcal{S}|\tilde{\Phi}_{2,s_1}\rangle)\ldots(c_0^-\mathcal{S}|\tilde{\Phi}_{2,s_N}\rangle)(c_0^-\mathcal{S}|\tilde{\Phi}_{1,\alpha_1}\rangle)]'\rangle''
\end{aligned} \tag{3.29}$$

The gauge invariance of the action $\bar{S}(\bar{\Psi})$ given in eq.(2.36) under the transformation (3.28) gives an equation identical to eq.(3.24) with $\tilde{B}^{(1)}$ and $\tilde{A}^{(3)}$ replaced by $\bar{B}^{(1)}$ and $\bar{A}^{(3)}$ respectively. Choosing the various external states as in eq.(3.26) we get,

$$\bar{A}^{(3)}_{\nu_2\tau_2\alpha_1} = 0 \tag{3.30}$$

Thus we see that eq.(3.16) is satisfied trivially when at least one of the external states is of the pure gauge type. Hence we need to verify this equation when all the external states are physical. In this case the left hand side of the equation is proportional to the physical three point amplitude calculated in the string field theory formulated around CFT$''$. On the other hand, the right hand side of the equation is proportional to the physical three point amplitude calculated from the action $\bar{S}(\bar{\Psi})$, or, equivalently, $\hat{S}(\hat{\Psi})$, which is related to $\bar{S}(\bar{\Psi})$ by a linear field redefinition. The equality of these two amplitudes was proved in complete detail in ref.[13]. Hence we reach the conclusion that eq.(3.16) is satisfied identically, thereby proving that it is possible to satisfy eq.(2.40) (and hence eq.(2.27)) by appropriately adjusting the coefficients $\bar{S}^{(1)}$.

Continuity of $S^{(0)}$, $S^{(1)}$ as functions of external momenta

The equations (3.6), (3.10) and (3.14), which determine the components $\bar{S}^{(1)}_{\alpha_2 rs}$ for different choices of the external states r, s, appear to be drastically different. This may be the cause of some alarm, since, if we choose a basis of states for a given momentum, and vary the momentum continuously, then some of the unphysical states in the basis for generic values of the momentum may become BRST invariant states for some special values of the momentum. Eqs.(3.6), (3.10) and (3.14) would then seem to indicate that the components of $\bar{S}^{(1)}$ change discontinuously at these special values of momenta. To be more concrete, let us consider the case when all the space-like flat directions have been compactified and have discrete momenta/winding number, so that the only continuous index is the momentum associated with the time like directions. Let us denote this by k^0. If we consider the set of states in $\mathcal{H}$ with a fixed value of k^0, then, for generic k^0, all the states have $\tilde{L}_0^+ \neq 0$, and hence can be divided into pure gauge and unphysical states. In this case we can choose the basis of unphysical and pure gauge states in such a way that as we vary k^0 continuously, the unphysical states change smoothly into unphysical states, and pure gauge states change smoothly into pure gauge states. This continues till we reach some special values of k^0 (say k^0_c) for which $\tilde{L}_0^+ = 0$ states appear in the basis, so that the basis may contain physical states. In this case, the basis of states may be so chosen that as we approach k^0_c from a neighbouring value of k^0, some of the unphysical states in the basis become physical, the rest remains unphysical; similarly some of the pure gauge states in the basis become physical, the rest remains pure gauge.[⋆] As a result, the coefficients

⋆ Note that the usual choice of gauge, $b_0^+ = 0$, is not a good gauge choice for a general analysis, since the basis of physical, unphysical, and pure gauge states in this gauge collapses at $\tilde{L}_+^0 = 0$ ($k^0 = k^0_c$). In particular, at $k^0 = k^0_c$, new unphysical states may appear which do not satisfy the $b_0^+ = 0$ gauge condition. In other words, if we choose a basis of unphysical and pure gauge states for $k^0 \neq k^0_c$, then it is not true that in the $k^0 \to k^0_c$ limit the set of unphysical (pure gauge) states divide themselves into unphysical (pure gauge) and physical states.

There, is, however, one case in which $b_0^+ = 0$ is a good gauge choice; this is when at $k^0 = k^0_c$ the only $\tilde{L}_0^+ = 0$ state that appears is a tachyonic physical state. In this case,

$\bar{S}^{(1)}$ determined from eqs.(3.6), (3.10) and (3.14) may jump discontinuously as the momentum k^0 associated with one of the external states approach k_c^0. We shall now show that such a situation may be avoided by a judicious choice of the quantities $L^{(1)}$ and $L'^{(1)}$ appearing in eqs.(3.6) and (3.10) respectively.

Let us start with eq.(3.6) and assume that the momentum associated with the state $|\tilde{\Phi}_{2,\gamma_2}\rangle$ approaches k_c^0, and also that in this limit the state $|\tilde{\Phi}_{2,\gamma_2}\rangle$ approaches a specific physical state $|\tilde{\Phi}_{2,l_2^{(0)}}\rangle$. In this case eq.(3.6) gives a value of $\bar{S}^{(1)}_{\alpha_2\beta_2 l_2^{(0)}}$. If we want this value to agree with the answer given in eq.(3.10), we must demand that,

$$L^{(1)}_{\delta_2 l_2^{(0)}\beta_2} = L^{(1)}_{\delta_2\beta_2 l_2^{(0)}} = \frac{1}{2}(\tilde{A}^{(3)}_{\delta_2\beta_2 l_2^{(0)}} - \bar{A}^{(3)}_{\delta_2\beta_2 l_2^{(0)}}) + \frac{3}{2}L'^{(1)}_{\delta_2\beta_2 l_2^{(0)}} \tag{3.31}$$

On the other hand, if the momentum associated with the state $|\tilde{\Phi}_{2,\alpha_2}\rangle$ approaches k_c^0, the matrix $\langle\tilde{\Phi}_{2,\alpha_2}|\tilde{Q}_B c_0^-|\tilde{\Phi}_{2,\delta_2}\rangle''$ acquires a zero eigenvalue with eigenvector $v_{\delta_2} = \delta_{\delta_2 l_2^{(0)}}$. Thus the corresponding eigenvalue of $M_{\alpha_2\delta_2}$ has a pole. In order that $\bar{S}^{(1)}$ given in eq.(3.6) is finite in this limit, we must have,

$$L^{(1)}_{l_2^{(0)}\alpha_2\beta_2} = -(\tilde{A}^{(3)}_{l_2^{(0)}\alpha_2\beta_2} - \bar{A}^{(3)}_{l_2^{(0)}\alpha_2\beta_2}) \tag{3.32}$$

We may now ask if eqs.(3.31) and (3.32) are compatible with the constraint (3.7) that $L^{(1)}$ must satisfy. From these two equations we get,

$$L^{(1)}_{\delta_2 l_2^{(0)}\beta_2} + L^{(1)}_{\beta_2\delta_2 l_2^{(0)}} + L^{(1)}_{l_2^{(0)}\beta_2\delta_2} = 3(L'^{(1)}_{\delta_2\beta_2 l_2^{(0)}} + L'^{(1)}_{\beta_2\delta_2 l_2^{(0)}}) \tag{3.33}$$

The right hand side of eq.(3.33) vanishes identically by eq.(3.11). Hence eqs.(3.31) and (3.32) are compatible with eq.(3.7).

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in the $\tilde{L}_+^0 \to 0$ limit, an unphysical state gets converted to a physical state, without any other rearrangement of the basis. The examples that will be considered in sect.4 are all of this type, hence the $b_0^+ = 0$ gauge choice in that section will not cause any problem in our analysis.

This also indicates that in order to extend our analysis of N-point vertices in sect.4 and that in ref.[13] beyond the tachyonic states, it may be more convenient to choose a gauge other than $b_0^+ = 0$.

Next we consider eq.(3.10) and take the limit when the momentum associated with the state $|\tilde{\Phi}_{2,\beta_2}\rangle$ approaches the critical value k_c^0. Let us suppose that in this limit the state $|\tilde{\Phi}_{2,\beta_2}\rangle$ approaches the physical state $|\tilde{\Phi}_{2,l_2^{(0)}}\rangle$. In this case eq.(3.10) gives us a value of $\bar{S}^{(1)}_{\delta_2\mu_2 l_2^{(0)}}$. In order that this value agree with that obtained from eq.(3.14), we must have,

$$L'^{(1)}_{\alpha_2 l_2^{(0)}\mu_2} = \tilde{A}^{(3)}_{\alpha_2 l_2^{(0)}\mu_2} - \bar{A}^{(3)}_{\alpha_2 l_2^{(0)}\mu_2} \tag{3.34}$$

On the other hand, if in eq.(3.10) we take the momentum associated with the state $|\tilde{\Phi}_{2,\delta_2}\rangle$ approach k_c^0, then, in order that $\bar{S}^{(1)}$ approaches a finite value as $k^0 \to k_c^0$, we must demand,

$$L'^{(1)}_{l_2^{(0)}\beta_2\mu_2} = -(\tilde{A}^{(3)}_{l_2^{(0)}\mu_2\beta_2} - \bar{A}^{(3)}_{l_2^{(0)}\mu_2\beta_2}) \tag{3.35}$$

We now see that eqs.(3.34) and (3.35) satisfy the constraint (3.11):

$$L'^{(1)}_{l_2^{(0)}\alpha_2\mu_2} + L'^{(1)}_{\alpha_2 l_2^{(0)}\mu_2} = 0 \tag{3.36}$$

Finally, note that in eq.(3.14), if we let the momentum associated with the state $|\tilde{\Phi}_{2,\delta_2}\rangle$ approach $k_c^{(0)}$, then, in order that $\bar{S}^{(1)}_{\delta_2\mu_2\nu_2}$ is finite in this limit, we must have,

$$\tilde{A}^{(3)}_{l_2^{(0)}\mu_2\nu_2} - \bar{A}^{(3)}_{l_2^{(0)}\mu_2\nu_2} = 0 \tag{3.37}$$

which is part of the consistency condition given in eq.(3.16).

Similar remark also holds for the components $S^{(0)}_{rs}$. Writing $S^{(0)}_{rs} = 1 + \lambda K^{(0)}_{rs} + \mathcal{O}(\lambda^2)$, we may express eq.(2.26) as,

$$\tilde{A}^{(2)}_{rs} - \hat{A}^{(2)}_{rs} = \lambda(K^{(0)}_{ur}\hat{A}^{(2)}_{us} + K^{(0)}_{us}\hat{A}^{(2)}_{ur}) + \mathcal{O}(\lambda^2) \tag{3.38}$$

whose solution is given by,

$$\lambda K^{(0)}_{\alpha_2\beta_2} = \frac{1}{2} M_{\alpha_2\delta_2}(\tilde{A}^{(2)}_{\delta_2\beta_2} - \hat{A}^{(2)}_{\delta_2\beta_2} + L^{(0)}_{\delta_2\beta_2}) \tag{3.39}$$

$$\lambda K^{(0)}_{\alpha_2\mu_2} = M_{\alpha_2\delta_2}(\tilde{A}^{(2)}_{\delta_2\mu_2} - \hat{A}^{(2)}_{\delta_2\mu_2}) \tag{3.40}$$

where $L^{(0)}$ is a tensor satisfying,

$$L^{(0)}_{\alpha_2\beta_2} + L^{(0)}_{\beta_2\alpha_2} = 0 \tag{3.41}$$

In addition, eq.(2.30) (which, combined with eq.(2.31) is more restrictive than eq.(2.26)) gives a further constraint on $K^{(0)}$:

$$\lambda K^{(0)}_{k_2\alpha_1} = \langle\tilde{\Phi}^c_{2,k_2}|(\hat{Q}_B - \tilde{Q}_B)|\tilde{\Phi}_{1,\alpha_1}\rangle'' \tag{3.42}$$

(Note that here $\tilde{A}^{(2)}_{\delta_2\mu_2} = 0$, but $\hat{A}^{(2)}_{\delta_2\mu_2} \sim \lambda$.) As before, if the momentum k^0 associated with the state $|\tilde{\Phi}_{2,\beta_2}\rangle$ approaches k^0_c, then, in order that eq.(3.39) smoothly approaches eq.(3.40), we must have,

$$L^{(0)}_{\delta_2 l_2^{(0)}} = \tilde{A}^{(2)}_{\delta_2 l_2^{(0)}} - \hat{A}^{(2)}_{\delta_2 l_2^{(0)}} \tag{3.43}$$

On the other hand, if the momentum k^0 associated with the state $|\tilde{\Phi}_{2,\alpha_2}\rangle$ approaches k^0_c, then, in order that the right hand side of eq.(3.39) is finite in this limit, we must have,

$$L^{(0)}_{l_2^{(0)}\beta_2} = -(\tilde{A}^{(2)}_{l_2^{(0)}\beta_2} - \hat{A}^{(2)}_{l_2^{(0)}\beta_2}) \tag{3.44}$$

From eqs.(3.43) and (3.44) we get,

$$L^{(0)}_{\beta_2 l_2^{(0)}} + L^{(0)}_{l_2^{(0)}\beta_2} = 0 \tag{3.45}$$

as is required by eq.(3.41).

Finally, if we take the momentum k^0 of the state $|\tilde{\Phi}_{2,\alpha_2}\rangle$ in eq.(3.40) to approach k^0_c, then, in order that the right hand side of this equation is finite in this limit, we must have,

$$\tilde{A}^{(2)}_{l_2^{(0)}\mu_2} - \hat{A}^{(2)}_{l_2^{(0)}\mu_2} = 0 \tag{3.46}$$

This equation was shown to be satisfied identically in ref.[12].

These examples illustrate, how, by adjusting the parameters $L^{(0)}$, $L^{(1)}$ and $L'^{(1)}$, we may maintain the continuity of the solutions $S^{(0)}$ and $S^{(1)}$ as function of the momenta k^0 of various external states.

