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DUALITY SYMMETRIES OF 4D HETEROTIC STRINGS

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ABSTRACT

Target space duality (T duality), which interchanges Kaluza–Klein and winding-mode excitations of the compactified heterotic string, is realized as a symmetry of a world-sheet action. Axion-dilaton duality (S duality), a conjectured nonperturbative SL(2,Z) symmetry of the same theory, plays an analogous role for five-branes. We describe a soliton spectrum possessing both duality symmetries and argue that the theory has an infinite number of dual string descriptions.

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1. Introduction

One of the major gaps in our understanding of string theory is the lack of a fundamental formulation of the nonperturbative theory. Many efforts have been made to gain insight into nonperturbative aspects of string theory in recent years. These include studies of matrix models, construction of soliton solutions such as black holes and magnetic monopoles, studies of string field theory, and much more. Another recent focus, which will be pursued here, is a proposed nonperturbative SL(2,Z) symmetry of the heterotic string theory compactified to four dimensions $[1-3]$. While such a symmetry is not yet definitively established, the evidence for it is certainly mounting $[4-6]$. To be concise, let us refer to this symmetry as *S duality.*

If present at all, S duality is necessarily nonperturbative, since it transforms the four-dimensional dilaton field, whose value determines the string loop expansion parameter (Newton's constant), nonlinearly. Despite this fact, it has many remarkable similarities with target-space duality (called *T duality*), which is also an infinite discrete group. (This group generalizes the well-known $R \to 1/R$ symmetry.) In the case of toroidal compactification of the heterotic string, in the manner originally proposed by Narain [7], G_T = O(6,22;Z). In general, the group G_T depends on the particular compactification chosen. Other examples that have been studied include certain orbifolds and Calabi–Yau spaces. Unlike G_T, the S duality group SL(2,Z) seems to be "universal" in the sense that it does not depend on the compactification chosen, at least if the choice preserves some supersymmetry in four dimensions. Of course, in the case of toroidal compactification (the only case we will consider explicitly), there is N = 4 supersymmetry in four dimensions.

Heuristically, one can describe the toroidally compactified heterotic string theory by an effective four-dimensional action, containing fields associated with massless quanta only. Effects due to finite string size and string loops are then represented as a double series expansion in the string scale α' and Newton's constant. Of course, these series do not converge, and there are important nonperturbative phenomena associated with both expansions. The leading term (in both senses), which is a classical N = 4, D = 4 field theory, has both dualities — O(6,22) and SL(2,R), but as usually formulated, there is an apparent asymmetry between them. Namely, O(6,22) is a manifest symmetry of the action, whereas SL(2,R) is a symmetry of the equations of motion only. However, in a recent paper [6] we showed that it is possible to recast the theory, by introducing suitable auxiliary fields, so that both symmetries are realized simultaneously in the action in essentially the same way. In certain cases, the price for doing this is that the action no longer has manifest general coordinate invariance, though this symmetry is still present. The way this works is that the formulas for general coordinate transformations of vector fields are modified from the usual ones by terms that vanish when the equations of motion are satisfied.

An analogous mathematical problem arises in understanding the T duality group O(6,22) in the 2D world-sheet theory, which underlies the α' expansion. Namely, in the usual formulation O(6,22) is a symmetry of the world-sheet field equations only, not the world-sheet action. In section 2, methods analogous to those employed for the 4D problem are used to find a new form of the world-sheet action possessing O(6,22) symmetry. (This generalizes previous work by Tseytlin [8], which contained many of the essential ideas.) The boundary condition on the world-sheet fields break this O(6,22) symmetry to O(6,22;Z). Thus, if there are no anomalies, the toroidally compactified heterotic string theory should have this symmetry order-by-order in Newton's constant, provided that at each order the full nonperturbative α' dependence is taken into account. It seems plausible that the corresponding statement can be made for the S duality symmetry SL(2,Z) when the role of the α' and Newton's constant expansions are interchanged, i.e., S duality should be true order-by-order in α' when the full nonperturbative Newton's constant structure is included at each order. This interchange in the roles of α' and Newton's constant corresponds roughly to what one gets by replacing the string theory by a dual theory [9] based on five-branes [10] [11]. This is only heuristic, however, since there is no well-defined quantum theory of five-branes as yet. In any case, we propose to refer to this expected symmetry between the roles of the two duality groups as *duality of dualities.*

In section 3 a mass formula for string solitons as a function of their electric and magnetic charges is described. By assuming that a Bogomol'nyi bound is saturated (as is expected for an N = 4 theory), the spectrum of soliton masses is shown to depend on the moduli in just the right way to ensure O(6,22;Z) $\otimes$ SL(2,Z) symmetry. The spectrum of charges corresponds to a 56-dimensional even self-dual lattice, whose properties ensure that the Dirac–Schwinger–Zwanziger–Witten [12] [13] (DSZW) quantization requirements are automatically satisfied. The states containing electric charges only are present in the perturbative spectrum, whereas all the others containing at least one non-zero magnetic charge must arise nonperturbatively. As a result, the perturbative spectrum is O(6,22;Z) invariant but not SL(2,Z) invariant.

