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QUANTUM BACKGROUND INDEPENDENCE OF CLOSED STRING FIELD THEORY

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ABSTRACT

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We prove local background independence of the complete quantum closed string field theory using the recursion relations for string vertices and the existence of connections in CFT theory space. Indeed, with this data we construct an antibracket preserving map between the state spaces of two nearby conformal theories taking the corresponding string field measures $d\mu e^{2S/\hbar}$ into each other. A geometrical construction of the map is achieved by introducing a Batalin-Vilkovisky (BV) algebra on spaces of Riemann surfaces, together with a map to the BV algebra of string functionals. The conditions of background independence show that the field independent terms of the master action arise from vacuum vertices $\mathcal{V}_{g,0}$, and that the overall $\hbar$-independent normalization of the string field measure involves the theory space connection. Our result puts on firm ground the widely believed statement that string theories built from nearby conformal theories are different states of the same theory.

1. Introduction and Summary

Closed string field theory can be formulated as a full quantum theory in a completely precise way. Indeed, the master action for closed string fields has finite $\hbar$-dependent terms which are determined by BRST symmetry in a way that is logically independent of the issue of perturbative divergences (see [1]). This is precisely in the spirit of Batalin-Vilkovisky (BV) quantization, where the aim is to find a well defined quantum master action, possibly $\hbar$-dependent, which determines the quantum theory. The situation in string field theory is in contrast with the situation in particle field theory where, typically, the issue of perturbative divergences becomes mixed with the problem of finding a suitable master action. Given that quantum closed string theory is precisely defined, and, its formulation requires a choice of a conformal field theory representing a background, the question of background independence can be addressed. In the present paper we define what we believe is the physically correct notion for quantum background independence of a theory written in the BV formalism. We then prove that quantum closed string field theory, for the case of nearby backgrounds related by marginal deformations, satisfies the proposed notion of background independence. This paper should be considered the sequel of Ref.[2] where we proved local background independence of classical closed string field theory. The main tool for the present work is a BV algebra that we define on subspaces of moduli spaces of Riemann surfaces, together with a map to the BV algebra of string functionals, respecting the algebraic structure. Proving background independence establishes that string theories constructed around nearby conformal theories are, in fact, different states of the same theory. This widely believed statement has therefore been put on a firm ground. We hope that the insights gained in proving background independence will be useful in constructing a manifestly background independent formulation of string field theory.

The notion of background independence of a classical field theory is familiar. A classical theory is manifestly background independent if it is written without having to choose a consistent background representing a classical solution. In such case no particular classical solution plays any special role. An example is provided by classical gravitation. Einstein's action is written for an arbitrary metric representing an arbitrary space-time background. No Ricci flat background plays a role in the formulation of the theory. Alternatively, the formulation of a theory may require choosing a background corresponding to a classical solution. In this case the theory is not manifestly background independent since, *a priori*, the theory depends on the background that has been used to formulate the theory. One may prove background independence by formulating the theory using two different backgrounds, and then showing that there is a classical solution in one of the theories such that, after a shift of the field by this classical solution, one obtains, up to a field redefinition, the theory formulated in the other background. This was the way background independence of classical closed string field theory was first discussed [3 − 5].

In Ref.[2] we used the BV formulation of classical closed string field theory to prove local background independence. We showed that there is a string field transformation that takes the classical master action in one background into the classical master action of the other background, while at the same time, the antibracket in one background is taken into the antibracket in the other background. Since a (classical) master action and an antibracket define a classical theory in the BV formulation, the field transformation maps one theory into the other. It then follows that the classical closed string actions (obtained by keeping the ghost number zero fields of the classical master action) in the two backgrounds are related by this field redefinition, restricted to the ghost number zero fields.

If we turn to the complete quantum theory it is less clear what is the physically significant notion of background independence. Perturbatively, quantum theories are usually defined only after gauge fixing. Is background independence therefore a relation between gauge fixed theories ? or is it a relation between effective actions ? Moreover, what is the physical significance of the shift of the field ? We should note that, while in classical closed string field theory conformal field theories are consistent backgrounds, in the quantum theory they need not be. Quantum closed string field theory formulated around conformal theories, contains elementary vertices giving rise to terms linear in the string field (at higher orders in $\hbar$). Therefore quantum closed string field theory is formulated on backgrounds around which the action is not stationary. We will see in this paper that in the BV approach we can define a satisfactory notion of background independence of the quantum theory.

This notion of background independence is based on the idea that there should be a formal equivalence between the theories formulated around the two different backgrounds. This formal equivalence, in turn, is proved by establishing the strict (non-formal) equivalence of appropriate action weighted measures in the two theories. This point is relevant even for the classical theory, where we showed the existence of a diffeomorphism mapping the actions and the symplectic structures. This is a strict (non-formal) statement. One may be tempted to conclude that this diffeomorphism proves that the observables of the two theories agree. This, however, is just a formal statement. It would be a strict statement for compact systems with a finite number of degrees of freedom, but is only a formal equivalence for typical field theories. The reason is that observables involve functional integrals over noncompact (infinite dimensional) spaces, and boundary conditions are typically necessary to make sense out of the integrals. A standard boundary condition is the asymptotic vanishing of the fields. This condition, however, cannot be imposed simultaneously on field configurations related by a diffeomorphism that involves a constant shift. This constant shift, representing the classical solution moving us from one background to the other is the obstruction to strict physical equivalence of the two backgrounds. Background independence is nevertheless a well-defined strict relation between two theories.

To state this relation explicitly, let us recall that a quantum theory in the BV formulation, is defined by the data $(M, \omega, d\mu, S)$, where M is the supermanifold of field/antifield configurations, ω is an odd symplectic form on M, $d\mu$ is a volume element on M leading to a nilpotent $\Delta_{d\mu}$ operator, and S is the master action, a function on M satisfying the master equation. For closed string field theory, M is a subspace $\widehat{\mathcal{H}}$ of the state space of a conformal field theory, the symplectic form ω, the measure $d\mu$ and the action S are all well-known. The problem of background independence of quantum closed string field theory must therefore begin with two conformal field theories x, and y and the data $(\widehat{\mathcal{H}}_x, \omega_x, d\mu_x, S_x)$ and $(\widehat{\mathcal{H}}_y, \omega_y, d\mu_y, S_y)$. Since background independence must imply a formal equivalence between the two string field theories, we must produce a diffeomorphism between the spaces $\widehat{\mathcal{H}}_x$ and $\widehat{\mathcal{H}}_y$ which establishes the equivalence. Experience with the classical master action [2] indicates that the diffeomorphism ought to be symplectic and therefore must carry ω_y to ω_x. A further guide are the results of Ref.[6], where it was seen that the symplectic diffeomorphism relating string field theories using different string vertices does not preserve the volume element $d\mu$ nor the action S. Such diffeomorphism, however, can be seen to preserve the action weighted measure $d\mu_S \equiv d\mu e^{2S/\hbar}$. This result suggests that the symplectic diffeomorphism, relating string field theories around two different backgrounds, must be required to map the measure $d\mu_y \, e^{2S_y/\hbar}$ to $d\mu_x \, e^{2S_x/\hbar}$. This is the condition of quantum background independence proposed in this paper. Since the distinction between the path integral measure and the action is ambiguous, it is not surprising that only a relevant combination of both is background independent.

The factor of two multiplying the action in $d\mu \, e^{2S/\hbar}$ is not an accident of normalization. This factor is unusual, but correct. We confirm this in several ways. First of all the object $d\mu_S = d\mu \, e^{2S/\hbar}$ is rather fundamental. It is implicit in [7] that, if $d\mu$ leads to a consistent delta operator $\Delta_{d\mu}$ (that is, a nilpotent one), then the action-weighted measure $d\mu_S$ also leads to a consistent operator $\Delta_{d\mu_S}$ whenever the action S satisfies the master equation. We also observe that the definition of observables does not require the separate existence of $d\mu$ and of S, but rather the existence of $d\mu_S$ (suggesting that there could be theories where only one consistent measure $d\mu_S$ exists). It has also been puzzling that, while the classical master equation implies the existence of gauge transformations for the classical master action, the quantum master equation only seemed to imply BRST transformations for the quantum theory. It has now been shown [8] that the quantum master equation implies the existence of gauge transformations leaving invariant the measure $d\mu_S$. These transformations agree with the gauge transformations of the classical theory in the limit $\hbar \to 0$. Finally, the main reason why $d\mu_S$ is the correct background independent object is that it guarantees the formal background independence of observables. In BV theory, observables arise by integration of suitable measures over a Lagrangian submanifold L of the full manifold M of field/antifield configurations. The dimension of the lagrangian submanifold L is half of that of M, and the measure $d\mu$ on M induces a measure $d\lambda$ on L by an operation involving a square root [7]. It then follows that the integrals over L defining observables use the measure $d\lambda_S \equiv d\lambda\, e^{S/\hbar}$. The symplectic diffeomorphism, if it maps $(d\mu_S)_x$ and $(d\mu_S)_y$ into each other, will map $(d\lambda_S)_x$ and $(d\lambda_S)_y$ into each other, and the (formal) background independence of observables will follow. The main objective of the present paper is the construction of the symplectic diffeomorphism implementing the background independence of quantum CSFT. We will present this construction for the case of backgrounds corresponding to nearby conformal field theories related by an exactly marginal perturbation.

Our success in establishing and proving a criterion for quantum background independence based on the master action provides further evidence of the deep significance of the BV formulation of string theory. A clear proof of background independence is of value, not only because the absence of background independence would be a catastrophe, but because such a proof explains how background independence is realized. Such understanding is likely to be necessary for further progress. For example, the background independence of $d\mu_S$ suggests that this is the object we should try to construct (on theory space) to achieve a manifestly background independent formulation of string field theory.

Our proof of background independence required completing the construction of the closed string field theory master action. The condition that the master action satisfies the master equation fixes this action up to field independent constants. This implies that when we construct a single string field theory, the measure $d\mu_S$

is only fixed up to a constant. This time, however, we are constructing string field theories over a CFT theory space, and therefore we must compare measures $(d\mu_S)_x$ at different points x. The normalization of these measures has to be fixed consistently if one is to have background independence. In the spirit of our analysis, where the string action is expanded as $S \sim \sum_{g,n} S_{g,n}\hbar^g$, the overall constant also has an $\hbar$ expansion. We show that the $\hbar$ dependent terms arise from the vacuum string vertices $\mathcal{V}_{g,0}$ (with $g \geq 2$) which in turn, are defined by the geometrical recursion relations of string field theory. Such role for vacuum vertices was anticipated in Ref.[1] where it was also pointed out that the minimal area methods are applicable to vacuum graphs. The case of the $\hbar$ independent overall constant is quite fascinating. When we choose basis sections $|\Phi_i\rangle_x$ for $\widehat{\mathcal{H}}_x$ throughout theory space, the measures $d\mu_x$ can be written as $d\mu_x = \rho(x)\prod_i d\psi_x^i$, where ψ^i denotes the coordinate associated to the basis vector $|\Phi_i\rangle$. In addition to $\rho(x)$, the term $S_{1,0}$ also affects the $\hbar$-independent part of the normalization of the measure $d\mu_S$. We show that the conditions of background independence can be satisfied if $\rho(x)$ is chosen to satisfy the equation $\frac{\partial \ln \rho}{\partial x^\mu} = \mathrm{str}\, \widetilde{\Gamma}_\mu$, where $\widetilde{\Gamma}_\mu$ is a connection on theory space, and $S_{1,0}$ is the integral of the one loop CFT partition function over part of the moduli space of tori. We find it thought provoking that some information about the connection is necessary to formulate a background independent action for string field theory. It suggests that a connection on theory space is more than just a technical tool to prove background independence of string field theory. We feel that the appearance of a connection in the formulation of quantum string field theory suggests that a dynamical connection might be necessary to achieve a manifestly background independent formulation of string field theory.

The proof of local background independence presented in [2] for classical closed string field theory made special use of the particular properties of polyhedra. Since the classical part of the quantum master action for closed string field theory cannot be built with polyhedra (it requires stubs), the ability to deal with general string vertices was necessary to address quantum background independence. The present formalism will achieve this goal. In doing so we reach the conclusion that background independence is simply a consequence of the geometrical consistency conditions of the string vertices (however they might be chosen) and of the existence of a theory space with a connection.

The main development that paved the way to a geometrical proof of background independence was the setting up of a formalism to deal efficiently with subspaces of moduli spaces of Riemann surfaces. In fact we introduce a complete BV structure acting on subspaces of moduli spaces. We introduce an antibracket $\{\, ,\, \}$ that, acting on two spaces of Riemann surfaces, produces a third space whose elements are all the surfaces obtained by twist-sewing two surfaces each in one of the original spaces. We also introduce an operator Δ, which, acting on a space of surfaces, gives a space of surfaces whose elements are all the surfaces obtained by twist-sewing two punctures in a surface in the original space. Defining carefully the orientation of the subspaces involved, this antibracket and delta operator are shown to satisfy all the formal properties associated to these objects in BV quantization. This correspondence is not accidental, for we show that these operations on moduli spaces are indeed represented by the BV antibracket and delta operator at the level of functions on $\widehat{\mathcal{H}}$. Finally, we introduce an operator $\widehat{\mathcal{K}}$ which, applied to a space of surfaces, gives us a space of surfaces whose elements are surfaces in the original set but with one additional puncture. This operator is intimately related to the computation of covariant derivatives of surface states using the special connection $\widehat{\Gamma}_\mu$ [9,10,11].

In analogy to the case of the classical theory, the field redefinition relating the two quantum string field theories is defined by vertices $\mathcal{B}_{g,n}$ that interpolate between the string vertices $\mathcal{V}_{g,n}$ and a new set of vertices $\mathcal{V}'_{g,n}$. A $\mathcal{V}'$ vertex contains all the surfaces obtained in three possible ways; surfaces arising from $\widehat{\mathcal{K}}$ acting on a $\mathcal{V}$ vertex, from Δ acting on a $\mathcal{B}$ vertex, or by twist sewing of a lower dimensional $\mathcal{B}$ vertex to a lower dimensional $\mathcal{V}$ vertex.

In a recent paper, Witten [12] has discussed a problem of quantum background independence in the context of superconformal field theories representing Calabi-

Yau backgrounds. In this context, in addition to parameters representing true changes of backgrounds, there is an extra set of parameters representing BRST-trivial deformations, that, through an anomaly, end up creating true deformations [13]. Further discussion of the obstructions to full background independence in this context has been given in Ref.[14], where the relevant closed string field theory was constructed. The methods of our present work may be useful to investigate if strict background independence can be achieved.

Since we have introduced the analog of anti-bracket and the Δ operator in the moduli space of Riemann surfaces, it is a natural question to ask if we can also find an operator $(\cdot)$ in the moduli space, that will be an analog of ordinary multiplication in the space of functions. It turns out that such an operation can indeed be defined, but we need to generalize the space by including moduli spaces of surfaces that can contain disconnected components. Furthermore, in this new space the string field theory vertices can be shown to generate a non-trivial cohomology element of a nilpotent operator $(\partial+\hbar\Delta)$, where ∂ denotes the operation of taking the boundary of a space.

Let us sketch briefly the contents of the present paper. In §2 we develop the techniques to deal efficiently with spaces of surfaces by introducing the operations $\{\, ,\, \}$, Δ, and $\widehat{\mathcal{K}}$ which act on subspaces of moduli spaces. This section is purely geometrical. In §3 we review the construction of the volume element and delta operator suitable for closed string field theory, as well as the construction of the quantum master action. We show how the operators introduced for moduli spaces, are represented in $\widehat{\mathcal{H}}$ by the standard BV antibracket and delta operators. In §4 we define our conditions for quantum background independence, discuss their physical and geometrical significance, and obtain the explicit form of the conditions for nearby backgrounds. Our proof of quantum background independence of the field dependent, as well as field independent but $\hbar$ dependent part of the action weighted measure is given in §5. We prove in §6 the background independence of the $\hbar$ independent normalization of the measure $d\mu_S$. In §7 we introduce the $(\cdot)$ operation on the moduli space of Riemann surfaces with disconected components, and rewrite the recursion relations of closed string field vertices as a cohomology condition in this space. We also reinterpret the recursion relations that determine the $\mathcal{B}$ spaces. We offer some concluding observations and remarks in §8.

2. Operations on Spaces of Riemann Surfaces

In the present section we introduce notation that will enable us to manipulate with ease spaces of Riemann surfaces. The spaces of surfaces we have in mind are made of formal sums (in the sense of homology) of oriented subspaces of moduli spaces. Given two spaces of surfaces $\mathcal{A}_1$ and $\mathcal{A}_2$, we define a third space of surfaces $\{\mathcal{A}_1, \mathcal{A}_2\}$, whose surfaces are obtained by sewing surfaces of the original two spaces. This sewing is done with sewing parameter $t = e^{i\theta}$ with $\theta \in [0, 2\pi]$ (twist-sewing). We discuss the properties of this bracket operation on spaces of surfaces, particularly with respect to the operation ∂ of taking the boundary of a space of surfaces. We also define an operation Δ, which acting on a space of surfaces, gives a new space of surfaces whose elements are obtained by twist-sewing two punctures in each surface of the original space. Finally, we introduce the operation $\widehat{\mathcal{K}}$, which acting on a space of surfaces, gives us a space of surfaces whose elements have one additional puncture. Our use of a bracket $\{\, ,\, \}$, and Δ to denote operations on spaces of surfaces is not accidental. These operations satisfy the formal properties expected from these objects in standard BV quantization. As we will see in §3, the action of the BV antibracket and delta operator on functions in the vector space $\widehat{\mathcal{H}}$ are, in fact, represented by the corresponding operations in moduli space. In §7 we complete the construction of a BV algebra on spaces of surfaces by introducing the relevant dot product.

2.1. Antibracket for Moduli Spaces

Let $\widehat{\mathcal{P}}_{g,n}$ denote, as usual, the moduli space of Riemann surfaces of genus g with n punctures, with a choice of a local coordinate, up to a phase, around each puncture. This space has the structure of a fiber bundle with base space the moduli space $\mathcal{M}_{g,n}$ of genus g Riemann surfaces with n punctures (without a choice of local coordinates). For any surface $\Sigma \in \mathcal{M}_{g,n}$, the fiber over Σ is the set of all points of $\widehat{\mathcal{P}}_{g,n}$ representing the surface Σ equipped with some choice of local coordinates at its punctures. In the present section we will develop the notation and basic results that will allow us to manipulate with ease subspaces of $\widehat{\mathcal{P}}_{g,n}$. In particular we will study what happens when we construct new subspaces by sewing operations.

Given two subspaces $\mathcal{A}_1 \subset \widehat{\mathcal{P}}_{g_1,n_1}$ and $\mathcal{A}_2 \subset \widehat{\mathcal{P}}_{g_2,n_2}$, we select a *fixed* labelled puncture P_1 on all surfaces of $\mathcal{A}_1$ (one out of n_1 possible choices), and a *fixed* labelled puncture P_2 on all surfaces of $\mathcal{A}_2$ (one out of n_2 possible choices). We then define the subspace $\{\mathcal{A}_1, \mathcal{A}_2\}' \subset \widehat{\mathcal{P}}_{g_1+g_2,n_1+n_2-2}$ as the set of surfaces obtained by twist sewing every surface in $\mathcal{A}_1$ to every surface in $\mathcal{A}_2$ using the selected punctures P_1 and P_2. If $\mathcal{A}_1$ is a subspace of dimensionality d_1, and $\mathcal{A}_2$ is a subspace of dimensionality d_2, then $\{\mathcal{A}_1, \mathcal{A}_2\}'$ is a subspace of dimensionality $d_1 + d_2 + 1$, where the one extra dimension arises from the twist angle. The bracket $\{\, ,\, \}'$ depends, in general, on the choice of labelled punctures, and that is why we have introduced the prime. If a subspace $\mathcal{A}$ is symmetric (in the sense of assignment of labelled punctures [2]) then the choice of a labelled puncture for sewing is irrelevant. Therefore the bracket of two symmetric subspaces is uniquely defined without specifying punctures. For this case we will introduce later an unprimed bracket, where in addition, the resulting space of surfaces will be symmetric. While only the unprimed bracket will be necessary for our applications, the primed bracket is a more basic object.

We need to define the orientation of the subspace $\{\mathcal{A}_1, \mathcal{A}_2\}'$. To this end we must define an ordered set of basis vectors $[\cdots]$ for the tangent space to $\{\mathcal{A}_1, \mathcal{A}_2\}'$ at any point. Consider a surface $\Sigma \in \{\mathcal{A}_1, \mathcal{A}_2\}'$ obtained by sewing a surface

$\Sigma_1 \in \mathcal{A}_1$ to a surface $\Sigma_2 \in \mathcal{A}_2$. Let $[\mathcal{A}_1]$ denote the orientation of $T_{\Sigma_1}\mathcal{A}_1$ and $[\mathcal{A}_2]$ denote the orientation of $T_{\Sigma_2}\mathcal{A}_2$. Each basis vector in $[\mathcal{A}_1]$ or $[\mathcal{A}_2]$ defines naturally a basis vector in $T_\Sigma\{\mathcal{A}_1, \mathcal{A}_2\}'$, since any deformation of the surfaces to be sewn determines a deformation of the sewn surface (when we keep the sewing parameter fixed).⋆ Let $\{\mathcal{A}_1\}$ denote the set of basis vectors in $T_\Sigma\{\mathcal{A}_1, \mathcal{A}_2\}'$ arising from $[\mathcal{A}_1]$, and $\{\mathcal{A}_2\}$ denote the set of basis vectors in $T_\Sigma\{\mathcal{A}_1, \mathcal{A}_2\}'$ arising from $[\mathcal{A}_2]$. We define the orientation of $T_\Sigma\{\mathcal{A}_1, \mathcal{A}_2\}'$ as $[\,\{\mathcal{A}_1\}, \frac{\partial}{\partial\theta}, \{\mathcal{A}_2\}\,]$, where $\frac{\partial}{\partial\theta}$ is the tangent vector associated to changes in the sewing twist angle θ, arising in $z_1 z_2 = \exp(i\theta)$, whenever we sew together the punctures associated to the local coordinates z_1 and z_2.†

For any region $\mathcal{A} \subset \widehat{\mathcal{P}}_{g,n}$, $\partial\mathcal{A}$ will denote the boundary of $\mathcal{A}$. The orientation of $\mathcal{A}$ induces an orientation on $\partial\mathcal{A}$ as usual. Given a point $p \in \partial\mathcal{A}$, a set of basis vectors $[v_1, \cdots v_k]$ of $T_p(\partial\mathcal{A})$ defines the orientation of $\partial\mathcal{A}$ if $[\,n, v_1, \cdots v_k\,]$, with n a basis vector of $T_p\mathcal{A}$ pointing outwards,‡ is the orientation of $\mathcal{A}$ at p. From this definition it is clear that

$$\partial\,\{\mathcal{A}_1, \mathcal{A}_2\}' = \{\partial\mathcal{A}_1, \mathcal{A}_2\}' + (-)^{\mathcal{A}_1+1}\{\mathcal{A}_1, \partial\mathcal{A}_2\}' \,. \tag{2.1}$$

The factor of $(-)^{\mathcal{A}_1+1}$ § in the second term arises because in order to compare the orientation of $\partial\{\mathcal{A}_1, \mathcal{A}_2\}'$, when $\mathcal{A}_2$ is at a boundary, with that of $\{\mathcal{A}_1, \partial\mathcal{A}_2\}'$ one must move the outward vector across the tangent vector $\partial/\partial\theta$, and across all the tangent vectors of $\mathcal{A}_1$. Therefore, ∂ acts as an odd derivation of the bracket, with a

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⋆ Strictly speaking, since the surfaces in $\mathcal{A}_1$ and $\mathcal{A}_2$ are equipped with local coordinates *up to a phase*, deforming these surfaces keeping the sewing parameter fixed is not a well defined concept. Thus the set of vectors $\{\mathcal{A}_1\}$ and $\{\mathcal{A}_2\}$ are defined up to addition of vectors proportional to $\frac{\partial}{\partial\theta}$. This ambiguity does not affect the orientation defined by $[\{\mathcal{A}_1\}, \frac{\partial}{\partial\theta}, \{\mathcal{A}_2\}]$.

