

# BLACK HOLE SOLUTIONS IN HETEROTIC STRING THEORY ON A TORUS

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## Abstract

We construct the general electrically charged, rotating black hole solution in the heterotic string theory compactified on a six dimensional torus and study its classical properties. This black hole is characterized by its mass, angular momentum, and a 28 dimensional electric charge vector. We recover the axion-dilaton black holes and Kaluza-Klein black holes for special values of the charge vector. For a generic black hole of this kind, the 28 dimensional magnetic dipole moment vector is not proportional to the electric charge vector, and we need two different gyromagnetic ratios for specifying the relation between these two vectors. We also give an algorithm for constructing a 58 parameter rotating dyonic black hole solution in this theory, characterized by its mass, angular momentum, a 28 dimensional electric charge vector and a 28 dimensional magnetic charge vector. This is the most general asymptotically flat black hole solution in this theory consistent with the no-hair theorem.

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# 1 Introduction and Summary

There has been a lot of activity in recent years towards construction of black hole solutions in string theory[1]. Since string theory is expected to provide a finite quantum theory of gravity, we hope that within the context of string theory we might be able to address various vexing questions in black hole physics related to the black hole evaporation and the consequent information loss puzzle. One suggestion in this direction has been made in refs.[2, 3]. It has also been proposed there that massive elementary string states themselves should be identified with black holes in string theory. Similar suggestions have also been made in refs.[4, 5, 6] but for different reasons.

In order to study the physics of black holes in string theory, we need to first construct the black hole solutions in string theory, and then study their properties. In particular, study of the relationship between black holes and elementary string states requires us to construct the most general electrically charged black hole solution in the theory. In this paper we undertake the task of constructing the most general black hole solution in one particular four dimensional string theory, – the heterotic string theory compactified on a six dimensional torus. There are various reasons for choosing this particular theory. One of them is that due to the existence of an  $N=4$  supersymmetry in this theory, there are various non-renormalization theorems which may make the quantum theory in this case more tractable compared to the other compactification schemes. Second, this is perhaps the best understood four dimensional string theory[7]. And finally, this theory is expected to possess a strong-weak coupling duality symmetry[8] which may make the study of the non-perturbative physics more feasible.

The theory under consideration has 28  $U(1)$  gauge fields, and thus one would expect that a generic electrically charged black hole should be labelled by a 28 dimensional charge vector. In section 2 we use the technique of  $O(d,d+p)$  transformations[9, 10, 11, 1] to explicitly construct a black hole solution characterized by an arbitrary 28 dimensional charge vector, as well as mass and angular momentum, and study its various properties.<sup>1</sup> The relevant transformations in the present case generate the group  $O(7,23)$ . The final solution is given in eqs.(2.29) - (2.37). For various special values of the parameters we can identify the solution to various known solutions *e.g.* the rotating charged dilaton black hole of ref.[13] and the rotating Kaluza-Klein black hole of refs.[14, 15]. One novel feature of a generic black hole solution is that the 28 dimensional vector representing its magnetic dipole moment is not, in general, parallel to the 28 dimensional vector representing its electric charge, and we need two gyromagnetic ratios for specifying the relation between these two vectors. Also, a generic black hole solution has all the 132 moduli fields, as well as the dilaton-axion field non-zero. We discuss the extremal limits, both for non-zero angular momentum and zero angular momentum. For non-rotating black holes, some of the extremal black holes saturate the Bogomol'nyi bound and hence have unbroken supersymmetry. But black holes carrying non-zero angular momentum never saturate Bogomol'nyi bound, and hence have no unbroken supersymmetry.

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<sup>1</sup>Although the general procedure for constructing such a solution was outlined in ref.[12], in the present paper we find a simple form of the  $O(7,23)$  transformations on the four dimensional fields that allows us to construct these solutions explicitly.