However, the spectrum of the perturbative five-branes, compactified on a six-dimensional torus, contains states that carry both magnetic and electric charges in such a way that the spectrum of charges is symmetric under SL(2,Z) transformations, but not under O(6,22;Z) transformations.

Since the string world-sheet theory does not have S duality, one obtains a different world-sheet theory by applying an SL(2,Z) transformation. Section 4 explains that the transformed theories can be interpreted as an infinite family of isomorphic theories, any one of which provides an equally good starting point for defining the full theory. The essential difference between different choices is which states in the spectrum belong to the perturbative spectrum and which ones arise nonperturbatively as solitons. This SL(2,Z) duality of string theories generalizes a Z_2 duality proposed for certain field theories by Montonen and Olive [14].

2. World-Sheet Action with Manifest O(6,22;Z) Symmetry

When the heterotic string is compactified on a 28-torus that is conjugate to an even self-dual lattice of signature (6,22), one obtains a consistent four-dimensional theory. The resulting 4D theory has N = 4 supersymmetry and contains the following massless bosons: graviton ($g_{\mu\nu}$), 4D dilaton (Φ), antisymmetric tensor ($B_{\mu\nu}$) – related by a duality transformation to the axion (χ), 28 abelian vector fields (A_μ^a) transforming as a vector of the group O(6,22), and scalars (or moduli) described by a matrix M^{ab}, which parametrizes the coset O(6,22)/O(6) $\times$ O(22). The matrix M is an arbitrary real symmetric 28×28 matrix belonging to the group O(6,22). The axion and dilaton can be combined into a complex field $\lambda = \chi + ie^{-\Phi} \equiv \lambda_1 + i\lambda_2$, which transforms under S duality according to $\lambda \to (a\lambda + b)/(c\lambda + d)$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ SL(2,Z). For "generic" values of the moduli this is the complete massless bosonic spectrum. However, for special values corresponding to various hypersurfaces in moduli space, there are additional massless states and nonabelian gauge symmetries. Certain parts of our analysis are not easily generalized to include nonabelian gauge symmetries, so we restrict the moduli to "generic" values. The lifting of this restriction is an important topic for future study. To keep formulas from becoming unwieldy, all fermions are dropped, though their inclusion would not be an essential complication.

Let us now consider the string world-sheet theory in the presence of all the bosonic fields listed above, each being a function of the four-dimensional space-time coordinate x^μ. Thus, the world-sheet theory contains as "couplings" exactly those fields that are included in the low-energy effective field theory. In the usual formulation of the world-sheet theory, the moduli described by M^{ab} appear in three distinct pieces corresponding to internal components of the ten-dimensional metric, antisymmetric tensor, and vectors (of which there are 16). This action certainly does not have O(6,22) symmetry. The equations of motion of the world-sheet theory can be recast in a manifestly O(6,22) symmetric form, however [15] [8] [16]. For this purpose one introduces 28 world-sheet fields $y^a(\sigma,\tau)$ to parametrize the 28-torus discussed earlier. Since the geometric data reside in the moduli, each y^a can be regarded as an angular coordinate for a circle of unit radius. The invariant metric of the group O(6,22) is conveniently taken to have the form

$$L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix} , \tag{1}$$

so that six eigenvalues are $+1$ and 22 are -1. Since $M^T L M = L$ and $M^T = M$, $M^{-1} = LML$. In terms of these quantities, it was shown in ref.[16] that the world-sheet field equations can be recast in the manifestly O(6,22) symmetric form*

$$D_0 y^a = -(ML)^a{}_b D_1 y^b \tag{2}$$

and

$$\begin{aligned} g_{\mu\nu}\partial^\alpha\partial_\alpha x^\nu + \Gamma_{\mu\nu\rho}\partial^\alpha x^\nu \partial_\alpha x^\rho &= -\frac{1}{2} D_1 y^a (L\partial_\mu M L)_{ab} D_1 y^b \\ &- \epsilon^{\alpha\beta}\partial_\alpha x^\nu F^a_{\mu\nu} L_{ab} D_\beta y^b + \frac{1}{2}\epsilon^{\alpha\beta} H_{\mu\nu\rho}\partial_\alpha x^\nu \partial_\beta x^\rho . \end{aligned} \tag{3}$$

In these equations

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \\ H_{\mu\nu\rho} &= \partial_\mu B_{\nu\rho} + \frac{1}{2} A^a_\mu L_{ab} F^b_{\nu\rho} + \text{cyc. perms.} \\ D_\alpha y^a &= \partial_\alpha y^a + A^a_\mu \partial_\alpha x^\mu \end{aligned} \tag{4}$$

and $\Gamma_{\mu\nu\rho}$ is the usual Christoffel connection. The $\int \Phi R^{(2)} d^2\sigma$ term has been dropped from the world-sheet action, since it is higher order in α'. Note that O(6,22) symmetry requires

$\star$ The vectors A_μ^{m+6} and y^{m+6} $(1 \le m \le 6)$ are related to the vectors $A^{(2)}_{m\mu}$ and y^{m+6} of ref.[16] by a minus sign.

regarding the coordinates y^a as a 28-vector. Since they describe a product of 28 circles, it is clearly only possible to rotate them with integer coefficients, so the group must be restricted to O(6,22;Z). Note also that $D_\alpha y^a$ is gauge invariant provided that under a gauge transformation, $\delta A^a_\mu = \partial_\mu \Lambda^a$, the internal coordinates transform as follows: $\delta y^a = -\Lambda^a$. Since the matrix ML has 22 eigenvalues that are -1 and 6 that are $+1$ the y equation of motion (2) describes 22 left-moving bosons and 6 right-moving bosons.