† In ref.[2] the bracket of two spaces of surfaces was indicated by the operation $\times$. In defining the orientation of $\mathcal{B}\times\mathcal{V}$, we had taken the ordering to be $[\,\{\mathcal{B}\}, \{\mathcal{V}\}, \frac{\partial}{\partial\theta}]$. Since the manifold $\mathcal{V}$ was always even dimensional, the present bracket agrees with $\times$ in the cases relevant to ref.[2].

‡ To obtain an outward vector one constructs a diffeomorphism between the neighborhood of p and a suitable half-space. The outward vector is the image under the diffeomorphism of the standard normal to the half space (see, for example [15]).

§ $(-)^{\mathcal{A}}$ stands for $(-)^{\dim\mathcal{A}}$, where $\dim\mathcal{A}$ is the real dimension of $\mathcal{A}$.

space of surfaces treated as an even object if it is even dimensional, and as an odd object if it is odd dimensional. It then follows from these rules, and the definition of the bracket that

$$\{\mathcal{A}_1, \mathcal{A}_2\}' = -\,(-)^{(\mathcal{A}_1+1)(\mathcal{A}_2+1)}\,\{\mathcal{A}_2, \mathcal{A}_1\}' . \tag{2.2}$$

This indeed coincides with the conventional exchange property of the BV antibracket. It is convenient to introduce a notation for sewing of more than two spaces of surfaces. We will denote by $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}'$ the space of surfaces whose elements are obtained as a result of twist-sewing a puncture of each surface Σ_1 in $\mathcal{A}_1$ to a puncture of each surface $\Sigma_2 \in \mathcal{A}_2$, and another puncture of Σ_2 to a puncture of each surface Σ_3 in $\mathcal{A}_3$. We then have that

$$\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}' = -\,(-)^{\mathcal{A}_1\mathcal{A}_2+\mathcal{A}_2\mathcal{A}_3+\mathcal{A}_3\mathcal{A}_1}\,\{\mathcal{A}_3, \mathcal{A}_2, \mathcal{A}_1\}' , \tag{2.3}$$

where the extra minus sign on the right hand side appears due to the exchange of the two $\partial/\partial\theta$'s. Since they are associated to tangent vectors corresponding to different sewing parameters, they must be moved through each other. Another relation that follows naturally from the above discussion and will be useful to us is,

$$\big\{\{\mathcal{A}_1, \mathcal{A}_2\}', \mathcal{A}_3\big\}' = \begin{cases} \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}' , \\ -(-)^{(\mathcal{A}_1+1)(\mathcal{A}_2+1)}\{\mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_3\}' , \end{cases} \tag{2.4}$$

according to whether $\Sigma_3 \in \mathcal{A}_3$ is sewn to a puncture on $\Sigma_2 \in \mathcal{A}_2$ (first case) or to a puncture on $\Sigma_1 \in \mathcal{A}_1$ (second case).

As mentioned before, if the spaces $\mathcal{A}_i$ are symmetric the primed bracket does not need the specification of the punctures to be sewn. The resulting space of surfaces, however, will not be symmetric in general. Let $\mathbf{S}$ denote a symmetrization operator, and introduce the unprimed bracket as

$$\{\mathcal{A}_1, \mathcal{A}_2\} \equiv \mathbf{S}\,\{\mathcal{A}_1, \mathcal{A}_2\}' . \tag{2.5}$$

When the resulting space of surfaces has n labelled punctures, the operator $\mathbf{S}$ adds the result of the primed bracket, for all possible ways of splitting the n labelled punctures among the two initial spaces of surfaces. Similarly, we define

$$\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\} \equiv \mathbf{S}\,\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}' \,, \tag{2.6}$$

where, if the resulting space of surfaces has n labelled punctures, the operator $\mathbf{S}$ adds the result of the primed bracket, for all possible ways of splitting the n labelled punctures among the three initial spaces of surfaces. If we use the unprimed bracket Eqn.(2.4) need not deal with separate cases. It becomes

$$\Big\{\{\mathcal{A}_1, \mathcal{A}_2\}\,,\, \mathcal{A}_3\Big\} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\} - (-)^{(\mathcal{A}_1+1)(\mathcal{A}_2+1)}\,\{\mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_3\}\,. \tag{2.7}$$

The previous identities satisfied by the primed bracket immediately imply the identities

$$\partial\,\{\mathcal{A}_1, \mathcal{A}_2\} = \{\partial\mathcal{A}_1, \mathcal{A}_2\} + (-)^{\mathcal{A}_1+1}\{\mathcal{A}_1, \partial\mathcal{A}_2\}\,, \tag{2.8}$$

$$\{\mathcal{A}_1, \mathcal{A}_2\} = -\,(-)^{(\mathcal{A}_1+1)(\mathcal{A}_2+1)}\,\{\mathcal{A}_2, \mathcal{A}_1\}\,, \tag{2.9}$$

$$\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\} = -\,(-)^{\mathcal{A}_1\mathcal{A}_2+\mathcal{A}_2\mathcal{A}_3+\mathcal{A}_3\mathcal{A}_1}\,\{\mathcal{A}_3, \mathcal{A}_2, \mathcal{A}_1\}\,. \tag{2.10}$$

As befits an antibracket, the Jacobi identity indeed holds,

$$(-)^{(A_1+1)(A_3+1)}\Big\{\{\mathcal{A}_1, \mathcal{A}_2\}\,,\, \mathcal{A}_3\Big\} \;+\; \text{cyclic} \;= 0\,, \tag{2.11}$$

as can be verified using (2.7) and (2.10). We have therefore defined a consistent antibracket acting on symmetric subspaces of moduli spaces of Riemann surfaces. There is also a delta operator on moduli space satisfying all requisite properties. This operator is the subject of next subsection.

2.2. Delta Operator on Moduli Spaces

In our analysis we shall also need an operator Δ. Acting on a subspace $\mathcal{A} \subset \widehat{\mathcal{P}}_{g,n}$, the operator Δ will be defined to give a subspace $\Delta\mathcal{A} \subset \widehat{\mathcal{P}}_{g+1,n-2}$, representing the surfaces obtained by twist sewing two fixed-label punctures for every surface in $\mathcal{A}$. The result $\Delta\mathcal{A}$ depends on the choice of labels, unless the subspace $\mathcal{A}$ is symmetric. Since in our analysis the vertex $\mathcal{A}$ will always be symmetric in the punctures that are sewed by the Δ operator, we shall include an explicit factor of $1/2$ in the definition of $\Delta\mathcal{A}$ in order to avoid double counting a given contribution to $\Delta\mathcal{A}$ arising from the interchange of the two punctures.[⋆] The orientation of $\Delta\mathcal{A}$ is defined to be $[\partial/\partial\theta, \{\mathcal{A}\}]$. The operator Δ is an odd operator, just as the corresponding object in BV quantization, and satisfies the following properties:

$$\Delta\partial\mathcal{A} = -\partial\Delta\mathcal{A}, \tag{2.12}$$

$$\Delta\{\mathcal{A}_1, \mathcal{A}_2\}' = \begin{cases} \{\Delta\mathcal{A}_1, \mathcal{A}_2\}', \\ (-)^{(\mathcal{A}_1+1)}\{\mathcal{A}_1, \Delta\mathcal{A}_2\}', \\ (-)^{\mathcal{A}_1}\mathcal{A}_1 \asymp \mathcal{A}_2, \end{cases} \tag{2.13}$$

depending on whether Δ sews two punctures of $\mathcal{A}_1$ (first case), two punctures of $\mathcal{A}_2$ (second case) or one puncture of $\mathcal{A}_1$ with one puncture of $\mathcal{A}_2$ (third case). In the last term, two of the punctures of $\mathcal{A}_1$ are sewed to two of the punctures of $\mathcal{A}_2$. For this term, the ordering of the tangent vectors is $[\{\mathcal{A}_1\}, \frac{\partial}{\partial\theta_\Delta}, \frac{\partial}{\partial\theta_{ab}}, \{\mathcal{A}_2\}]$, where θ_Δ is the sewing parameter associated with the Δ operator and θ_{ab} is the sewing parameter associated with the antibracket on the left hand side.

In all of our analysis, the vertices $\mathcal{A}_i$ that we shall encounter will be symmetric in the relevant legs, namely, the ones which are sewn by the bracket or the Δ

⋆ Note that this convention may leave an explicit factor of $1/2$ in the definition of $\Delta\mathcal{A}_{0,3} \subset \widehat{\mathcal{P}}_{1,1}$. This is the case, for example, when $\mathcal{A}_{0,3} \in \widehat{\mathcal{P}}_{0,3}$ contains a single point in $\widehat{\mathcal{P}}_{0,3}$. Thus $\Delta\mathcal{A}_{0,3}$ is typically a formal space of surfaces. When doing integrals this multiplicative factor in the definition of the space is simply converted to a multiplicative factor for the integrand.

operator. If $\mathcal{A}_1$ and $\mathcal{A}_2$ in Eqn.(2.13) denote symmetric vertices, then the last term on the right hand side of this equation vanishes. To see this let us denote the pairs of *unlabelled* punctures that are sewn by (P_1, P_2) and (Q_1, Q_2), with P_1, Q_1 lying on a surface $\Sigma_1 \in \mathcal{A}_1$ and P_2, Q_2 lying on a surface $\Sigma_2 \in \mathcal{A}_2$. Moreover, let $\partial/\partial\theta_P$ be the tangent vector associated with the sewing of the P punctures, and $\partial/\partial\theta_Q$ be the tangent vector associated with the sewing of the Q punctures. Assume now that the labelling of the punctures is such that the surfaces $\Sigma = \Delta\{\Sigma_1, \Sigma_2\}'$ are constructed with the antibracket sewing P_1 to P_2, and Δ sewing Q_1 to Q_2.(Recall that bracket and delta must sew punctures of fixed labels.) The orientation of the space at Σ, as explained above, will contain the tangent vectors $[\frac{\partial}{\partial\theta_Q}, \frac{\partial}{\partial\theta_P}]$ in this order. The symmetry of the spaces $\mathcal{A}_1$ and $\mathcal{A}_2$ implies that $\Delta\{\mathcal{A}_1, \mathcal{A}_2\}'$ contains surfaces $\Sigma' = \Delta\{\Sigma_1', \Sigma_2'\}'$ where Σ_1' differs from Σ_1 only by the exchange of the labels on the fixed punctures P_1 and Q_1, and Σ_2' differs from Σ_2 only by the exchange of the labels on the fixed punctures P_2 and Q_2. This time the antibracket will end up sewing the Q punctures, and Δ will sew the P punctures. The resulting Σ' will contain the same surfaces as Σ but this time the orientation will contain the sewing tangent vectors in the opposite order $[\frac{\partial}{\partial\theta_P}, \frac{\partial}{\partial\theta_Q}]$. Therefore Σ and Σ' contribute to $\Delta\{\mathcal{A}_1, \mathcal{A}_2\}'$ with opposite orientations and hence cancel. This gives, for symmetric vertices, and with the unprimed bracket

$$\Delta\{\mathcal{A}_1, \mathcal{A}_2\} = \ \{\Delta\mathcal{A}_1, \mathcal{A}_2\} + (-)^{\mathcal{A}_1+1}\{\mathcal{A}_1, \Delta\mathcal{A}_2\}\ . \tag{2.14}$$

This is in correspondence to the fact that in BV quantization, the delta operator defines a derivation of the antibracket. The same symmetry argument shows that for a symmetric vertex $\mathcal{A}$,

$$\Delta(\Delta\mathcal{A}) = \Delta^2\mathcal{A} = 0\,, \tag{2.15}$$

due to the antisymmetry under the exchange of the tangent vectors associated with the two sewing angles. This result is again in correspondence with the nilpotency of the delta operator in BV quantization.

For our analysis, it will be useful to introduce the complex

$$\widehat{\mathcal{P}} = \oplus_{g,n} \widehat{\mathcal{P}}_{g,n} \quad \text{with} \begin{cases} n \geq 3 \text{ for } g = 0, \\ n \geq 1 \text{ for } g = 1 \\ n \geq 0 \text{ for } g \geq 2\,. \end{cases} \tag{2.16}$$

whose elements are formal sums of the form $\sum_{g,n} a_{g,n}\mathcal{A}_{g,n}$, where $a_{g,n}$ are real numbers and $\mathcal{A}_{g,n} \subset \widehat{\mathcal{P}}_{g,n}$ are subspaces which do not include surfaces arbitrarily close to degeneration. Here $\widehat{\mathcal{P}}_{g,0} \equiv \mathcal{M}_{g,0}$, as we have no punctures. The definition of the Δ and the $\{\,,\,\}$ operators is extended to this complex by treating them as linear and bilinear operators respectively:

$$\Delta(a_1\mathcal{A}_1 + a_2\mathcal{A}_2) = a_1\Delta\mathcal{A}_1 + a_2\Delta\mathcal{A}_2\,, \tag{2.17}$$

$$\begin{aligned} \{\, a_1\mathcal{A}_1 + a_2\mathcal{A}_2\,,\, b_1\mathcal{B}_1 + b_2\mathcal{B}_2 \,\} = \;& a_1b_1\{\mathcal{A}_1, \mathcal{B}_1\} + a_1b_2\{\mathcal{A}_1, \mathcal{B}_2\} \\ & + a_2b_1\{\mathcal{A}_2, \mathcal{B}_1\} + a_2b_2\{\mathcal{A}_2, \mathcal{B}_2\}\,. \end{aligned} \tag{2.18}$$

The above notation will now be used to write down the geometrical recursion relations of closed string field theory. The closed string field theory vertices $\mathcal{V}_{g,n}$ ($n \geq 3$ for $g = 0$, $n \geq 1$ for $g = 1$, $n \geq 0$ for $g \geq 2$) are $d_{g,n} = (6g + 2n - 6)$ dimensional subspaces of $\widehat{\mathcal{P}}_{g,n}$ and satisfy the recursion relations [16,1]

$$\begin{aligned} \partial\mathcal{V}_{g,n} &= -\tfrac{1}{2} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n+2}} \big\{\mathcal{V}_{g_1,n_1}\,,\, \mathcal{V}_{g_2,n_2}\big\} \;-\; \Delta\mathcal{V}_{g-1,n+2}\,, \\ &= -\tfrac{1}{2} \sum_{g_1=0}^{g} \sum_{m=1}^{n+1} \big\{\mathcal{V}_{g-g_1,n-m+2}\,,\, \mathcal{V}_{g_1,m}\big\} - \Delta\mathcal{V}_{g-1,n+2}\,, \end{aligned} \tag{2.19}$$

In writing the above equation we have defined

$$\mathcal{V}_{0,n} \equiv 0 \quad \text{for } n \leq 2\,, \tag{2.20}$$

otherwise the range of summation over m will depend on g. This definition is consistent with (2.19) since, for $n \leq 2$, $\partial\mathcal{V}_{0,n}$ computed with the help of this equation is indeed zero. Although the form of the field independent part of the action, given by $\mathcal{V}_{g,0}$, is not specified by the master equation, we shall take $\mathcal{V}_{g,0}$ for $g \geq 2$ to satisfy the above equation[1]. The case of $g = 1, n = 0$ is special, and will be discussed in §6.

We now define a special vector in the complex $\widehat{\mathcal{P}}$. We define

$$\mathcal{V} \equiv \sum_{g,n} \hbar^g \mathcal{V}_{g,n} \quad \text{with} \quad \begin{cases} n \geq 3 \text{ for } g = 0, \\ n \geq 1 \text{ for } g = 1, \\ n \geq 0 \text{ for } g \geq 2\,. \end{cases} \tag{2.21}$$

Note that we have included the vacuum vertices $\mathcal{V}_{g,0}$ for $g \geq 2$, but not a genus one vacuum vertex. It then follows from (2.19) that the recursion relations can be written as

$$\partial \mathcal{V} + \hbar \Delta \mathcal{V} + \tfrac{1}{2}\{\mathcal{V}, \mathcal{V}\} = 0\,. \tag{2.22}$$

An even nicer reformulation of the recursion relations will be given in §7.

2.3. The Operator $\widehat{\mathcal{K}}$ on Moduli Spaces

Consider a genus g Riemann surface Σ with n punctures and equipped with local coordinates determined by coordinate curves around the punctures. We consider the general case when g and n can take arbitrary values. Assume also that the coordinate curves do not intersect, and therefore Σ minus the unit disks $D_i^{(1)}$ (bounded by the coordinate curves) is a nonvanishing region of the surface. We now define a subspace $\widehat{\mathcal{K}}(\Sigma) \in \widehat{\mathcal{P}}_{g,n+1}$ which will contain the $n+1$ punctured surfaces corresponding to Σ with an extra puncture lying anywhere on the region $\Sigma - \cup_i D_i^{(1)}$. If the surface Σ has no punctures, the extra puncture can lie anywhere on it. The coordinate at the extra puncture will be fixed arbitrarily, but continuously as it moves on the surface. It follows that $\widehat{\mathcal{K}}(\Sigma)$ is a section over the subspace of $\mathcal{M}_{g,n+1}$ defined by the surface Σ with the extra puncture lying somewhere on the surface minus its unit disks. This section is two dimensional, and its orientation is defined by $[V(v_1), V(v_2)]$, where $[v_1, v_2]$ define the standard orientation of

Σ, and $V(v)$ denotes the tangent (in $\widehat{\mathcal{P}}_{g,n+1}$) representing the deformation induced by moving the extra puncture in the direction indicated by v. While we have said that the coordinate at the extra puncture can be fixed arbitrarily, it will be useful to fix it explicitly. One way to do it is as follows. On Σ we find the minimal area metric, satisfying the requirement that all homotopically non-trivial closed curves on the surface have length $\geq 2\pi$ [1]. The coordinate curves are then some curves C_i on the surface (that need not correspond to critical trajectories nor geodesics). Then at every point p on Σ minus the disks bounded by C_i we shall take the coordinate curve associated with the puncture at p as the locus of points at a distance a from p. If $a < \pi$ this rule determines well defined coordinate disks that vary continuously as p moves on Σ.

Given a subspace $\mathcal{A} \subset \widehat{\mathcal{P}}_{g,n}$ we define $\widehat{\mathcal{K}}\mathcal{A}$ to be the subspace of $\widehat{\mathcal{P}}_{g,n+1}$ representing the surfaces $\widehat{\mathcal{K}}(\Sigma)$ for all surfaces $\Sigma \in \mathcal{A}$. The space $\widehat{\mathcal{K}}\mathcal{A}$ is connected if $\mathcal{A}$ is, if the local coordinate at the extra puncture changes continuously as we move in $\mathcal{A}$. This condition is automatically satisfied if the local coordinate at the extra puncture is chosen using the prescription given at the end of the last paragraph. Let $[\mathcal{A}]$ denote the orientation of $\mathcal{A}$, and $\{\mathcal{A}\}$ denote the tangent vectors on $\widehat{\mathcal{P}}_{g,n+1}$ arising from the tangent vectors in $[\mathcal{A}]$. (These are defined up to the addition of linear combinations of the vectors $V(v_1)$, $V(v_2)$.) The orientation of $\widehat{\mathcal{K}}\mathcal{A}$ is defined to be $[V(v_1), V(v_2), \{\mathcal{A}\}\,]$, where, again, $[v_1, v_2]$ is the orientation of the Riemann surface.

We shall now introduce a new symbol $\approx$ to express relations that hold up to the local coordinate at the extra puncture. More precisely, we say $\mathcal{A}_{g,n+1} \approx \mathcal{B}_{g,n+1}$ if $\pi_f(\mathcal{A}_{g,n+1}) = \pi_f(\mathcal{B}_{g,n+1})$, where π_f is the projection map that forgets about the coordinate at the special puncture, that is, π_f is the map from $\widehat{\mathcal{P}}_{g,n+1}$ to the space $\widehat{\mathcal{P}}_{g,n,1}$ of surfaces of genus g and $n+1$ punctures, with local coordinates up to phases around n of the punctures. For a space $\mathcal{A}$ with a special puncture, and having a tangent vector representing a deformation of the coordinate at the special puncture, we define $\pi_f(\mathcal{A}) = 0$. The space $\pi_f(\mathcal{A})$, if nonvanishing will therefore have the same dimensionality of $\mathcal{A}$. In this case, the orientation will be defined by

$[\pi_f(\widehat{V}_1), \cdots, \pi_f(\widehat{V}_n)]$, where $[\widehat{V}_1, \cdots, \widehat{V}_n]$ define the orientation of $\mathcal{A}$.

The following properties are easily verified

$$\widehat{\mathcal{K}}\,(\,\{\mathcal{A}_1, \mathcal{A}_2\}\,) \,\approx\, \{\widehat{\mathcal{K}}\mathcal{A}_1, \mathcal{A}_2\} \,+\, \{\mathcal{A}_1, \,\widehat{\mathcal{K}}\mathcal{A}_2\}\,, \tag{2.23}$$

$$\widehat{\mathcal{K}}\,(\Delta\,\mathcal{A}) \approx \Delta\,(\widehat{\mathcal{K}}\mathcal{A})\,. \tag{2.24}$$

The above equations state that $\widehat{\mathcal{K}}$ is an even derivation of the antibracket, and it commutes with the Δ operator. In these equations the special punctures created by $\widehat{\mathcal{K}}$ are *never* used for sewing; one must think of these punctures as ones where we insert a marginal operator. The above equations, with equality up to local coordinates at the special puncture, is all we shall need for carrying out our proof of background independence. Nevertheless, our definition of the local coordinate via minimal area metrics, plus the compatibility of minimal area with sewing, guarantees that for subspaces $\mathcal{A}$ of minimal area sections [1] in $\widehat{\mathcal{P}}$, the above equations hold strictly (including the local coordinate at the extra puncture). An important result gives us the boundary of the space $\widehat{\mathcal{K}}\mathcal{A}$

$$\partial\,\big(\widehat{\mathcal{K}}\mathcal{A}\big) \approx \widehat{\mathcal{K}}\big(\partial\mathcal{A}\big) -\, \{\mathcal{V}'_{0,3}\,, \mathcal{A}\}\,, \tag{2.25}$$

or more abstractly, as the operator equation

$$[\,\partial\,,\,\widehat{\mathcal{K}}\,] \,\approx\, -\{\,\mathcal{V}'_{0,3}\,,\;\}\,, \tag{2.26}$$

where the left hand side denotes the standard commutator. We must explain our notation in (2.25). The second term in the right hand side involves twist sewing the three punctured sphere $\mathcal{V}'_{0,3}$ to the surfaces in $\mathcal{A}$. This three punctured sphere $\mathcal{V}'_{0,3} \equiv \mathcal{V}'_3$, introduced in [2], is symmetric under the exchange of two punctures, but does not have any further exchange symmetry. The third, and special puncture of $\mathcal{V}'_3$, is the puncture associated with the operation $\widehat{\mathcal{K}}$ in the above equation. It is a puncture that stands on a different footing from all other punctures, since a marginal operator is always to be inserted there. Whenever our antibracket, whose definition included symmetrization, contains a surface with a special puncture, that puncture *will not be symmetrized over*. That puncture cannot be used for sewing, and must be considered as if it was filled. The bracket $\{\mathcal{V}'_{0,3}\,, \mathcal{A}\}$ effectively sews $\mathcal{V}'_{0,3}$ to each of the punctures of each of the surfaces in $\mathcal{A}$.