If we allow the solution to carry magnetic charges also, then it should be characterized by 58 parameters, the mass, the angular momentum, 28 electric charges, and 28 magnetic charges. In section 3 we give an algorithm to construct this 58 parameter rotating dyonic black hole solution. This is done by using a combination of the S-duality transformations[16, 12] belonging to the group  $SL(2,R)$  and the  $O(7,23)$  transformations[17, 18, 19, 20, 6]. These two transformations together generate the group  $O(8,24)$ [21, 22]. This algorithm in fact naturally produces a 59 parameter solution with singular metric, belonging to the Taub-NUT family. We recover the non-singular black hole solutions by imposing one restriction on these 59 parameters, which sets the NUT charge to zero.

We conclude the paper in section 4 by pointing out some amusing coincidences between the properties of black holes and those of elementary string states in the extremal limit.

## 2 Rotating Electrically Charged Black Holes

The massless fields in heterotic string theory compactified on a six dimensional torus consist of the metric  $G_{\mu\nu}$ , the anti-symmetric tensor field  $B_{\mu\nu}$ , twenty eight  $U(1)$  gauge fields  $A_\mu^{(a)}$  ( $1 \leq a \leq 28$ ), the scalar dilaton field  $\Phi$ , and a  $28 \times 28$  matrix valued scalar field  $M$  satisfying,

$$MLM^T = L, \quad M^T = M. \quad (2.1)$$

Here  $L$  is a  $28 \times 28$  symmetric matrix with 22 eigenvalues  $-1$  and 6 eigenvalues  $+1$ . For definiteness we shall take  $L$  to be,

$$L = \begin{pmatrix} -I_{22} & \\ & I_6 \end{pmatrix}, \quad (2.2)$$

where  $I_n$  denotes an  $n \times n$  identity matrix. The action describing the effective field theory of these massless bosonic fields is given by[23],

$$S = C \int d^4x \sqrt{-\det G} e^{-\Phi} \left[ R_G + G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{8} G^{\mu\nu} Tr(\partial_\mu ML \partial_\nu ML) - \frac{1}{12} G^{\mu\mu'} G^{\nu\nu'} G^{\rho\rho'} H_{\mu\nu\rho} H_{\mu'\nu'\rho'} - G^{\mu\mu'} G^{\nu\nu'} F_{\mu\nu}^{(a)} (LML)_{ab} F_{\mu'\nu'}^{(b)} \right], \quad (2.3)$$

where,

$$F_{\mu\nu}^{(a)} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)}, \quad (2.4)$$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + 2A_\mu^{(a)} L_{ab} F_{\nu\rho}^{(b)} + \text{cyclic permutations of } \mu, \nu, \rho, \quad (2.5)$$

and  $R_G$  denotes the scalar curvature associated with the metric  $G_{\mu\nu}$ .  $C$  is an arbitrary constant which does not affect the equations of motion and can be absorbed into the dilaton field  $\Phi$ . This action is invariant under an  $O(6,22)$  transformation,

$$M \rightarrow \Omega M \Omega^T, \quad A_\mu^{(a)} \rightarrow \Omega_{ab} A_\mu^{(b)}, \quad \Phi \rightarrow \Phi, \quad G_{\mu\nu} \rightarrow G_{\mu\nu}, \quad B_{\mu\nu} \rightarrow B_{\mu\nu}, \quad (2.6)$$

where  $\Omega$  is a  $28 \times 28$  matrix satisfying,

$$\Omega L \Omega^T = L. \quad (2.7)$$

This is a symmetry of the full string theory if we also rotate the 28 dimensional lattice  $\Lambda$  of electric charges by  $\Omega$ [7]. On the other hand, for fixed  $\Lambda$ , only a discrete subgroup  $O(6,22;Z)$  of  $O(6,22)$  is a symmetry of the full string theory.