Following Tseytlin [8] (whose work we are generalizing here), it is possible to find an action based on the world-sheet coordinates x^μ and y^a that has manifest O(6,22) symmetry. The Lagrangian (for flat world-sheet metric) that gives the equations of motion (2) and (3) is

$$\begin{aligned}\mathcal{L} = &\frac{1}{2} g_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu - \frac{1}{2} D_0 y^a L_{ab} D_1 y^b - \frac{1}{2} D_1 y^a (LML)_{ab} D_1 y^b \\ &+ \frac{1}{2}\epsilon^{\alpha\beta}[B_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu - A^a_\mu \partial_\alpha x^\mu L_{ab} D_\beta y^b] \ .\end{aligned} \tag{5}$$

The $[U(1)]^{28}$ gauge invariance of this formula involves an interplay between the last two terms, since $\delta B_{\mu\nu} = -\frac{1}{2} F^a_{\mu\nu} L_{ab}\Lambda^b$.

To understand this theory better, it is important to exhibit the coupling to a world-sheet metric $h_{\alpha\beta}$ that gives 2D Weyl invariance and reparametrization invariance. This is achieved by replacing the first term (as usual) by

$$\mathcal{L}'_1 = \frac{1}{2}\sqrt{-h} h^{\alpha\beta} g_{\mu\nu}(x) \partial_\alpha x^\mu \partial_\beta x^\nu \ , \tag{6}$$

and the third term by

$$\mathcal{L}'_3 = -\frac{1}{2\sqrt{-h} h^{00}} D_1 y^a (LML)_{ab} D_1 y^b - \frac{h^{01}}{2h^{00}} D_1 y^a L_{ab} D_1 y^b \ . \tag{7}$$

The other three terms are not modified. The last two terms are reparametrization and Weyl invariant as they stand. The second term ($\mathcal{L}_2$) is not reparametrization invariant, but it turns out the sum $\mathcal{L}_2 + \mathcal{L}'_3$ is. The construction used here is a specialization to two dimensions of the general method introduced in ref.[6]. Reparametrization invariance is achieved by modifying the usual rule $\delta y^a = \xi^1 \partial_1 y^a + \xi^0 \partial_0 y^a$. Specifically, the term $\partial_0 y^a$ should be replaced by the expression that it equals as a result of the y^a equation of motion. All other transformations are the usual ones.

The action can now be varied with respect to $h^{\alpha\beta}$ to give the symmetric traceless energy–momentum tensor $T_{\alpha\beta}$. The requirement that $T_{\alpha\beta}$ vanishes gives the usual Virasoro conditions, which are rather simple in the $h_{\alpha\beta} = \eta_{\alpha\beta}$ gauge. Alternatively, if one wishes, the $h_{\alpha\beta}$ equations of motion can be solved and used to eliminate $h_{\alpha\beta}$ from the action, thereby obtaining the "Nambu form." This Nambu form still has reparametrization symmetry, which can be used to impose the Virasoro conditions as gauge conditions that supplement the equations of motion given previously.

The T duality group O(6,22) relates Kaluza–Klein excitations of the compactified string to winding-mode excitations. From the point of view of the conventional 2D world-sheet field theory, the KK excitations can be understood perturbatively (in the α' expansion), whereas the winding-mode excitations are nonperturbative solitons. If the characteristic size of the compact dimensions is called R, these statements are reflected in the fact that the masses of Kaluza–Klein excitations are proportional to $1/R$, whereas those of winding-mode excitations are proportional to R/α'. Thus, the latter become infinitely heavy in the weak coupling limit $\alpha' \to 0$, a characteristic feature of solitons. Given these facts, it seems remarkable that the O(6,22) symmetry is realized on the action! Clearly, this requires some explanation. The internal components of the metric and the other moduli are of order R^2/α' (times dimensionless numbers). In the usual string action only terms proportional to R^2/α' or $1/\alpha'$ appear. However, the matrix M^{ab} is constructed out of the internal metric and its inverse. Thus, it has pieces proportional to $(R^2/\alpha')^n$ for $n = -1, 0, +1$. To understand the symmetries in question perturbatively in α', we must consider R^2 to be of order α'. This means that the y^2 terms in the O(6,22) symmetric action (5) are strongly coupled and must be treated exactly. Fortunately, since the y dependence in eq. (5) is quadratic, this is possible and explains why a symmetry that relates perturbative excitations to solitons can be realized in the action.