Note that in eq.(3.40), by taking the limit where the momentum k^0 associated with the state $|\tilde{\Phi}_{2,\alpha_2}\rangle$ approaches k^0_c, we can evaluate $K^{(0)}_{l_2^{(0)}\mu_2}$. (An independent expression for $K^{(0)}_{l_2^{(0)}\alpha_1}$ is obtained from eq.(3.42), but it is easy to check that this agrees with the expression for $K^{(0)}_{l_2^{(0)}\alpha_1}$ obtained from eq.(3.40) by taking $k^0 \to k^0_c$ limit and setting $|\tilde{\Phi}_{2,\mu_2}\rangle = |\tilde{\Phi}_{2,\alpha_1}\rangle$.) On the other hand, an expression for $K^{(0)}_{l_2^{(0)}k_2}$ was found in ref.[13] (eq.(3.60)) from the analysis of three and higher point functions. Thus we must ensure that these two results agree with each other. A similar situation arises also in case of $\bar{S}^{(1)}_{k_2\mu_2\nu_2}$. As we shall see in the next section, analysis of four and higher point functions determines the quantities $\bar{S}^{(1)}_{k_2\mu_2\nu_2}$; hence we must ensure that this value agrees with the value of $\bar{S}^{(1)}_{l_2^{(0)}\mu_2\nu_2}$ obtained by taking the $k^0 \to k^0_c$ limit of $\bar{S}^{(1)}_{\delta_2\mu_2\nu_2}$, where k^0 is the momentum associated with the state $|\tilde{\Phi}_{2,\delta_2}\rangle$.

A general argument showing that this must be the case may be given as follows. As we have seen, by adjusting $L^{(1)}$, $L'^{(1)}$ and $L^{(0)}$, we can ensure that none of the components of $S^{(0)}$ and $S^{(1)}$ has any singularity as a function of the momenta of the external states. Furthermore, the coefficients $\tilde{A}^{(N)}$ and $A^{(N)}$ appearing in eqs.(2.26)-(2.28) are also smooth functions of the momenta of the external states. Thus if we can find a set of solutions $S^{(N)}$ which satisfy eqs.(2.26)-(2.28) for generic values of the external momenta ($k^0 \neq k^0_c$), and set $S^{(N)}(k^0_c) = \lim_{k^0\to k^0_c} S^{(N)}(k^0)$, then this value of $S^{(N)}(k^0_c)$ must satisfy eqs.(2.26)-(2.28). Since $S^{(0)}_{k_2l_2}$ and $S^{(1)}_{k_2\mu_2\nu_2}$ determined in ref.[13] and sect.4 respectively give solutions to eqs.(2.26)-(2.28), they must be compatible with the results obtained by taking the $k^0 \to k^0_c$ limit of $S^{(0)}_{\delta_2l_2}$ and $S^{(1)}_{\delta_2\mu_2\nu_2}$, where k^0 is the momentum associated with the state $|\tilde{\Phi}_{2,\delta_2}\rangle$.

Another set of consistency conditions are obtained by starting from eq.(3.42) and going to a limit when the state $|\tilde{\Phi}_{2,\alpha_1}\rangle = \tilde{Q}_B|\tilde{\Phi}_{1,\alpha_1}\rangle$ approaches a physical state $|\tilde{\Phi}_{2,m_2^{(0)}}\rangle$ after appropriate normalization. This would happen if in this limit $|\tilde{\Phi}_{1,\alpha_1}\rangle$ reduces to a physical state so that $\tilde{Q}_B|\tilde{\Phi}_{1,\alpha_1}\rangle$ vanishes in this limit unless it is divided by an appropriate normalization factor before we take the limit. A consistency condition is then obtained by demanding that $K^{(0)}_{k_2 m_2^{(0)}}$ found this way agrees with the answer given in eq.(3.60) of ref.[13]. The analysis of appendix A shows that this is indeed the case. (Since physical states of ghost number 1 appear only in the zero momentum sector, we need to study only the $k=0$ sector for this analysis.)

There is another related point that we must mention before concluding this section. In proving the equality of the physical amplitudes calculated from the actions $\tilde{S}(\tilde{\Psi})$ and $\bar{S}(\bar{\Psi})$, we had assumed in ref.[13] that the physical states are of the form $c_1\bar{c}_1|\tilde{V}\rangle$, where $|\tilde{V}\rangle$ is a dimension (1,1) primary state in CFT$''$. While this is true for generic values of momenta (see refs.[17][18] and references therein) there is a physical state at zero momentum which cannot be expressed in this form. This is the zero momentum dilaton state, given by,

$$|D\rangle = (c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})|0\rangle \tag{3.47}$$

Since we must show the equality of the two sides of eq.(3.16) for *all* states, we must also prove this for the case where one or more of the external states correspond to the state $|D\rangle$ given in eq.(3.47). We have shown in appendix A that this is indeed what happens, thereby completing the proof that eq.(3.16) is satisfied for all values of μ_2, ν_2 and τ_2.

4. ANALYSIS OF FOUR AND HIGHER POINT TERMS OF THE ACTION

In this section we shall sketch the analysis of eq.(2.41) for $N \geq 4$, but shall not give a complete proof of existence of solutions to these equations. Let us start with eq.(2.41) for $N = 4$. Following the same analysis as in the previous section, one can show that this equation can be satisfied by adjusting the coefficients $\bar{S}^{(2)}$ when at least one of the indices r_i correspond to unphysical states. In order to avoid repetition, we shall not go through the whole analysis, but only consider a single case as an example. Consider the case,

$$|\tilde{\Phi}_{2,r_1}\rangle = |\tilde{\Phi}_{2,\mu_2}\rangle, \quad |\tilde{\Phi}_{2,r_2}\rangle = |\tilde{\Phi}_{2,\alpha_2}\rangle, \quad |\tilde{\Phi}_{2,r_3}\rangle = |\tilde{\Phi}_{2,\beta_2}\rangle, \quad |\tilde{\Phi}_{2,r_4}\rangle = |\tilde{\Phi}_{2,\gamma_2}\rangle \tag{4.1}$$

In this case eq.(2.41) for $n = 4$ is satisfied if we choose,

$$\begin{aligned}
\bar{S}^{(2)}_{\delta_2\beta_2\gamma_2\mu_2} =& \frac{1}{3} M_{\delta_2\alpha_2} [\tilde{A}^{(4)}_{\alpha_2\beta_2\gamma_2\mu_2} - \bar{A}^{(4)}_{\alpha_2\beta_2\gamma_2\mu_2} - (\bar{A}^{(3)}_{s\alpha_2\beta_2}\bar{S}^{(1)}_{s\gamma_2\mu_2} + \bar{A}^{(3)}_{s\alpha_2\gamma_2}\bar{S}^{(1)}_{s\beta_2\mu_2} + \bar{A}^{(3)}_{s\beta_2\gamma_2}\bar{S}^{(1)}_{s\alpha_2\mu_2} \\
& + \bar{A}^{(3)}_{s\alpha_2\mu_2}\bar{S}^{(1)}_{s\beta_2\gamma_2} + \bar{A}^{(3)}_{s\beta_2\mu_2}\bar{S}^{(1)}_{s\alpha_2\gamma_2} + \bar{A}^{(3)}_{s\gamma_2\mu_2}\bar{S}^{(1)}_{s\alpha_2\beta_2}) \\
& - \tilde{A}^{(2)}_{ss'}(\bar{S}^{(1)}_{s\alpha_2\beta_2}\bar{S}^{(1)}_{s'\gamma_2\mu_2} + \bar{S}^{(1)}_{s\beta_2\gamma_2}\bar{S}^{(1)}_{s'\alpha_2\mu_2} + \bar{S}^{(1)}_{s\alpha_2\gamma_2}\bar{S}^{(1)}_{s'\beta_2\mu_2}) + L^{(2)}_{\alpha_2\beta_2\gamma_2\mu_2}]
\end{aligned} \tag{4.2}$$

where $L^{(2)}_{\alpha_2\beta_2\gamma_2\mu_2}$ is an arbitrary tensor which is symmetric in β_2 and γ_2 and satisfies the relation:

$$L^{(2)}_{\alpha_2\beta_2\gamma_2\mu_2} + L^{(2)}_{\beta_2\gamma_2\alpha_2\mu_2} + L^{(2)}_{\gamma_2\alpha_2\beta_2\mu_2} = 0 \tag{4.3}$$

Similar analysis can be done for other choices of indices. The only possible problem comes from the case where all the indices r_i correspond to BRST invariant states, since in this case the terms involving $\bar{S}^{(2)}$ drop out of eq.(2.41) for $N = 4$. More precisely, for the choice,

$$|\tilde{\Phi}_{2,r_1}\rangle = |\tilde{\Phi}_{2,\mu_2}\rangle, \quad |\tilde{\Phi}_{2,r_2}\rangle = |\tilde{\Phi}_{2,\nu_2}\rangle, \quad |\tilde{\Phi}_{2,r_3}\rangle = |\tilde{\Phi}_{2,\tau_2}\rangle, \quad |\tilde{\Phi}_{2,r_4}\rangle = |\tilde{\Phi}_{2,\sigma_2}\rangle \tag{4.4}$$

eq.(2.41) for $N = 4$ takes the form:

$$\begin{aligned}\tilde{A}^{(4)}_{\mu_2\nu_2\tau_2\sigma_2} =& \bar{A}^{(4)}_{\mu_2\nu_2\tau_2\sigma_2} + \tilde{A}^{(2)}_{ss'}(\bar{S}^{(1)}_{s\mu_2\nu_2}\bar{S}^{(1)}_{s'\tau_2\sigma_2} + \bar{S}^{(1)}_{s\mu_2\tau_2}\bar{S}^{(1)}_{s'\nu_2\sigma_2} + \bar{S}^{(1)}_{s\mu_2\sigma_2}\bar{S}^{(1)}_{s'\nu_2\tau_2}) \\ &+ (\bar{A}^{(3)}_{s\mu_2\nu_2}\bar{S}^{(1)}_{s\sigma_2\tau_2} + \bar{A}^{(3)}_{s\mu_2\sigma_2}\bar{S}^{(1)}_{s\nu_2\tau_2} + \bar{A}^{(3)}_{s\mu_2\tau_2}\bar{S}^{(1)}_{s\nu_2\sigma_2} + \bar{A}^{(3)}_{s\nu_2\tau_2}\bar{S}^{(1)}_{s\mu_2\sigma_2} \\ &+ \bar{A}^{(3)}_{s\nu_2\sigma_2}\bar{S}^{(1)}_{s\mu_2\tau_2} + \bar{A}^{(3)}_{s\tau_2\sigma_2}\bar{S}^{(1)}_{s\mu_2\nu_2})\end{aligned} \tag{4.5}$$

In the last set of terms on the right hand side of eq.(4.5), the sum over s can be broken up into sum over physical, unphysical, and pure gauge states. Of these, the sum over pure gauge states vanishes, since, as we have seen in the last section, $\bar{A}^{(3)}_{s\mu_2\nu_2}$ vanish for such states. In the second set of terms on the right hand side of eq.(4.5) the sum over s and s' can be restricted to unphysical states, since $\tilde{A}_{ss'}$ vanishes for other set of states. Using eq.(3.14), and the fact that $M_{\alpha_2\beta_2}$ is the inverse of $\tilde{A}_{\alpha_2\beta_2}$, we may bring eq.(4.5) to the form:

$$\begin{aligned}&\tilde{A}^{(4)}_{\mu_2\nu_2\tau_2\sigma_2} - \tilde{A}^{(3)}_{\mu_2\nu_2\alpha_2}M_{\alpha_2\beta_2}\tilde{A}^{(3)}_{\beta_2\tau_2\sigma_2} - \tilde{A}^{(3)}_{\mu_2\tau_2\alpha_2}M_{\alpha_2\beta_2}\tilde{A}^{(3)}_{\beta_2\nu_2\sigma_2} - \tilde{A}^{(3)}_{\mu_2\sigma_2\alpha_2}M_{\alpha_2\beta_2}\tilde{A}^{(3)}_{\beta_2\nu_2\tau_2} \\ =&[\bar{A}^{(4)}_{\mu_2\nu_2\tau_2\sigma_2} - \bar{A}^{(3)}_{\mu_2\nu_2\alpha_2}M_{\alpha_2\beta_2}\bar{A}^{(3)}_{\beta_2\tau_2\sigma_2} - \bar{A}^{(3)}_{\mu_2\tau_2\alpha_2}M_{\alpha_2\beta_2}\bar{A}^{(3)}_{\beta_2\nu_2\sigma_2} - \bar{A}^{(3)}_{\mu_2\sigma_2\alpha_2}M_{\alpha_2\beta_2}\bar{A}^{(3)}_{\beta_2\nu_2\tau_2}] \\ &+ (\bar{A}^{(3)}_{k_2\mu_2\nu_2}\bar{S}^{(1)}_{k_2\sigma_2\tau_2} + \bar{A}^{(3)}_{k_2\mu_2\sigma_2}\bar{S}^{(1)}_{k_2\nu_2\tau_2} + \bar{A}^{(3)}_{k_2\mu_2\tau_2}\bar{S}^{(1)}_{k_2\sigma_2\nu_2} + \bar{A}^{(3)}_{k_2\nu_2\tau_2}\bar{S}^{(1)}_{k_2\mu_2\sigma_2} \\ &+ \bar{A}^{(3)}_{k_2\nu_2\sigma_2}\bar{S}^{(1)}_{k_2\mu_2\tau_2} + \bar{A}^{(3)}_{k_2\tau_2\sigma_2}\bar{S}^{(1)}_{k_2\mu_2\nu_2})\end{aligned} \tag{4.6}$$