This identity arises as follows. The boundary of $\widehat{\mathcal{K}}\mathcal{A}$ arises from two sources. One is the boundary of $\mathcal{A}$ itself. This contribution is given by operating $\widehat{\mathcal{K}}$ on $\partial\mathcal{A}$ (first term on the right hand side of Eqn.(2.25)). The second contribution comes from the boundary of the region of integration over the location of the new puncture p. This is given by configurations where p is inserted on the coordinate curves $\mathcal{C}_i$ of the original set of punctures. If we disregard the choice of the local coordinate system at the special puncture on $\widehat{\mathcal{K}}\mathcal{A}$, these contributions are given by twist sewing the vertex $\mathcal{V}'_{0,3}$ to the original surfaces in $\mathcal{A}$ at all punctures. This gives rise to the second set of terms on the right hand side of Eqn.(2.25). The minus sign arises as follows. When twist sewing $\mathcal{V}'_{0,3}$ to a surface Σ in $\mathcal{A}$ the extra puncture ends up traveling counterclockwise along each of the coordinate curves of Σ. The boundary of the region of integration $\Sigma - \cup_i D_i^{(1)}$, however, gives the coordinate curves of Σ, with an orientation corresponding to clockwise travel.

For subspaces of minimal area sections, the above equation holds strictly if we define the local coordinate at the special puncture of $\mathcal{V}'_{0,3}$ in a suitable way. This is done as follows. $\mathcal{V}'_{0,3}$ is mapped to an infinite cylinder of circumference 2π with the symmetric punctures at the two points at infinity. The local coordinates at those punctures are fixed by taking as a common coordinate curve an arbitrary geodesic circle on the cylinder. On that circle we fix a point to be the special puncture, and define its coordinate disk to be the set of points at a distance $\leq a$. This can be seen to yield the same local coordinate on the special puncture on the surfaces in $\{\mathcal{V}'_{0,3}, \mathcal{A}\}$ as the construction of $\widehat{\mathcal{K}}$ does when the special puncture lies at the boundary of the region of integration.

For spaces of overlap surfaces (spaces where each surface Σ is equal to the disjoint union of the unit disks around the punctures) the identity in (2.25) reduces to $\{\mathcal{V}'_{0,3}\,,\mathcal{A}\} \approx 0$, by virtue of $\widehat{\mathcal{K}}\mathcal{A} = 0$. This identity, obtained in [2] (where instead of using a bracket, we used the symbol $\times$) played a crucial role in the proof of background independence. In the same way as this equation holds strictly for the case of polyhedra, our present equation (2.25) holds strictly for minimal area sections. For carrying out the proof of background independence we only need relations up to a choice of local coordinates at the special puncture, since a dimension (0,0) primary state $|\widehat{\mathcal{O}}_\mu\rangle \equiv |c\bar{c}\mathcal{O}_\mu\rangle$ will be inserted there.

3. Representation in $\widehat{\mathcal{H}}$ and Quantum Master Action

In the present section we begin by discussing the construction of the volume element on $\widehat{\mathcal{H}}$ necessary to define the BV delta operator Δ. We then give representation formulae on $\widehat{\mathcal{H}}$ for the geometrical operations $\partial, \Delta\,, \{\,,\,\}$, and $\widehat{\mathcal{K}}$ introduced in the previous section. Here we see why the moduli space operators Δ and $\{\,,\,\}$, correspond to the delta and antibracket of BV quantization. The operator $\widehat{\mathcal{K}}$ turns out to be related to the action of covariant derivatives $D_\mu(\,\widehat{\Gamma}\,)$, with $\widehat{\Gamma}$ the canonical connection of Refs.[9,10,11]. We then review, in this new language, the geometrical recursion relations of closed string field theory, and the basics of the construction of the quantum closed string master action. While presenting the material in this (and later) sections, we shall assume some familiarity with Ref.[2], whose notation we follow.

3.1. Anti-bracket, Volume Element and Delta Operator

In Ref.[2] we have described in detail the symplectic form ω on the state space $\widehat{\mathcal{H}}$ relevant to the formulation of closed string field theory. This space is the subspace of the complete state space $\mathcal{H}$, and is spanned by the states $|\Phi_i\rangle$ annihilated by b_0^- and L_0^-. The string field is written as $|\Psi\rangle = \sum_i |\Phi_i\rangle\,\psi^i$, where the target space fields ψ^i play the role of coordinates in $\widehat{\mathcal{H}}$. The symplectic form $\langle\omega|$, and its inverse $|\mathcal{S}\rangle$, in $\widehat{\mathcal{H}}$ are given by

$$\begin{aligned} \langle\omega_{12}| &= \langle R'_{12}|c_0^{-(2)} \equiv -\ {}_1\langle\Phi^i|\ \omega_{ij}(x)\ \ {}_2\langle\Phi^j|, \\ |\mathcal{S}_{12}\rangle &= b_0^{-(1)}|R'_{12}\rangle \equiv\ |\Phi_i\rangle_1\ (-)^{j+1}\omega^{ij}(x)\ \ |\Phi_j\rangle_2\,. \end{aligned} \tag{3.1}$$

Given two functions A and B in $\widehat{\mathcal{H}}$ one can show that

$$\{A\,,B\,\} \equiv \frac{\partial_r A}{\partial\psi^i}\omega^{ij}\frac{\partial_l B}{\partial\psi^j}\ =\ (-)^{B+1}\frac{\partial\,A}{\partial\,|\Psi\rangle}\,\frac{\partial\,B}{\partial\,|\Psi\rangle}\,|\mathcal{S}\rangle\,, \tag{3.2}$$

where the sewing ket $|\mathcal{S}\rangle$ is gluing the two state spaces left open by the differentiation with respect to the string field. There is no need to specify left or right derivatives because the string field is even. The kinetic term of closed string field theory uses the symplectic form. It is given by

$$S_{0,2} = \tfrac{1}{2}\,\langle\omega_{12}|Q^{(2)}|\Psi\rangle_1|\Psi\rangle_2\,. \tag{3.3}$$

We must now describe the volume element and the construction of the BV delta operator Δ, relevant to the formulation of quantum closed string field theory. In general, in a BV theory, the volume element $d\mu$ is written as

$$d\mu = \rho(\psi)\prod_i d\psi^i, \tag{3.4}$$

where the product runs over all values of i. The volume element is not a differential form[⋆] and does not arise from the symplectic form ω (since $\omega\wedge\omega\,=\,0$). It is independent data that must be specified. In the same sense as $\sqrt{g}dx\wedge\cdots dx$, in Riemannian geometry, it is a coordinate invariant object. Therefore, an explicit

⋆ The volume element acts on a set of vectors, if we transform this set by a linear transformation A, the volume element transforms with a factor sdet$A$. Such transformation law cannot be achieved with a linear differential form.

computation of $\rho(\psi)$ requires a choice of local coordinates. Associated to the volume element $d\mu$, the delta operator $\Delta_{d\mu}$ is defined by

$$\Delta_{d\mu} A \equiv \tfrac{1}{2}\,\mathrm{div}_\rho V_A \equiv \frac{1}{2\rho}(-)^i \frac{\partial_l}{\partial\psi^i}\Big(\rho\,\omega^{ij}\frac{\partial_l}{\partial\psi^j}A\Big), \tag{3.5}$$

where A is a scalar function, and $\Delta_{d\mu} A$ is the scalar obtained by taking (one-half of) the divergence of the hamiltonian vector field V_A. There is an important consistency condition, one must choose $\rho(\psi)$ such that $\Delta_{d\mu}^2 = 0$ [7]. As usual, ∂_l and ∂_r denote left and right derivatives respectively.

For the case of closed string field theory, in the chosen basis for $\widehat{\mathcal{H}}_x$, we take $\rho = \rho(x)$, which is field independent, but can, in general, depend on the coordinate x labeling the particular CFT around which string field theory is being formulated,

$$d\mu = \rho(x)\prod_i d\psi_x^i, \quad \to \quad \Delta_{d\mu} A = \tfrac{1}{2}(-)^i\frac{\partial_l}{\partial\psi_x^i}\Big(\omega^{ij}\frac{\partial_l}{\partial\psi_x^j}A\Big) = \tfrac{1}{2}\,\mathrm{str}\Big[\frac{\partial_l}{\partial\psi_x^k}\Big(\omega^{ij}\frac{\partial_l}{\partial\psi_x^j}A\Big)\Big]. \tag{3.6}$$

where $\mathrm{str}(\widetilde{A}_k{}^i) \equiv (-1)^i \widetilde{A}_i{}^i$. Since the ω^{ij}'s are constants with statistics $(-)^{(i+j+1)}$, and satisfy $\omega^{ji} = -(-)^{(i+1)(j+1)}\omega^{ij}$, one readily verifies that $\Delta_{d\mu}$ is nilpotent. While the field independent $\rho(x)$ factor in the measure drops out of the delta operator, it will be relevant for us since we will be comparing theories at different points x, and there is no a priori reason why this scale factor should be a constant on theory space. With the same methods used in Ref.[2] to prove (3.2) we can verify that for any function U on $\widehat{\mathcal{H}}$,

$$\Delta U = \tfrac{1}{2}(-)^{U+1}\Big(\frac{\partial}{\partial|\Psi\rangle}\frac{\partial}{\partial|\Psi\rangle}U\Big)|\mathcal{S}\rangle\,. \tag{3.7}$$

In the BV formalism observables are obtained by integration, with suitable measures, over lagrangian submanifolds (of the supermanifold M of field/antifield configurations). A lagrangian submanifold L is defined by the condition that at any point $p \in L$, for any two tangent vectors $e_i, e_j \in T_pL$, we have $\omega(e_i, e_j) = 0$.

The volume element $d\mu$ in M then induces a volume element $d\lambda$ on L as follows [7]. Let $p \in L$, and $(e_1, \ldots, e_n)$ be a basis of T_pL. One then defines

$$d\lambda(e_1, \cdots, e_n) \equiv \left[d\mu(e_1, \cdots, e_n, f^1, \cdots, f^n) \right]^{1/2} , \tag{3.8}$$

where the vectors f are any set of vectors satisfying $\omega(e_i, f^j) = \delta_i^j$. This condition fixes the vectors f^j up to the transformation $f^j \to f^j + C^{ji} e_i$. The right hand side of (3.8), however, is invariant under this transformation, since it corresponds to a transformation of the complete basis $(\{e_i\}; \{f^j\})$ by a matrix of unit superdeterminant.

3.2. Representation on $\widehat{\mathcal{H}}$.

Basic Representation Formulae. In conformal field theory surfaces are represented by surface states, and spaces of surfaces can therefore be represented by integrals of surface states. The operators ∂, Δ and $\{\, ,\, \}$, discussed above, act on spaces of surfaces. They can, as a consequence, be represented by operators acting on integrals of surface states. We are familiar with the correspondence $\partial \leftrightarrow Q$ leading to

$$\int\limits_{\mathcal{A}_{g,n}} \langle \Omega^{(k)g,n} | \sum_{i=1}^{n} Q^{(i)} = (-1)^k \int\limits_{\partial \mathcal{A}_{g,n}} \langle \Omega^{(k-1)g,n} |. \tag{3.9}$$

Here $\langle \Omega_\Sigma^{(k)g,n} |$ denotes (upon contraction with n arbitraty states in $\widehat{\mathcal{H}}$), a $(6g+2n-6+k)$-form on the moduli space $\widehat{\mathcal{P}}_{g,n}$. We generally omit from the form the label Σ corresponding to the surface. These forms are explicitly given by [1,17]*

$$\langle \Omega^{(k)g,n} | (V_1, \cdots, V_{6g+2n-6+k}) = (-2\pi i)^{(3-n-3g)} \langle \Sigma | \, \mathbf{b}(\mathbf{v}_1) \cdots \mathbf{b}(\mathbf{v}_{6g+2n-6+k}). \tag{3.10}$$

The Schiffer vector $\mathbf{v}_r = (v_r^{(1)}(z), \cdots v_r^{(n)}(z))$ creates the deformation of the surface Σ specified by the tangent V_r, and the antighost insertions are given by ($\oint dz/z =$

$\star$ The normalization factor includes an extra minus sign that went unnoticed in Ref.[1].

$\oint d\bar{z}/\bar{z} = 2\pi i$)

$$\mathbf{b}(\mathbf{v}) = \sum_{i=1}^{n} \left(\oint b^{(i)}(z_i) v^{(i)}(z_i) \frac{dz_i}{2\pi i} + \oint \overline{b}^{(i)}(\overline{z}_i) \overline{v}^{(i)}(\overline{z}_i) \frac{d\overline{z}_i}{2\pi i} \right). \tag{3.11}$$

The sewing ket $|\mathcal{S}\rangle$ implements, at the level of states, the action of the bracket $\{\,,\,\}'$, or Δ at the level of surfaces. If the two punctures to be sewn are on different surfaces we have (see refs.[1,2] for details):

$$\Big(\int_{\mathcal{A}_{g_1,n_1}} \langle \Omega^{(k_1) g_1, n_1} | \int_{\mathcal{A}_{g_2,n_2}} \langle \Omega^{(k_2) g_2, n_2} | \Big) |\mathcal{S}\rangle \;=\; (-)^{k_2} \int_{\{\mathcal{A}_{g_1,n_1}\,,\,\mathcal{A}_{g_2,n_2}\}'} \langle \Omega^{(k_1+k_2-1) g_1+g_2, n_1+n_2-2} | \,, \tag{3.12}$$

where the sewing ket contracts with the state spaces corresponding to the punctures being sewn by the bracket. The factor $(-)^{k_2}$ arises from the necessity of moving the tangent vector associated with the sewing angle through the tangent vectors of $\mathcal{A}_{g_2,n_2}$. When the punctures to be sewn are on the same surface we have

$$\frac{1}{2} \int_{\mathcal{A}_{g,n}} {}_{1\cdots n}\langle \Omega^{(k) g,n} | \mathcal{S}_{12} \rangle \;=\; (-)^k \int_{\Delta \mathcal{A}_{g,n}} {}_{3\cdots n}\langle \Omega^{(k-1) g+1, n-2} | \,. \tag{3.13}$$

For the case of $A_{0,3}$, a space whose elements are three punctured spheres, the definition of Δ implies that the space $\Delta\mathcal{A}_{0,3}$ is formally $\frac{1}{2}$ of a space $\mathcal{A}'_{1,1}$ in $\widehat{\mathcal{P}}_{1,1}$. The integral on the right hand side of the equation should be interpreted as $\frac{1}{2}\int_{\mathcal{A}'_{1,1}}$. This factor of $\frac{1}{2}$ multiplying an integral over a space of surfaces is natural on account of the observation of Ref.[18] that a punctured torus has a Z_2 group of diffeomorphism (generated by $z \to -z$), and hence the expression for the one loop partition function (or one point function) should automatically carry a factor of $\frac{1}{2}$. This factor of $\frac{1}{2}$ will be relevant in our analysis of §6.

<u>Representation of the BV algebra.</u> In order to make manifest why our operations on moduli spaces become the standard BV operations on $\widehat{\mathcal{H}}$, we consider a canonical procedure that, given a subspace of surfaces, associates to this subspace a functions

on $\widehat{\mathcal{H}}$. Let $\mathcal{A}^{(k)}_{g,n}$ denote a (basic) subspace of $\widehat{\mathcal{P}}_{g,n}$ of real dimensionality $(6g-6+2n+k)$. We define the associated function $f(\mathcal{A}^{(k)}_{g,n})$ as

$$f(\mathcal{A}^{(k)}_{g,n}) \equiv \frac{1}{n!} \int\limits_{\mathcal{A}^{(k)}_{g,n}} \langle \Omega^{(k)g,n} |\Psi\rangle_1 \cdots |\Psi\rangle_n \ , \quad n \geq 1 \, . \tag{3.14}$$

where, for convenience of writing, we do not include $|\Psi\rangle$ as an argument of f. This is clearly a very natural operation, we are simply integrating the canonical forms on moduli space over the subspace of surfaces. This definition is complete for spaces of surfaces of genus zero with three or more punctures, and for spaces of surfaces of higher genus with one or more punctures. For $g \geq 2, n = 0$, we define,

$$f(\mathcal{A}^{(k)}_{g,0}) \equiv \int\limits_{\mathcal{A}^{(k)}_{g,0}} \Omega^{(k)g,0}_{\Sigma} \, , \tag{3.15}$$

where,

$$\Omega^{(k)g,0}_{\Sigma} \equiv (-2\pi i) \langle \Omega^{(k-2)g,1}_{\widehat{\Sigma}} | 0 \rangle \, . \tag{3.16}$$

Here Σ is a genus g surface without punctures, and $\Omega^{g,0}_{\Sigma}$ is a $(6g-6+k)$ form on the tangent space to $\mathcal{M}_{g,0}$ at Σ. To construct this form, we must first construct a surface $\widehat{\Sigma} \in \widehat{\mathcal{P}}_{g,1}$, by introducing a puncture (with a local coordinate) on Σ. Then, as indicated in the above equation, we use a form on $T_{\widehat{\Sigma}}\widehat{\mathcal{P}}_{g,1}$ to define the form $\Omega^{(k)g,0}_{\Sigma}$. More precisely, given $(6g-6+k)$ tangents $(\vec{V}_1, \cdots \vec{V}_{6g-6+k}) \in T_{\Sigma}\mathcal{M}_{g,0}$, we must choose $(6g-6+k)$ tangents $(\widehat{V}_1, \cdots \widehat{V}_{6g-6+k}) \in T_{\widehat{\Sigma}}\widehat{\mathcal{P}}_{g,1}$, which project down to the original tangents as we forget about the extra puncture. The definition in (3.16) then reads

$$\Omega^{(k)g,0}_{\Sigma}(\vec{V}_1, \cdots \vec{V}_{6g-6}) \equiv (-2\pi i) \langle \Omega^{(k-2)g,1}_{\widehat{\Sigma}} | 0 \rangle (\widehat{V}_1, \cdots \widehat{V}_{6g-6}) \tag{3.17}$$

It is known that this definition is independent of the choice of surface $\widehat{\Sigma}$ projecting to Σ, and of the tangents in $T_{\widehat{\Sigma}}\widehat{\mathcal{P}}_{g,1}$ projecting to the given tangents in $T_{\Sigma}\mathcal{M}_{g,0}$

[19]. Also, since $\langle\Omega^{(k-2)g,1}_{\widehat{\Sigma}}|$ carries ghost number $6-k$, it is clear from Eqn.(3.16) that $\Omega^{(k)g,0}$ vanishes unless $k=0$. This gives,

$$f(\mathcal{A}^{(k)}_{g,0}) = 0 \quad \text{for } k \neq 0\,. \tag{3.18}$$

There is no completely natural definition of f for $g=1, n=0$, but we shall define an analog of the function f in §6.

The definition of f on the complete complex $\widehat{\mathcal{P}}$ of surfaces follows from

$$f\Big(\sum_i \; a_i \mathcal{A}^{(k_i)}_{g_i,n_i}\Big) = \sum_i \; a_i\, f\,(\mathcal{A}^{(k_i)}_{g_i,n_i})\,. \tag{3.19}$$

We now claim that for $\mathcal{A},\mathcal{B}\in\widehat{\mathcal{P}}$, the following representation identities hold:

$$\begin{aligned} f\big(\Delta\mathcal{A}\big) &= -\Delta f(\mathcal{A}) \\ f\big(\{\mathcal{A},\mathcal{B}\}\big) &= -\{\, f(\mathcal{A}), f(\mathcal{B})\}\,. \end{aligned} \tag{3.20}$$

On the left hand side of the above equations, the delta operator and the antibracket act on the spaces of surfaces, on the right they act on the functions in $\widehat{\mathcal{H}}$. These equations justify our claim that the bracket and delta operators on moduli space give rise, by representation on functions in $\widehat{\mathcal{H}}$ to the BV antibracket and delta operators. More precisely, these define a homomorphism of the Riemann surface algebra of Δ and the antibracket, to the corresponding algebra of string functionals. Eqns.(3.20) are first proven for basic spaces, and then are easily extended to generalized spaces. Let us prove the second of these equations as an illustration. Making use of (3.2) we find

$$\begin{aligned} \big\{ f(\mathcal{A}^{(k_1)}_{g_1,n_1})\,, f(\mathcal{A}^{(k_2)}_{g_2,n_2})\,\big\} &= (-)^{k_2+1}\, \frac{\partial f(\mathcal{A}^{(k_1)}_{g_1,n_1})}{\partial|\Psi\rangle}\, \frac{\partial f(\mathcal{A}^{(k_2)}_{g_2,n_2})}{\partial|\Psi\rangle}\, |\mathcal{S}\rangle\,, \\ &= (-)^{k_2+1} \int\limits_{\mathcal{A}^{(k_1)}_{g_1,n_1}} \langle\Omega^{(k_1)g_1,n_1}|\, \frac{|\Psi\rangle^{n_1-1}}{(n_1-1)!} \int\limits_{\mathcal{A}^{(k_2)}_{g_2,n_2}} \langle\Omega^{(k_2)g_2,n_2}|\, \frac{|\Psi\rangle^{n_2-1}}{(n_2-1)!}|\mathcal{S}\rangle\,, \\ &= -\int\limits_{\{\mathcal{A}^{(k_1)}_{g_1,n_1},\mathcal{A}^{(k_2)}_{g_2,n_2}\}'} \langle\Omega^{(k_1+k_2-1)g_1+g_2,n_1+n_2-2}| \frac{|\Psi\rangle^{n_1+n_2-2}}{(n_1-1)!(n_2-1)!}\,, \end{aligned} \tag{3.21}$$

where in the last step we made use of (3.12). From the relation between the primed antibracket and the standard antibracket (Eqn.(2.5)) we now obtain

$$
\begin{aligned}
\big\{ f(\mathcal{A}^{(k_1)}_{g_1,n_1})\,,\, f(\mathcal{A}^{(k_2)}_{g_2,n_2})\,\big\} &= -\int\limits_{\{\,\mathcal{A}^{(k_1)}_{g_1,n_1}\,,\mathcal{A}^{(k_2)}_{g_2,n_2}\}} \langle \Omega^{(k_1+k_2-1)g_1+g_2,n_1+n_2-2} | \frac{|\Psi\rangle^{n_1+n_2-2}}{(n_1+n_2-2)!}\,, \\
&= -f\big(\{\,\mathcal{A}^{(k_1)}_{g_1,n_1}\,,\mathcal{A}^{(k_2)}_{g_2,n_2}\}\big)\,,
\end{aligned}
\tag{3.22}
$$

as we wanted to show. The first identity in Eqn.(3.20) follows in a similar manner through the use of Eqn.(3.13). One more identity will be relevant to our analysis. From Eqns.(3.2) and (3.9) it follows that

$$
\{\,S_{0,2}, f(\mathcal{A})\} = -\,f\big(\partial\mathcal{A}\big)\,,
\tag{3.23}
$$

where $S_{0,2}$ is the kinetic term of closed string field theory, as given in Eqn.(3.3). Thus the operation of taking boundary is represented in $\widehat{\mathcal{H}}$ by a canonical transformation induced by the kinetic term of closed string field theory.