3. Bogomol'nyi Bound, Soliton Spectrum, and Five-Branes

The effective field theory of massless bosonic fields for heterotic string theory compactified on a Narain torus at a generic point in the moduli space can be written in various classically equivalent forms. One form that has manifest T duality and general coordinate invariance is

$$S = \frac{1}{32\pi}\int d^4x\sqrt{-g}\Big[R - \frac{1}{2(\lambda_2)^2}g^{\mu\nu}\partial_\mu\lambda\partial_\nu\bar{\lambda} - \sum_{a,b=1}^{28}\frac{\lambda_2}{4}F^a_{\mu\nu}(LML)_{ab}F^{b\mu\nu} + \frac{\lambda_1}{4}\sum_{a,b=1}^{28}F^a_{\mu\nu}L_{ab}\tilde{F}^{b\mu\nu} + \frac{1}{8}g^{\mu\nu}Tr(\partial_\mu ML\partial_\nu ML)\Big]. \tag{8}$$

The overall multiplicative factor of $1/32\pi$ is irrelevant for classical analysis, and was omitted in ref.[6], but it provides a convenient normalization of the action when discussing charge quantization, breaking of SL(2,R) symmetry to SL(2,Z), and the Bogomol'nyi bound [4] [5]. Although there are no massless charged fields in this theory, the full string theory does contain massive charged states, as well as soliton states carrying magnetic charges. The electric and magnetic charges q^a_{el} and q^a_{mag} of a state are defined by*

$$2q^a_{el} = \lim_{r\to\infty} rx^iF^a_{0i}, \quad 2q^a_{mag} = \lim_{r\to\infty} rx^i\tilde{F}^a_{0i}. \tag{9}$$

The Bogomol'nyi lower bound [17] on the mass squared of a state for a given value of (q^a_{el}, q^a_{mag}) is given by [18] [5]

$$m^2 \geq \frac{\lambda_2^{(0)}}{16}\Big(q^a_{el}(LM^{(0)}L+L)_{ab}q^b_{el} + q^a_{mag}(LM^{(0)}L+L)_{ab}q^b_{mag}\Big) \equiv (m_0)^2, \tag{10}$$

where the superscript $^{(0)}$ denotes the asymptotic value of the corresponding field.

In ref.[5] the expression for m_0 in eq.(10) was shown to be SL(2,Z) invariant. In order to rewrite it in a manifestly SL(2,Z) invariant form, let us express q^a_{el} and q^a_{mag} in terms of vectors α^a_0 and β^a_0 [5]

$$q^a_{el} = \frac{1}{\lambda_2^{(0)}}M^{(0)}_{ab}(\alpha^b_0 + \lambda_1^{(0)}\beta^b_0), \quad q^a_{mag} = L_{ab}\beta^b_0, \tag{11}$$

where both α^a_0 and β^a_0 belong to a reference lattice P_0, which is even and self-dual with

$\star$ These definitions differ from those of ref.[4][5] by a factor of two due to different normalization conventions for the gauge fields.

respect to the metric L. Now eq.(10) may be rewritten as

$$(m_0)^2 = \frac{1}{16} \left(\alpha_0^a \quad \beta_0^a \right) \mathcal{M}^{(0)} (M^{(0)} + L)_{ab} \begin{pmatrix} \alpha_0^b \\ \beta_0^b \end{pmatrix}, \tag{12}$$

where we define

$$\mathcal{M} = \frac{1}{\lambda_2} \begin{pmatrix} 1 & \lambda_1 \\ \lambda_1 & |\lambda|^2 \end{pmatrix} \quad \text{and} \quad \mathcal{L} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{13}$$

$\mathcal{M}$ and $\mathcal{L}$ play the same role for SL(2,Z) that M and L do for O(6,22;Z). Eq.(12) is manifestly invariant under SL(2,Z) transformations ω and under O(6,22;Z) transformations Ω:†

$$\begin{aligned} M \to& \Omega^T M \Omega, \quad \alpha_0^a \to (\Omega^{-1})_{ab} \alpha_0^b, \quad \beta_0^a \to (\Omega^{-1})_{ab} \beta_0^b \\ \mathcal{M} \to& \omega^T \mathcal{M} \omega, \quad \begin{pmatrix} \alpha_0^a \\ \beta_0^a \end{pmatrix} \to \omega^{-1} \begin{pmatrix} \alpha_0^a \\ \beta_0^a \end{pmatrix}. \end{aligned} \tag{14}$$

Eq.(12) suggests that it is natural to combine the vectors α_0^a and β_0^a into a single 56-dimensional vector $\xi = \begin{pmatrix} \alpha_0^a \\ \beta_0^a \end{pmatrix}$ which now belongs to a 56-dimensional lattice Γ. The new lattice Γ is self-dual not only with respect to the metric L, but also with respect to the metric $\hat{\mathcal{L}} = \mathcal{L} \otimes L$. The latter condition says that, for any two vectors $\xi = (\alpha_0^a, \beta_0^a)$ and $\xi' = (\alpha_0'^a, \beta_0'^a)$ belonging to the lattice Γ,‡

$$\xi^T \hat{\mathcal{L}} \xi' = \alpha_0^a L_{ab} \beta_0'^b - \alpha_0'^a L_{ab} \beta_0^b = \text{integer}. \tag{15}$$

In our normalization this is just the DSZW quantization condition for the magnetic charge.