We shall first show that the above equation can be satisfied when one of the external states (say $|\tilde{\Phi}_{2,\mu_2}\rangle$) is of the form of a pure gauge state. Let us take $|\tilde{\Phi}_{2,\mu_2}\rangle = |\tilde{\Phi}_{2,\alpha_1}\rangle = \tilde{Q}_B|\tilde{\Phi}_{1,\alpha_1}\rangle = \tilde{B}^{(0)}_{r\alpha_1}|\tilde{\Phi}_{2,r}\rangle$. In this case, in eq.(4.6), all factors of $\tilde{A}^{(N)}_{\mu_2 r_1\ldots r_{N-1}}$ and $\bar{A}^{(N)}_{\mu_2 r_1\ldots r_{N-1}}$ are replaced by $\tilde{A}^{(N)}_{rr_1\ldots r_{N-1}}\tilde{B}^{(0)}_{r\alpha_1}$ and $\bar{A}^{(N)}_{rr_1\ldots r_{N-1}}\tilde{B}^{(0)}_{r\alpha_1}$ respectively. The left hand side of eq.(4.6) then takes the form:

$$(\tilde{A}^{(4)}_{r\nu_2\tau_2\sigma_2} - \tilde{A}^{(3)}_{r\nu_2\alpha_2}M_{\alpha_2\beta_2}\tilde{A}^{(3)}_{\beta_2\tau_2\sigma_2} - \tilde{A}^{(3)}_{r\tau_2\alpha_2}M_{\alpha_2\beta_2}\tilde{A}^{(3)}_{\beta_2\nu_2\sigma_2} - \tilde{A}^{(3)}_{r\sigma_2\alpha_2}M_{\alpha_2\beta_2}\tilde{A}^{(3)}_{\beta_2\nu_2\tau_2})\tilde{B}^{(0)}_{r\alpha_1} \tag{4.7}$$

Choosing $r_1 = \nu_2$, $r_2 = \alpha_2$ in eq.(3.24) we get,

$$\tilde{A}^{(2)}_{\alpha_2\beta_2}\tilde{B}^{(1)}_{\beta_2\alpha_1\nu_2} + \tilde{A}^{(3)}_{\nu_2\alpha_2 r}\tilde{B}^{(0)}_{r\alpha_1} = 0 \tag{4.8}$$

Using this and similar equations with ν_2 replaced by τ_2, σ_2, and the relation

$\tilde{A}^{(2)}_{\alpha_2\beta_2} M_{\beta_2\gamma_2} = \delta_{\alpha_2\gamma_2}$, eq.(4.7) may be brought into the form:

$$\tilde{A}^{(4)}_{r\nu_2\tau_2\sigma_2}\tilde{B}^{(0)}_{r\alpha_1} + \tilde{A}^{(3)}_{\beta_2\tau_2\sigma_2}\tilde{B}^{(1)}_{\beta_2\alpha_1\nu_2} + \tilde{A}^{(3)}_{\beta_2\nu_2\sigma_2}\tilde{B}^{(1)}_{\beta_2\alpha_1\tau_2} + \tilde{A}^{(3)}_{\beta_2\nu_2\tau_2}\tilde{B}^{(1)}_{\beta_2\alpha_1\sigma_2} \tag{4.9}$$

On the other hand, choosing $r_1 = \nu_2$, $r_2 = \tau_2$, $r_3 = \sigma_2$ in eq.(3.25), and using the relations $\tilde{A}^{(2)}_{r\tau_2} = \tilde{A}^{(2)}_{r\sigma_2} = \tilde{A}^{(2)}_{r\nu_2} = 0$, we get,

$$\tilde{A}^{(4)}_{r\nu_2\sigma_2\tau_2}\tilde{B}^{(0)}_{r\alpha_1} + \tilde{A}^{(3)}_{r\tau_2\sigma_2}\tilde{B}^{(1)}_{r\alpha_1\nu_2} + \tilde{A}^{(3)}_{r\nu_2\sigma_2}\tilde{B}^{(1)}_{r\alpha_1\tau_2} + \tilde{A}^{(3)}_{r\tau_2\nu_2}\tilde{B}^{(1)}_{r\alpha_1\sigma_2} = 0 \tag{4.10}$$

Comparing the left hand side of eq.(4.10) with the expression given in eq.(4.9) we see that they differ from each other only in the fact that the sum over r in eq.(4.10) runs over all states, whereas the sum over β_2 in eq.(4.9) runs over unphysical states only. Since $\tilde{A}^{(3)}_{r\tau_2\sigma_2}$ etc. vanish by eq.(3.27) if $|\tilde{\Phi}_{2,r}\rangle$ is a pure gauge state, we see that the only extra terms on the left hand side of eq.(4.10) involves terms where $|\tilde{\Phi}_{2,r}\rangle$ corresponds to a physical state. In other words, using eq.(4.10), the expression (4.9) may be brought into the form:

$$-\tilde{A}^{(3)}_{k_2\tau_2\sigma_2}\tilde{B}^{(1)}_{k_2\alpha_1\nu_2} - \tilde{A}^{(3)}_{k_2\nu_2\sigma_2}\tilde{B}^{(1)}_{k_2\alpha_1\tau_2} - \tilde{A}^{(3)}_{k_2\nu_2\tau_2}\tilde{B}^{(1)}_{k_2\alpha_1\sigma_2} \tag{4.11}$$

Using a similar analysis, the terms inside the square bracket on the right hand side of eq.(4.6) may be brought into the form of eq.(4.11) with $\tilde{A}^{(3)}$ and $\tilde{B}^{(1)}$ replaced by $\bar{A}^{(3)}$ and $\bar{B}^{(1)}$ respectively. Finally, the first three terms inside the parantheses in the right hand side of eq.(4.6) vanish by eq.(3.27) for the choice $|\tilde{\Phi}_{2,\mu_2}\rangle = |\tilde{\Phi}_{2,\alpha_1}\rangle$. The equation (4.6) then takes the form:

$$\begin{aligned}
&\bar{A}^{(3)}_{k_2\nu_2\tau_2}\bar{S}^{(1)}_{k_2\alpha_1\sigma_2} + \bar{A}^{(3)}_{k_2\nu_2\sigma_2}\bar{S}^{(1)}_{k_2\alpha_1\tau_2} + \bar{A}^{(3)}_{k_2\sigma_2\tau_2}\bar{S}^{(1)}_{k_2\alpha_1\nu_2} \\
=&\bar{B}^{(1)}_{k_2\alpha_1\nu_2}\bar{A}^{(3)}_{k_2\tau_2\sigma_2} + \bar{B}^{(1)}_{k_2\alpha_1\tau_2}\bar{A}^{(3)}_{k_2\nu_2\sigma_2} + \bar{B}^{(1)}_{k_2\alpha_1\sigma_2}\bar{A}^{(3)}_{k_2\tau_2\nu_2} \\
-&\tilde{B}^{(1)}_{k_2\alpha_1\nu_2}\tilde{A}^{(3)}_{k_2\tau_2\sigma_2} - \tilde{B}^{(1)}_{k_2\alpha_1\tau_2}\tilde{A}^{(3)}_{k_2\nu_2\sigma_2} - \tilde{B}^{(1)}_{k_2\alpha_1\sigma_2}\tilde{A}^{(3)}_{k_2\tau_2\nu_2}
\end{aligned} \tag{4.12}$$

Using the consistency condition $\bar{A}^{(3)}_{k_2\tau_2\sigma_2} = \tilde{A}^{(3)}_{k_2\tau_2\sigma_2}$ we see that eq.(4.12) is satisfied if we choose,

$$\bar{S}^{(1)}_{k_2\alpha_1\sigma_2} = \bar{B}^{(1)}_{k_2\alpha_1\sigma_2} - \tilde{B}^{(1)}_{k_2\alpha_1\sigma_2} \tag{4.13}$$

Thus we see that eq.(4.6) can be satisfied by appropriately adjusting $\bar{S}^{(1)}_{k_2\alpha_1\sigma_2}$ if at least one of the external states is pure gauge. It now remains to analyze eq.(4.6) for the case where the states $|\tilde{\Phi}_{2,\mu_2}\rangle$, $|\tilde{\Phi}_{2,\nu_2}\rangle$, $|\tilde{\Phi}_{2,\tau_2}\rangle$ and $|\tilde{\Phi}_{2,\sigma_2}\rangle$ are all physical. We consider two cases separately. First we shall analyze the case where the external momenta are such that no physical state can appear as intermediate state either in s, t or u channel. This, in fact, is the generic situation, since for generic values of external momenta, the states allowed by momentum conservation as the intermediate states in the s, t and u channel have non-zero L_0^+ eigenvalue. In this case, $\bar{A}^{(3)}_{k_2\mu_2\nu_2}$, $\bar{A}^{(3)}_{k_2\mu_2\tau_2}$, $\bar{A}^{(3)}_{k_2\mu_2\sigma_2}$, $\bar{A}^{(3)}_{k_2\nu_2\tau_2}$, $\bar{A}^{(3)}_{k_2\nu_2\sigma_2}$, $\bar{A}^{(3)}_{k_2\sigma_2\tau_2}$ vanish for all k_2, and hence the last set of terms on the right hand side of eq.(4.6) vanish. Also, in this case, $M_{\alpha_2\beta_2}$, being the inverse of $\tilde{A}^{(2)}_{\alpha_2\beta_2}$ in the subspace of unphysical states (which in this case coincides with the subspace of states which are not pure gauge) may be interpreted as the propagator of the gauge fixed theory, both, for the theory described by the action $\bar{S}(\bar{\psi})$ and the theory described by the action $\tilde{S}(\tilde{\psi})$. As a result, the left and the right hand side of eq.(4.6) are proportional to the amputated four point Green's functions in the theories described by the actions $\tilde{S}(\tilde{\Psi})$ and $\bar{S}(\bar{\Psi})$ respectively, with BRST invariant external states. Thus in this case the problem reduces to showing that the on-shell four point functions in the two theories are identical.

This is precisely the problem that we analyzed in ref.[13], where we proved that the on-shell amplitudes in these two theories with arbitrary number of tachyonic external legs are indeed identical.* Although the proof of this result could not be extended to the case of arbitrary external states due to some technical difficulties, we believe that the result does hold for arbitrary external states, particularly in view of the fact that many of the S-matrix elements involving higher excited states

$\star$ In ref.[13] we worked with the action $\hat{S}(\hat{\Psi})$ instead of $\bar{S}(\bar{\Psi})$, but since they are related by a linear field redefinition, they have the same on-shell S-matrix elements.

in string theory can be determined by studying the S-matrix elements involving tachyonic states near the poles.

We now turn to the case of special values of momenta, which allow physical external states to propagate in the s, t or u channel. We shall assume, for simplicity, that this happens only in one channel, say the $\mu_2\nu_2 \to \sigma_2\tau_2$ channel, although our analysis can be easily extended to the case where this happens in more than one channel simultaneously. Also, in order to avoid technical difficulties similar to the one that arose in the analysis of ref.[13] for arbitrary external states, we shall restrict ourselves to the case where all the external states are tachyonic, and the physical states that appear in the intermediate channel are also tachyonic. In this case, among the last set of terms on the right hand side of eq.(4.6), only the terms involving $\bar{A}^{(3)}_{k_2\mu_2\nu_2}$ and $\bar{A}^{(3)}_{k_2\sigma_2\tau_2}$ survive, and this equation can be written as,

$$\begin{aligned}
&\tilde{A}^{(4)}_{\mu_2\nu_2\tau_2\sigma_2} - \tilde{A}^{(3)}_{\mu_2\nu_2\alpha_2} M_{\alpha_2\beta_2} \tilde{A}^{(3)}_{\beta_2\tau_2\sigma_2} - \tilde{A}^{(3)}_{\mu_2\tau_2\alpha_2} M_{\alpha_2\beta_2} \tilde{A}^{(3)}_{\beta_2\nu_2\sigma_2} - \tilde{A}^{(3)}_{\mu_2\sigma_2\alpha_2} M_{\alpha_2\beta_2} \tilde{A}^{(3)}_{\beta_2\nu_2\tau_2} \\
&- [\bar{A}^{(4)}_{\mu_2\nu_2\tau_2\sigma_2} - \bar{A}^{(3)}_{\mu_2\nu_2\alpha_2} M_{\alpha_2\beta_2} \bar{A}^{(3)}_{\beta_2\tau_2\sigma_2} - \bar{A}^{(3)}_{\mu_2\tau_2\alpha_2} M_{\alpha_2\beta_2} \bar{A}^{(3)}_{\beta_2\nu_2\sigma_2} - \bar{A}^{(3)}_{\mu_2\sigma_2\alpha_2} M_{\alpha_2\beta_2} \bar{A}^{(3)}_{\beta_2\nu_2\tau_2}] \\
=&(\bar{A}^{(3)}_{k_2\mu_2\nu_2} \bar{S}^{(1)}_{k_2\sigma_2\tau_2} + \bar{A}^{(3)}_{k_2\tau_2\sigma_2} \bar{S}^{(1)}_{k_2\mu_2\nu_2})
\end{aligned} \tag{4.14}$$

It is shown in appendix B that if $\bar{A}^{(3)}_{\mu_2\nu_2k_2}$ and $\bar{A}^{(3)}_{\sigma_2\tau_2k_2}$ vanish for all k_2, then the left hand side of eq.(4.14) vanishes. Thus the left hand side of eq.(4.14) must be of the form:

$$\bar{A}^{(3)}_{\mu_2\nu_2k_2} \Sigma^{(1)}_{k_2\tau_2\sigma_2} + \bar{A}^{(3)}_{\tau_2\sigma_2k_2} \Sigma^{(1)}_{k_2\mu_2\nu_2} \tag{4.15}$$

for some tensor $\Sigma^{(1)}$. Thus eq.(4.14) may be satisfied by choosing,

$$\bar{S}^{(1)}_{k_2\sigma_2\tau_2} = \Sigma^{(1)}_{k_2\sigma_2\tau_2} \tag{4.16}$$

Note that $\bar{S}^{(1)}_{k_2\sigma_2\tau_2}$, which was left undetermined during the analysis of three point functions, is determined during the analysis of four point functions. The situation is analogous to the one encountered in refs.[12][13], where the components $S^{(0)}_{k_2l_2}$ were left undetermined in the analysis of two point functions in ref.[12], but were determined during the analysis of on-shell three point functions in ref.[13].