<u>Representation identities for $\widehat{\mathcal{K}}$.</u> A specific connection $\widehat{\Gamma}$ in the CFT theory space was introduced in Refs.[9,10,11] and further explored in Ref.[2]. It is defined by the relation

$$
D_\mu(\,\widehat{\Gamma}\,)\langle\,\Sigma\,| = -\frac{1}{\pi}\int\limits_{\Sigma-\cup_i D_i^{(1)}} d^2z\,\langle\,\Sigma; z\,|\mathcal{O}_\mu\rangle\,.
\tag{3.24}
$$

It follows from this relation that for spaces of surfaces and string field forms $\langle\Omega|$ we have

$$
D_\mu(\,\widehat{\Gamma}\,)\int\limits_{\mathcal{A}_{g,n}}\langle\Omega^{(k)g,n}| = \int\limits_{\widehat{\mathcal{K}}\mathcal{A}_{g,n}}\langle\Omega^{(k)g,n+1}|\widehat{\mathcal{O}}_\mu\rangle_{n+1}\,,
\tag{3.25}
$$

where the $(n+1)$-th puncture in the right hand side is the special puncture of $\widehat{\mathcal{K}}\mathcal{A}_{g,n}$. Equation (3.25) is verified using the fact that the connection $\widehat{\Gamma}$ does not act on antighost insertions, the relation between string forms and surface states (Eqn.(3.10)), and the relation $dx\wedge dy\, b(\partial/\partial x)b(\partial/\partial y)|\widehat{\mathcal{O}}_\mu\rangle = 2i dx\wedge dy\,|\mathcal{O}_\mu\rangle$, which requires the use of the definition given in Eqn.(3.11).

We define now the map f_μ which acts on spaces of surfaces of two types; surfaces with one special puncture, or surfaces with no special puncture. Acting on such spaces it gives us a function in $\widehat{\mathcal{H}}$. We take

$$f_\mu(\mathcal{A}^{(k)}_{g,n+1}) \equiv \frac{1}{n!} \int\limits_{\mathcal{A}^{(k)}_{g,n+1}} \langle \Omega^{(k)g,n+1} |\Psi\rangle_1 \cdots |\Psi\rangle_n |\widehat{\mathcal{O}}_\mu\rangle \,. \tag{3.26}$$

For spaces of surfaces with a special puncture, the state $|\widehat{\mathcal{O}}_\mu\rangle$ is inserted at that puncture. For spaces with no special puncture, $|\widehat{\mathcal{O}}_\mu\rangle$ can be inserted at any puncture (since anyway all our spaces are symmetric). It follows from this definition and Eqn.(3.25) that

$$\mathrm{D}_\mu(\widehat{\Gamma}\,)\, f(\mathcal{A}) = f_\mu\left(\widehat{\mathcal{K}}\mathcal{A}\right). \tag{3.27}$$

where the operation $\mathrm{D}_\mu(\widehat{\Gamma})$, acting on a function of Ψ, has been defined in Eqn.(3.5) of Ref.[2]. Here $\mathcal{A}$ is a space of surfaces without a special puncture, and $\widehat{\mathcal{K}}\mathcal{A}$ is a space with one special puncture. Whereas the validity of Eqn.(3.27) for $\mathcal{A} \subset \widehat{\mathcal{P}}_{g,n}$ with $n \geq 1$, is an immediate consequence of Eqn.(3.25), the extension to the case $n = 0$ involves more work. It has been shown in appendix A that,

$$\partial_\mu \int\limits_{\mathcal{A}_{g,0}} \Omega^{(g,0)}_\Sigma = \int\limits_{\widehat{\mathcal{K}}\mathcal{A}_{g,0}} \langle \Omega^{(0)g,1} |\widehat{\mathcal{O}}_\mu\rangle \tag{3.28}$$

where $\Omega^{(g,0)}_\Sigma \equiv \Omega^{(0)g,0}_\Sigma$ is the top form in $\mathcal{M}_{g,0}$ whose definition was given in Eqn.(3.16). Since acting on $|\Psi\rangle$ independent terms $\mathrm{D}_\mu \equiv \partial_\mu$, Eqn.(3.28) implies that Eqn.(3.27) holds also for $\mathcal{A} \subset \widehat{\mathcal{P}}_{g,0}(\equiv \mathcal{M}_{g,0})$, and hence for any $\mathcal{A} \in \widehat{\mathcal{P}}$.

It is also straigthforward to verify that equations analogous to (3.20) hold for the function f_μ. We find that

$$\begin{aligned} f_\mu\big(\Delta\mathcal{A}\big) &= -\Delta f_\mu(\mathcal{A}) \\ f_\mu\big(\{\mathcal{A}, \underline{\mathcal{B}}\}\big) &= -\{\, f(\mathcal{A}), f_\mu(\mathcal{B})\}\,, \end{aligned} \tag{3.29}$$

where in the second equation the space $\mathcal{B}$ on the left hand side was underlined to denote the fact that this space has a special puncture.

There is an odd hamiltonian function on $\widehat{\mathcal{H}}$ that by a canonical transformation inserts a special puncture. We define

$$\mathbf{U}_{\mu(0,2)} \equiv \langle \omega_{12} \, | \widehat{\mathcal{O}}_\mu \rangle_1 | \Psi \rangle_2 \ , \tag{3.30}$$

and one can readily prove that for a space $\mathcal{A}$ without a special puncture

$$f_\mu(\mathcal{A}) = \{ \, f(\mathcal{A}) \, , \, \mathbf{U}_{\mu(0,2)} \, \} \ . \tag{3.31}$$

Two additional equations are obtained easily. We have

$$\{ \, S_{0,2} \, , \, f_\mu(\mathcal{A}) \} = - \, f_\mu(\partial \mathcal{A}) \, , \tag{3.32}$$

which is the exact analog of Eqn.(3.23), and

$$\{ \, S_{0,2} \, , \, \mathbf{U}_{\mu(0,2)} \} = 0 \, , \tag{3.33}$$

which follows as a result of $Q|\widehat{\mathcal{O}}_\mu\rangle = 0$. Finally, we can show that for a space of surfaces $\mathcal{A}$ with one special puncture (and any number of ordinary ones)

$$\mathcal{A} \approx 0 \quad \rightarrow \quad f_\mu(\mathcal{A}) = 0 \, . \tag{3.34}$$

This is obtained as follows. By definition $\mathcal{A} \approx 0$ means $\pi_f(\mathcal{A}) = 0$, where π_f was defined above Eqn.(2.23). Since $L_{n\geq 0}$ and $b_{n\geq 0}$, together with their antiholomorphic counterparts, annihilate $|\widehat{\mathcal{O}}_\mu\rangle$, it follows that the form $\langle \Omega | \Psi \rangle \cdots | \widehat{\mathcal{O}}_\mu \rangle$ on $\mathcal{A}$ which enters in the definition of f_μ, descends to a well defined form in $\pi_f(\mathcal{A})$. Since π_f must be locally a diffeomorphism (otherwise $\mathcal{A}$ must have some tangent vector representing change of coordinates around the special puncture, and the corresponding antighost insertion will give $b_n|\widehat{\mathcal{O}}_\mu\rangle$ for $n \geq 0$, making f_μ vanish trivially) the integral over $\mathcal{A}$ defining f_μ can be written as an integral over $\pi_f(\mathcal{A})$, and therefore vanishes since $\pi_f(\mathcal{A}) = 0$. This establishes (3.34).

3.3. Master Action of Closed String Field Theory

The complete quantum master action for closed string field theory is given by the sum of the kinetic term plus geometrical terms corresponding to integrals over string vertices. Using the map f introduced earlier we simply have

$$S = S_{0,2} + f(\mathcal{V}) + \hbar S_{1,0}\,, \tag{3.35}$$

where $\mathcal{V}$ and $S_{0,2}$ were given in Eqns.(2.21) and (3.3). Here we have included the vacuum vertices for $g \geq 2$ in $\mathcal{V}$, but have written down the contribution of the $g = 1, n = 0$ term separately. The kinetic term of the theory is easily shown to satisfy [1]

$$\{S_{0,2}\,,\, S_{0,2}\,\} = 0\,, \quad \text{and,} \quad \Delta S_{0,2} = 0. \tag{3.36}$$

Our work proving representation identities allows us to give a straightforward proof that the action satisfies the master equation. Using Eqns.(3.35) and (3.36) we find

$$\begin{aligned} \hbar\Delta S + \tfrac{1}{2}\{S,S\} &= \hbar\Delta f(\mathcal{V}) + \{\, S_{0,2}\,,\, f(\mathcal{V})\,\} + \tfrac{1}{2}\{f(\mathcal{V})\,,\, f(\mathcal{V})\,\}\,, \\ &= -\hbar f\,(\Delta\mathcal{V}) - f\,(\partial\mathcal{V}) - \tfrac{1}{2} f\,(\,\{\mathcal{V},\mathcal{V}\}\,)\,, \\ &= -f\,(\hbar\Delta\mathcal{V} + \partial\mathcal{V} + \tfrac{1}{2}\,\{\mathcal{V},\mathcal{V}\}\,) = 0\,, \end{aligned} \tag{3.37}$$

where we used Eqns.(3.20) and (3.23), and the last expression vanishes by virtue of the geometrical recursion relations (2.22).⋆

Our proof of background independence will require the computation of the covariant derivative $\mathrm{D}_\mu(\widehat{\Gamma}\,)S$. We first compute the derivative of the kinetic term

$$\begin{aligned} \mathrm{D}_\mu(\,\widehat{\Gamma}\,)S_{0,2} &= \tfrac{1}{2}\Big(D_\mu(\widehat{\Gamma})\langle\omega_{12}|Q^{(2)}\Big)|\Psi\rangle_1|\Psi\rangle_2\,, \\ &= \tfrac{1}{2}\langle V'^{(0,3)}_{123}|\widehat{\mathcal{O}}_\mu\rangle_3\,|\Psi\rangle_1|\Psi\rangle_2\,, \\ &= f_\mu(\mathcal{V}'_{0,3})\,, \end{aligned} \tag{3.38}$$

where we made use of Eqn.(3.23) of Ref.[2], the definition of f_μ in Eqn.(3.26),

⋆ Note that $\Delta\mathcal{V}_{g,0}$ vanish identically, and $\mathcal{V}_{g,0}$ does not contribute to $\{\mathcal{V},\mathcal{V}\}$. Furthermore, $\partial\mathcal{V}_{g,0}$ is of dimension $(6g-7)$, and hence $f(\partial\mathcal{V}_{g,0})$ vanishes by Eqn.(3.18). This shows that the vacuum graphs are irrelevant to the verification of the master equation.

and the definition $\langle V'^{(0,3)}| \equiv \langle \Omega^{(0)0,3}|_{\mathcal{V}'_{0,3}}$. The complete result for the action now follows from this result for the kinetic term and Eqn.(3.27). We have

$$\mathrm{D}_\mu(\,\widehat{\Gamma}\,)S = f_\mu(\mathcal{V}'_{0,3}) + \mathrm{D}_\mu(\,\widehat{\Gamma}\,)f(\mathcal{V}) + \hbar\partial_\mu S_{1,0} = f_\mu(\mathcal{V}'_{0,3} + \widehat{\mathcal{K}}\mathcal{V}) + \hbar\partial_\mu S_{1,0}\,. \quad (3.39)$$

where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$. Since $S_{1,0}$ is string field independent, we have replaced $\mathrm{D}_\mu(\widehat{\Gamma})S_{1,0}$ by $\partial_\mu S_{1,0}$.

4. Setup for Quantum Background Independence

The objective of the present section is to give the conditions for quantum background independence of closed string field theory. The physical and geometrical motivation for the proposed conditions will be given, and we will derive their explicit form for the case of nearby backgrounds. The main point that we make here is that the action weighted measure $d\mu_S \equiv d\mu e^{2S/\hbar}$ is the background independent object. As studied recently in Ref.[8] this measure encodes all of the physics of the theory, and quantum gauge transformations leave it invariant. For a discussion of $d\mu_S$ from a somewhat different perspective see Ref.[20].

4.1. The Conditions of Quantum Background Independence

The question of quantum background independence of string field theory is formulated as follows. Let x and y be two conformal field theories, and let $\widehat{\mathcal{H}}_x$ and $\widehat{\mathcal{H}}_y$ be their respective state spaces, equipped with symplectic structures and measures $(\omega_x, d\mu_x)$ and $(\omega_y, d\mu_y)$ respectively. Moreover, we have master actions $S_x : \widehat{\mathcal{H}}_x \to R$ and $S_y : \widehat{\mathcal{H}}_y \to R$ satisfying the respective master equations. We demand the existence of a diffeomorphism

$$F_{y,x} : \widehat{\mathcal{H}}_x \to \widehat{\mathcal{H}}_y\,, \quad (4.1)$$

between the corresponding spaces $\widehat{\mathcal{H}}_x$ and $\widehat{\mathcal{H}}_y$ such that

$$\omega_x = {F_{y,x}}^*\,\omega_y\;, \quad (4.2)$$

$$d\mu_x \, e^{2S_x/\hbar} = {F_{y,x}}^* \left(d\mu_y \, e^{2S_y/\hbar} \right) \,, \tag{4.3}$$

with ${F_{y,x}}^*$ denoting the pullback performed using the diffeomorphism $F_{y,x}$. The first of these equations implies that the diffeomorphism preserves the antibracket, and the second equation indicates that the action-weighted volume elements $(d\mu_S)_x$ and $(d\mu_S)_y$ must be taken into each other. The question of quantum background independence is simply the question whether such symplectic diffeomorphism exists.

<u>Geometrical Interpretation</u> The above conditions for background independence might seem a bit strange at first sight. A Batalin-Vilkovisky (BV) manifold is defined by $(M, \omega, d\mu)$ where M is a supermanifold, ω is a symplectic form, and $d\mu$ is a volume element. The volume element must lead to a nilpotent $\Delta_{d\mu}$. The space $\widehat{\mathcal{H}}$ relevant for string field theory is such a BV manifold; the symplectic structure was given in [2], and the volume element $d\mu = \rho \prod_i d\psi^i$ leads to a nilpotent $\Delta_{d\mu}$. One might be tempted to assume that a proof of background independence at the quantum level would have meant the existence of a diffeomorphism mapping the BV manifolds $(\widehat{\mathcal{H}}_x, \omega_x, d\mu_x)$ and $(\widehat{\mathcal{H}}_y, \omega_y, d\mu_y)$ into each other, and at the same time taking one master action into the other. This is not the case. One cannot find a symplectic map that preserves the volume element and the action separately. In fact, the existence of such map is not necessary physically, as will be explained below. There is, however, a simple sense in which quantum background independence is strictly a statement that two spaces are identical BV manifolds.

Assume we have a volume element $d\mu$ leading to a nilpotent $\Delta_{d\mu}$. Consider now the measure $d\mu_S = d\mu \, e^{2S/\hbar}$. The associated delta operator is given by

$$\Delta_{d\mu_S} = \Delta_{d\mu} + \frac{1}{\hbar} \{ S \,, \, \cdot \, \} \tag{4.4}$$

and then, one can show (following Schwarz[7])

$$\Delta^2_{d\mu_S} = \frac{1}{\hbar^2} \left\{ \hbar \, \Delta_{d\mu} S + \frac{1}{2} \{ S \,, \, S \} \,, \, \cdot \, \right\} , \tag{4.5}$$

which shows that $\Delta^2_{d\mu_S}$ is a linear operator, in fact, a hamiltonian vector field. This equation also shows that $d\mu_S$ is a consistent measure, leading to a nilpotent $\Delta_{d\mu_S}$, if $d\mu$ is consistent, and S satisfies the quantum master equation. This implies that given a BV manifold $(\widehat{\mathcal{H}}, \omega, d\mu)$, we can define another BV manifold $(\widehat{\mathcal{H}}, \omega, d\mu_S)$. Quantum background independence is the statement that $(\widehat{\mathcal{H}}_x, \omega_x, [d\mu_S]_x)$, and $(\widehat{\mathcal{H}}_y, \omega_y, [d\mu_S]_y)$ are the same BV manifold.

Physical Interpretation The conditions of background independence should imply that the two string field theories, formulated around different conformal backgrounds, are just two different descriptions of the same underlying theory. More precisely, the conditions of background independence imply a *formal* equivalence between the theories formulated using the different backgrounds. The emphasis on the word formal is necessary, the physics around different backgrounds can, in general, be different. As will be discussed below, this is due to the constant shift in the diffeomorphism relating the two theories.

Let us see why the conditions of background independence imply the formal equivalence of the two theories. The observables in a theory are defined by

$$\langle A \rangle \equiv \int\limits_L d\lambda e^{S/\hbar} A \,, \tag{4.6}$$

where L denotes a Lagrangian submanifold, and A, in order to be a physical operator, must be a function of fields/antifields satisfying $\hbar\,\Delta_{d\mu} A + \{S, A\} = 0$. This condition, with the help of Eqn.(4.4) reads $\Delta_{d\mu_S} A = 0$. This is important, as it shows that knowledge of the measure $d\mu_S$ suffices to define physical operators.

Consider now the case when we have two string field theories formulated around conformal theories x and y. Let A_y, satisfying $\Delta_{d\mu_{S_y}} A_y = 0$, be an observable of the string theory at y. Since condition (4.3) makes $\Delta_{d\mu_S}$ a scalar operator, $A_x \equiv {F_{y,x}}^* A_y$ satisfies $\Delta_{d\mu_{S_x}} A_x = 0$, and is therefore a physical operator of the theory at x. Since ${F_{y,x}}^*$ is symplectic, the preimage $L_x \equiv {F_{y,x}}^* L_y$ of L_y is a lagrangian submanifold in $\widehat{\mathcal{H}}_x$. Finally, it follows from (4.3) and (3.8) that

$$d\lambda_x \, e^{S_x/\hbar} = F_{y,x}{}^* \left(d\lambda_y \, e^{S_y/\hbar} \right) \; . \tag{4.7}$$

All this put together implies the formal equality of observables in the two string theories

$$\langle A_x \rangle \equiv \int\limits_{L_x} d\lambda_x e^{S_x/\hbar} \, A_x \; = \int\limits_{F^*_{y,x} L_y} F_{y,x}{}^* \left(d\lambda_y \, e^{S_y/\hbar} A_y \, \right) = \int\limits_{L_y} d\lambda_y e^{S_y/\hbar} \, A_y \; = \langle A_y \rangle \; . \tag{4.8}$$

This proves that Eqns.(4.2) and (4.3) guarantee the formal equivalence of the theories formulated around different backgrounds.

Does the existence of a solution of Eqns.(4.2) and (4.3) imply that string theories formulated around different points in the CFT moduli space correspond to the same quantum theory? To answer this question, we begin with the observation that in computing correlation functions of observables using the path integral formalism, we need not only the action weighted measure, but also the asymptotic boundary conditions on the fields. More than one possible boundary conditions may be allowed in general, giving rise to different consistent quantum theories with different sets of correlation functions.⋆ These different correlation functions may be interpreted as expectation values of observables calculated in different ground states. Thus the question of uniqueness of the correlation functions is related to the uniqueness of possible allowed asymptotic boundary conditions on the fields, or, equivalently, uniqueness of the ground state of the theory. If there are different allowed ground states of the theory, we may refer to them as different phases of the same theory.

Normally, when we work with an action $S_x(|\Psi_x\rangle)$, it is implicitly assumed that we use the boundary condition $|\Psi_x\rangle \to 0$ asymptotically. Existence of an $F_{y,x}$ satisfying Eqns.(4.2) and (4.3) guarantee that doing a path integral with

⋆ Examples of such theories already exist in the context of $c \leq 1$ string theories[21,22].

measure $d\lambda_x e^{S_x}$ and asymptotic boundary condition $|\Psi_x\rangle \to 0$ is equivalent to doing a path integral with measure $d\lambda_y e^{S_y}$ with asymptotic boundary condition $|\Psi_y\rangle \to F_{y,x}(\,|\Psi_x\rangle = 0\,)$. The result of such a path integration will, in general, be different from the one where we use the same action weighted measure, but put the asymptotic boundary condition $|\Psi_y\rangle \to 0$. In this sense the perturbative correlation functions, computed around two different CFT backgrounds x and y will be different. This would seem to imply that we must necessarily have different phases of string theory associated with different points x in the CFT moduli space, since for each point x we have an associated asymptotic boundary condition $|\Psi_x\rangle \to 0$. This, however, could be an artifact of perturbation theory, since $|\Psi_x\rangle \to 0$ is an allowed boundary condition only in perturbation theory. It may turn out that in full quantum string theory, only a subset of these are allowed asymptotic boundary conditions. This would be the case, for example, if the degeneracy between the different ground states at the classical level is lifted by quantum corrections. In fact, for bosonic string theory, probably none of the above choices of boundary conditions give rise to a consistent quantum theory due to the presence of the tachyon. For superstring theories, on the other hand, each of the above boundary condition seems to give rise to a consistent quantum theory, at least to all orders in perturbation theory, if the corresponding tree level string theory has unbroken $N = 1$ space-time supersymmetry.

To summarise, the question we address in our analysis is the background independence of the action weighted measure $d\mu_S$. The question of what are the possible allowed asymptotic boundary conditions, or, equivalently, what are the possible ground states of the theory, is a dynamical question, and is beyond the scope of the present analysis.