The statement that the spectrum of electric and magnetic charges in the theory remains invariant under SL(2,Z) transformations [4] can now be translated to the statement that the lattice Γ is invariant under SL(2,Z) transformations. This follows from eq.(14) and the fact that both $\vec{\alpha}_0$ and $\vec{\beta}_0$ belong to the lattice P_0. Similarly, T duality invariance of the spectrum is the statement that Γ is invariant under O(6,22;Z) transformations. This follows from the invariance of the lattice P_0 under such transformations.

† The SL(2,Z) and O(6,22;Z) transformation laws of the vectors α_0^a and β_0^a can be read off from eqs.(9), (11), and the known transformation laws [2] [6] of the fields λ, M and $F^a_{\mu\nu}$ under these transformations.

‡ In fact, the terms are separately integers, since P_0 is even and self-dual. This reflects the fact that there are states in the spectrum without magnetic charge.

To summarize, we have expressed the mass squared of supersymmetric states in the theory in a form that it is manifestly invariant under the SL(2,Z) and O(6,22;Z) transformations. Furthermore, these two transformations appear on an equal footing. In this formalism, both the SL(2,Z) and O(6,22;Z) invariances of the allowed spectrum of charges correspond to the invariance of the lattice Γ under the corresponding transformations. However, in string theory, there is a fundamental difference between these two transformations. An O(6,22;Z) transformation relates Kaluza–Klein modes to string winding modes, and hence transforms perturbative string excitations to perturbative string excitations, whereas an SL(2,Z) transformation transforms perturbative string excitations to monopole (or dyon) solutions in string theory. Thus, the spectrum of perturbative string excitations has O(6,22;Z) symmetry, but not SL(2,Z) symmetry. This can be seen explicitly by noting that the perturbative string spectrum contains charge vectors $\vec{\xi}$ of the form $\begin{pmatrix} \alpha_0^a \\ 0 \end{pmatrix}$. States with $\beta_0^a \neq 0$ are solitons, and their masses diverge in the weak coupling limit.

Assuming that SL(2,Z) is a genuine symmetry of string theory, it is reasonable to ask if there is some dual formulation of the theory for which the spectrum of perturbative excitations has SL(2,Z) invariance, and O(6,22;Z) symmetry of the spectrum becomes manifest only after including the soliton solutions of this dual theory. Let us now look for such a possibility among p-brane theories in ten dimensions. When a ten-dimensional p-brane theory is compactified on a torus to four dimensions, the spectrum includes the usual Kaluza–Klein modes, which can be identified with the Kaluza–Klein modes in string theory. But there are also excitations that correspond to the p-brane wrapped around the six-torus, which are required to be the SL(2,Z) transforms of the Kaluza–Klein modes, just as the string winding modes are O(6,22;Z) transforms of Kaluza–Klein modes.

We shall now show that if such a scenario holds, p must be five. From eq.(12), taking $\lambda_1^{(0)} = 0$ (i.e., vanishing θ angle), the ratio of the masses of a purely electrically charged particle to a purely magnetically charged particle is given by

$$1/\lambda_2^{(0)} = \lim_{r\to\infty} e^{\Phi^{(10)}} \left(\det G_{Smn}^{(10)}\right)^{-1/2} \propto R^{-6}, \tag{16}$$

where $\Phi^{(10)}$ is the ten-dimensional dilaton field, $G_{Smn}^{(10)}$ denotes the internal components of the ten-dimensional string metric, and $R \propto \sqrt{G_{Smn}^{(10)}}$ denotes the linear scale of the internal manifold measured in this metric. Let us now consider the dependence of this mass ratio on R for a fixed value of $\Phi^{(10)}$.

For a p-brane theory compactified on a six-torus, the masses of the Kaluza–Klein modes are proportional to $1/R'$, whereas those of the p-brane winding modes, which are supposed to be identified with the string theory monopoles, are proportional to R'^p. Here R' denotes the radius of the internal manifold computed in the p-brane metric, which can differ from the string metric by a multiplicative factor involving the dilaton field. In order to study the dependence of the mass ratio on R for fixed value of the dilaton field, we can take R' to be proportional to R. The mass ratio is then proportional to R^{-p-1}. Comparison with the calculation based on string theory in eq.(16) then gives $p = 5$. This shows that if there exists a dual version of string theory for which the perturbative spectrum is manifestly SL(2,Z) invariant, it must be a theory of five-branes.