Let us now turn to the analysis of five and higher point functions. The relevant equation to be satisfied is eq.(2.41). The term involving $\bar{S}^{(N-2)}$ on the right hand side of this equation is of the form:

$$(\tilde{A}^{(2)}_{r_1 s}\bar{S}^{(N-2)}_{s r_2 \ldots r_N} + \text{ Permutations of } r_1, \ldots r_N) \tag{4.17}$$

Generalizing the analysis for three and four point functions, we see that if at least one of the indices r_i is unphysical, then eq.(2.41) may be satisfied by adjusting $\bar{S}^{(N-2)}_{\alpha_2 s_1 \ldots s_{N-1}}$. Thus the only non-trivial constraint comes when all the indices $r_1, \ldots r_N$ correspond to BRST invariant states. Let us take,

$$|\tilde{\Phi}_{2,r_i}\rangle = |\tilde{\Phi}_{2,\mu_2^{(i)}}\rangle, \qquad 1 \le i \le N \tag{4.18}$$

Then eq.(2.41) takes the form:

$$\begin{aligned}\tilde{A}^{(N)}_{\mu_2^{(1)} \ldots \mu_2^{(N)}} =& \bar{A}^{(N)}_{\mu_2^{(1)} \ldots \mu_2^{(N)}} + \sum_{\substack{n, M, N_1, \ldots N_n \\ 2 \le M < N, 1 \le n \le M, N_i \ge 2, M + \sum N_i - n = N}} \frac{1}{n!(M-n)!\prod_i N_i!} \\ & \times \Big(\bar{A}^{(M)}_{\mu_2^{(1)} \ldots \mu_2^{(M-n)} s_1 \ldots s_n} \bar{S}^{(N_1-1)}_{s_1 \mu_2^{(M-n+1)} \ldots \mu_2^{(M-n+N_1)}} \ldots \bar{S}^{(N_n-1)}_{s_n \mu_2^{(N-N_n+1)} \ldots \mu_2^{(N)}} \\ & + \text{ All permutations of } \mu_2^{(1)} \ldots \mu_2^{(N)} \Big)\end{aligned} \tag{4.19}$$

Let us first consider the case where the external momenta are such that no physical state can propagate in the intermediate channel. In this case, the sum over all the s_i's in eq.(4.19) may be taken to be over unphysical states only. Since the analysis of eq.(2.41) for a given value of N determines $\bar{S}^{(N-2)}_{\alpha_2 r_1 \ldots r_{N-1}}$ in terms of the coefficients $\tilde{A}$, $\bar{A}$, $M_{\alpha_2\beta_2}$ and $\bar{S}^{(M-1)}$ for $M < N$, we can eliminate all the $\bar{S}$'s appearing on the right hand side of eq.(4.19) in terms of $\bar{A}$ and $M_{\alpha_2\beta_2}$. The resulting equation is an expression involving the coefficients $\tilde{A}^{(M)}$, $\bar{A}^{(M)}$ and $M_{\alpha_2\beta_2}$. We shall now give a general argument to show that this equation must be of the form,

$$\bar{\Gamma}^{(N)}_{\mu_2^{(1)}\ldots\mu_2^{(N)}} = \tilde{\Gamma}^{(N)}_{\mu_2^{(1)}\ldots\mu_2^{(N)}} \tag{4.20}$$

where $\tilde{\Gamma}$ is proportional to the amputated Greens function calculated with the propagator $iM_{\alpha_2\beta_2}$ and vertices $i\tilde{A}^{(N)}_{r_1\ldots r_N}$; for definiteness we choose the proportionality factor in such a way that $\tilde{A}^{(N)}$ appears with coefficient unity in the expression for $\tilde{\Gamma}^{(N)}$. Similarly, $\bar{\Gamma}^{(N)}$ is proportional to the amputated Green's function calculated with the propagator $iM_{\alpha_2\beta_2}$ and vertices $i\bar{A}^{(N)}_{r_1\ldots r_N}$.

Although it should be possible, in principle, to give a detailed combinatoric proof of eq.(4.20), we shall give an indirect argument here to show that this is indeed the case. From general arguments based on path integrals (or combinatoric analysis of diagrams) one can show that if two theories are related by a field redefinition, then they must have the same on-shell S-matrix elements. Since the set of equations (4.19) are the only set of constraints required for showing the equivalence of the theories described by the actions $\tilde{S}(\tilde{\Psi})$ and $\bar{S}(\bar{\Psi})$, they must include the condition that the on-shell S-matrix elements in the two theories are identical. This would happen only if the combinarotic factors work out correctly, so that eq.(4.19) reduces to eq.(4.20).

Once eq.(4.19) is expressed in the form of eq.(4.20), we see that the left and the right hand sides of this equation vanish identically due to gauge invariance if any of the external states is of the pure gauge type. Thus we only need to consider the case when all the external states are physical. Again, the problem was analyzed in ref.[13], where eq.(4.20) was proved to be true for all tachyonic external states. The same technical problems that plague the analysis of four point amplitude also prevents us from proving eq.(4.20) for a general set of external states. But the result is expected to hold for general external states also in view of the fact that amplitudes involving tachyonic states in string theory also contain most of the information about the on-shell amplitudes involving higher excited states.

Finally we turn to the case when one or more intermediate channels admit physical states propagating in them. We would expect that in this case eq.(4.19) may be satisfied by appropriate choice of the coefficients $\bar{S}^{(M-1)}_{k_2 r_1 \ldots r_M}$, as in the case of three and four point functions. We shall not analyze the general case, but illustrate this through an example. Let us consider eq.(4.19) for $N=5$, and let us suppose that physical ($L_0^+ = 0$) states can appear in the $\mu_2^{(1)}\mu_2^{(2)} \to \mu_2^{(3)}\mu_2^{(4)}\mu_2^{(5)}$ channel. We may now eliminate $\bar{S}^{(1)}_{\alpha_2 rs}$ and $\bar{S}^{(2)}_{\alpha_2 rst}$ by eqs.(3.6), (3.10), (3.14) and analogs of eq.(4.2). The final result is of the form:

$$
\begin{aligned}
&\tilde{\Gamma}^{(5)}_{\mu_2^{(1)}\ldots\mu_2^{(5)}} - \bar{\Gamma}^{(5)}_{\mu_2^{(1)}\ldots\mu_2^{(5)}} \\
=&\bar{\Gamma}^{(4)}_{\mu_2^{(3)}\mu_2^{(4)}\mu_2^{(5)}k_2} \bar{S}^{(1)}_{k_2\mu_2^{(1)}\mu_2^{(2)}} + \bar{A}^{(3)}_{\mu_2^{(1)}\mu_2^{(2)}k_2} (\bar{S}^{(2)}_{k_2\mu_2^{(3)}\mu_2^{(4)}\mu_2^{(5)}} - \bar{S}^{(1)}_{k_2\alpha_2\mu_2^{(3)}} M_{\alpha_2\beta_2} \bar{A}^{(3)}_{\beta_2\mu_2^{(4)}\mu_2^{(5)}} \\
&- \bar{S}^{(1)}_{k_2\alpha_2\mu_2^{(4)}} M_{\alpha_2\beta_2} \bar{A}^{(3)}_{\beta_2\mu_2^{(3)}\mu_2^{(5)}} - \bar{S}^{(1)}_{k_2\alpha_2\mu_2^{(5)}} M_{\alpha_2\beta_2} \bar{A}^{(3)}_{\beta_2\mu_2^{(3)}\mu_2^{(4)}})
\end{aligned}
\tag{4.21}
$$

In writing down the above equation, we have used the gauge invariance of three and four point amplitudes to eliminate terms involving $\bar{S}^{(2)}_{\alpha_1 rst}$ and $\bar{S}^{(1)}_{\alpha_1 rs}$. We now consider, as before, only those configurations for which the only $\tilde{L}_0^+ = 0$ states which can propagate in the intermediate state in the $\mu_2^{(1)}\mu_2^{(2)} \to \mu_2^{(3)}\mu_2^{(4)}\mu_2^{(5)}$ channel are tachyonic. In this case, by going through an analysis similar to the one discussed in appendix B, one can see that the left hand side of eq.(4.21) vanishes if $\bar{A}^{(3)}_{\mu_2^{(1)}\mu_2^{(2)}k_2}$ and $\bar{\Gamma}^{(4)}_{\mu_2^{(3)}\mu_2^{(4)}\mu_2^{(5)}k_2}$ vanish. Hence this has the form:

$$
\bar{A}^{(3)}_{\mu_2^{(1)}\mu_2^{(2)}k_2} \Sigma^{(2)}_{k_2\mu_2^{(3)}\mu_2^{(4)}\mu_2^{(5)}} + \bar{\Gamma}^{(4)}_{\mu_2^{(3)}\mu_2^{(4)}\mu_2^{(5)}k_2} \Sigma^{(1)}_{k_2\mu_2^{(1)}\mu_2^{(2)}}
\tag{4.22}
$$

Note that we have taken the coefficient of $\bar{\Gamma}^{(4)}$ in eq.(4.22) to be the same tensor $\Sigma^{(1)}$ as the one that appeared in eq.(4.15). This is due to the fact that the coefficient of the $\bar{\Gamma}^{(4)}$ term is expected to be determined solely by the structure of the states $|\tilde{\Phi}_{2,\mu_2^{(1)}}\rangle$, $|\tilde{\Phi}_{2,\mu_2^{(2)}}\rangle$ and the propagator $M_{\alpha_2\beta_2}$, and hence is expected to be equal to the coefficient of the $\bar{A}^{(3)}$ term on the left hand side of eq.(4.14) with appropriate external indices.⋆ Using eq.(4.16) we see that the term involving $\bar{\Gamma}^{(4)}$ in eq.(4.22)

⋆ This is similar to the result of ref.[13] where the coefficient $S^{(0)}_{k_2 l_2}$ required to satisfy the equality of on-shell N-point amplitudes in the two theories turned out to be independent of N.

is identical to the term involving $\bar\Gamma^{(4)}$ on the right hand side of eq.(4.21). Thus eq.(4.21) may be satisfied by choosing,

$$\begin{aligned}&(\bar S^{(2)}_{k_2\mu_2^{(3)}\mu_2^{(4)}\mu_2^{(5)}} - \bar S^{(1)}_{k_2\alpha_2\mu_2^{(3)}} M_{\alpha_2\beta_2}\bar A^{(3)}_{\beta_2\mu_2^{(4)}\mu_2^{(5)}} - \bar S^{(1)}_{k_2\alpha_2\mu_2^{(4)}} M_{\alpha_2\beta_2}\bar A^{(3)}_{\beta_2\mu_2^{(3)}\mu_2^{(5)}}\\ &- \bar S^{(1)}_{k_2\alpha_2\mu_2^{(5)}} M_{\alpha_2\beta_2}\bar A^{(3)}_{\beta_2\mu_2^{(3)}\mu_2^{(4)}}\\ =&\Sigma^{(2)}_{k_2\mu_2^{(3)}\mu_2^{(4)}\mu_2^{(5)}}\end{aligned} \tag{4.23}$$

The coefficients $\bar S^{(2)}_{k_2\mu_2^{(3)}\mu_2^{(4)}\mu_2^{(5)}}$ and $\bar S^{(1)}_{k_2\alpha_2\mu_2}$ had not been determined previously. Hence we can choose these coefficients appropriately so as to satisfy eq.(4.23), and hence eq.(4.21).

The analysis of this and the previous section leaves the coefficients $\bar S^{(N-1)}_{\alpha_1 r_1\ldots r_N}$ completely undetermined. This ambiguity in determining $\bar S$ can be traced to the fact that a field redefinition of the form

$$b_0^-|\tilde\Psi\rangle \to b_0^-|\tilde\Psi\rangle + \lambda\Big(\tilde Q_B b_0^-|f(\tilde\Psi)\rangle + \sum_{N=3}^{\infty}\frac{g^{N-2}}{(N-2)!}[(|f(\tilde\Psi)\rangle)\tilde\Psi^{N-2}]''\Big) \tag{4.24}$$

leaves the action $\tilde S(\tilde\Psi)$ invariant to order λ for any choice of $|f(\tilde\Psi)\rangle$. This induces a transformation of the coefficients $\bar S^{(N-1)}_{sr_1\ldots r_N}$ which is reflected in the ambiguity in determining the coefficients $\bar S^{(N-1)}_{\alpha_1 r_1\ldots r_N}$.

Finally, note that if we want to evaluate the right hand side of eqs.(2.27), (2.28), using the values of $S^{(N-1)}_{s_1 r_1\ldots r_N}$ determined in this and the previous sections, and compare them with the left hand sides of these equations, we need to perform infinite sum over the indices s_i. These sums can be performed using the techniques of refs.[10][11], as was done in refs.[19].

Thus we have demonstrated in this section, through some specific examples, how the analysis of sect.3 might be extended to construct appropriate field redefinitions which converts the full classical string field theory action $S(\Psi)$ formulated around the conformal field theory CFT to the string field theory action $\tilde S(\tilde\Psi)$ formulated around the conformal field theory CFT$''$.

5. BACKGROUNDS SEPARATED BY FINITE DISTANCE AND RELATED BY MARGINAL DEFORMATION

So far we have considered backgrounds which are infinitesimally close to each other. In this section we shall show how to use the results that we have obtained so far to relate backgrounds that are not necessarily close to each other, but are still connected to each other by a series of marginal deformations. An intuitive undestanding of the resullts of this section may be obtained by noting that if, to first order in the perturbation parameter λ, the string field theories formulated around CFT and CFT$''$ are related by field redefinition, then the result is also expected to be valid for finite λ, since we can deform CFT to CFT$''$ by marginal deformation in infinitesimal steps.

To be more specific, let us now consider a one parameter family of allowed backgrounds, such that any two neighbouring members of the family are related to each other by a marginal perturbation. Let τ denote the parameter labelling these backgrounds. Then, given the conformal field theories at τ and $\tau+\delta\tau$, the two dimensional action of these two conformal field theories are related by,

$$\mathcal{S}_{\tau+\delta\tau} = \mathcal{S}_\tau + f(\tau)\delta\tau \int d^2 z \varphi(z,\bar{z}) \tag{5.1}$$

where $\varphi(z,\bar{z})$ is a properly normalized dimension (1,1) primary field. $f(\tau)$ is some function of τ that can be set to 1 by appropriately reparametrizing τ. For simplicity, we shall assume that this has been done.