4.2. The Conditions of Local Background Independence

Let us now find the explicit form that the background independence conditions take when we consider string theories formulated around nearby backgrounds. From the expressions for the measures $d\mu_x = \rho(x) \prod_i d\psi_x^i$, and $d\mu_y = \rho(y) \prod_i d\psi_y^i$, we readily find

$$F_{y,x}{}^{*} \left(d\mu_y \right) = \frac{\rho(y)}{\rho(x)} \,\text{sdet} \left[\frac{\partial_l \psi_y}{\partial \psi_x} \right] \cdot d\mu_x \,. \tag{4.9}$$

This result, back into condition (4.3) of background independence gives

$$\exp\Bigl(\frac{2S(\psi_x, x)}{\hbar}\Bigr) = \exp\Bigl(\frac{2S(\psi_y, y)}{\hbar}\Bigr) \cdot \frac{\rho(y)}{\rho(x)} \cdot \text{sdet} \left[\frac{\partial_l \psi_y}{\partial \psi_x} \right] . \tag{4.10}$$

This is an exact equation constraining the map relating theories at x and y. We now consider, as we did in [2], the infinitesimal diffeomorphism relating a theory at x to one at $y = x + \delta x$. We write

$$\psi^i_{x+\delta x} = F^i \left(\psi_x, x, x + \delta x \right) = \psi^i_x + \delta x^\mu \, f^i_\mu(\psi_x, x) + \mathcal{O}(\delta x^2) \,. \tag{4.11}$$

In this case Eqn.(4.10) expressing the background independence of the measure $d\mu_S$ reduces to the condition

$$\frac{\partial S(\psi, x)}{\partial x^\mu} + \frac{\partial_r S(\psi_x, x)}{\partial \psi^i_x} f^i_\mu + \tfrac{1}{2}\, \hbar \left[\frac{\partial \ln \rho}{\partial x^\mu} + \text{str}\Bigl(\frac{\partial_l f^i_\mu}{\partial \psi^j} \Bigr) \right] \;=\; 0 \,. \tag{4.12}$$

In order to give a geometrical formulation for the background independence condition, it is convenient to define the object U^i_μ through the relation

$$f^i_\mu \equiv - \widehat{\Gamma}_{\mu j}{}^{i} \, \psi^j \; - \; U^i_\mu \,, \tag{4.13}$$

where we have separated out a term proportional to the connection $\widehat{\Gamma}_\mu$. Using (4.13), we rewrite (4.12) in the following form

$$\mathrm{D}_\mu(\widehat{\Gamma}) \, S - \frac{\partial_r S}{\partial \psi^i} \, U^i_\mu \; + \; \tfrac{1}{2} \, \hbar \left[\frac{\partial \ln \rho}{\partial x^\mu} - \text{str} \, \widehat{\Gamma}_\mu \; - \; \text{str} \left(\frac{\partial_l U^i_\mu}{\partial \psi^j} \right) \right] \; = 0 \,, \tag{4.14}$$

Using the covariant constancy of the symplectic form $D_\mu(\widehat{\Gamma}) \langle \omega | \; = \; 0$ [2], the condition that F^i be a symplectic map reduces to the condition that U^i_μ be a symplectic diffeomorphism. This, in turn, implies that there is an odd hamiltonian function $\mathbf{U}_\mu$ such that

$$U^i_\mu = \{\, \psi^i \,,\, \mathbf{U}_\mu \,\} \;\leftrightarrow\; U^i_\mu = \omega^{ij} \frac{\partial_l \mathbf{U}_\mu}{\partial \psi^j} \,. \tag{4.15}$$

This implies that (4.14) can be written as

$$\mathrm{D}_\mu(\,\widehat{\Gamma}\,)\, S \;-\; \big(\hbar \Delta \mathbf{U}_\mu \;+ \{\, S \,, \mathbf{U}_\mu \,\}\big) \;=\; \tfrac{1}{2}\, \hbar \left[\, \mathrm{str}\, \widehat{\Gamma}_\mu \;-\; \frac{\partial \ln \rho}{\partial x^\mu} \,\right] . \tag{4.16}$$

This is our final form for the condition of local quantum background independence. Notice that the right hand side contains only field independent terms, and only at first power of $\hbar$. Proving quantum background independence will amount to finding an odd hamiltonian $\mathbf{U}_\mu$ such that Eqn.(4.16) holds. The diffeomorphism implementing background independence will then read

$$|\Psi\rangle_{x+\delta x} = {}_{x+\delta x}\mathcal{I}_x \left[\, |\Psi\rangle - \delta x^\mu \,(\, \widehat{\Gamma}_\mu |\Psi\rangle + |U_\mu\rangle \,)\, \right] , \tag{4.17}$$

where $|U_\mu\rangle \equiv |\Phi_i\rangle\, U^i_\mu$. Finding the odd hamiltonian $\mathbf{U}_\mu$ is our aim.

4.3. The Conditions of Background Independence in Geometric Language

In this subsection we shall first rewrite the condition (4.16) of background independence in a geometric language, and then separate out the terms carrying different powers of $\hbar$ and the string field $|\Psi\rangle$. As we shall see in the later sections, while the study of condition (4.16) will be efficiently carried out in geometrical terms, the field independent terms at $\mathcal{O}(\hbar)$ will require special attention.

We start with the observation that the part of $\mathbf{U}_\mu$ which is $\hbar$ independent and is linear in $|\Psi\rangle$, should give rise to a constant shift in $|\Psi\rangle$ proportional to $|\widehat{\mathcal{O}}_\mu\rangle$, and, as in Ref.[2], is given by $\mathbf{U}_{\mu(0,2)}$ defined in (3.30). Using the insight obtained from the analysis of Ref.[2], we start with the following ansatz for $\mathbf{U}_\mu$:

$$\mathbf{U}_\mu = \mathbf{U}_{\mu\,(0,2)} - f_\mu(\mathcal{B})\,, \tag{4.18}$$

where,

$$\mathcal{B} \equiv \sum_{g,n} \hbar^g \mathcal{B}_{g,n} \quad \text{with} \quad \begin{cases} \mathcal{B}_{0,n} \equiv 0 \text{ for } n \leq 2, \\ \mathcal{B}_{g,n} \equiv 0 \text{ for } n \leq 1\,. \end{cases} \tag{4.19}$$

Each space $\mathcal{B}_{g,n}$ will be a $(6g + 2n - 6) + 1$ dimensional subspace of $\widehat{\mathcal{P}}_{g,n}$ (in our notation we should write $\mathcal{B}^{(1)}_{g,n}$, but we will omit the superscript, as $\mathcal{B}$ spaces will always have one real dimension more than the corresponding moduli space). The function f_μ was defined in Eqn.(3.26). $\mathcal{B}$ spaces will always have one special puncture (for inserting $|\widehat{\mathcal{O}}_\mu\rangle$) and will be symmetric in all other punctures. Since any field independent term in the hamiltonian $\mathbf{U}_\mu$ would drop out of the background independence condition, the $\mathcal{B}$ spaces will have at least one puncture besides the special one, where the string field will be inserted. This is reflected in the range of n given in Eqn.(4.19). The proof of background independence now amounts to showing the existence of the subspaces $\mathcal{B}_{g,n}$ such that Eqn.(4.16) is satisfied. Note that $\mathcal{B}_{g,2}$ for $g \geq 1$ determine the quantum corrections to the classical shift generated by $\mathbf{U}_{\mu(0,2)}$.

We now evaluate various terms in Eqn.(4.16). We have

$$\Delta\mathbf{U}_\mu = \Delta\mathbf{U}_{\mu(0,2)} - \Delta f_\mu(\mathcal{B}) \, = f_\mu(\Delta\mathcal{B})\,, \tag{4.20}$$

where use was made of Eqn.(3.29). Furthermore, since $S_{1,0}$ is field independent,

$$\begin{aligned} \{S, \mathbf{U}_\mu\} &= \{\, S_{0,2} + f(\mathcal{V})\,,\, \mathbf{U}_{\mu(0,2)} - f_\mu(\mathcal{B})\,\}\,, \\ &= -\{S_{0,2}\,, f_\mu(\mathcal{B})\} + \{f(\mathcal{V})\,, \mathbf{U}_{\mu(0,2)}\} - \{f(\mathcal{V})\,,\, f_\mu(\mathcal{B})\}\,, \\ &= \, f_\mu\big(\partial\mathcal{B} + \underline{\mathcal{V}} + \{\mathcal{V}, \mathcal{B}\}\big)\,, \end{aligned} \tag{4.21}$$

where we made use of Eqns.(3.29)–(3.33).* The two equations above, together with

* Recall that the underline is there to remind us that the space has one special puncture.

Eqn.(3.39) can now be substituted into the background independence condition (4.16). We then obtain

$$f_\mu\Big(\mathcal{V}'_{0,3}+\widehat{\mathcal{K}}\mathcal{V}-\partial\mathcal{B}-\hbar\Delta\mathcal{B}-\{\mathcal{V},\mathcal{B}\}-\underline{\mathcal{V}}\Big) = \tfrac{1}{2}\,\hbar\left[\operatorname{str}\widehat{\Gamma}_\mu - \frac{\partial\ln\rho}{\partial x^\mu} - 2\frac{\partial S_{1,0}}{\partial x^\mu}\right]. \quad (4.22)$$

This is the final geometrical version of the conditions for background independence.

<u>Power Series Expansion</u> We can now demand that the above equation holds separately for terms containing different powers of $\hbar$ and $|\Psi\rangle$. The field independent, order $\hbar$ contribution to the above equation takes the form:

$$\partial_\mu S_{1,0} = \tfrac{1}{2}\left[\operatorname{str}(\,\widehat{\Gamma}_\mu\,) - \frac{\partial\ln\rho}{\partial x^\mu}\right] + f_\mu(\Delta\mathcal{B}_{0,3}) + f_\mu(\underline{\mathcal{V}}_{1,1})\,. \quad (4.23)$$

Field independent terms of order $\hbar^g$ ($g\geq 2$) give

$$f_\mu\Big(\widehat{\mathcal{K}}\mathcal{V}_{g,0} - \Delta\mathcal{B}_{g-1,3} - \sum_{g_1=1}^{g-1}\{\mathcal{V}_{g_1,1},\mathcal{B}_{g-g_1,2}\} - \underline{\mathcal{V}}_{g,1}\Big) = 0\,. \quad (4.24)$$

Finally, order $\hbar^g|\Psi\rangle^N$ terms give,

$$f_\mu(\mathcal{V}'_{g,N+1} - \partial\mathcal{B}_{g,N+1} - \underline{\mathcal{V}}_{g,N+1}) = 0\,, \quad \begin{cases} N\geq 2, \text{ for } g=0 \\ N\geq 1, \text{ for } g\geq 1, \end{cases} \quad (4.25)$$

where,

$$\mathcal{V}'_{g,N+1} = \widehat{\mathcal{K}}\mathcal{V}_{g,N} - \sum_{g_1=0}^{g}\sum_{m=1}^{N+1}\{\mathcal{V}_{g_1,m}\,,\,\mathcal{B}_{g-g_1,N-m+3}\} - \Delta\mathcal{B}_{g-1,N+3} \quad \begin{cases} N\geq 3, \text{ for } g=0 \\ N\geq 1, \text{ for } g\geq 1\,. \end{cases} \quad (4.26)$$

Note that we have specified to the right of the equation the restrictions on values of N for this equation to hold. At genus zero, $N\geq 3$ because $\mathcal{V}'_{0,3}$ was already defined, and Eqn.(4.26), for $N=2$, would give $\mathcal{V}'_{0,3}=0$.

Looking at these equations, it is clear that the conditions (4.24) and (4.25) are purely geometrical in the sense of Riemann surfaces, whereas the condition (4.23) involves extra theory space elements. In the next section we shall show how to find the spaces $\mathcal{B}_{g,n}$ satisfying Eqns.(4.24) and (4.25), leaving the analysis of Eqn.(4.23) to §6.

5. Background Independence of $d\mu_S$ up to $\hbar$-Independent Normalization

In this section we will establish the background independence of $d\mu_S$ up to an overall $\hbar$ independent normalization factor. This will involve finding subspaces $\mathcal{B}_{g,N+1}$ of $\widehat{\mathcal{P}}_{g,N+1}$ satisfying Eqns.(4.24) and (4.25). In fact, we shall first solve Eqns.(4.25) to find $\mathcal{B}_{g,N+1}$, and then show that these $\mathcal{B}_{g,N+1}$ automatically satisfy Eqn.(4.24).

We shall now carry out a consistency check of the set of Eqns.(4.22). Ignoring the order $\hbar$ constant terms, and using Eqn.(3.34), we write (4.22) as

$$\partial\mathcal{B} \simeq \mathcal{V}' - \underline{\mathcal{V}}\,, \quad \text{with} \quad \mathcal{V}' \equiv \mathcal{V}'_{0,3} + \widehat{\mathcal{K}}\mathcal{V} - \hbar\Delta\mathcal{B} - \{\mathcal{V},\mathcal{B}\}\,. \tag{5.1}$$

Here the symbol $\simeq$ is used to denote that the equality holds up to the choice of local coordinate at the special puncture, *and* up to the addition of terms of order $\hbar$ representing surfaces of genus one carrying only the special puncture, but no ordinary puncture. Such surfaces will produce order $\hbar$ constant terms when acted on by f_μ. If we now apply the ∂ operator again on this equation, then the left hand side of this equation vanishes. Thus for consistency, the right hand side must also vanish up to order $\hbar$ terms which do not carry any ordinary puncture. The verification of this condition is straightforward. We have,

$$\partial^2\mathcal{B} \simeq \partial\widehat{\mathcal{K}}\mathcal{V} + \hbar\Delta\partial\mathcal{B} - \{\partial\mathcal{V},\mathcal{B}\} + \{\mathcal{V},\partial\mathcal{B}\} - \partial\underline{\mathcal{V}} \tag{5.2}$$

Using Eqn.(5.1) for $\partial\mathcal{B}$, and the various identities derived in §2, we find,*

$$
\begin{aligned}
\partial^2\mathcal{B} \simeq\ & -\{\widehat{\mathcal{K}}\mathcal{V},\mathcal{V}\} - \hbar\Delta\widehat{\mathcal{K}}\mathcal{V} - \{\mathcal{V}'_{0,3},\mathcal{V}\} \\
& + \hbar(\Delta\mathcal{V}'_{0,3} + \Delta\widehat{\mathcal{K}}\mathcal{V} - \{\Delta\mathcal{V},\mathcal{B}\} + \{\mathcal{V},\Delta\mathcal{B}\} - \Delta\underline{\mathcal{V}}) \\
& + \tfrac{1}{2}\{\{\mathcal{V},\mathcal{V}\},\mathcal{B}\} + \hbar\{\Delta\mathcal{V},\mathcal{B}\} \\
& + \{\mathcal{V}\,,\,\mathcal{V}'_{0,3} + \widehat{\mathcal{K}}\mathcal{V} - \hbar\Delta\mathcal{B} - \{\mathcal{V},\mathcal{B}\} - \underline{\mathcal{V}}\} \\
& + \{\underline{\mathcal{V}},\mathcal{V}\} + \hbar\Delta\underline{\mathcal{V}}\,,
\end{aligned}
\tag{5.3}
$$

where, for the help of the reader, each term in (5.2) has been written as one line in (5.3). Using the symmetry properties of $\{,\}$, and the Jacobi identity,

$$
\tfrac{1}{2}\{\{\mathcal{V},\mathcal{V}\}\,,\,\mathcal{B}\} - \{\mathcal{V}\,,\,\{\mathcal{V},\mathcal{B}\}\} = 0\,,
\tag{5.4}
$$

it is easily checked that all terms in the right hand side cancel out. The term $\hbar\Delta\mathcal{V}'_{0,3}$ is of order $\hbar$ and does not carry any ordinary puncture; hence it must be dropped in the present calculation. This finishes the verification of the consistency condition. The analysis in the later part of this section will involve showing how, using this consistency condition, we can construct the vertices $\mathcal{B}_{g,N+1}$ following a recursive procedure.

5.1. Background Independence of the Field Dependent Part of $d\mu_S$

We now turn our attention to the construction of explicit solutions of Eqn.(4.25). This equation is satisfied, provided we find a set of vertices $\mathcal{B}_{g,N+1}$ satisfying,

$$
\partial\mathcal{B}_{g,N+1} \approx \mathcal{V}'_{g,N+1} - \mathcal{V}_{g,N+1}\,,
\tag{5.5}
$$

where $\approx$ denotes equality up to a choice of the coordinate system at the special puncture. In arriving at this equation we made use of (3.34). The existence of

* Note that we can use Eqn.(5.1) because the terms in $\partial\mathcal{B}$ of order $\hbar$ and only one (special) puncture, which we do not keep track, do not contribute to the right hand side of Eqn.(5.2).

$\mathcal{B}_{g,N+1}$ satisfying Eqns.(5.5) and (4.26) can be proved by induction as follows. Let $b_{g,N} \equiv (6g+2N-6)+1$ denote the dimension of the space $\mathcal{B}_{g,N}$. We assume that the spaces $\mathcal{B}_{g,N+1}$ satisfying Eqns.(5.5) and (4.26) have been constructed for all g and N satisfying $1 \leq b_{g,N+1} \leq b_0$ for some odd $b_0 \geq 1$, and then show that $\mathcal{V}'_{g_0,M+1}$ defined through Eqn.(4.26) satisfies the consistency condition

$$\partial\mathcal{V}'_{g_0,M+1} \approx \partial\mathcal{V}_{g_0,M+1}\,, \tag{5.6}$$

for $b_{g_0,M+1} = b_0+2$. (Remember that $b_{g,N}$ is always odd.) Once Eqn.(5.6) is satisfied, the space of surfaces $\mathcal{B}_{g_0,M+1}$ can be defined as a symmetric homotopy between $\mathcal{V}_{g_0,M+1}$ and $\mathcal{V}'_{g_0,M+1}$.⋆ This allows us to define $\mathcal{B}_{g_0,M+1}$ satisfying Eqn.(5.5) for all g_0 and M for which $b_{g_0,M+1} = b_0+2$. Equation.(5.6) is satisfied trivially for $b_{g,N+1} = 1$. Indeed, this case corresponds to $g=0, M=2$, and both $\mathcal{V}'_{0,3}$ and $\mathcal{V}_{0,3}$ have zero dimension, and therefore zero boundary. The lowest dimensional $\mathcal{B}$ space is therefore $\mathcal{B}_{0,3}$ which is a symmetric homotopy between $\mathcal{V}_{0,3}$ and $\mathcal{V}'_{0,3}$. The induction argument will then imply that consistent $\mathcal{B}_{g,N}$'s can be constructed for all values of $N \geq 3$ for $g=0$, and all values of $N \geq 2$ for $g \geq 1$.

Thus the task that remains is to calculate $\partial(\mathcal{V}'-\underline{\mathcal{V}})_{g_0,M+1}$ using the expression (4.26) for $\mathcal{V}'_{g_0,M+1}$, and show that it vanishes. This expression contains $\partial\mathcal{B}_{g,n}$ for different g and n on the right hand side, but all of these $\mathcal{B}_{g,n}$'s are of dimension $\leq b_{g_0,M+1}-2 = b_0$. Hence we can replace these $\partial\mathcal{B}_{g,n}$'s by $\mathcal{V}'_{g,n}-\mathcal{V}_{g,n}$, with $\mathcal{V}'_{g,n}$ defined through Eqn.(4.26). It is clear, however that the resulting expression is precisely equal to the term on the right hand side of Eqn.(5.3) which is of order $\hbar^{g_0}$ and carries a total of $M+1$ punctures (including the special puncture). We have already shown that the right hand side of Eqn.(5.3) vanishes except for a $g_0=1$, $M=0$ term. This, in turn, shows that our expression indeed vanishes as long as $(g_0,M) \neq (1,0)$. This condition is automatically satisfied, since $\mathcal{B}_{g_0,M+1}$ is defined only in the range $M \geq 2$ for $g_0=0$, and $M \geq 1$ for $g_0 \geq 1$. This completes the proof by induction, and the complete construction of the $\mathcal{B}$ spaces.

⋆ See Ref.[2] for a canonical procedure to construct a symmetric homotopy.

5.2. Background Independence of the $\hbar$-Dependent Normalization of $d\mu_S$

In this subsection we verify that the $\mathcal{B}_{g,N+1}$, constructed in the previous subsection, does satisfy Eqn.(4.24). This equation can be rewritten as,

$$f_\mu(\mathcal{V}'_{g,1} - \underline{\mathcal{V}}_{g,1}) = 0\,, \tag{5.7}$$

where, as the notation suggests,

$$\mathcal{V}'_{g,1} \equiv \widehat{\mathcal{K}}\mathcal{V}_{g,0} - \sum_{g_1=1}^{g-1}\{\mathcal{V}_{g_1,1}\,,\,\mathcal{B}_{g-g_1,2}\} - \Delta\mathcal{B}_{g-1,3}\ , \tag{5.8}$$

is the subspace of $\mathcal{V}'$ of genus g and one special puncture. In order to satisfy Eqn.(5.7), it suffices if $\mathcal{V}'_{g,1} \approx \mathcal{V}_{g,1}$. By definition, this is equivalent to $\pi_f\,(\mathcal{V}_{g,1}) = \pi_f\,(\mathcal{V}'_{g,1})$, where π_f is now the projection from $\widehat{\mathcal{P}}_{g,1}$ to $\mathcal{M}_{g,1}$. Both $\pi_f(\mathcal{V}_{g,1})$ and $\pi_f(\mathcal{V}'_{g,1})$ are subspaces of $\mathcal{M}_{g,1}$ of codimension 0. Furthermore, neither contains degenerate surfaces. This implies that $\pi_f\,(\mathcal{V}_{g,1}) = \pi_f\,(\mathcal{V}'_{g,1})$ will hold provided that $\partial\,\pi_f(\mathcal{V}_{g,1}) = \partial\,\pi_f(\mathcal{V}'_{g,1})$. This last condition is equivalent to $\pi_f\,(\partial\mathcal{V}_{g,1}) = \pi_f\,(\partial\mathcal{V}'_{g,1})$, which by definition is equivalent to $\partial\mathcal{V}_{g,1} \approx \partial\mathcal{V}'_{g,1}$. Thus, if we can show that,

$$\partial\,(\mathcal{V}'_{g,1} - \mathcal{V}_{g,1}) \approx 0\,, \tag{5.9}$$

we would have established background independence of the order $\hbar^{g-1}$ constant terms in the action-weighted measure. Using the fact that all $\mathcal{B}$ spaces satisfy Eqns.(5.5) and (4.26), it is easy to verify that the left hand side of the above equation can be identified to the order $\hbar^g$ terms on the right hand side of Eqn.(5.3) which do not carry any ordinary puncture. We have already seen that this vanishes for $g \geq 2$. This establishes background independence of order $\hbar^{g-1}$ field independent terms in the action weighted measure for all $g \geq 2$.

6. Background Independence of the $\hbar$-Independent Normalization of $d\mu_S$

We shall now turn to the analysis of the terms that control the $\hbar$-independent part of the normalization of the physical measure $d\mu_S$. There are two ingredients to this normalization. One is the genus one part $S_{1,0}$ of the master action. As pointed out earlier, there is no *a priori* geometric construction of $S_{1,0}$, and the problem here is to construct an appropriate $S_{1,0}$ using the requirement of background independence. The other ingredient is the scale factor $\rho(x)$ of the measure (Eqn.(3.6)). Since both $S_{1,0}$ and $\rho(x)$ determine the $\hbar$ independent part of the normalization of the measure $d\mu_S$, we need to discuss the construction of $S_{1,0}$ and $\rho(x)$ together. We begin with some preliminary observations relevant to the problem at hand. Then, we discuss the relevant equations, and determine both $S_{1,0}$ and $\rho(x)$. Our determination of $\rho(x)$ is somewhat indirect, and is based on the background independence of the free energy of string field theory.