This result can be made more concrete by identifying the quantum numbers α_0^m and β_0^m ($1 \le m \le 6$) with the internal momenta and winding numbers of the five-brane wrapped around a six-torus. In this analysis all fields that arise from the dimensional reduction of the 16 ten-dimensional gauge fields are set to zero, and we only consider states that do not carry any charge associated with these gauge fields. In this case, the indices a, b in eq.(8) can be taken to run from 1 to 12, α_0^a, β_0^a can be regarded as 12-dimensional vectors, and M and L can be taken to be 12×12 matrices of the form

$$M = \begin{pmatrix} \hat{G}^{-1} & \hat{G}^{-1}\hat{B} \\ -\hat{B}\hat{G}^{-1} & \hat{G} - \hat{B}\hat{G}^{-1}\hat{B} \end{pmatrix}, \quad L = \begin{pmatrix} 0 & I_6 \\ I_6 & 0 \end{pmatrix}, \tag{17}$$

where $\hat{G}$ and $\hat{B}$ are internal components of the metric and antisymmetric tensor fields, respectively. As was shown in ref.[6], the gauge field dependent part of the action (8) can be replaced by

$$-\frac{1}{128\pi}\int d^4x\sqrt{-g}\sum_{m,n=1}^{6}\Big[F^{(m,\alpha)}_{\mu\nu}\hat{G}_{mn}(\mathcal{L}^T\mathcal{M}\mathcal{L})_{\alpha\beta}F^{(n,\beta)\mu\nu} + F^{(m,\alpha)}_{\mu\nu}\hat{B}_{mn}\mathcal{L}_{\alpha\beta}\tilde{F}^{(n,\beta)\mu\nu}\Big]. \tag{18}$$

The precise relation between the fields $F^{(m,\alpha)}_{\mu\nu}$ and $F^a_{\mu\nu}$ can be found by using the manifestly O(6,6)×SL(2,R) form of the action given in ref.[6] and the equations of motion derived from that action. This form of the action contains 24 field strengths $F^{(a,\alpha)}_{\mu\nu}$ ($1 \le a \le 12$, $1 \le \alpha \le 2$), with the identification

$$F^{(a,1)}_{\mu\nu} = F^a_{\mu\nu}. \tag{19}$$

The equations of motion relate $F^{(a,2)}$ to $F^{(a,1)}$. In particular,

$$\begin{aligned} F^{(m,2)}_{\mu\nu} =& \lambda_1 F^{(m,1)}_{\mu\nu} + \lambda_2 \hat{G}^{mn} \tilde{F}^{(n+6,1)}_{\mu\nu} + \lambda_2 \hat{G}^{mn} \hat{B}_{np} \tilde{F}^{(p,1)}_{\mu\nu} \\ =& \lambda_1 F^{m}_{\mu\nu} + \lambda_2 \hat{G}^{mn} \tilde{F}^{n+6}_{\mu\nu} + \lambda_2 \hat{G}^{mn} \hat{B}_{np} \tilde{F}^{p}_{\mu\nu}, \qquad 1 \le m \le 6. \end{aligned} \tag{20}$$

Let us now consider adding source terms of the form

$$\frac{1}{4}\int d^4x \sqrt{-g}(A^{(m,1)}_\mu J^\mu_m + A^{(m,2)}_\mu \tilde{J}^\mu_m) \tag{21}$$

to the action (18). For asymptotically Minkowskian metric $g_{\mu\nu}$ the gauge field equations of motion derived from the combined action (18), (21) give rise to the following form of Gauss's law after we use eqs.(9), (11), (19) and (20)

$$\begin{aligned} \int d^3x \sqrt{-g} J^0_m =& \frac{1}{2}\lim_{r\to\infty} r x^i \Big(\frac{|\lambda|^2}{\lambda_2}\hat{G}_{mn} F^{(n,1)}_{0i} - \frac{\lambda_1}{\lambda_2}\hat{G}_{mn} F^{(n,2)}_{0i} + \hat{B}_{mn}\tilde{F}^{(n,2)}_{0i}\Big) = \alpha^m_0, \\ \int d^3x \sqrt{-g} \tilde{J}^0_m =& \frac{1}{2}\lim_{r\to\infty} r x^i \Big(\frac{1}{\lambda_2}\hat{G}_{mn} F^{(n,2)}_{0i} - \frac{\lambda_1}{\lambda_2}\hat{G}_{mn} F^{(n,1)}_{0i} - \hat{B}_{mn}\tilde{F}^{(n,1)}_{0i}\Big) = \beta^m_0. \end{aligned} \tag{22}$$

This shows that the quantum numbers α^m_0 and β^m_0 are the total charges coupled to the gauge fields $A^{(m,1)}_\mu$ and $A^{(m,2)}_\mu$, respectively. Since these gauge fields couple naturally to the five-brane [6], the contribution to these charges from a given configuration of the five-brane can be calculated. To do this, let us introduce the world-volume metric γ_{rs} $(0 \le r, s \le 5)$ and ten-dimensional fields $G^{(10)}_{FMN}$, $\mathcal{A}^{(10)}_{M_1 \ldots M_6}$ $(0 \le M, N, M_i \le 9)$ that couple naturally to the five-brane [10] [11], and write the five-brane σ-model action in terms of these background fields:

$$\int d^6\xi \Big[\frac{1}{2}\sqrt{-\gamma}\gamma^{rs} G^{(10)}_{FMN}\partial_r Z^M \partial_s Z^N - 2\sqrt{-\gamma} + \frac{1}{6!}\mathcal{A}_{M_1\ldots M_6}\epsilon^{r_1\ldots r_6}\partial_{r_1} Z^{M_1}\ldots\partial_{r_6} Z^{M_6}\Big]. \tag{23}$$