Let $\{|\Phi_{n,r}(\tau)\rangle\}$ be an appropriate set of basis states in the conformal field theory corresponding to a given value of τ, and $b_0^-|\Psi(\tau)\rangle = \sum_r \psi_r(\tau)|\Phi_{2,r}(\tau)\rangle$ be the string field. Identifying the conformal field theory corresponding to the point τ to CFT$''$, that corresponding to the point $\tau+\delta\tau$ as CFT, and using eqs.(2.10), (2.17), we see that $\psi_r(\tau)$ and $\psi_r(\tau+\delta\tau)$ are related by a functional relationship of the form:

$$\psi_s(\tau+\delta\tau) = V^{[2]}_{sr}(\tau)(\psi^{(0)}_r(\tau) + \sum_{N=1}^{\infty} \frac{1}{N!} S^{(N-1)}_{rs_1\ldots s_N} \psi_{s_1}(\tau)\ldots\psi_{s_N}(\tau)) \tag{5.2}$$

Since $\psi_s(\tau+\delta\tau)$ and $\psi_s(\tau)$ are expected to differ by an amount of order $\delta\tau$ (which was called λ in the previous sections), we may rewrite eq.(5.2) as,

$$\psi_s(\tau+\delta\tau) = \psi_s(\tau) + \delta\tau \sum_{N=0}^{\infty} \frac{1}{N!} T^{(N)}_{sr_1\ldots r_N}(\tau)\psi_{r_1}(\tau)\ldots\psi_{r_N}(\tau) \tag{5.3}$$

Note that we have included a τ dependence in $T^{(N)}$ since these coefficients depend on the correlation functions in the conformal field theory labelled by the parameter τ.

Let τ_0 be some fixed reference point in the τ space. We shall now try to show that there is a field redefinition of the form:

$$\psi_s(\tau) = \sum_{N=0}^{\infty} \frac{1}{N!} R^{(N)}_{sr_1\ldots r_N}(\tau,\tau_0)\psi_{r_1}(\tau_0)\ldots\psi_{r_N}(\tau_0) \tag{5.4}$$

which relates the string field theory actions formulated around the points τ and τ_0 even when τ and τ_0 are not close to each other. Using eqs.(5.4) we get,

$$\psi_s(\tau+\delta\tau) - \psi_s(\tau) = \delta\tau \sum_{N=0}^{\infty} \frac{1}{N!} \frac{dR^{(N)}_{sr_1\ldots r_N}(\tau,\tau_0)}{d\tau}\psi_{r_1}(\tau_0)\ldots\psi_{r_N}(\tau_0) \tag{5.5}$$

On the other hand, replacing $\psi_{r_i}(\tau)$ in the right hand side of eq.(5.3) in terms of $\{\psi_{r_i}(\tau_0)\}$ given in eq.(5.4) we get,

$$\begin{aligned}
&\psi_s(\tau+\delta\tau) - \psi_s(\tau) \\
=&\delta\tau \sum_{N=0}^{\infty} \frac{1}{N!} T^{(N)}_{ss_1\ldots s_N}(\tau) \Big(\sum_{M_1=0}^{\infty} \frac{1}{M_1!} R^{(M_1)}_{s_1 r_1\ldots r_{M_1}} \psi_{r_1}\ldots\psi_{r_{M_1}} \Big) \\
&\ldots \Big(\sum_{M_N=0}^{\infty} \frac{1}{M_N!} R^{(M_N)}_{s_N r_{(M_1+\ldots M_{N-1}+1)}\ldots r_{(M_1+\ldots M_N)}} \psi_{r_{(M_1+\ldots M_{N-1}+1)}} \ldots \psi_{r_{(M_1+\ldots M_N)}} \Big)
\end{aligned} \tag{5.6}$$

Comparing eqs.(5.5) and (5.6) we get,

$$\frac{dR^{(N)}_{sr_1\ldots r_N}(\tau,\tau_0)}{d\tau} = \sum_n \frac{1}{n!} T^{(n)}_{ss_1\ldots s_n}(\tau)\Big[\sum_{\{M_i\},\sum M_i=N}\Big(\prod_{i=1}^n \frac{1}{M_i!}\Big) R^{(M_1)}_{s_1 r_1\ldots r_{M_1}} \\ R^{(M_2)}_{s_2 r_{(M_1+1)}\ldots r_{(M_1+M_2)}}\ldots R^{(M_n)}_{s_N r_{(N-M_n+1)}\ldots r_N} + \text{ Permutations of } r_1,\ldots r_N\Big] \tag{5.7}$$

These equations, together with the initial condition,

$$R^{(N)}_{sr_1\ldots r_N}(\tau,\tau_0)|_{\tau=\tau_0} = \delta_{N1}\delta_{sr_1} \tag{5.8}$$

which follows from eq.(5.4), determines $R^{(N)}_{sr_1\ldots r_N}(\tau,\tau_0)$ for general values of τ which are not necessarily close to τ_0.

In order to see how eq.(5.7) may be used for practical computation, let us consider a special case, $N = 0$. This gives,

$$\frac{dR^{(0)}_s(\tau,\tau_0)}{d\tau} = \sum_n \frac{1}{n!} T^{(n)}_{ss_1\ldots s_n}(\tau) R^{(0)}_{s_1}(\tau,\tau_0)\ldots R^{(0)}_{s_n}(\tau,\tau_0) \tag{5.9}$$

From eq.(5.4) we see that $R^{(0)}_s(\tau,\tau_0)$ gives the value of $\psi_s(\tau)$ corresponding to the point $\{\psi_r(\tau_0) = 0\}$. Since the point $\{\psi_r(\tau_0) = 0\}$ describes the background corresponding to the conformal field theory labelled by τ_0, we see that $R^{(0)}_s(\tau,\tau_0)$ gives us the coordinate of this particular background in the coordinate system $\{\psi_s(\tau)\}$. In other words,

$$b_0^-|\Psi\rangle \equiv \sum_s R^{(0)}_s(\tau,\tau_0)|\Phi_{2,s}(\tau)\rangle \tag{5.10}$$

is a solution of the classical equation of motion in string field theory formulated around the point τ that describes the background labelled by τ_0.

We shall now derive an expression for $R^{(0)}_s(\tau,\tau_0)$ to order $(\tau-\tau_0)^2$ by solving eq.(5.9), and compare this with the results of ref.[19] where a general algorithm was given for constructing solutions of the classical equations of motion in string field theory to all orders in the deformation parameter for (nearly) marginal perturbation. Since for $\tau \simeq \tau_0$, $R_s^{(0)} \propto (\tau - \tau_0)$, and since we are interested only in the value of $R_s^{(0)}$ to order $(\tau - \tau_0)^2$, we may express eq.(5.9) as,

$$\frac{dR_s^{(0)}(\tau, \tau_0)}{d\tau} = T_s^{(0)}(\tau) + T_{sr}^{(1)}(\tau) R_r^{(0)}(\tau) + \mathcal{O}\big((\tau - \tau_0)^2\big) \tag{5.11}$$

Let us denote by $|\varphi\rangle$ the dimension (1,1) primary state in the internal conformal field theory representing the marginal deformation, and assume, for simplicity, that $c_1 \bar{c}_1 |\varphi\rangle$ is the only physical state $|\tilde{\Phi}_{2,k_2}\rangle \equiv |\Phi_{2,k_2}(\tau)\rangle$ at zero momentum. If we take $s = k_2$ in eq.(5.11), we get,

$$\frac{dR_{k_2}^{(0)}(\tau, \tau_0)}{d\tau} = T_{k_2}^{(0)}(\tau) + T_{k_2 r}^{(1)}(\tau) R_r^{(0)}(\tau) + \mathcal{O}\big((\tau - \tau_0)^2\big) \tag{5.12}$$

Using eqs.(2.29), (5.2), (5.3), and the fact that $V_{sr}^{[2]} = \delta_{sr} + \mathcal{O}(\lambda)$, we get $T_{k_2}^{(0)} = \sqrt{2}/g$, $T_{\alpha_1}^{(0)} = T_{\beta_2}^{(0)} = 0$. This gives,

$$R_{k_2}^{(0)}(\tau, \tau_0) = \frac{\sqrt{2}}{g}(\tau - \tau_0) + \mathcal{O}\Big((\tau - \tau_0)^2\Big) \tag{5.13}$$

We shall now carry out a further reparametrization of τ such that the right hand side of eq.(5.13) is exactly given by $\sqrt{2}(\tau - \tau_0)/g$. Thus $R_{k_2}^{(0)}$ takes the form:

$$R_{k_2}^{(0)}(\tau, \tau_0) = \frac{\sqrt{2}}{g}(\tau - \tau_0) \tag{5.14}$$

Let us now concentrate on the components $R_{\alpha_1}^{(0)}$ and $R_{\beta_2}^{(0)}$. From eq.(5.11), and the fact that $T_{\alpha_1}^{(0)} = T_{\beta_2}^{(0)} = 0$, we see that $R_{\alpha_1}^{(0)}$ and $R_{\beta_2}^{(0)}$ are of order $(\tau - \tau_0)^2$. From eq.(5.10) we see that $R_{\alpha_1}^{(0)}$ can be set to zero to order $(\tau - \tau_0)^2$ by a gauge transformation with the parameter $-R^{(0)}_{\alpha_1}|\Phi_{1,\alpha_1}(\tau)\rangle$. Thus $R^{(0)}_{\beta_2}$ are the only relevant components to be computed. From eq.(5.11) we get,

$$\frac{dR^{(0)}_{\beta_2}}{d\tau} = T^{(1)}_{\beta_2 r}(\tau) R^{(0)}_r + \mathcal{O}\Big((\tau-\tau_0)^2\Big) \tag{5.15}$$

Note that the τ reparametrization needed to make eq.(5.14) an exact equation is of the form $\tau \to \tau + \mathcal{O}\big((\tau-\tau_0)^2\big)$, and hence changes the right hand side of eq.(5.15) only by a factor of order $(\tau-\tau_0)^2$. Using eq.(5.13) and the fact that $R^{(0)}_{k_2}$ is the only non-vanishing component of $R^{(0)}_r$ to order $(\tau-\tau_0)$, we get,

$$R^{(0)}_{\beta_2} = \frac{1}{\sqrt{2}g} T^{(1)}_{\beta_2 k_2}(\tau)(\tau-\tau_0)^2 + \mathcal{O}\Big((\tau-\tau_0)^3\Big) \tag{5.16}$$

It thus remains to determine $T^{(1)}_{\beta_2 k_2}$. It has been shown in appendix C that,

$$T^{(1)}_{\beta_2 k_2} = -\sqrt{2}\{(\tilde{\Phi}^c_{3,\beta_2})(c_0^- c_1\bar{c}_1|\varphi\rangle)(c_0^- c_1\bar{c}_1|\varphi\rangle)\}'' \tag{5.17}$$

Hence the classical solution to order $(\tau-\tau_0)^2$ may be written as,

$$\begin{aligned} b_0^-|\Psi\rangle =& \frac{\sqrt{2}}{g}\Big((\tau-\tau_0)c_1\bar{c}_1|\varphi\rangle - \frac{1}{\sqrt{2}}(\tau-\tau_0)^2|\tilde{\Phi}_{2,\beta_2}\rangle\{(\tilde{\Phi}^c_{3,\beta_2})(c_0^- c_1\bar{c}_1|\varphi\rangle)(c_0^- c_1\bar{c}_1|\varphi\rangle)\}'' \\ &+ \mathcal{O}\Big((\tau-\tau_0)^3\Big) \end{aligned} \tag{5.18}$$

It can be easily verified that this agrees with the result computed in ref.[19] to this order up to a gauge transformation and possible reparametrization of τ.[⋆]

⋆ The notation for the choice of basis given in ref.[19] is different from the one used here. In order to translate the result of ref.[19] to the present notation, we must make the replacement $\langle\Phi^c_{n,\tilde{k}_n}| \to \langle\Phi^c_{6-n,k_{6-n}}|$, $\langle\Phi^c_{n,\tilde{\alpha}_n}| \to \langle\Phi^c_{6-n,\alpha_{5-n}}|$, and $\langle\Phi^c_{n,\tilde{\alpha}_{n-1}}| \to \langle\Phi^c_{6-n,\alpha_{6-n}}|$ for all n.

6. DISCUSSION

In this paper we have shown that string field theories constructed around two nearby conformal field theories that are related by marginal perturbation are actually the same, in the sense that they are related to each other by field redefinition. Although the result has been proved in complete detail only for the quadratic and cubic terms in the action, we have shown, by analyzing several special cases, that the result is likely to be valid even for the quartic and higher order terms in the action. Finally, we have also shown that the result holds beyond leading order in the perturbation parameter λ; as a result, string field theories formulated around two different conformal field theories which are not necessarily close to each other but are still connected to each other by a series of marginal deformations, are also related to each other through a field redefinition. In this case, the appropriate field redefinition is found by solving an infinite set of differential equations.

A question that naturally arises at this stage is whether it is possible to formulate string field theory in a way where the background independence is manifest. For open string field theory such a formulation was given in reference [20] where it was shown that starting from a purely cubic action which is independent of the background, one can obtain string field theories around different backgrounds by shifting the field by a classical solution. The action expanded in terms of the shifted field has a background dependent kinetic term, but a background independent interaction term. Such a simple notion of background independence cannot, however, be implemented in non-polynomial closed string field theory, since, by the very nature of the non-polynomial interaction, a shift in the string field will modify all the interaction vertices.

We would like to point out that the analysis of this paper may be applied not only to study the equivalence of string field theories formulated around two different backgrounds, but also to study the equivalence of different formulations of string field theory around the same background. The result of this paper tells us that if two string field theories have the same kinetic terms, linearized gauge invariance, and on-shell S-matrix elements (with special definition of 'subtracted' S-matrix elements when physical states can propagate in the intermediate channel), then they can be related to each other by field redefinition.

Finally, we would like to mention that the analysis of this paper (and that of refs.[12][13]) have dealt with string field theory at the classical level. A complete quantum string field theory has been constructed [7] using the Batalin-Vilkovisky (BV) formalism [21], which requires adding new terms to the string field theory action at the loop level. A proof of background independence of the complete quantum string field theory action will then involve showing that the actions of BV quantized string field theories formulated around different backgrounds are related to each other by appropriate field redefinitions, after taking into account the change of path integral measure due to this field redefinition. We hope to return to this question in the near future.