6.1. Preliminary Observations

Basis Independence. We may ask if observables are affected by a change of basis in $\widehat{\mathcal{H}}$. It is clear that observables will be basis independent if the construction of closed string field theory is fully basis independent. The string field can be written as

$$|\Psi\rangle = \sum |\Phi_i\rangle\, \psi^i \;=\; \sum |\Phi'_i\rangle\, \psi'^i \,, \tag{6.1}$$

where we have used two different basis, related by some invertible transformation $|\Phi'_i\rangle = |\Phi_j\rangle A^j{}_i$. Basis independence of the string field theory requires that $d\mu_S(\psi)$ and $d\mu_S(\psi')$ agree when $\psi'^i = (A^{-1})^i{}_j\psi^j$. It is important to note that the construction of the string field action is completely basis independent, since the string field $|\Psi\rangle$ is basis independent and the string field vertices $\langle V^{(g,N)}|$ are also basis independent. Our choice of measure $d\mu$ given by $d\mu = \rho(x)\prod d\psi^i$ with some fixed $\rho(x)$, appears to be basis dependent. In order to keep $d\mu$ basis independent, $\rho(x)$

must transform under a change of basis: $\rho(x) \to \rho(x)\,\mathrm{sdet}(A)$. Operationally, we choose a basis in $\widehat{\mathcal{H}}$, determine a suitable scale factor $\rho(x)$, and formulate string field theory. If another basis is desired, one must readjust $\rho(x)$ using the condition of invariance of the measure. Interpreted this way, the physical action weighted measure $d\mu_S \equiv (\rho(x) \prod d\psi_x^i) \exp(2S_x/\hbar)$ is manifestly basis independent.

<u>What can be Determined?</u> In separating the measure factor ρ from $S_{1,0}$ there is an ambiguity. The term $S_{1,0}$ of the string field action enters in $d\mu_S$ as an overall ($\hbar$ independent) constant. Thus, we could, in principle, absorb $\rho(x)$ into $S_{1,0}$ in the form $S_{1,0} = \frac{1}{2} \ln \rho(x) + \cdots$ where the dots would indicate basis independent terms. We will not do so. It is natural to demand that $S_{1,0}$ be a scalar, since it is part of the action, which in the BV formulation is taken to be a scalar. This requirement, of course, is not enough for splitting unambiguously the scale factor between ρ and $S_{1,0}$, as we can always shift scalar factors back and forth between them. In satisfying the demands of background independence, we will be able to take for $S_{1,0}$ a Riemann surface geometrical object, very much as we did for the higher vacuum vertices $S_{g,0}$ ($g \geq 2$). We will then get an equation for $\partial_\mu \ln \rho$ involving a theory space connection. This equation encodes naturally the required basis dependence of ρ. Working in any given basis, the overall x-independent multiplicative factor of ρ is not determined by the condition of background independence. We must use an observable to fix this constant. The one-loop free energy of string field theory is the appropriate observable. It is guaranteed by the master equation to be gauge independent. Working at some specific point x_0 in CFT theory space, we demand that $\rho(x_0)$ be chosen in such a way that the one loop free energy from string field theory around x_0 is equal to the partition function of CFT_{x_0}, integrated over one copy of the full moduli space of the torus.* The same result would then hold for string field theory formulated around any CFT in the space as a consequence of background independence of string field theory, and the result[23]

$\star$ It is best to do it in a basis in which the one loop vacuum graph is finite up to tachyon divergence, since in such a basis $\rho(x_0)$ will also be finite. Later in this section we shall discuss how this can be done.

that the difference in one loop free energy of string field theory expanded around two nearby classical solutions correctly accounts for the change in CFT partition function under a marginal deformation.

On Non-canonical and Canonical Transformations String field theory can be formulated with stubs on all the vertices and with a standard kinetic term, or alternatively it can be written with a cutoff propagator, by adding factors of $\exp(2aL_0^+)$ in the kinetic term, together with vertices that have no stubs. If both formulations use the same measure, they yield exactly the same observables for processes involving external legs. Nevertheless they do not yield the same vacuum graph; using the same basis and the same gauge, it is clear that the respective propagators have different traces. This appears to be a somewhat surprising result at the first sight because the two actions can easily be seen to be related by a field redefinition of the form $\psi^i \to exp(a\gamma_i)\psi^i$, where γ_i is the conformal dimension of the state $|\Phi_i\rangle$. Since the fields and antifields, which have opposite statistics, are scaled by the same amount, one gets $\mathrm{sdet}(A) = 1$ under this transformation, and hence no change in ρ. In order to understand the origin of the change in the vacuum graph, we note that the transformation $\psi^i \to exp(a\gamma_i)\psi^i$ is not canonical, since the fields and the antifields are not scaled in appropriately correlated way. Hence if we want this transformation to generate an equivalent theory, we must transform the symplectic form ω in addition to the action. This change in ω gives rise to an extra factor in the construction of $d\lambda$ from $d\mu$ using Eqn.(3.8), and compensates for the change in the vacuum graph. In contrast, we note that a change of cell decomposition in closed string field theory, induced by changing the three string and higher string vertices, is always implemented by a *canonical* transformation [6]. This transformation left all observables, including one loop free energy, unchanged. Under an infinitesimal canonical transformation generated by the hamiltonian ϵ, we have that $d\mu \exp(2S/\hbar) \to d\mu(1 + 2\Delta\epsilon) \exp\bigl(2[S + \{S, \epsilon\}]/\hbar\bigr)$. In Ref.[6] the genus zero contribution to ϵ was cubic in the string field and therefore $\Delta\epsilon$ cannot give an $\hbar$ independent and field independent contribution. Nor can $\{S, \epsilon\}$, since the classical action is at least quadratic in the string field. The invariance of the

$\hbar$-independent normalization of the measure is consistent with the fact that the kinetic term is unchanged under the canonical transformation, which, in turn, leaves the one loop vacuum graph unchanged. The one loop free energy, which is a sum of the vacuum graph and half of the logarithm of the $\hbar$ independent normalization of the measure, is then left unchanged, as it should be.

6.2. Equations for $S_{1,0}$ and $\ln \rho$

Now that we have given an overview of most of the relevant issues we can proceed to the analysis of equation (4.23), which reads

$$\partial_\mu S_{1,0} = -\tfrac{1}{2}\,\frac{\partial \ln \rho}{\partial x^\mu} \,+\, \tfrac{1}{2}\,\mathrm{str}\,(\widehat{\Gamma}_\mu) + f_\mu(\Delta\mathcal{B}_{0,3}) + f_\mu(\mathcal{V}_{1,1})\,, \tag{6.2}$$

As written, Eqn.(6.2) is not manifestly free of divergences. For example, the term $f_\mu(\Delta\mathcal{B}_{0,3})$ involves integration over the space $\Delta\mathcal{B}_{0,3}$. Since $\mathcal{B}_{0,3}$ (recall $\partial\mathcal{B}_{0,3} = \mathcal{V}'_{0,3} - \mathcal{V}_{0,3}$) is an interpolation from the closed string vertex $\mathcal{V}_{0,3}$ to the auxiliary vertex $\mathcal{V}'_{0,3}$, and $\Delta\mathcal{V}'_{0,3}$ gives singular tori, the integral above is potentially divergent. Also $\mathrm{str}\,\widehat{\Gamma}_\mu$ is potentially divergent. We will now show that with an appropriate choice of basis, $\frac{1}{2}\mathrm{str}\,(\widehat{\Gamma}_\mu) + f_\mu(\Delta\mathcal{B}_{0,3})$ can be made finite. To this end we introduce a connection $\widetilde{\Gamma}$, defined through the relation:

$$\langle \omega_{12}|(\widetilde{\Gamma}_\mu)^{(1)} = \langle \omega_{12}|(\widehat{\Gamma}_\mu)^{(1)} + \int\limits_{\widetilde{\mathcal{B}}_{0,3}} \langle \Omega^{(1)0,3}|\widehat{\mathcal{O}}_\mu\rangle\,, \tag{6.3}$$

where $\widetilde{\mathcal{B}}_{0,3}$ is a path in $\widehat{\mathcal{P}}_{0,3}$ interpolating between $\mathcal{V}'_{0,3}$ and another point $\widetilde{\mathcal{V}}_{0,3}$:

$$\partial\widetilde{\mathcal{B}}_{0,3} = \widetilde{\mathcal{V}}_{0,3} - \mathcal{V}'_{0,3}. \tag{6.4}$$

The new auxiliary vertex $\widetilde{\mathcal{V}}_{0,3}$ is taken to be a sphere punctured at $z = 0$ with local coordinate $w_1 = e^a z$, at $z = \infty$, with local coordinate $w_2 = e^a/z$, and at $z = 1$, with a local coordinate w_3 which we shall keep arbitrary for the time being. The constant a is some number ≥ 0. We can now use Eqns.(3.26) and (6.3) to write

$$\begin{aligned} \tfrac{1}{2}\,\mathrm{str}\,(\widehat{\Gamma}_\mu) + f_\mu(\Delta\mathcal{B}_{0,3}) &= \tfrac{1}{2}\,\mathrm{str}\,(\,\widetilde{\Gamma}_\mu) \;-\tfrac{1}{2}\int\limits_{\widetilde{\mathcal{B}}_{0,3}} \langle\Omega^{(1)0,3}|\mathcal{S}\rangle\cdot|\widehat{\mathcal{O}}_\mu\rangle + f_\mu(\Delta\mathcal{B}_{0,3}) \\ &= \tfrac{1}{2}\,\mathrm{str}\,(\,\widetilde{\Gamma}_\mu) \;+ \int\limits_{\Delta(\mathcal{B}_{0,3}+\widetilde{\mathcal{B}}_{0,3})} \langle\Omega^{(0)1,1}|\widehat{\mathcal{O}}_\mu\rangle\,. \end{aligned} \tag{6.5}$$

where we have used $\mathrm{str}\,A = \langle\omega_{12}|A^{(1)}|\mathcal{S}_{12}\rangle$, which holds for any operator A. (The superscript $^{(1)}$ denotes that the operator acts on the first state space.) Here $\mathcal{B}_{0,3}+\widetilde{\mathcal{B}}_{0,3}$ is a path from $\mathcal{V}_{0,3}$ to $\widetilde{\mathcal{V}}_{0,3}$ made by joining together the path $\mathcal{B}_{0,3}$ going from $\mathcal{V}_{0,3}$ to $\mathcal{V}'_{0,3}$, and the path $\widetilde{\mathcal{B}}_{0,3}$ going from $\mathcal{V}'_{0,3}$ to $\widetilde{\mathcal{V}}_{0,3}$. The crucial point is that the integral above is independent of the path we take! For two paths $\mathcal{C}_{0,3}$ and $\mathcal{C}'_{0,3}$ with identical endpoints, so that there is a $\mathcal{D}_{0,3}$ with $\partial\mathcal{D}_{0,3} = \mathcal{C}_{0,3} - \mathcal{C}'_{0,3}$, we have from Eqn.(3.9)

$$\begin{aligned} \Big(\int\limits_{\mathcal{C}_{0,3}} - \int\limits_{\mathcal{C}'_{0,3}}\Big)\langle\Omega^{(1)0,3}|\mathcal{S}_{12}\rangle\cdot|\widehat{\mathcal{O}}_\mu\rangle_3 &= \int\limits_{\partial\mathcal{D}_{0,3}} \langle\Omega^{(1)0,3}|\mathcal{S}_{12}\rangle\cdot|\widehat{\mathcal{O}}_\mu\rangle_3 \\ &= \int\limits_{\mathcal{D}_{0,3}} \langle\Omega^{(2)0,3}|(Q^{(1)}+Q^{(2)}+Q^{(3)})|\mathcal{S}_{12}\rangle\cdot|\widehat{\mathcal{O}}_\mu\rangle_3 = 0\,, \end{aligned} \tag{6.6}$$

since $(Q^{(1)}+Q^{(2)})|\mathcal{S}_{12}\rangle = 0$, and $Q|\widehat{\mathcal{O}}_\mu\rangle = 0$. Path independence allows us to replace in (6.5) the problematic path $(\mathcal{B}_{0,3}+\widetilde{\mathcal{B}}_{0,3})$ by a new path $\mathcal{B}'_{0,3}$ from $\mathcal{V}_{0,3}$ to $\widetilde{\mathcal{V}}_{0,3}$ which avoids $\mathcal{V}'_{0,3}$

$$\partial\mathcal{B}'_{0,3} = \widetilde{\mathcal{V}}_{0,3} - \mathcal{V}_{0,3}\,. \tag{6.7}$$

We choose this path such that for every surface in this path, the Δ operator gives a regular torus. This is guaranteed if the coordinates curves of the symmetric punctures do not touch. Since this is the case both for $\mathcal{V}_{0,3}$ and $\widetilde{\mathcal{V}}_{0,3}$ there is clearly a homotopy satisfying this condition. This shows that the second term on the right hand side of Eqn.(6.5) is finite.[⋆] While this term is basis independent, the first term $\mathrm{str}(\widetilde{\Gamma}_\mu)$ is basis dependent, and can be made finite by an appropriate choice of basis. In particular, if the basis states are chosen such that they are parallel transported by the connection $\widetilde{\Gamma}_\mu$, then in this basis $\widetilde{\Gamma}_\mu$ vanishes, and hence $\mathrm{str}(\widetilde{\Gamma}_\mu) = 0$.[†] We shall not choose basis states at this stage. Equation (6.2), with the help of (6.5) can now be written as

$$\partial_\mu S_{1,0} - \int\limits_{\mathcal{V}_{1,1}+\Delta\mathcal{B}'_{0,3}} \langle \Omega^{(0)1,1} | \widehat{\mathcal{O}}_\mu \rangle = \tfrac{1}{2}\Big(\mathrm{str}\,(\,\widetilde{\Gamma}_\mu\,) \,-\, \partial_\mu \ln\rho \Big)\,. \tag{6.8}$$

We will solve this equation by requiring both

$$\partial_\mu S_{1,0} = \int\limits_{\mathcal{V}_{1,1}+\Delta\mathcal{B}'_{0,3}} \langle \Omega^{(0)1,1} | \widehat{\mathcal{O}}_\mu \rangle\,, \tag{6.9}$$

and,

$$\partial_\mu \ln\rho = \mathrm{str}(\widetilde{\Gamma}_\mu)\,. \tag{6.10}$$

We must now show that the above two equations can be solved for $S_{1,0}$ and $\ln\rho$.

6.3. Explicit Determination of $S_{1,0}$

Let us first discuss Eqn.(6.9). Since the position of the puncture in a torus is irrelevant, the set of surfaces $\mathcal{V}_{1,1}+\Delta\mathcal{B}'_{0,3}$, determines a set of surfaces $\mathcal{V}_{1,0} \subset \mathcal{M}_{1,0}$

⋆ Since our proof of path independence assumes finiteness of the quantities involved, we should interpret our construction as a way to make manifest a cancellation of infinite quantities (regularization). There should be a way to eliminate every reference to $\mathcal{V}'_{0,3}$ from the start.

† It is always possible to arrange that the basis states at $x+\delta x$ are related to the basis states at x via parallel transport by the connection $\widetilde{\Gamma}_\mu$ to first order in δx. This is all we need to set $\widetilde{\Gamma}_\mu = 0$ at x.

by the simple operation of deleting the puncture

$$\mathcal{V}_{1,0} \equiv \pi_F \left(\mathcal{V}_{1,1} + \Delta\mathcal{B}'_{0,3}\right), \tag{6.11}$$

where π_F denotes the operation of forgetting the puncture. We now claim that

$$S_{1,0} = \int\limits_{\mathcal{V}_{1,0}} \Omega^{(0)1,0}, \quad \text{with} \quad \Omega^{(0)1,0}_{\Sigma} \equiv \frac{d\tau_1 \wedge d\tau_2}{\tau_2}\, Z^G Z^M_x(\tau,\bar\tau), \tag{6.12}$$

where we have defined the volume two-form $\Omega^{(0)1,0}_{\Sigma}$ (on $T_\Sigma(\mathcal{M}_{1,0})$) where the surface Σ is a torus with modular parameter τ. Here $Z^G \equiv Tr_{\rm ghost}\big((-1)^{n_G} e^{2\pi i\tau L_0^G} e^{-2\pi i\bar\tau \bar L_0^G} b_0 c_0 \bar b_0 \bar c_0\big) = (\eta(\tau))^2(\overline{\eta(\tau)})^2$ is the ghost partition function (computed after removing the ghost zero modes on the torus), and $Z^M_x(\tau,\bar\tau)$ denotes the partition function of the matter CFT at some point x. It follows from the analysis of ref.[23] that

$$\partial_\mu Z^M_x = -\frac{\tau_2}{\pi}\langle\mathcal{O}_\mu\rangle^M_{g=1}, \quad \partial_\mu Z^G = 0, \tag{6.13}$$

where $\langle\mathcal{O}_\mu\rangle^M_{g=1}$ denotes the one point function of $\mathcal{O}_\mu$ on the torus in the matter CFT. In computing this correlation function, the local coordinate at the point of insertion of the marginal operator $\mathcal{O}_\mu$ on the torus must be taken to be the natural complex coordinate w on the torus, in terms of which the torus looks like a parallelogram with the identification $w \equiv w+1 \equiv w+\tau$. We can now compute $\partial_\mu S_{1,0}$ using (6.12) and (6.13). We immediately find

$$\partial_\mu S_{1,0} = -\frac{1}{\pi}\int\limits_{\mathcal{V}_{1,0}} d\tau_1\wedge d\tau_2\; Z^G\langle\mathcal{O}_\mu\rangle^M_{g=1}\,. \tag{6.14}$$

It will be shown in appendix B that

$$\frac{1}{\pi}\int\limits_{\mathcal{V}_{1,0}} d\tau_1\wedge d\tau_2\; Z^G\langle\mathcal{O}_\mu\rangle^M_{g=1} = -\int\limits_{\mathcal{V}_{1,1}+\Delta\mathcal{B}'_{0,3}} \langle\Omega^{(0)1,1}|\widehat{\mathcal{O}}_\mu\rangle\,. \tag{6.15}$$

The last two equations imply that our ansatz for $S_{1,0}$ (Eqn.(6.12)) satisfies the condition given in Eqn.(6.9). We point out that, as desired, $S_{1,0}$ is a geometrical basis independent object (a scalar on $\widehat{\mathcal{H}}$). We can get to understand what is the region of integration $\mathcal{V}_{1,0}$ of the genus one vacuum vertex defining $S_{1,0}$ by computing the boundary of $\mathcal{V}_{1,0}$. It follows from (6.11) that

$$\partial\mathcal{V}_{1,0} = \pi_F\,(\partial\mathcal{V}_{1,1} - \Delta\partial\mathcal{B}'_{0,3}) \;=\; -\,\pi_F\,\Delta\widetilde{\mathcal{V}}_{0,3}\,, \tag{6.16}$$

where use was made of Eqns.(2.19) and (6.7). Since $\widetilde{\mathcal{V}}_{0,3}$ has stubs of length a in the coordinate system in which the string has length 2π, this means that $S_{1,0}$ is given by the integral of the partition function over $\mathcal{M}_{1,0} - R$, where R, in the usual representation of $\mathcal{M}_{1,0}$ in the τ plane, is the region $-\frac{1}{2} \leq \tau_1 < \frac{1}{2}\,,\;\; \tau_2 > \frac{a}{\pi}$.

6.4. Determination of ρ

We must now discuss the equation $\partial_\mu\rho = \mathrm{str}(\widetilde{\Gamma}_\mu)$. This equation is sensible as both sides of the equality transform in the same way under a change of basis. It is also intuitively clear why the theory dependence of ρ should be affected by a connection implementing background independence. After expanding an SFT around a nontrivial background we get a theory that gives the same observables as the SFT formulated on a CFT corresponding to that background. Nevertheless to relate the expanded theory to the theory on the second background we need to perform, to first order, a change of basis encoded by a connection. Thus properties of this connection must arise in comparing the measures.

It would be desirable to give a direct proof that $\partial_\mu\rho = \mathrm{str}(\widetilde{\Gamma}_\mu)$ is an integrable equation. This would involve showing that $\mathrm{str}(\partial_\mu\widetilde{\Gamma}_\nu - \partial_\nu\widetilde{\Gamma}_\mu) = 0$. We are not quite able to do this at present. An argument that the equation is integrable can be given if we are cavalier about divergences. From Eqn.(6.3) we have that

$$\mathrm{str}(\,\widetilde{\Gamma}_\mu) = \mathrm{str}(\widehat{\Gamma}_\mu) - 2\int\limits_{\Delta\mathcal{B}_{0,3}} \langle\Omega^{(0)1,1}|\widehat{\mathcal{O}}_\mu\rangle\,. \tag{6.17}$$

Since the connection $\widehat{\Gamma}$ is flat, we can choose a basis in which it vanishes in a neighborhood of a point x_0. (There are of course problems with infinities here).

By the result of appendix B, the second term is given by $\left(-2\partial_\mu \int_{\pi_F(\Delta\mathcal{B}_{0,3})} \Omega^{(0)1,0}\right)$. Hence $\mathrm{str}(\widetilde{\Gamma}_\mu)$ is derivative of a quantity, making the integrability of the ρ equation clear (this quantity, however, is divergent due to the presence of singular tori in $\Delta\mathcal{B}_{0,3}$). This argument suggests that if we could regulate properly everything we should not have problems establishing integrability. What we will do next, however, is to find $\rho(x)$ directly from the condition of background independence as it applies to observables. The observable in question will be the free energy of string field theory.

Let SFT_x denote the closed string field theory formulated around CFT_x. Moreover, let $F_x(y)$ denote the free energy calculated from SFT_x expanded around a nontrivial classical solution representing CFT_y. In this notation $F_x(x)$ is simply the free energy calculated from SFT_x using the trivial background $|\Psi_x\rangle = 0$. The computation of $F_x(x)$ involves computing a one loop vacuum graph, and including appropriately the measure factor ρ and the constant $S_{1,0}$. We work in the Siegel gauge, which is fixed as follows. We split the full set of string fields that appear in the BV formalism into two groups. Let $\{|\Phi_r^{(n)}\rangle\}$ denote a set of basis states in $\widehat{\mathcal{H}}$ of ghost number n and annihilated by b_0^+, and $\{|\widetilde{\Phi}_r^{(n)}\rangle\}$ denote the set of basis states in $\widehat{\mathcal{H}}$ of ghost number n and annihilated by c_0^+, normalized in such a way that

$$\langle\omega_{12}\,|\Phi_s^{(m)}\rangle_{(1)}|\widetilde{\Phi}_r^{(5-n)}\rangle_{(2)} = \delta_{rs}\delta_{nm}\,. \tag{6.18}$$

(This choice of basis still leaves freedom to be exploited later.) If we denote by $\psi^r_{(n)}$ and $\widetilde{\psi}^r_{(n)}$ the coefficients of $\{|\Phi_r^{(n)}\rangle\}$ and $\{|\widetilde{\Phi}_r^{(n)}\rangle\}$ in the expansion of the string field, then with the choice of normalization we have given, the pair $\left(\psi^r_{(n)}\,,\widetilde{\psi}^r_{(5-n)}\right)$ is a properly normalized field/antifield pair. The original BV measure now can be expressed as $\rho(x)\prod_\alpha d\psi^\alpha d\widetilde{\psi}^\alpha$ where the index α runs over both r and n. The Lagrangian submanifold is now taken to be the surface $\widetilde{\psi}^\alpha = 0$. From Eqn.(3.8) we get $d\lambda = \sqrt{\rho(x)}\prod_\alpha d\psi^\alpha$. The free energy of SFT_x is now given by,

$$
\begin{aligned}
\exp\bigl(F_x(x)\bigr) &= \sqrt{\rho(x)}\,\prod_\alpha d\psi^\alpha \exp\left(\frac{1}{\hbar}S\left(\psi^\alpha, \widetilde{\psi}^\alpha = 0\right)\right), \\
&= \exp\Bigl(\tfrac{1}{2}\ln \mathrm{sdet} K(x) + \tfrac{1}{2}\ln\rho(x) + S_{1,0}(x) + \mathcal{O}(\hbar)\Bigr),
\end{aligned}
\tag{6.19}
$$

where K denotes the gauge fixed kinetic term matrix in the chosen basis for the string field. The first term on the last line of the above equation is the so called vacuum graph.