In writing this equation, the coupling of the ten-dimensional dilaton field $\Phi^{(10)}$ to the five-brane has been ignored, but this will not affect the analysis. Let us now consider backgrounds characterized by non-zero values of

$$G^{(10)}_{Fmn}, \quad G^{(10)}_{F\mu\nu}, \quad \mathcal{A}^{(10)}_{m_1\ldots m_6} = \lambda_1 \epsilon_{m_1\ldots m_6}, \quad 1 \le m, n \le 6, \quad \mu, \nu = 0, 7, 8, 9 \tag{24}$$

with all other components of all the fields set to zero. Denoting the internal coordinates by

Y^m and the space-time coordinates by X^μ, the action can be written as

$$\int d^6\xi[\frac{1}{2}\sqrt{-\gamma}\gamma^{rs}(G^{(10)}_{Fmn}\partial_r Y^m\partial_s Y^n + G^{(10)}_{F\mu\nu}\partial_r X^\mu\partial_s X^\nu) + \frac{\lambda_1}{6!}\epsilon_{m_1\ldots m_6}\epsilon^{r_1\ldots r_6}\partial_{r_1}Y^{m_1}\ldots\partial_{r_6}Y^{m_6}]. \tag{25}$$

Taking the background fields to be independent of the internal coordinates Y^m, this theory has the following two conserved world-volume current densities corresponding to internal momentum and winding-number densities of the five-brane

$$\begin{aligned} j^r_m =&(\sqrt{-\gamma}\gamma^{rs}G^{(10)}_{Fmn}\partial_s Y^n + \frac{\lambda_1}{5!}\epsilon^{rr_2\ldots r_6}\epsilon_{mm_2\ldots m_6}\partial_{r_2}Y^{m_2}\ldots\partial_{r_6}Y^{m_6}),\\ \tilde{j}^r_m =&\frac{1}{5!}\epsilon^{rr_2\ldots r_6}\epsilon_{mm_2\ldots m_6}\partial_{r_2}Y^{m_2}\ldots\partial_{r_6}Y^{m_6}. \end{aligned} \tag{26}$$

Let us now introduce background fields $G^{(10)}_{Fm\mu}$ and $\mathcal{A}^{(10)}_{\mu m_2\ldots m_6}$ and write down the extra terms that appear in the world-volume action to linear order in these fields. Using the identifications [6]

$$G^{(10)}_{Fm\mu} = G^{(10)}_{Fmn}A^{(n,1)}_\mu, \quad \mathcal{A}^{(10)}_{\mu m_2\ldots m_6} = \epsilon_{mm_2\ldots m_6}(-A^{(m,2)}_\mu + \lambda_1 A^{(m,1)}_\mu), \tag{27}$$

the extra terms in the world-volume action take the form

$$\int d^6\xi\Big(A^{(m,1)}_\mu j^r_m\partial_r X^\mu - A^{(m,2)}_\mu \tilde{j}^r_m\partial_r X^\mu\Big). \tag{28}$$

Let us now work in the static gauge $X^0 = \xi^0$. Comparing eqs.(21) and (28), and using eqs.(22), gives

$$\alpha^m_0 = 4\int j^0_m d^5\xi, \quad \beta^m_0 = -4\int \tilde{j}^0_m d^5\xi, \quad 1 \le m \le 6. \tag{29}$$

This shows that α^m_0 and β^m_0 are proportional to the total internal momenta and winding numbers of the five-brane, respectively. In other words, the spectrum of perturbative five-brane excitations, which contains both Kaluza–Klein states and five-brane winding states, is characterized by states for which the first six components of the twelve-dimensional vectors α^a and β^a are non-zero. Since this is an SL(2,Z) invariant set, the spectrum of allowed charges for perturbative five-brane excitations is SL(2,Z) invariant.

This analysis does not prove definitively that the mass spectrum of perturbative five-brane states is SL(2,Z) invariant. A complete answer to this question requires a better understanding of the five-brane mass spectrum. However, the five-brane theory is characterized by the same space-time supersymmetry algebra that is responsible for the Bogomol'nyi bound (12). (The presence of central charges corresponding to five-brane winding numbers in the supersymmetry algebra was established in ref.[19].) Hence the masses of the perturbative five-brane states are expected to satisfy the same lower bound as given in eq.(12). It remains to be proved that there are five-brane states that saturate this bound.

We conclude this section with the observation that the perturbative five-brane spectrum does not contain string winding modes; these must appear as soliton solutions in the five-brane theory. Hence T duality is not a symmetry of the perturbative five-brane spectrum.