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APPENDIX A

In this appendix we shall try to verify eq.(3.16) when one or more of the external states $|\tilde{\Phi}_{2,\mu_2}\rangle$, $|\tilde{\Phi}_{2,\nu_2}\rangle$, and $|\tilde{\Phi}_{2,\tau_2}\rangle$ correspond to a zero momentum dilaton

$$|D\rangle = (c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})|0\rangle \tag{A.1}$$

and the rest of the physical states are of the form $c_1\bar{c}_1|\tilde{V}\rangle$, $|\tilde{V}\rangle$ being a dimension (1,1) primary state in CFT$''$.

Let us define,

$$|s(k)\rangle = (c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})|k\rangle + c_1 \bar{c}_1 \eta_{\mu\nu} \alpha^{\mu}_{-1} \bar{\alpha}^{\nu}_{-1}|k\rangle \tag{A.2}$$

The starting point of our analysis will be the identity:

$$|s(k)\rangle = \Big(\eta_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{n.k}\Big) c_1\bar{c}_1 \alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}|k\rangle - \tilde{Q}_B \frac{n_\mu}{n.k}(c_1\alpha^\mu_{-1} - \bar{c}_1\bar{\alpha}^\mu_{-1})|k\rangle \quad \text{(A.3)}$$

where k is an arbitrary momentum satisfying $k^2 = 0$, α^μ_n are the matter oscillators associated with the flat directions, and n^μ is any vector. This shows that $|s(k)\rangle$ has the form of $c_1\bar{c}_1|\tilde{V}\rangle$ + pure gauge state. Hence for amplitudes involving such states, $\tilde{A}^{(3)}$ and $\bar{A}^{(3)}$ have the same values. If we now take the $k \to 0$ limit of such amplitudes, then, if the $k \to 0$ limit is smooth, and is identical to the amplitude for $k = 0$, this would imply that $\tilde{A}^{(3)}$ and $\bar{A}^{(3)}$ have the same values for amplitudes involving external states of the form $(c_1c_{-1} - \bar{c}_1\bar{c}_{-1})|0\rangle + c_1\bar{c}_1\eta_{\mu\nu}\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}|0\rangle$. But the state $c_1\bar{c}_1\eta_{\mu\nu}\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}|0\rangle$ has the form $c_1\bar{c}_1|\tilde{V}\rangle$, and hence, by following the analysis of ref.[13] one can see that $\tilde{A}^{(3)}$ and $\bar{A}^{(3)}$ have the same values for such external states. This, in turn, would then imply that $\tilde{A}^{(3)}$ and $\bar{A}^{(3)}$ are identical even when one or more external physical states are of the form $(c_1c_{-1} - \bar{c}_1\bar{c}_{-1})|0\rangle$.

Thus it remains to show that the $k \to 0$ limit of the amplitudes involving the state $|s(k)\rangle$ is smooth, both in $\tilde{A}^{(3)}$ and $\bar{A}^{(3)}$, and is identical to the corresponding amplitudes involving $|s(0)\rangle$. For $\tilde{A}^{(3)}$ such amplitudes are obtained simply by taking the three point correlation functions of appropriate vertex operators. The $k \to 0$ limit of such an amplitude is manifestly smooth. On the other hand, if we take $|\tilde{\Phi}_{2,\mu_2}\rangle = |s(k)\rangle$, then,

$$\begin{aligned} \bar{A}^{(3)}_{\mu_2\nu_2\tau_2} =& \hat{A}^{(3)}_{rst} S^{(0)}_{r\mu_2} S^{(0)}_{s\nu_2} S^{(0)}_{s\tau_2} \\ =& \{(c_0^- \mathcal{S}|s(k)\rangle)(c_0^- \mathcal{S}|\tilde{\Phi}_{2,\nu_2}\rangle)(c_0^- \mathcal{S}|\tilde{\Phi}_{2,\tau_2}\rangle)\} \\ & + \{(c_0^-|s(k)\rangle)(c_0^-|\tilde{\Phi}_{2,\nu_2}\rangle)(c_0^-|\tilde{\Phi}_{2,\tau_2}\rangle)(|\Psi^{(0)}\rangle)\} + \mathcal{O}(\lambda^2) \end{aligned} \quad \text{(A.4)}$$

The second term on the right hand side of eq.(A.4) has well defined $k \to 0$ limit and is identical to the $k = 0$ answer. The first term, in principle, can have a value whose $k \to 0$ limit does not agree with the $k = 0$ answer, since it involves the operator $\mathcal{S} = 1 + \lambda K + \mathcal{O}(\lambda^2)$. K is determined by the set of eqations discussed in ref.[12][13], and the algorithm given there was not manifestly smooth in the $k \to 0$ limit, since it involved treating the physical, unphysical, and pure gauge states on a different footing. We shall now demonstrate explicitly that the $k \to 0$ limit of the operator K is actually smooth. Once this is established, it would imply that the first term on the right hand side of eq.(A.4) also has smooth $k \to 0$ limit that agrees with the $k = 0$ result.

Thus we now need to show that the state $K|s(k)\rangle$ reduces, in the $k \to 0$ limit, to $K|s(0)\rangle$. From the analysis of refs.[12][13] one can see that since $|s(k)\rangle$ is a physical state, only the components of the form $\langle\tilde{\Phi}_{2,\alpha_2}|c_0^- \tilde{Q}_B K|s(k)\rangle''$ and $\langle\tilde{\Phi}_{3,k_3}|c_0^- K|s(k)\rangle''$ are determined. The equation determining $\langle\tilde{\Phi}_{2,\alpha_2}|c_0^- \tilde{Q}_B K|s(k)\rangle''$ is,

$$\lambda\langle\tilde{\Phi}_{2,\alpha_2}|c_0^- \tilde{Q}_B K|s(k)\rangle'' = -\langle\tilde{\Phi}_{2,\alpha_2}|c_0^- \Delta Q_B|s(k)\rangle'' + \mathcal{O}(\lambda^2) \tag{A.5}$$

where,

$$\begin{aligned}\langle\Phi_1|c_0^- \Delta Q_B|\Phi_2\rangle = &-\sqrt{2}\lambda\{(c_0^-|\Phi_1\rangle)(c_0^- c_1\bar{c}_1|\varphi\rangle)(c_0^-|\Phi_2\rangle)\} \\ &- \lambda\langle\Phi_1|c_0^- \oint\limits_{|z|=\epsilon} \big(dz\bar{c}(\bar{z})\varphi(z,\bar{z}) - d\bar{z}c(z)\varphi(z,\bar{z})\big)|\Phi_2\rangle\end{aligned} \tag{A.6}$$

$|\varphi\rangle$ being the same state that appeared in eq.(2.29), and ϵ is the short distance cut-off used to define the correlation functions in CFT$''$ in terms of those in CFT. Since $|s(k)\rangle$ is a state built by the α^μ_{-n}, $\bar{\alpha}^\nu_{-n}$ and the ghost oscillators on the state $|k\rangle$, it is clear from eqs.(A.6) that the only states $\langle\tilde{\Phi}_{2,\alpha_2}|$ for which the right hand side of eq.(A.5) does not vanish trivially are those built by the α^μ_n, $\bar{\alpha}^\nu_n$ and the ghost oscillators on $\langle\varphi|\otimes\langle -k|$. A simple analysis of BRST cohomology shows that the number of unphysical states built on $\langle\varphi|\otimes\langle -k|$ does have a smooth limit as $k \to 0$ since no new physical state appears at this value of the momentum.* Thus it is possible to choose the basis of unphysical states $\langle\tilde{\Phi}_{2,\alpha_2}|$ for momentum k in such a way that as $k \to 0$, these states reduce to the unphysical states in the $k = 0$

$\star$ Note that we are taking the limit keeping $k^2 = 0$ all the time.

sector. In this case the $k \to 0$ limit of the right hand side of eq.(A.5) agrees with its $k = 0$ value. This, in turn, shows that the components $\langle\tilde\Phi_{2,\alpha_2}|c_0^-\tilde Q_B K|s(k)\rangle''$ has smooth limit as $k \to 0$, and this limit agrees with the corresponding expression with $|s(k)\rangle$ replaced by $|s(0)\rangle$.

Let us now analyze the components $\langle\tilde\Phi_{3,k_3}|c_0^- K|s(k)\rangle''$. Using eq.(3.60) of ref.[13], and eq.(A.3) we see that,

$$\begin{aligned}&\langle\tilde\Phi_{3,k_3}|c_0^- K(|s(k)\rangle'' + \tilde Q_B \frac{n^\mu}{n.k}(c_1\alpha^\mu_{-1} - \bar c_1\bar\alpha^\mu_{-1})|k\rangle'')\\ =&-2\ln(\frac{f_1'(0)}{\epsilon})\langle\tilde\Phi_{3,k_3}|c_0^-\varphi(1)(\eta_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{n.k})c_1\bar c_1\alpha^\mu_{-1}\bar\alpha^\nu_{-1}|k\rangle'' + \mathcal{O}(\lambda)\end{aligned} \tag{A.7}$$

The only physical state $\langle\tilde\Phi_{3,k_3}|$ for which eq.(A.7) is non-trivial is $\langle -k|\otimes\langle\varphi|c_{-1}\bar c_{-1}c_0^+$. But the right hand side of eq.(A.7) vanishes for such a state. Using the equation $\lambda[K,\tilde Q_B] = \Delta Q_B$, we get,

$$\lambda\langle\tilde\Phi_{3,k_3}|c_0^- K|s(k)\rangle'' = -\langle\tilde\Phi_{3,k_3}|c_0^-\Delta Q_B\frac{n^\mu}{n.k}(c_1\alpha^\mu_{-1} - \bar c_1\bar\alpha^\mu_{-1})|k\rangle'' + \mathcal{O}(\lambda^2) \tag{A.8}$$

We shall now evaluate $\lambda\langle\tilde\Phi_{3,k_3}|c_0^- K|s(0)\rangle''$ and compare with the right hand side of eq.(A.8) in the $k \to 0$ limit. Using eq.(3.60) of ref.[13] we get,

$$\begin{aligned}&\langle\tilde\Phi_{3,k_3}|c_0^- K c_1\bar c_1\alpha^\mu_{-1}\bar\alpha^\nu_{-1}\eta_{\mu\nu}|0\rangle''\\ =&-2\ln(\frac{f_1'(0)}{\epsilon})\langle\varphi|c_{-1}\bar c_{-1}c_0^+c_0^-\varphi(1)c_1\bar c_1\alpha^\mu_{-1}\bar\alpha^\nu_{-1}\eta_{\mu\nu}|0\rangle'' + \mathcal{O}(\lambda)\\ =&\mathcal{O}(\lambda)\end{aligned} \tag{A.9}$$

Using eqs.(A.2), (A.8), and (A.9) we see that the $k \to 0$ limit of $\langle\tilde\Phi_{3,k_3}|c_0^- K|s(k)\rangle''$ matches $\langle\tilde\Phi_{3,k_3}|c_0^- K|s(0)\rangle''$ to order λ, if,

$$\begin{aligned}&\langle 0|\otimes\langle\varphi|c_{-1}\bar c_{-1}c_0^+c_0^- K(c_1c_{-1} - \bar c_1\bar c_{-1})|0\rangle''\\ =&-\frac{1}{\lambda}\lim_{k\to 0}\langle -k|\otimes\langle\varphi|c_{-1}\bar c_{-1}c_0^+c_0^-\Delta Q_B\frac{n^\mu}{n.k}(c_1\alpha^\mu_{-1} - \bar c_1\bar\alpha^\mu_{-1})|k\rangle'' + \mathcal{O}(\lambda)\\ =&-\frac{f_1''(0)}{\sqrt{2}(f_1'(0))^2}\langle 0|\otimes\langle\varphi|c_{-1}\bar c_{-1}c_0^+c(f_2(1))\bar c(f_2(1))\varphi(f_2(1))(c_1 - \bar c_1)|0\rangle'' + \mathcal{O}(\lambda)\end{aligned} \tag{A.10}$$

We now note that the matrix element appearing on the left hand side of eq.(A.10) has not been determined previously. Hence we can use eq.(A.10) to define this matrix element. This, in turn, guarantees that in the $k \to 0$ limit $K|s(k)\rangle$ reduces to the state $K|s(0)\rangle$, and hence proves that eq.(3.16) is satisfied even when one or more of the external states correspond to the zero momentum dilaton state $(c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})|0\rangle$.

The result of this appendix may also be interpreted in the following way. Note that the pure gauge state,

$$|\tilde{\Phi}_{2,\alpha_1}\rangle \equiv \tilde{Q}_B \frac{n_\mu}{n \cdot k}(c_1 \alpha^\mu_{-1} - \bar{c}_1 \bar{\alpha}^\mu_{-1})|k\rangle \tag{A.11}$$

reduces to a physical state $|\tilde{\Phi}_{2,m_2^{(0)}}\rangle$ in the $k \to 0$ limit. Our result shows that,

$$\lim_{k\to 0}\langle\tilde{\Phi}_{3,k_3}|c_0^- K|\tilde{\Phi}_{2,\alpha_1}\rangle = \langle\tilde{\Phi}_{3,k_3}|c_0^- K|\tilde{\Phi}_{2,m_2^{(0)}}\rangle\big|_{k=0} \tag{A.12}$$

thereby showing that eq.(3.42) can be made compatible with eq.(3.60) of ref.[13].