As has already been pointed out before, Eqn.(6.10) leaves an overall multiplicative constant in ρ undetermined. To fix this constant we must specify a value for ρ at some point x_0 in the CFT theory space. We will choose $\rho(x_0)$ in such a way that the free energy $F_{x_0}(x_0)$ of the string field theory SFT_{x_0} is equal to the partition function of CFT_{x_0} integrated over $\mathcal{M}_{1,0}$

$$
F_{x_0}(x_0) = \int\limits_{\mathcal{M}_{1,0}} Z_{CFT_{x_0}}\,. \tag{6.20}
$$

In view of the last two equations, we have that

$$
\ln\rho(x_0) = \Bigl(2\int\limits_{\mathcal{M}_{1,0}} Z_{CFT_{x_0}} - \ln \mathrm{sdet} K(x_0) - 2S_{1,0}(x_0)\Bigr)\,. \tag{6.21}
$$

The quantity $\ln \mathrm{sdet} K(x_0)$ can be computed for a given choice of basis, and $S_{1,0}(x_0)$ is given by Eqn.(6.12). Thus we can use Eqn.(6.21) to fix $\rho(x_0)$. Later we shall see how to choose the basis states so that $\rho(x_0)$ (and also $\rho(x)$) turns out to be finite.

It was proven in Ref.[23], that Eqn.(6.20) guarantees that the free energy calculated with SFT_{x_0}, expanded around a classical solution representing CFT_x, gives the integral of the partition function of CFT_x:

$$
F_{x_0}(x) = \int\limits_{\mathcal{M}_{1,0}} Z_{CFT_x}\,. \tag{6.22}
$$

This analysis rests on two basic results, one giving the variation of the partition function when we deform a CFT using a marginal operator, and the other being the geometrical recursion relation $\partial\mathcal{V}_{1,1} = -\Delta\mathcal{V}_{0,3}$.

Background independence implies that the physics of SFT_x can be equally obtained from SFT_{x_0} expanded around a classical solution representing CFT_x. Since free energy is an observable in string field theory, we must have that $F_x(x) = F_{x_0}(x)$, and therefore, in view of the above equation

$$F_x(x) = \int\limits_{\mathcal{M}_{1,0}} Z_{CFT_x}\,. \tag{6.23}$$

Thus background independence demands that not only at x_0, but for any x in a neighborhood of x_0, $\rho(x)$ must be chosen so that $F_x(x)$ calculated using Eqn.(6.19) equals the right hand side of the above equation. Thus we have to choose

$$\rho(x) = \widehat{\rho}(x) \equiv \exp\Big[\, 2\Big(\int\limits_{\mathcal{M}_{1,0}} Z_{CFT_x} - S_{1,0}(x) - \tfrac{1}{2}\ln \mathrm{sdet} K(x)\Big)\Big]\,. \tag{6.24}$$

For a given choice of basis states around x_0, $K(x)$ is fixed, and hence the above equation determines $\rho(x)$ in a finite neighborhood of x_0 uniquely.[⋆]

We now argue that the above choice of ρ is also sufficient to ensure background independence, in particular it must satisfy the required equation $\partial_\mu \ln\widehat{\rho} = \mathrm{str}(\widetilde{\Gamma}_\mu)$. Let x be any point in the CFT theory space, and n^μ be an arbitrary tangent vector at x. Then we need to prove that $n^\mu(\partial_\mu \ln\widehat{\rho} - \mathrm{str}(\widetilde{\Gamma}_\mu)) = 0$. To this end, let us introduce a curve C through x whose tangent vector at x is proportional to n^μ. Let t parametrize the curve, such that $x(t=0) = x$. We need to prove that

$$\big(\partial_t \ln\widehat{\rho}(x(t)) - \mathrm{str}(\widetilde{\Gamma}_t)\big)\big|_{t=0} = 0\,. \tag{6.25}$$

The choice $\rho(x) = \widehat{\rho}(x)$ guarantees that $F_{x(t)}(x(t)) = \int_{\mathcal{M}_{1,0}} Z_{CFT_{x(t)}}$. On the other hand, the result of Ref.[23] together with the fact that $F_x(x) = \int_{\mathcal{M}_{1,0}} Z_{CFT_x}$,

⋆ The one loop vacuum graph and the integral of CFT partition function over $\mathcal{M}_{1,0}$, have tachyon divergences. These divergences cancel out and do not affect the determination of ρ.

guarantees that $F_x(x(t)) = \int_{\mathcal{M}_{1,0}} Z_{CFT_{x(t)}}$ at least to first order in t. This gives,

$$\partial_t \big(F_{x(t)}(x(t)) - F_x(x(t)) \big)\big|_{t=0} = 0\,. \tag{6.26}$$

Let us now assume that $\widehat{\rho}(x)$ does not satisfy Eqn.(6.25). We define,

$$\epsilon(t) = \partial_t \ln \widehat{\rho}(x(t)) - \mathrm{str}(\widetilde{\Gamma}_t)\,. \tag{6.27}$$

Let us also define $\widetilde{\rho}(t)$ to be a solution of the equation

$$\partial_t \ln \widetilde{\rho}(t) - \mathrm{str}(\widetilde{\Gamma}_t) = 0\,, \quad \widetilde{\rho}(t=0) = \rho(x)\,, \tag{6.28}$$

along the curve C. Note that $\widetilde{\rho}(t)$ is defined only along the curve $C(t)$, and hence we are not assuming the integrability of the general set of equations (6.10). In fact, we shall need to assume the existence of $\widetilde{\rho}(t)$ only to first order in t. If we choose ρ to be equal to $\widetilde{\rho}(t)$ along the curve C, then by Eqn.(6.10) the theory is background independent along this curve. Let $\widetilde{F}_t(t)$ denote the free energy of $\mathrm{SFT}_{x(t)}$ around the trivial background, calculated with this choice of ρ. We then have

$$\partial_t \big(\widetilde{F}_t(t) - F_x(x(t)) \big|_{t=0} = 0\,. \tag{6.29}$$

(With a slight abuse of notation, we are using t to denote the point $x(t)$, as arguments of functions that are defined only on C.) Eqs.(6.26) and (6.29) give,

$$\partial_t \big(F_{x(t)}(x(t)) - \widetilde{F}_t(t) \big)\big|_{t=0} = 0\,. \tag{6.30}$$

We now note that the computation of $\widetilde{F}_t(t)$ and that of $F_{x(t)}(x(t))$ differ from each other only in the choice of ρ. Furthermore $\frac{1}{2} \ln \rho$ appears as an additive term in the expression for $F_x(x)$. Thus we have

$$\partial_t \big(F_{x(t)}(x(t)) - \widetilde{F}_t(t) \big)\big|_{t=0} = \tfrac{1}{2}\, \partial_t \big(\ln \widehat{\rho}(x(t)) - \ln \widetilde{\rho}(t) \big)\big|_{t=0} = \tfrac{1}{2}\, \epsilon(0)\,, \tag{6.31}$$

where in the last step we have made use of Eqns.(6.27) and (6.28). Comparing Eqns.(6.30) and (6.31) we see that $\epsilon(0)$ must vanish. This, together with the definition of $\epsilon(t)$ given in Eqn.(6.27) proves Eqn.(6.25).

Finiteness of ρ. Let $\{|\Phi_i\rangle\}$ denote a basis for $\widehat{\mathcal{H}}$ comprised at each ghost number n by sets of basis vectors $\{|\Phi_r^{(n)}\rangle\}$, and $\{|\widetilde{\Phi}_r^{(n)}\rangle\}$, annihilated by b_0^+ and c_0^+ respectively, and satisfying the normalization condition (6.18). In the subspace of $\widehat{\mathcal{H}}$ defined by the states annihilated by b_0^+, the bra $\langle G_{12}| \equiv \langle R'_{12}|(c_0^- c_0^+)^{(2)}$ provides a canonical, nondegenerate, symmetric metric pairing states whose ghost numbers add up to four. The basis $\{|\Phi_i\rangle\}$ will be said to be of standard normalization if we further have that $\langle G_{12}|\Phi_r^{(n)}\rangle|\Phi_s^{(m)}\rangle = \delta_{rs}\delta_{m+n-4}$. If we expand the string field in this basis $|\Psi\rangle = \sum_i |\Phi_i\rangle\psi^i$, and choose the string field theory measure to be $\rho(x)\prod_i d\psi^i$, then the propagator in the $b_0^+ = 0$ gauge is proportional to $(L_0^+)^{-1}$, and the corresponding one loop vacuum graph is divergent.

Indeed, with the conventional kinetic term, the corresponding vacuum graph is proportional to $\mathrm{str}(\ln L_0^+)$ in the subspace of states annihilated by b_0 and $\bar{b}_0$. This quantity is manipulated using the relation

$$\ln(L_0^+) = -\int_0^\infty \frac{d\alpha}{\alpha} e^{-\alpha L_0^+}\,, \tag{6.32}$$

which holds up to an infinite constant that is immaterial to the computation of the supertrace. We will adjust the basis so that the resulting kinetic term will be of the form $\frac{1}{g(L_0^+)} L_0^+$ with g chosen so that

$$\ln\Big[\frac{1}{g(L_0^+)} L_0^+\Big] = -\int_{2a}^\infty \frac{d\alpha}{\alpha} e^{-\alpha L_0^+}\,. \tag{6.33}$$

The right hand side of this equation, upon taking the supertrace, leads to the picture of a vacuum graph built including tori of length parameter greater than or equal to $2a$ (when the circumference is 2π). The function g necessary to have (6.33) is given by

$$g(w) = w\,\exp\Big(\int_{2a}^\infty \frac{d\alpha}{\alpha} e^{-\alpha w}\Big)\,, \tag{6.34}$$

as can be verified by direct substitution.[⋆] We therefore introduce the basis states $\{|\Phi'_i\rangle\}$ related to the properly normalized basis $|\Phi_i\rangle$ through the relation

$$|\Phi'_i\rangle = (g(L_0^+))^{G_i - 5/2}|\Phi_i\rangle\,, \tag{6.35}$$

where G_i denotes the ghost number carried by the state $|\Phi_i\rangle$. The particular form of the scaling was chosen so that two fields with the same L_0^+ eigenvalue and having ghost numbers that add up to five would scale in precisely opposite ways. Since these are the states coupled by the symplectic form, such form will remain invariant under this transformation. Indeed, it is straightforward to verify that $\omega'_{ij} \equiv (-)\langle\omega_{12}|\Phi'_i\rangle_{(1)}|\Phi'_j\rangle_{(2)} = (-)\langle\omega_{12}|\Phi_i\rangle_{(1)}|\Phi_j\rangle_{(2)} \equiv \omega_{ij}$. We now expand the string field as $|\Psi\rangle = \sum_i |\Phi'_i\rangle\psi'^i$ and take the string field theory measure to be of the form $\rho'(x)\prod_i d\psi'^i$. The kinetic term now takes the form

$$(-)^{G_i}\langle R'_{12}|(g(L_0^+))^{-1}(c_0^- Q)^{(2)}|\Phi_i\rangle|\Phi_j\rangle\,\psi'^i\psi'^j\,, \tag{6.36}$$

and, upon gauge fixing in the $b_0^+ = 0$ gauge, the kinetic operator in this primed basis is given by $(g(L_0^+))^{-1}L_0^+$, as desired. The vacuum graph calculated from this will be finite (up to tachyon divergence), and will therefore lead to a finite value for ρ in the primed basis through the use of Eqn.(6.24).

⋆ The function $g(w)$ is well defined for $w \geq 0$ (with a finite limit as $w \to 0$), but must be defined by analytic continuation for $w < 0$ (as is always the case when tachyons are present). The analytic continuation can be done as follows. Let us define $\phi(w) = \int_{2a}^{\infty}\frac{d\alpha}{\alpha}e^{-\alpha w}$. Then $\frac{d\phi(w)}{dw} = -\frac{e^{-2aw}}{w}$, and we may write $\phi(w) = -\int_{\infty}^{w}\frac{e^{-2az}}{z}dz$. For $w > 0$, the contour of integration can be taken along the real line. For $w < 0$, we take the contour from ∞ to w in the upper / lower half plane, avoiding the origin. The difference between taking the contour in the upper and lower half plane is given by $\int_{C_0}\frac{e^{-2az}}{z}dz = 2\pi i$, where C_0 is a contour around the origin. This gives rise to an additive ambiguity of $2\pi i$ in the definition of $\phi(w)$. But $g(w) = we^{\phi(w)}$ is well defined. This shows that $g(w)$ has a well defined analytic continuation to $w < 0$.

7. A Batalin Vilkovisky Algebra on Spaces of Riemann Surfaces

We have introduced in §2 the antibracket and Δ operators on a complex $\widehat{\mathcal{P}}$ spanned by symmetric subspaces of moduli spaces. In the present section we will enlarge the complex, and introduce a graded commutative and associative dot product. A suitably generalized Δ operator in this new complex will be seen to give the structure of a BV algebra. We will be able to give a succint formulation of the recursion relations for string vertices in terms of the cohomology of an odd operator $\delta \equiv \partial + \hbar\Delta$ satisfying $\delta^2 = 0$. The recursion relations for the $\mathcal{B}$ spaces implementing background independence can also be written briefly and clearly in terms of a nilpotent operator $\delta_{\mathcal{V}} \equiv \delta + \{\mathcal{V}, \}$.

7.1. Dot Product

It is well known that in the space of functions on a supermanifold, the antibracket is an object that can be derived if we are provided with a second order nilpotent derivation Δ and a graded-commutative associative product $(\cdot)$ [24]. Given two functions A and B one has

$$(-)^A\{A, B\} = \Delta(A \cdot B) - (\Delta A) \cdot B - (-)^A A \cdot (\Delta B) \ . \tag{7.1}$$

In addition the antibracket acts as a derivation of the dot product, namely

$$\{A, B \cdot C\} = \{A, B\} \cdot C + (-)^{(A+1)B} B \cdot \{A, C\} \, . \tag{7.2}$$

Since we have found analogs of the antibracket and the delta operator defined on symmetric subspaces of moduli spaces, one can ask if there is an analog for the dot product $(\cdot)$ leading to identities of the above form. Such object exists, but we have to generalize a bit our definition of a space of surfaces by including surfaces that can have disconnected components. Here we shall only sketch the construction of $(\cdot)$ leaving some details for a future publication[25]. Let us denote by

$\widehat{\mathcal{P}}_{(g_1,n_1)\ldots(g_r,n_r)}$ the space of surfaces with r disconnected components, with genera $g_1,\ldots g_r$ respectively, and carrying $n_1,\ldots n_r$ punctures equipped with local coordinates up to phases. Then $\widehat{\mathcal{P}}_{(g_1,n_1)\ldots(g_r,n_r)} \equiv \widehat{\mathcal{P}}_{g_1,n_1} \otimes \ldots \otimes \widehat{\mathcal{P}}_{g_r,n_r}$. We denote by $(\Sigma_1,\ldots \Sigma_r)$ a specific point in this space where $\Sigma_i \in \widehat{\mathcal{P}}_{g_i,n_i}$. Similarly, if $\mathcal{A}_i \subset \widehat{\mathcal{P}}_{g_i,n_i}$, we denote by $(\mathcal{A}_1,\ldots \mathcal{A}_r)$ the subspace $\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_r$ of $\widehat{\mathcal{P}}_{(g_1,n_1)\ldots(g_r,n_r)}$. If $\{\mathcal{A}_i\}$ denote the tangent vectors in $\widehat{\mathcal{P}}_{(g_1,n_1)\ldots(g_r,n_r)}$ induced by the tangent vectors $[\mathcal{A}_i]$ of $\mathcal{A}_i \subset \widehat{\mathcal{P}}_{g_i,n_i}$, then the orientation of $(\mathcal{A}_1,\ldots \mathcal{A}_r)$ is taken to be $[\{\mathcal{A}_1\},\ldots\{\mathcal{A}_r\}]$. From this it is clear that under the exchange of $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$, the orientation of $(\mathcal{A}_1,\ldots \mathcal{A}_r)$ picks up a factor of $(-)^{\mathcal{A}_i\mathcal{A}_{i+1}}$. Let us denote by $\widehat{\mathcal{C}}$ the full complex $\oplus\widehat{\mathcal{P}}_{(g_1,n_1)\ldots(g_r,n_r)}$. Thus $\widehat{\mathcal{C}}$ is the generalization of the complex $\widehat{\mathcal{P}}$ for surfaces with disconnected components. The operation of addition has a natural definition in this complex, and satisfies $(\mathcal{A},\mathcal{B})+(\mathcal{A},\mathcal{C}) = (\mathcal{A},\mathcal{B}+\mathcal{C})$. From the direct product structure of $\widehat{\mathcal{P}}_{(g_1,n_1)\ldots(g_r,n_r)}$ it is clear that $(\mathcal{A}_1,\ldots \mathcal{A}_n)$ is a zero element of the complex if any of the $\mathcal{A}_i$ is an empty set.

We now define $\mathcal{A}_1 \circ \mathcal{A}_2 \equiv (\mathcal{A}_1,\mathcal{A}_2)$, and, more generally,

$$(\mathcal{A}_1,\ldots,\mathcal{A}_n)\circ(\mathcal{A}_{n+1},\ldots\mathcal{A}_{n+m}) \equiv (\mathcal{A}_1,\ldots,\mathcal{A}_{n+m})\,. \tag{7.3}$$

The $\circ$ operation is bilinear in its argument, and is associative and graded commutative. However, note that even if the original elements $\mathcal{B}_1$ and $\mathcal{B}_2$ of the complex are symmetric in the external punctures, $\mathcal{B}_1 \circ \mathcal{B}_2$ is not. Given two symmetric elements $\mathcal{B}_1,\mathcal{B}_2 \in \widehat{\mathcal{C}}$, we now introduce the symmetric product:

$$\mathcal{B}_1 \cdot \mathcal{B}_2 \equiv \mathbf{S}(\mathcal{B}_1 \circ \mathcal{B}_2) \tag{7.4}$$

where $\mathbf{S}$ is the same symmetrization operator that appeared in Eqn.(2.5). This is the graded commutative associative product in the complex.

The Δ and the anti-bracket operations are defined on spaces of surfaces with disconnected components in a manner identical to the corresponding operations on spaces of connected surfaces. In particular, the action of Δ can result in two possibilities; the two punctures may lie on the same connected component of the surface, or they may lie on two different components of the surface which are disconnected. It is then clear that if $\mathcal{B}_1$ and $\mathcal{B}_2$ denote two spaces of surfaces (containing surfaces with disconnected components in general), then the object

$$\Delta(\mathcal{B}_1 \cdot \mathcal{B}_2) - (\Delta\mathcal{B}_1) \cdot \mathcal{B}_2 - (-)^{\mathcal{B}_1}\mathcal{B}_1 \cdot (\Delta\mathcal{B}_2) \ , \tag{7.5}$$

only contains the surfaces that arise when one surface of $\mathcal{B}_1$ is sewn to another surface in $\mathcal{B}_2$, and therefore is clearly related to $\{B_1, \mathcal{B}_2\}$, as would be expected from (7.1). In relating (7.5) to $\{\mathcal{B}_1, \mathcal{B}_2\}$ we need to move the tangent vector associated with the sewing parameter through the tangent vectors of $\mathcal{B}_1$, thereby giving rise to the factor of $(-)^{\mathcal{B}_1}$, as is expected from the left hand side of Eqn.(7.1). It is also clear that for the spaces of surfaces (with disconnected components) the analog of (7.2) holds. The algebra of the operations $\{\ ,\ \}$, Δ and $(\cdot)$ on $\widehat{\mathcal{C}}$ therefore satisfy the axioms of a BV algebra [26,27,28].

7.2. Representation of the Dot Product On the Space of Functions

In §3 we have shown that the Δ and the $\{,\}$ operations on the moduli space induce the corresponding opearations in the function space. It is natural to ask if the same holds for the dot product. To test this, we first need to extend the definition of f to moduli space of disconnected Riemann surfaces. Given a subspace $(\mathcal{A}^{(k_1)}_{g_1,n_1}, \dots \mathcal{A}^{(k_r)}_{g_r,n_r})$ of $\widehat{\mathcal{P}}_{(g_1,n_1)\dots(g_r,n_r)}$, we define

$$f\Big((\mathcal{A}^{(k_1)}_{g_1,n_1}, \dots \mathcal{A}^{(k_r)}_{g_r,n_r})\Big) = \frac{1}{\Big(\sum_{i=1}^r n_i\Big)!}\prod_{i=1}^r \Big(\int\limits_{\mathcal{A}^{(k_i)}_{g_i,n_i}} \langle \Omega^{(k_i)g_i,n_i}|\Psi\rangle_1 \cdots |\Psi\rangle_{n_i}\Big) \tag{7.6}$$

With the help of this equation, and the definition of the symmetrized product $(\cdot)$, it is easy to verify that for any two elements $\mathcal{B}_1$, $\mathcal{B}_2$ of the complex $\widehat{\mathcal{C}}$, we have

$$f(\mathcal{B}_1 \cdot \mathcal{B}_2) = f(\mathcal{B}_1) \cdot f(\mathcal{B}_2) \tag{7.7}$$

Using this identity, and the corresponding identities involving Δ and $\{,\}$, we see that the BV algebra of Δ, $\{,\}$ and $(\cdot)$ on the space of functions is an immediate consequence of the BV algebra of the corresponding operations in the complex $\widehat{\mathcal{C}}$.