4. SL(2,Z) Transformed World-Sheet Theories

The preceding section described a spectrum of electric and magnetic charge excitations in terms of 56-component vectors $(\vec{\alpha}_0, \vec{\beta}_0)$ that is consistent with the S and T dualities of the toroidally compactified theory. It was obtained by saturating the Bogomol'nyi bound and is consistent with the most general DSZW quantization requirements. In perturbation theory (i.e., the expansion in Newton's constant), all of the electrically charged states $(\vec{\alpha}_0, 0)$ are present in the spectrum, whereas none of the magnetically charged states $(\vec{\beta}_0 \neq 0)$ appear. All $\vec{\beta}_0 \neq 0$ states must arise nonperturbatively as solitons. The world-sheet theory of section 2 accounts for all the electrically charged states. As was explained there, some of these are perturbative and some are solitons from the world-sheet (first quantization) viewpoint. However, they are all perturbative from the space-time (second quantization) viewpoint. Since the S duality group SL(2,Z) relates electrically charged states to magnetically charged states, it relates perturbative states and nonperturbative states of the space-time theory, just as the T duality group O(6,22;Z) did for the world-sheet theory. Specifically, the SL(2,Z) group element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps states with charges $(\vec{\alpha}_0, 0)$ to ones with charges $(a\vec{\alpha}_0, c\vec{\alpha}_0)$. It is possible to find group elements for any pair of relatively prime integers a and c.

Now let us consider applying the SL(2,Z) transformation directly to the world-sheet theory. This transformation acts nontrivially on the background gauge fields $A^a_\mu(x)$, as well as the antisymmetric tensor $B_{\mu\nu}(x)$ and the dilaton $\Phi(x)$. In the form that the theory has been written, these are complicated nonlocal transformations. However, this doesn't really matter; the formulas are not required. In terms of the transformed fields $\tilde{A}(x), \tilde{B}(x)$, and $\tilde{\Phi}(x)$, the transformed world-sheet theory is isomorphic to the original one expressed in terms of A, B, and Φ. Therefore, this gives an S transformed dual formulation of the world-sheet theory, for which the excitation spectrum has charge vectors of the form $(a\vec{\alpha}_0, c\vec{\alpha}_0)$, as measured by the original gauge fields $A^a_\mu(x)$.⋆ Of course, from the viewpoint of the transformed potentials $\tilde{A}^a_\mu$ these are electrically charged states as before. In terms of the transformed world-sheet theory, all states with charges that are not of the form $(a\vec{\alpha}_0, c\vec{\alpha}_0)$ must arise as solitons of the associated space-time theory. Therefore there are an infinite number of equivalent dual starting points for defining the theory, which can be labeled by pairs of relatively prime integers (a, c). This generalizes the proposal of Montonen and Olive [14] (in another context) that there should be two dual formulations, which could be called "electric strings" $(a = 1, c = 0)$ and "magnetic strings" $(a = 0, c = 1)$. Note that distinct dual formulations are labeled by pairs (a, c) rather than by SL(2,Z) elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The reason for this is that the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which corresponds to a quantized shift of the axion field, is a symmetry of the world-sheet theory. Only group elements with $c \neq 0$ act nontrivially on the electric string.

If one considers five-branes, on the other hand, one finds that the perturbative excitations can be described as having six nontrivial electric charges (the first six components of $\vec{\alpha}_0$) and six nontrivial magnetic charges (the first six components of $\vec{\beta}_0$). This charge spectrum does have SL(2,Z) symmetry, but it is not O(6,22;Z) invariant. Dual formulations of the five-brane world-volume theory can be reached by acting with T dualities in the manner described above for S dualities and the string world-sheet theory.

5. Discussion

By assuming that the heterotic string compactified on a torus to four dimensions has S duality, this Letter has shown that an attractive picture, satisfying a number of consistency tests, emerges. It seems likely that this symmetry is sufficiently robust that it is applicable even for more realistic compactification schemes. A deeper understanding of S duality should be helpful for understanding crucial features of realistic models, such as the origin of supersymmetry breaking together with the absence of a cosmological constant. It might

⋆ For long strings, explicit soliton solutions representing these dual strings, and some properties of these solutions were discussed in ref.[20].

even provide helpful clues for constructing a better string field theory. Of course, we are still very far from such a level of understanding.

There are some more modest, but still challenging, problems that may be appropriate to study first: One is to generalize our analysis to nongeneric values of the moduli for which there is unbroken nonabelian gauge symmetry. Another (possibly related) one is to explore whether it is possible to construct an effective four-dimensional space-time theory with S duality symmetry when charged states are included. As has been explained, perturbative string excitations should be related to nonperturbative solitons by the S duality symmetry. In Ref.[6] a space-time action with SL(2,Z) symmetry was constructed, but this was done only for the low-energy field theory without charged particle excitations. If any of them are added, then the magnetically charged states that they transform into would need to be added, too. One reason for thinking that this might be possible is the example of the world-sheet theory. As we have shown, the world-sheet theory can be recast so as to incorporate the T duality symmetry that relates perturbative Kaluza–Klein excitations to winding-mode solitons.

Some related issues have been discussed in recent papers by Kallosh and Ortin, Binétruy, and Duff and Khuri [21]. We gratefully acknowledge useful discussions with P. Binétruy, M. Duff, and A. Strominger.

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