APPENDIX B

In this appendix we shall analyze the left hand side of eq.(4.14) and show that it can be brought into the form of eq.(4.15). Using eq.(2.37) and (2.33) and the fact [12] [13] that when CFT and CFT$''$ differ by a marginal perturbation, then $\mathcal{S}$ has the form:

$$\mathcal{S} = 1 + \lambda K + \mathcal{O}(\lambda^2) \tag{B.1}$$

we may express $\bar{A}^{(N)}_{\mu_2^{(1)}\ldots\mu_2^{(N)}}$ as,

$$\bar{A}^{(N)}_{\mu_2^{(1)}\ldots\mu_2^{(N)}} = \hat{A}^{(N)}_{\mu_2^{(1)}\ldots\mu_2^{(N)}} + \lambda \sum_{i=1}^{N} K^{(0)}_{r\mu_2^{(i)}} \hat{A}^{(N)}_{r\mu_2^{(1)}\ldots\mu_2^{(i-1)}\mu_2^{(i+1)}\mu_2^{(N)}} \tag{B.2}$$

where $K^{(0)}_{rs}$ is defined through the relation,

$$K|\tilde{\Phi}_{2,r}\rangle = \sum_s K^{(0)}_{sr}|\tilde{\Phi}_{2,s}\rangle \tag{B.3}$$

i.e.

$$K_{rs}^{(0)} = \langle \tilde{\Phi}_{2,r}^c | K | \tilde{\Phi}_{2,s} \rangle'' \tag{B.4}$$

Let us define,

$$\tilde{\Delta}_{rs} = \delta_{r\alpha_2} \delta_{s\beta_2} M_{\alpha_2\beta_2} \tag{B.5}$$

The left hand side of eq.(4.14) may now be written as,

$$\begin{aligned}
L =& [\tilde{A}^{(4)}_{\mu_2\nu_2\tau_2\sigma_2} - \tilde{A}^{(3)}_{\mu_2\nu_2 t}\tilde{\Delta}_{ts}\tilde{A}^{(3)}_{s\tau_2\sigma_2} - \tilde{A}^{(3)}_{\mu_2\tau_2 t}\tilde{\Delta}_{ts}\tilde{A}^{(3)}_{s\nu_2\sigma_2} - \tilde{A}^{(3)}_{\mu_2\sigma_2 t}\tilde{\Delta}_{ts}\tilde{A}^{(3)}_{s\nu_2\tau_2}] \\
& - [\hat{A}^{(4)}_{\mu_2\nu_2\tau_2\sigma_2} - \hat{A}^{(3)}_{\mu_2\nu_2 t}\tilde{\Delta}_{ts}\hat{A}^{(3)}_{s\tau_2\sigma_2} - \hat{A}^{(3)}_{\mu_2\tau_2 t}\tilde{\Delta}_{ts}\hat{A}^{(3)}_{s\nu_2\sigma_2} - \hat{A}^{(3)}_{\mu_2\sigma_2 t}\tilde{\Delta}_{ts}\hat{A}^{(3)}_{s\nu_2\tau_2}] \\
& - \lambda\big[K^{(0)}_{r\mu_2}(\hat{A}^{(4)}_{r\nu_2\tau_2\sigma_2} - \hat{A}^{(3)}_{r\nu_2 s}\tilde{\Delta}_{st}\hat{A}^{(3)}_{t\tau_2\sigma_2} - \hat{A}^{(3)}_{r\tau_2 s}\tilde{\Delta}_{st}\hat{A}^{(3)}_{t\nu_2\sigma_2} - \hat{A}^{(3)}_{r\sigma_2 s}\tilde{\Delta}_{st}\hat{A}^{(3)}_{t\tau_2\nu_2}) \\
& + \text{ Cyclic permutations of } \mu_2, \nu_2, \tau_2, \sigma_2\big] \\
& + \lambda\big[(\hat{A}^{(3)}_{\mu_2\nu_2 r}K^{(0)}_{rs}\tilde{\Delta}_{st}\hat{A}^{(3)}_{t\tau_2\sigma_2} + \hat{A}^{(3)}_{\mu_2\nu_2 s}\tilde{\Delta}_{st}K^{(0)}_{rt}\hat{A}^{(3)}_{r\tau_2\sigma_2}) \\
& + \text{ Other pairings of } \mu_2, \nu_2, \tau_2, \sigma_2\big] + \mathcal{O}(\lambda^2)
\end{aligned} \tag{B.6}$$

We assume that the only physical states that can appear in the $\mu_2\nu_2 \to \tau_2\sigma_2$ channel are tachyonic. This, in turn, shows that if k is the momentum flowing in this channel, then the corresponding vertex operator $e^{ik.X}$ has dimension $h > 0$. As a result, the only possible $\tilde{L}_0^+ = 0$ states propagating in this channel are of the form $c_1\bar{c}_1 e^{ik.X(0)}|\tilde{V}\rangle$, where $|\tilde{V}\rangle$ is a dimension $(1-h, 1-h)$ primary state in the internal conformal field theory. In other words, the only possible $\tilde{L}_+^0 = 0$ states propagating in this channel are the tachyonic physical states.

This, in turn, implies that if $\tilde{A}^{(3)}_{\mu_2\nu_2 k_2}$ and $\tilde{A}^{(3)}_{\tau_2\sigma_2 k_2}$ vanish for all k_2, then the only possible intermediate states $|\tilde{\Phi}_{2,t}\rangle$, $|\tilde{\Phi}_{2,s}\rangle$ appearing in eq.(B.6) are the $\tilde{L}_0^+ \neq 0$ states. (We have already assumed that the momenta flowing in the u and t channels are such that they do not allow $\tilde{L}_0^+ = 0$ states to appear as intermediate states in these channels). We shall now show that the right hand side of eq.(B.6) vanishes in this case. In the $\tilde{L}_0^+ \neq 0$ sector, a basis of unphysical states can be chosen all of which are annihilated by b_0^+ (see, for example, ref.[22]). This gives rise to the standard expression for the propagator $\tilde{\Delta}_{rs}$:

$$\tilde{\Delta}_{rs} = \langle \tilde{\Phi}^c_{2,r} | b_0^- b_0^+ (\tilde{L}_0^+)^{-1} | \tilde{\Phi}^c_{2,s} \rangle'' \tag{B.7}$$

Computation of the terms inside the first square bracket in eq.(B.6) then reduces to the standard computation of a four point function in string field theory defined by the action $\tilde{S}(\tilde{\Psi})$ with physical external states. Standard manipulations given in ref.[13] then shows that this amplitude is identical to the corresponding amplitude calculated with the action $\hat{S}(\hat{\Psi})$. In the analysis of ref.[13] the kinetic term of $\hat{S}(\hat{\Psi})$ was split into two pieces:

$$\langle \hat{\Psi} | \hat{Q}_B b_0^- | \hat{\Psi} \rangle = \langle \hat{\Psi} | Q_B b_0^- | \hat{\Psi} \rangle + \langle \hat{\Psi} | (\hat{Q}_B - Q_B) b_0^- | \hat{\Psi} \rangle \tag{B.8}$$

The inverse of the first term on the right hand side of eq.(B.8) in the $b_0^+ = 0$ gauge, i.e.,

$$\Delta_{rs} = \langle \Phi'^c_{2,r} | b_0^- b_0^+ (L_0^+)^{-1} | \Phi'^c_{2,s} \rangle \tag{B.9}$$

was taken as the propagator, whereas the second term on the right hand side of eq.(B.8) was taken as the interaction term. Here $\{\langle \Phi'^c_{2,r}|\}$ is the basis of states conjugate to $\{|\tilde{\Phi}_{2,r}\rangle\}$ with respect to the inner product $\langle | \rangle$, i.e.,

$$\langle \Phi'^c_{2,r} | \tilde{\Phi}_{2,s} \rangle = \delta_{rs} \tag{B.10}$$

Thus, by repeating the analysis of ref.[13] one can show that the contribution from the set of terms inside the first square bracket on the right hand side of eq.(B.6) takes the form:

$$\begin{aligned} & S^{(0)}_{r\mu_2} S^{(0)}_{s\nu_2} S^{(0)}_{t\tau_2} S^{(0)}_{u\sigma_2} \big\{ \hat{A}^{(4)}_{rstu} - (\Delta_{r't'} - \lambda \Delta_{r'u'} \Lambda_{u's'} \Delta_{s't'}) \\ & \times (\hat{A}^{(3)}_{r'rs} \hat{A}^{(3)}_{t'tu} + \hat{A}^{(3)}_{r'rt} \hat{A}^{(3)}_{t'su} + \hat{A}^{(3)}_{r'ru} \hat{A}^{(3)}_{t'st}) \big\} + \mathcal{O}(\lambda^2) \end{aligned} \tag{B.11}$$

where,

$$\lambda \Lambda_{rs} = \langle \tilde{\Phi}_{2,r} | (\hat{Q}_B - Q_B) c_0^- | \tilde{\Phi}_{2,s} \rangle \tag{B.12}$$

Let us define,

$$\lambda\Omega_{rs} = \tilde{\Delta}_{rs} - \Delta_{rs} \tag{B.13}$$

Using eqs.(2.33), (B.1), (B.4), (B.6), and that the first set of terms inside the square bracket in eq.(B.6) can be expressed as eq.(B.11), we get,

$$\begin{aligned} L =&\lambda\big\{\hat{A}^{(3)}_{\mu_2\nu_2 r}\hat{A}^{(3)}_{\tau_2\sigma_2 s} + \hat{A}^{(3)}_{\mu_2\sigma_2 r}\hat{A}^{(3)}_{\tau_2\nu_2 s} + \hat{A}^{(3)}_{\mu_2\tau_2 r}\hat{A}^{(3)}_{\nu_2\sigma_2 s}\big\} \\ &\times \big\{\Omega_{rs} + \Delta_{rr'}\Lambda_{r's'}\Delta_{s's} + K^{(0)}_{rr'}\tilde{\Delta}_{r's} + \tilde{\Delta}_{rr'}K^{(0)}_{sr'}\big\} + \mathcal{O}(\lambda^2) \end{aligned} \tag{B.14}$$

We now note that,

$$\begin{aligned} \lambda\Omega_{rs} =&\langle\tilde{\Phi}^c_{2,r}|b_0^- b_0^+(\tilde{L}_0^+)^{-1}|\tilde{\Phi}^c_{2,s}\rangle'' - \langle\Phi'^c_{2,r}|b_0^- b_0^+(L_0^+)^{-1}|\Phi'^c_{2,s}\rangle \\ =&\langle\tilde{\Phi}^c_{2,r}|b_0^- b_0^+\big((\tilde{L}_0^+)^{-1} - (L_0^+)^{-1}\big)|\tilde{\Phi}^c_{2,s}\rangle'' \\ &+ \langle\tilde{\Phi}^c_{2,r}|b_0^- b_0^+(L_0^+)^{-1}|\tilde{\Phi}^c_{2,s}\rangle'' - \langle\tilde{\Phi}^c_{2,r}|b_0^- b_0^+(L_0^+)^{-1}|\tilde{\Phi}^c_{2,s}\rangle \\ &- \langle\delta\Phi^c_{2,r}|b_0^- b_0^+(L_0^+)^{-1}|\tilde{\Phi}^c_{2,s}\rangle - \langle\tilde{\Phi}^c_{2,r}|b_0^- b_0^+(L_0^+)^{-1}|\delta\Phi^c_{2,s}\rangle + \mathcal{O}(\lambda^2) \end{aligned} \tag{B.15}$$

where,

$$\langle\delta\Phi^c_{2,r}| = \langle\Phi'^c_{2,r}| - \langle\tilde{\Phi}^c_{2,r}| \tag{B.16}$$

From eqs.(3.18), (B.10) and (B.16), we see that,

$$\langle\delta\Phi^c_{2,r}|\tilde{\Phi}_{2,s}\rangle = \langle\tilde{\Phi}^c_{2,r}|\tilde{\Phi}_{2,s}\rangle'' - \langle\tilde{\Phi}^c_{2,r}|\tilde{\Phi}_{2,s}\rangle \tag{B.17}$$

Since $\{|\tilde{\Phi}_{2,s}\rangle\}$ forms a complete basis of states, eq.(B.17) is valid with $|\tilde{\Phi}_{2,s}\rangle$ replaced by any state $|A\rangle$. Choosing $|A\rangle = b_0^- b_0^+(L_0^+)^{-1}|\tilde{\Phi}^c_{2,s}\rangle$, we see that the second, third, and the fourth terms on the right hand side of eq.(B.15) cancel. On the other hand, the first term on the right hand side of eq.(B.15) may be expressed as,

$$\begin{aligned} &-\langle\tilde{\Phi}^c_{2,r}|b_0^- b_0^+(\tilde{L}_0^+)^{-1}(\tilde{L}_0^+ - L_0^+)(\tilde{L}_0^+)^{-1}|\tilde{\Phi}^c_{2,s}\rangle'' + \mathcal{O}(\lambda^2) \\ =&-\lambda\Delta_{rr'}P_{r's'}\Delta_{s's} + \mathcal{O}(\lambda^2) \end{aligned} \tag{B.18}$$

where,

$$\lambda P_{r's'} = \langle\tilde{\Phi}_{2,r'}|(\tilde{Q}_B - Q_B)c_0^-|\tilde{\Phi}_{2,s'}\rangle \tag{B.19}$$

Finally, since $b_0^- b_0^+(L_0^+)^{-1}$ is hermitian with respect to the inner product $\langle|\rangle$, using eq.(B.17) the last term on the right hand side of eq.(B.15) may be written as,

$$\begin{aligned}&-\langle\delta\Phi^c_{2,s}|b_0^-b_0^+(L_0^+)^{-1}|\tilde{\Phi}^c_{2,r}\rangle\\=&-\langle\tilde{\Phi}^c_{2,s}|b_0^-b_0^+(L_0^+)^{-1}|\tilde{\Phi}^c_{2,r}\rangle''+\langle\tilde{\Phi}^c_{2,s}|b_0^-b_0^+(L_0^+)^{-1}|\tilde{\Phi}^c_{2,r}\rangle+\mathcal{O}(\lambda^2)\end{aligned}\tag{B.20}$$

Eq.(B.15) may now be written as,

$$\begin{aligned}\lambda\Omega_{rs}=&-\lambda\Delta_{rr'}P_{r's'}\Delta_{s's}-\langle\tilde{\Phi}^c_{2,s}|b_0^-b_0^+(L_0^+)^{-1}|\tilde{\Phi}^c_{2,r}\rangle''\\&+\langle\tilde{\Phi}^c_{2,s}|b_0^-b_0^+(L_0^+)^{-1}|\tilde{\Phi}^c_{2,r}\rangle+\mathcal{O}(\lambda^2)\end{aligned}\tag{B.21}$$