7.3. Recursion relations as cohomology conditions

We shall define the action of the boundary operator ∂ on a space of surfaces with disconnected components in a manner identical to the corresponding operation on the space of connected surfaces. This gives

$$\partial(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n) \equiv (\partial\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n) + (-)^{\mathcal{A}_1} (\mathcal{A}_1 , \partial\mathcal{A}_2, \cdots, \mathcal{A}_n) + \cdots , \quad (7.8)$$

and it follows from this definition that ∂ is a derivation of the dot product

$$\partial\,(\mathcal{A}_1 \cdot \mathcal{A}_2) = (\partial\mathcal{A}_1 \cdot \mathcal{A}_2) + (-)^{\mathcal{A}_1} (\mathcal{A}_1 \cdot \partial\mathcal{A}_2)\,. \quad (7.9)$$

We can also define the exponential function of an even element $\mathcal{A} \in \widehat{\mathcal{C}}$ by the usual power series

$$\exp(\mathcal{A}) \equiv 1 + \mathcal{A} + \frac{1}{2}\mathcal{A}\cdot\mathcal{A} + \frac{1}{3!}\mathcal{A}\cdot\mathcal{A}\cdot\mathcal{A} + \cdots\,. \quad (7.10)$$

The object $\exp(\mathcal{A})$ contains disconnected surfaces even when $\mathcal{A}$ is a basic space of surfaces. It follows from the last two equations that

$$\partial\left[\exp(\mathcal{A})\right] = \partial\mathcal{A}\cdot\exp(\mathcal{A})\,, \quad (7.11)$$

and moreover, using Eqn.(7.1) we find

$$\Delta\exp(\mathcal{A}) = \big(\Delta\mathcal{A} + \tfrac{1}{2}\{\mathcal{A},\mathcal{A}\}\big)\exp(\mathcal{A})\,. \quad (7.12)$$

Let us now introduce the odd operator δ defined as follows

$$\delta \equiv \partial + \hbar\Delta\,. \quad (7.13)$$

From the last three equations we get,

$$\delta\exp(\mathcal{A}) = \big(\partial + \hbar\Delta\big)\exp(\mathcal{A}) = \big(\partial\mathcal{A} + \hbar\Delta\mathcal{A} + \tfrac{1}{2}\hbar\{\mathcal{A},\mathcal{A}\}\big)\exp(\mathcal{A})\,. \quad (7.14)$$

Making use of Eqn.(7.14) we see that we can now write the recursion relations

(2.22) in the simple form

$$\delta \exp(\mathcal{V}/\hbar) = 0\,. \tag{7.15}$$

It is interesting that δ is actually nilpotent,

$$\delta^2 = \big(\partial + \hbar\Delta\big)^2 = \partial^2 + \hbar(\partial\Delta + \Delta\partial) + \hbar^2\Delta^2 = 0\,. \tag{7.16}$$

Since $\mathcal{V}$ begins as a zero dimensional space $\mathcal{V}_{0,3}$, it is fairly clear that $\exp(\mathcal{V}/\hbar)$ is not δ trivial. A consistent set of closed string vertices therefore define a cohomology class of δ in the vector space spanned by subspaces of moduli spaces of Riemann surfaces with disconnected components. It can be shown that the change in $\mathcal{V}$ induced by a change of the cell decomposition in the moduli space [6] gives rise to a change in $exp(\mathcal{V}/\hbar)$ that is δ trivial [25]. Finally, note that if we define the operator:

$$\delta_{\mathcal{V}} \equiv \delta + \{\mathcal{V},\,\cdot\,\}\,, \tag{7.17}$$

then

$$(\delta_{\mathcal{V}})^2 = \Big\{\partial\mathcal{V} + \hbar\Delta\mathcal{V} + \tfrac{1}{2}\{\mathcal{V},\mathcal{V}\}\,,\,\cdot\Big\}\,. \tag{7.18}$$

Thus we see that the recursion relations may also be interpreted as the criteria of nilpotence of the operator $\delta_{\mathcal{V}}$.

7.4. Recursion Relations for $\mathcal{B}$ spaces

We want to observe that the language developed in the previous subsections enables us to obtain a better understanding of the recursion relations that were used to give an inductive construction of the $\mathcal{B}$ spaces. We had in Eqn.(5.1) that

$$\partial\mathcal{B} \simeq \mathcal{V}'_{0,3} + \widehat{\mathcal{K}}\mathcal{V} - \hbar\Delta\mathcal{B} - \{\mathcal{V},\mathcal{B}\} - \underline{\mathcal{V}}\,. \tag{7.19}$$

Using the definition of $\delta_{\mathcal{V}}$ in (7.17) we can now rewrite the above relation as

$$\delta_{\mathcal{V}}\,\mathcal{B} \simeq \mathcal{V}'_{0,3} + \widehat{\mathcal{K}}\mathcal{V} - \underline{\mathcal{V}}\,. \tag{7.20}$$

This is a simple equation telling us essentially, that $\widehat{\mathcal{K}}\mathcal{V} - \underline{\mathcal{V}}$ is $\delta_{\mathcal{V}}$ trivial. The object

$\widehat{\mathcal{K}}\mathcal{V} - \underline{\mathcal{V}}$ represents, for any fixed genus and number of punctures, the difference between the string vertex and the object obtained by adding one more puncture via $\widehat{\mathcal{K}}$ to a vertex having one less puncture.

We verify in a straightforward way the obvious consistency condition

$$\delta_{\mathcal{V}}^2 \mathcal{B} \simeq \delta_{\mathcal{V}} \mathcal{V}'_{0,3} + \delta_{\mathcal{V}} \widehat{\mathcal{K}}\mathcal{V} - \delta_{\mathcal{V}} \underline{\mathcal{V}} \simeq 0 \,, \tag{7.21}$$

which follows from a little calculation using $\delta_{\mathcal{V}} \underline{\mathcal{V}} = 0$, and Eqn.(2.25).

8. Conclusion

In this paper we have set up the criteria for background independence of the full quantum closed string field theory, or, in fact, the criteria for background independence of any quantum theory formulated in the BV formalism. We then proved that closed string field theory is a locally background independent quantum theory. More precisely, we have shown that there is a symplectic diffeomorphism that maps the appropriate action weighted measure of the string field theory formulated around a particular conformal field theory, to the action weighted measure of the string field theory formulated around a nearby conformal field theory (a theory related to the original conformal theory via an infinitesimal marginal deformation).

While we have carried out our analysis in the context of closed string field theory, the case of Witten's open string field theory can be studied similarly. If we ignore subtle issues of regularization that seem not to have been completely resolved [29], quantum background independence would be established by the symplectic diffeomorphism constructed in §9 of [2]. This can be seen by noting that i) subtleties aside, for Witten's open string theory the classical master action coincides with the quantum master action, ii) there is a symplectic field redefinition that relates classical master actions obtained from nearby backgrounds [2], and iii) this field redefinition is linear in the fields, so that it leaves the path integral measure $d\mu$ invariant up to a constant multiplicative factor. This extra multiplicative factor, which may be infinite, and the subtleties alluded to above, have their origin in the appearance of closed strings in open string field theory. A physical way to regulate everything is to work in the context of open-closed covariant string field theory [30]. Here the open string sector is only homotopy associative, and closed string field interactions must be added in order to satisfy, in a manifestly finite way, the master equation. It may be of interest to apply our methods to this theory.

Finally, our analysis of background independence has been restricted to infinitesimal deformations of conformal field theories. Given two conformal field theories a finite distance apart, and string field theories formulated around them, one may be able to integrate the field redefinition for infinitesimal deformations to find out the finite field redefinition. As discussed in ref.[2], we may, in principle, encounter infinities during the process of integration. We must prove the absence of infinities, and while there are reasons that suggest this could be the case [2], a detailed analysis seems important. This is possibly the most relevant open question in our whole analysis of background independence. A proper answer may require developing efficient techniques to deal with spaces of conformal theories. Of course, on a more fundamental level, we should focus on how to use the insights obtained proving background independence in order to construct a manifestly background independent formulation of string field theory. Perhaps most striking was the finding of a BV algebra at the level of Riemann surfaces, and a natural map (a homomorphism) to the BV algebra of string functionals. This development could open new directions for investigation.

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APPENDIX A: Proof of Eqn.(3.28).

Consider an arbitrary surface $\Sigma \in \mathcal{M}_{g,0}$. It follows from (3.16) that

$$\begin{aligned}\partial_\mu \Omega_\Sigma^{g,0} &= (-2\pi i)\, D_\mu(\,\widehat{\Gamma}\,)\, \langle \Omega_{\widehat{\Sigma}}^{(-2)g,1}|0\rangle \\ &= (-2\pi i)\Big[\big(\, D_\mu(\,\widehat{\Gamma}\,)\, \langle \Omega_{\widehat{\Sigma}}^{(-2)g,1}|\big)|0\rangle + \langle \Omega_{\widehat{\Sigma}}^{(-2)g,1}|\, \big(D_\mu(\,\widehat{\Gamma}\,)\,|0\rangle\big)\Big]\,.\end{aligned} \tag{A.1}$$

The first term in the right hand side can be written as

$$\big(\, D_\mu(\,\widehat{\Gamma}\,)\, \langle \Omega_{\widehat{\Sigma}}^{(-2)g,1}|\big)|0\rangle = \int\limits_{\widehat{\mathcal{K}}(\widehat{\Sigma})} \langle \Omega^{(-2)g,2}|\widehat{\mathcal{O}}_\mu\rangle|0\rangle\,, \tag{A.2}$$

where we have used (3.25). In this right hand side the vacuum state is to be contracted into the original puncture of $\widehat{\Sigma}$, and $\widehat{\mathcal{K}}(\widehat{\Sigma})$, as usual, denotes the set of surfaces built by introducing an extra puncture anywhere outside the unit disk around the only puncture of $\widehat{\Sigma}$. In order to simplify this we now argue that in general the vacuum state deletes punctures not only for surfaces, but also for the case of forms

$$\langle \Omega_\Sigma^{(k)g,n+1}|0\rangle \,=\, (-2\pi i)\, \langle \Omega_{\pi(\Sigma)}^{(k+2)g,n}| \tag{A.3}$$

where Σ is a genus g surface with $n+1$ punctures, the vacuum state refers to one of the punctures, and π denotes the operation of forgetting that puncture. As an equality of forms it holds when on the left hand side we act on a set of tangent vectors that deform Σ while on the right hand side we act on the set of tangent vectors representing the deformations of $\pi(\Sigma)$ induced by the first set of vectors, as we forget about the chosen puncture. This relation is proved as follows. The left hand side is given by

$$\langle \Omega_\Sigma^{(k)g,n+1}|0\rangle(\widehat{V}_1,\cdots \widehat{V}_p) = N_{g,n+1}\langle \Sigma\,|\mathbf{b}(\mathbf{v}_1)\cdots \mathbf{b}(\mathbf{v}_p)\,|0\rangle\,, \tag{A.4}$$

where $N_{g,n} = (-2\pi i)^{3-n-3g}$, the $\mathbf{v}$'s are Schiffer vectors, and $p = k + \dim(\mathcal{M}_{g,n+1})$. As reviewed in [1], all deformations of the underlying unpunctured Riemann surfaces can be performed with Schiffer variations around any puncture. We therefore choose a puncture different from the special one to represent such tangents. All deformations having to do with moving punctures and changing local coordinates must use Schiffer vectors based at those punctures. Therefore, we can arrange so that in (A.4) any antighost insertion referring to the special puncture only moves it or deforms its local coordinate. Since the vacuum is annihilated by $b_{-1}, b_0, b_1 \cdots$ (and the antiholomorphic analogs), all such antighost insertions dissappear. We denote the leftover Schiffer vectors by $\pi(\mathbf{v})$. We can now push the vacuum state without obstruction all the way into the surface state $\langle\Sigma|$ to find

$$\begin{aligned} \langle\Omega^{(k)g,n+1}_{\Sigma}|0\rangle(\widehat{V}_1,\cdots\widehat{V}_p) &= N_{g,n+1}\langle\Sigma\,|0\rangle\mathbf{b}(\pi(\mathbf{v}_1))\cdots\mathbf{b}(\pi(\mathbf{v}_p))\ , \\ &= (-2\pi i)^{-1}\, N_{g,n}\,\langle\pi(\Sigma)\,|\mathbf{b}(\pi(\mathbf{v}_1))\cdots\mathbf{b}(\pi(\mathbf{v}_p)) \\ &= (-2\pi i)^{-1}\,\langle\Omega^{(k+2)g,n}_{\pi(\Sigma)}\,|\,(\,\pi(\widehat{V}_1),\cdots\pi(\widehat{V}_p)\,)\ , \end{aligned} \tag{A.5}$$

where we made use of $\langle\Sigma|0\rangle = \langle\pi(\Sigma)|$. The final equality is the equality we wanted to establish (Eqn.(A.3)).

Back to Eqn. (A.2) we now have

$$\big(\,D_\mu(\,\widehat{\Gamma}\,)\,\langle\Omega^{(-2)g,1}_{\widehat{\Sigma}}|\big)|0\rangle \,=\, (-2\pi i)^{-1}\int\limits_{\pi\left(\widehat{\mathcal{K}}(\widehat{\Sigma})\right)}\langle\Omega^{(0)g,1}|\widehat{\mathcal{O}}_\mu\rangle\,. \tag{A.6}$$

The region of integration $\pi\big(\widehat{\mathcal{K}}(\widehat{\Sigma})\big)$ may be simplified as follows. We are integrating over the space of two-punctured surfaces $\widehat{\mathcal{K}}(\widehat{\Sigma})$, but with the original puncture (defined in $\widehat{\Sigma}$) deleted. This space is nothing else than $\widehat{\mathcal{K}}\big(\pi(\widehat{\Sigma})\big) = \widehat{\mathcal{K}}(\Sigma)$, minus the space of one-punctured surfaces $D(\widehat{\Sigma})$ comprising the set of surfaces Σ with one puncture anywhere in the (would be) unit disk around the original puncture of of $\widehat{\Sigma}$. This gives

$$\pi\big(\widehat{\mathcal{K}}(\widehat{\Sigma})\big) = \widehat{\mathcal{K}}(\Sigma) - D(\widehat{\Sigma}) \tag{A.7}$$

Having evaluated the first term on the right hand side of Eqn.(A.1) we now claim that the second term is given by

$$\langle\Omega^{(-2)g,1}_{\widehat{\Sigma}}|\, \big(D_\mu(\,\widehat{\Gamma}\,)\,|0\rangle\big) \;=\; (-2\pi i)^{-1} \int\limits_{D(\widehat{\Sigma})} \langle\Omega^{(0)g,1}|\widehat{\mathcal{O}}_\mu\rangle\,. \tag{A.8}$$

If so, the last three equations, back in (A.1) give us

$$\partial_\mu\Omega^{g,0}_{\Sigma} = \int\limits_{\widehat{\mathcal{K}}(\Sigma)} \langle\Omega^{(0)g,1}|\widehat{\mathcal{O}}_\mu\rangle\,, \tag{A.9}$$

and, as a consequence

$$\partial_\mu \int\limits_{\mathcal{A}_{g,0}} \Omega^{(g,0)} = \int\limits_{\widehat{\mathcal{K}}\mathcal{A}_{g,0}} \langle\Omega^{(0)g,1}|\widehat{\mathcal{O}}_\mu\rangle\,. \tag{A.10}$$

This is precisely Eqn.(3.28). It only remains to justify (A.8). It is simple to see intuitively why this equation holds. The vacuum state $|0\rangle$ can be written as $|0\rangle = \langle 0|R\rangle$ where R the symmetric reflector satisfying $D_\mu(\widehat{\Gamma})|R\rangle = 0$, and where the bra vacuum $\langle 0|$ can be represented by a one punctured sphere with local coordinate $z_1(z) = z$ at $z = 0$ and with nothing elsewhere (in particular in $z = \infty$). Therefore, the unit disk around this coordinate covers 'half' the sphere, and in particular, when we differentiate with the $\widehat{\Gamma}$ connection, we get the integral of $\mathcal{O}_\mu$ over the other half of the sphere. When we sew back the result into the Riemann surface $\widehat{\Sigma}$ we get the integral of $\mathcal{O}_\mu$ over the unit disk around the puncture in $\widehat{\Sigma}$. Quantitatively, we have

$$\begin{aligned} D_\mu(\,\widehat{\Gamma}\,)|0\rangle_{(1)} &= \big(D_\mu(\,\widehat{\Gamma}\,)\langle 0|_{(2)}\big)|R_{12}\rangle = -\frac{1}{\pi}\int d^2z\,\langle 0_{(2)}, z_{(3)}|\mathcal{O}_\mu\rangle_{(3)}|R_{12}\rangle \\ &= \frac{1}{-2\pi i}\int dx\wedge dy\;\langle 0_{(2)}, z_{(3)}|R_{12}\rangle\; b(\frac{\partial}{\partial x})\,b(\frac{\partial}{\partial y})|\widehat{\mathcal{O}}_\mu\rangle_3\,, \end{aligned} \tag{A.11}$$

where we used the covariant constancy of $|R_{12}\rangle$. The bra $\langle 0_{(2)}, z_{(3)}|$ is the surface state corresponding to a two punctured sphere (at $0, z$), and the subscripts label state spaces. In the last step we also made use of the remarks below (3.25). As we sew this into the form in (A.8) we must bring $\langle 0_{(2)}, z_{(3)}|R_{12}\rangle$ all the way to the surface state. On the way stand antighost insertions (in the state space (1)). Using the reflector they can be rewritten in the state space (2) and using $\langle 0_{(2)}, z_{(3)}|$ they can be written in the state space (3). After that they can be pushed to the right. When $\langle 0_{(2)}, z_{(3)}|R_{12}\rangle$ hits the surface state $\langle\widehat{\Sigma}|$ the extra puncture gets deleted and a new puncture, in the state space (3) appears. In this manner we obtain (with the correct coefficient) what the right hand side of (A.8) stands for. This concludes our proof of (3.28).

APPENDIX B: Proof of Eqn.(6.15)

In this appendix we shall give a proof of Eqn.(6.15). We begin by recalling that $\langle\mathcal{O}_\mu\rangle^M_{g=1}$ in this equation needs to be computed in a specific coordinate system w, in which the torus is described by the identification $w \equiv w+1 \equiv w+\tau$. The surface state $\langle\Sigma^{(1,1)}|$ corresponding to a one punctured torus with modular parameter τ may be defined as

$$ {}_3\langle\Sigma^{(1,1)}(\tau,\bar\tau)| \;=\; {}_{123}\langle V'^{(0,3)}|e^{2\pi i\tau L_0^{(1)}}e^{-2\pi i\bar\tau\bar L_0^{(1)}}|R_{12}\rangle \tag{B.1} $$

The choice of local coordinate at the puncture of $\Sigma^{(1,1)}$ is induced by the corresponding choice of local coordinate at the special puncture of $\langle V'^{(0,3)}|$, but this choice is irrelevant for our analysis since the state $|\widehat{\mathcal{O}}_\mu\rangle$ will be inserted there. $\langle\Omega^{(0)1,1}|$ is constructed from $\langle\Sigma^{(1,1)}| \equiv (-2\pi i)\langle\Omega^{(-2)1,1}|$ using the descent equations given in refs.[1,2]:

$$ d\langle\Omega^{(k-1)1,1}| = (-)^k\langle\Omega^{(k)1,1}|Q\,, \tag{B.2} $$

Using this equation and the definition

$$ {}_3\langle\Omega^{(-2)1,1}| = -\frac{1}{2\pi i}\;{}_3\langle\Sigma^{(1,1)}(\tau,\bar\tau)| = -\frac{1}{2\pi i}\;{}_{123}\langle V'^{(0,3)}|e^{2\pi i\tau L_0^{(1)}}e^{-2\pi i\bar\tau\overline{L}_0^{(1)}}|R_{12}\rangle\,, \tag{B.3} $$

we get,

$$ {}_3\langle\Omega^{(0)1,1}| = -\frac{1}{2\pi i}\cdot 4\pi^2\cdot d\tau\wedge d\bar\tau\ {}_{123}\langle V'^{(0,3)}|e^{2\pi i\tau L_0^{(1)}}e^{-2\pi i\bar\tau\overline{L}_0^{(1)}}b_0^{(1)}\bar b_0^{(1)}|R_{12}\rangle\,. \quad \text{(B.4)} $$

We shall now use Eqn.(B.4) to calculate $\langle\Omega^{(0)1,1}|\widehat{\mathcal{O}}_\mu\rangle$. Although the result is independent of the choice of coordinate system at the puncture, we shall choose a specific coordinate system which will allow us to express the result in terms of the coordinate dependent object $\langle\mathcal{O}_\mu\rangle^M_{g=1}$. It turns out that the required choice of coordinates is the same one introduced in §2.3. $\mathcal{V}'_{0,3}$ is mapped to an infinite cylinder of circumference 2π with the symmetric punctures at the two points at infinity. The local coordinates at those punctures are fixed by taking as a common coordinate curve an arbitrary geodesic circle on the cylinder. On that circle we fix a point to be the special puncture, and define its coordinate curve to be the set of points at a distance 2π. Let w denote this coordinate.* Then the torus is described in the w coordinate system by the identification $w\simeq w+1\simeq w+\tau$. The coordinate w is related to the local coordinate $z^{(1)}$ around the puncture 1 through the relation $z^{(1)}=e^{2\pi i w}$. Let us denote the ghost fields in the w coordinate system by c^w, $\bar c^w$, and the field $\mathcal{O}_\mu$ in the w coordinate system by $\mathcal{O}^w_\mu$. The ghost fields c^w, $\bar c^w$ now have expansions

$$ c^w(w) = \frac{1}{2\pi i}\sum_n c_n^{(1)}e^{2\pi i n w}, \qquad \bar c^w(\bar w) = -\frac{1}{2\pi i}\sum_n \bar c_n^{(1)}e^{-2\pi i n\bar w}\,. \quad \text{(B.5)} $$

where, as usual, the superscript $^{(1)}$ denotes that these operators act on the state space 1. From the above description of $\langle V'^{(3)}|$ we see that

$$ {}_{123}\langle V'^{(3)}|\widehat{\mathcal{O}}_\mu\rangle_3 = \langle R_{12}|c^w(w=0)\bar c^w(w=0)\mathcal{O}^w_\mu(w=0)\,. \quad \text{(B.6)} $$

Eqn.(B.4) now gives

$$ \langle\Omega^{(0)1,1}|\widehat{\mathcal{O}}_\mu\rangle = -\frac{1}{2\pi i}\cdot d\tau\wedge d\bar\tau\ \langle R_{12}|e^{2\pi i\tau L_0^{(1)}}e^{-2\pi i\bar\tau\overline{L}_0^{(1)}}b_0^{(1)}\bar b_0^{(1)}\sum_n c_n^{(1)}\sum_m \bar c_m^{(1)}\mathcal{O}^w_\mu(w=0)|R_{12}\rangle\,. \quad \text{(B.7)} $$

$\star$ Note that with this choice of normalization the coordinate disc covers the parts of the torus more than once. In fact, the curve $|w|=1/2$ just touches itself.

Only the $m = n = 0$ term in the sum in the above equation contributes. The right hand side of Eqn.(C.8) has the form $\langle R_{12}|A^{(1)}|R_{12}\rangle$, which, using the definition of R_{12}, reduces to the form $\sum_i (-1)^{\Phi_i} \langle \Phi^i|A|\Phi_i\rangle$. We can choose the basis of states in such a way that either $|\Phi_i\rangle$ is annihilated by b_0 from the left, or $\langle \Phi^i|$ is annihilated by b_0 from the right. Thus if A contains a b_0 without any accompanying c_0, the matrix element $\langle \Phi^i|A|\Phi_i\rangle$ vanishes identically. This shows that only the $n = 0$ term contributes. A similar argument involving $\bar{b}_0$ shows that only the $m = 0$ term will contribute. Using the definition of Z^G and $\langle \mathcal{O}_\mu\rangle^M_{g=1}$, we get

$$\int\limits_{\mathcal{V}_{1,1}+\Delta\mathcal{B}'_{0,3}} \langle \Omega^{(0)1,1}|\widehat{\mathcal{O}}_\mu\rangle = -\frac{1}{\pi} \int\limits_{\mathcal{V}_{1,0}} d\tau_1 \wedge d\tau_2 \; Z^G \langle \mathcal{O}_\mu\rangle^M_{g=1} \; . \tag{B.8}$$

This is precisely Eqn.(6.15).

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