Next we use the relation,

$$\tilde{\Delta}_{\beta_2 s}\langle\tilde{\Phi}_{2,s}|\tilde{Q}_Bc_0^-|\tilde{\Phi}_{2,\alpha_2}\rangle=\delta_{\alpha_2\beta_2}\tag{B.22}$$

$$\langle\tilde{\Phi}_{2,\beta_2}|\tilde{Q}_Bc_0^-|\tilde{\Phi}_{2,s}\rangle\tilde{\Delta}_{s\alpha_2}=\delta_{\alpha_2\beta_2}\tag{B.23}$$

to write,

$$K^{(0)}_{\alpha_2 r'}\tilde{\Delta}_{r'\beta_2}+\tilde{\Delta}_{\alpha_2 r'}K^{(0)}_{\beta_2 r'}=\tilde{\Delta}_{\alpha_2 s}R_{st}\tilde{\Delta}_{t\beta_2}\tag{B.24}$$

where,

$$R_{st}=\langle\tilde{\Phi}_{2,s}|\tilde{Q}_Bc_0^-K|\tilde{\Phi}_{2,t}\rangle''+\langle\tilde{\Phi}_{2,t}|\tilde{Q}_Bc_0^-K|\tilde{\Phi}_{2,s}\rangle''\tag{B.25}$$

Using the relation [12] [13],

$$\lambda[\tilde{Q}_B,K]=-\Delta Q_B=-\hat{Q}_B+\tilde{Q}_B\tag{B.26}$$

we may express eq.(B.25) as,

$$\begin{aligned}\lambda R_{st}=&\langle\tilde{\Phi}_{2,s}|(\tilde{Q}_B-\hat{Q}_B)c_0^-|\tilde{\Phi}_{2,t}\rangle''\\&-\langle\tilde{\Phi}_{2,s}|c_0^-K\tilde{Q}_B|\tilde{\Phi}_{2,t}\rangle''+\langle\tilde{\Phi}_{2,t}|\tilde{Q}_Bc_0^-K|\tilde{\Phi}_{2,s}\rangle''\end{aligned}\tag{B.27}$$

We now note that in eq.(B.14) the sum over r and s run over unphysical states only, since $\hat{A}^{(3)}_{\mu_2\nu_2 r}$ etc. are taken to be non-vanishing only for such states.* Taking

$\star$ Actually, it is $\tilde{A}^{(3)}_{\mu_2\nu_2 r}$ etc. which vanish for $r\neq\alpha_2$, but this implies that $\hat{A}^{(3)}_{\mu_2\nu_2 r}$ is of order λ for $r\neq\alpha_2$.

$r = \alpha_2$ and $s = \beta_2$, and using eqs.(B.12), (B.21), (B.19), (B.24), and (B.27), we get,

$$\begin{aligned}L =& \big(\hat{A}^{(3)}_{\mu_2\nu_2\alpha_2}\hat{A}^{(3)}_{\tau_2\sigma_2\beta_2} + \hat{A}^{(3)}_{\mu_2\sigma_2\alpha_2}\hat{A}^{(3)}_{\tau_2\nu_2\beta_2} + \hat{A}^{(3)}_{\mu_2\tau_2\alpha_2}\hat{A}^{(3)}_{\nu_2\sigma_2\beta_2}\big) \\ &\times \big\{ - \langle\tilde{\Phi}^c_{2,\beta_2}|b_0^- b_0^+ (L_0^+)^{-1}|\tilde{\Phi}^c_{2,\alpha_2}\rangle'' + \langle\tilde{\Phi}^c_{2,\beta_2}|b_0^- b_0^+ (L_0^+)^{-1}|\tilde{\Phi}^c_{2,\alpha_2}\rangle \\ &- \lambda\tilde{\Delta}_{\alpha_2 s}\langle\tilde{\Phi}_{2,s}|c_0^- K\tilde{Q}_B|\tilde{\Phi}_{2,t}\rangle''\tilde{\Delta}_{t\beta_2} + \lambda\tilde{\Delta}_{\beta_2 t}\langle\tilde{\Phi}_{2,t}|\tilde{Q}_B c_0^- K|\tilde{\Phi}_{2,s}\rangle''\tilde{\Delta}_{s\alpha_2}\big\} + \mathcal{O}(\lambda^2)\end{aligned} \tag{B.28}$$

Summing over a complete set of states, the last two terms in the curly bracket may be expressed as,

$$\lambda\langle\tilde{\Phi}^c_{2,\alpha_2}|b_0^- b_0^+ (\tilde{L}_0^+)^{-1} c_0^- K b_0^- |\tilde{\Phi}^c_{2,\beta_2}\rangle'' + \lambda\langle\tilde{\Phi}^c_{2,\beta_2}|K b_0^- b_0^+ (\tilde{L}_0^+)^{-1}|\tilde{\Phi}^c_{2,\alpha_2}\rangle'' \tag{B.29}$$

Using eqs.(2.31) and (B.1), and that $b_0^- b_0^+ (\tilde{L}_0^+)^{-1}$ is hermitian with respect to the inner product $\langle|\rangle''$, we can express eq.(B.29) as,

$$\langle\tilde{\Phi}^c_{2,\beta_2}|b_0^- b_0^+ (\tilde{L}_0^+)^{-1}|\tilde{\Phi}^c_{2,\alpha_2}\rangle'' - \langle\tilde{\Phi}^c_{2,\beta_2}|b_0^- b_0^+ (\tilde{L}_0^+)^{-1}|\tilde{\Phi}^c_{2,\alpha_2}\rangle + \mathcal{O}(\lambda^2) \tag{B.30}$$

Thus the terms inside the curly bracket in eq.(B.28) take the form:

$$\begin{aligned}&\langle\tilde{\Phi}^c_{2,\beta_2}|b_0^- b_0^+ ((\tilde{L}_0^+)^{-1} - (L_0^+)^{-1})|\tilde{\Phi}^c_{2,\alpha_2}\rangle'' \\ &- \langle\tilde{\Phi}^c_{2,\beta_2}|b_0^- b_0^+ ((\tilde{L}_0^+)^{-1} - (L_0^+)^{-1})|\tilde{\Phi}^c_{2,\alpha_2}\rangle + \mathcal{O}(\lambda^2)\end{aligned} \tag{B.31}$$

Note that $(\tilde{L}_0^+)^{-1}$ and $(L_0^+)^{-1}$ differ by a term of order λ. Also, $\langle|\rangle''$ and $\langle|\rangle$ differ by a term of order λ. Thus the expression given in eq.(B.31) is of order λ^2.

This shows that if $\tilde{A}^{(3)}_{\mu_2\nu_2 k_2}$ and $\tilde{A}^{(3)}_{\sigma_2\tau_2 k_2}$ vanish, then the left hand side of eq.(4.14) vanishes. This, in turn, implies that the left hand side of eq.(4.14) must be of the form given in eq.(4.15). Although the quantities $\Sigma^{(1)}_{k_2\mu_2\nu_2}$ and hence $S^{(1)}_{k_2\mu_2\nu_2}$ may be determined by careful analysis (similar to the one in ref.[13] which determined $\langle\tilde{\Phi}^c_{2,k_2}|K|\tilde{\Phi}_{2,l_2}\rangle$), we shall not carry out that analysis here.

APPENDIX C

In this appendix we shall derive an expression for $T^{(1)}_{\beta_2 k_2}$ appearing on the right hand side of eq.(5.16). From eqs.(5.2), (5.3), we get,

$$T^{(1)}_{\beta_2 k_2} = \lim_{\lambda\to 0}(K^{(0)}_{\beta_2 k_2} + W^{[2]}_{\beta_2 k_2}) \tag{C.1}$$

where $K^{(0)}$ is defined in eqs.(B.1), (B.3), and $W^{[n]}$ is defined through the relation,

$$V^{[n]}_{sr} = \delta_{sr} + \lambda W^{[n]}_{sr} + \mathcal{O}(\lambda^2) \tag{C.2}$$

We now use eqs.(B.4), (B.26), and the relation,

$$b_0^- |\tilde{\Phi}^c_{2,\beta_2}\rangle = -\tilde{Q}_B b_0^- |\tilde{\Phi}^c_{3,\beta_2}\rangle \tag{C.3}$$

to express $K^{(0)}_{\beta_2 k_2}$ as,

$$K^{(0)}_{\beta_2 k_2} = \langle\tilde{\Phi}^c_{3,\beta_2}|[\tilde{Q}_B, K]|\tilde{\Phi}_{2,k_2}\rangle'' = \frac{1}{\lambda}\langle\tilde{\Phi}^c_{3,\beta_2}|(\tilde{Q}_B - \hat{Q}_B)|\tilde{\Phi}_{2,k_2}\rangle'' \tag{C.4}$$

Proof of eq.(C.3): To prove this equation let us note that $|\tilde{\Phi}^c_{2,\beta_2}\rangle$ is defined through the equations:

$$\langle\tilde{\Phi}^c_{2,\beta_2}|\tilde{\Phi}_{2,\alpha_1}\rangle'' = 0, \quad \langle\tilde{\Phi}^c_{2,\beta_2}|\tilde{\Phi}_{2,k_2}\rangle'' = 0, \quad \langle\tilde{\Phi}^c_{2,\beta_2}|\tilde{\Phi}_{2,\alpha_2}\rangle = \delta_{\alpha_2\beta_2} \tag{C.5}$$

It is clear that $|\tilde{\Phi}^c_{2,\beta_2}\rangle$ given in eq.(C.3) satisfy the first two sets of equations given in eq.(C.5). To verify the last set of equations in eq.(C.5) let us note that with $|\tilde{\Phi}^c_{2,\beta_2}\rangle$ as defined in eq.(C.3), we get,

$$\begin{aligned}\langle\tilde{\Phi}^c_{2,\beta_2}|\tilde{\Phi}_{2,\alpha_2}\rangle'' =& - \langle\tilde{Q}_B b_0^- \tilde{\Phi}^c_{3,\beta_2}|c_0^-|\tilde{\Phi}_{2,\alpha_2}\rangle'' \\ =&\langle\tilde{\Phi}^c_{3,\beta_2}|\tilde{Q}_B b_0^- c_0^-|\tilde{\Phi}_{2,\alpha_2}\rangle'' = \langle\tilde{\Phi}^c_{3,\beta_2}|\tilde{\Phi}_{3,\alpha_2}\rangle'' = \delta_{\alpha_2\beta_2}\end{aligned} \tag{C.6}$$

This shows that $|\tilde{\Phi}^c_{2,\beta_2}\rangle$ defined in eq.(C.3) also satisfies the last set of equations (C.5).

Let us now compute $W^{[2]}_{\beta_2 k_2}$. From eqs.(2.8) and (C.2) we see that,

$$|\tilde{\Phi}_{n,r}\rangle = |\Phi_{n,r}\rangle + \lambda W^{[n]}_{sr}|\Phi_{n,s}\rangle + \mathcal{O}(\lambda^2) \tag{C.7}$$

Let us divide the basis $\{|\Phi_{n,r}\rangle\}$ into physical $\{|\Phi_{n,k_n}\rangle\}$, unphysical $\{|\Phi_{n,\alpha_n}\rangle\}$, and pure gauge $\{|\Phi_{n,\alpha_{n-1}}\rangle = Q_B|\Phi_{n-1,\alpha_{n-1}}\rangle\}$ as in eqs.(3.1)-(3.3), with respect to the BRST charrge Q_B, and let $\{\langle\Phi^c_{n,r}|\}$ be the conjugate basis defined with respect to the BPZ inner product $\langle|\rangle$ in CFT. In this case, $\{b_0^-|\Phi^c_{n,\alpha_n}\rangle\}$, $\{b_0^-|\Phi^c_{n,k_n}\rangle\}$ and $\{b_0^-|\Phi^c_{n,\alpha_{n-1}}\rangle\}$ correspond to the basis of pure gauge, physical, and unphysical states respectively of ghost number $(5-n)$ [13].

We now note that in the generic case, the basis of unphysical states can be taken to be the same in CFT and CFT$''$; these are the states that are not annihilated by Q_B or $\tilde{Q}_B$. Using this, let us take,

$$b_0^-|\Phi^c_{n,\alpha_{n-1}}\rangle = b_0^-|\tilde{\Phi}^c_{n,\alpha_{n-1}}\rangle \tag{C.8}$$

Using eq.(C.8) and the analog of eq.(C.3) in CFT, we get,

$$b_0^-|\Phi^c_{2,\alpha_2}\rangle = -Q_B b_0^-|\Phi^c_{3,\alpha_2}\rangle = -Q_B b_0^-|\tilde{\Phi}^c_{3,\alpha_2}\rangle \tag{C.9}$$

Eq.(C.7) now gives,

$$\lambda W^{[2]}_{\beta_2 k_2} = \langle\Phi^c_{2,\beta_2}|\tilde{\Phi}_{2,k_2}\rangle + \mathcal{O}(\lambda^2) = \langle\tilde{\Phi}^c_{3,\beta_2}|Q_B|\tilde{\Phi}_{2,k_2}\rangle + \mathcal{O}(\lambda^2) \tag{C.10}$$

Using eqs.(C.1), (C.4), and (C.10), and the relation $\tilde{Q}_B|\tilde{\Phi}_{2,k_2}\rangle = 0$, we get,

$$T^{(1)}_{\beta_2 k_2} = \lim_{\lambda\to 0}\left(\frac{1}{\lambda}\langle\tilde{\Phi}^c_{3,\beta_2}|(Q_B - \hat{Q}_B)|\tilde{\Phi}_{2,k_2}\rangle\right) \tag{C.11}$$

Using eqs.(2.22) and (2.29) and that $|\tilde{\Phi}_{2,k_2}\rangle = c_1\bar{c}_1|\varphi\rangle + \mathcal{O}(\lambda)$, we get,

$$T^{(1)}_{\beta_2 k_2} = -\sqrt{2}\{(\tilde{\Phi}^c_{3,\beta_2})(c_0^- c_1\bar{c}_1|\varphi\rangle)(c_0^- c_1\bar{c}_1|\varphi\rangle)\}'' \tag{C.12}$$

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