Let us now consider backgrounds that are independent of the time coordinate  $t$ . In this case the action is expected to have an  $O(7,23)$  symmetry[11, 1]. To see how this appears, let us define new variables as follows:

$$\begin{aligned} \bar{A}_i^{(a)} &= A_i^{(a)} - (G_{tt})^{-1} G_{ti} A_t^{(a)}, \quad 1 \leq a \leq 28, \quad 1 \leq i \leq 3, \\ \bar{A}_i^{(29)} &= \frac{1}{2} (G_{tt})^{-1} G_{ti}, \\ \bar{A}_i^{(30)} &= \frac{1}{2} B_{ti} + A_t^{(a)} L_{ab} \bar{A}_i^{(b)}, \end{aligned} \quad (2.8)$$

$$\bar{M} = \begin{pmatrix} M + 4(G_{tt})^{-1} A_t A_t^T & -2(G_{tt})^{-1} A_t & 2M L A_t + 4(G_{tt})^{-1} A_t (A_t^T L A_t) \\ -2(G_{tt})^{-1} A_t^T & (G_{tt})^{-1} & -2(G_{tt})^{-1} A_t^T L A_t \\ 2A_t^T L M + 4(G_{tt})^{-1} A_t^T (A_t^T L A_t) & -2(G_{tt})^{-1} A_t^T L A_t & G_{tt} + 4A_t^T L M L A_t \\ & & + 4(G_{tt})^{-1} (A_t^T L A_t)^2 \end{pmatrix}, \quad (2.9)$$

$$\bar{G}_{ij} = G_{ij} - (G_{tt})^{-1} G_{ti} G_{tj}, \quad (2.10)$$

$$\bar{B}_{ij} = B_{ij} + (G_{tt})^{-1} (G_{ti} A_j^{(a)} - G_{tj} A_i^{(a)}) L_{ab} A_t^{(b)} + \frac{1}{2} (G_{tt})^{-1} (B_{ti} G_{tj} - B_{tj} G_{ti}), \quad (2.11)$$

$$\bar{\Phi} = \Phi - \frac{1}{2} \ln(-G_{tt}), \quad (2.12)$$

and,

$$\bar{L} = \begin{pmatrix} L & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2.13)$$

It can be verified that for time independent field configurations, the action (2.3) can be rewritten as,

$$\begin{aligned} S &= C \int dt \int d^3x \sqrt{\det \bar{G}} e^{-\bar{\Phi}} \left[ R_{\bar{G}} + \bar{G}^{ij} \partial_i \bar{\Phi} \partial_j \bar{\Phi} + \frac{1}{8} \bar{G}^{ij} Tr(\partial_i \bar{M} \bar{L} \partial_j \bar{M} \bar{L}) \right. \\ &\quad \left. - \frac{1}{12} \bar{G}^{ii'} \bar{G}^{jj'} \bar{G}^{kk'} \bar{H}_{ijk} \bar{H}_{i'j'k'} - \bar{G}^{ii'} \bar{G}^{jj'} \bar{F}_{ij}^{(\bar{a})} (\bar{L} \bar{M} \bar{L})_{\bar{a}\bar{b}} \bar{F}_{i'j'}^{(\bar{b})} \right], \end{aligned} \quad (2.14)$$

where,

$$\bar{F}_{ij}^{(\bar{a})} = \partial_i \bar{A}_j^{(\bar{a})} - \partial_j \bar{A}_i^{(\bar{a})}, \quad 1 \leq \bar{a} \leq 30, \quad (2.15)$$

and,

$$\bar{H}_{ijk} = \partial_i \bar{B}_{jk} + 2\bar{A}_i^{(a)} \bar{L}_{\bar{a}\bar{b}} \bar{F}_{jk}^{(\bar{b})} + \text{cyclic permutations of } i, j, k. \quad (2.16)$$

This action has an  $O(7,23)$  symmetry:

$$\bar{M} \rightarrow \bar{\Omega} \bar{M} \bar{\Omega}^T, \quad \bar{A}_i^{(\bar{a})} \rightarrow \bar{\Omega}_{\bar{a}\bar{b}} \bar{A}_i^{(\bar{b})}, \quad \bar{\Phi} \rightarrow \bar{\Phi}, \quad \bar{G}_{ij} \rightarrow \bar{G}_{ij}, \quad \bar{B}_{ij} \rightarrow \bar{B}_{ij}, \quad (2.17)$$

where  $\bar{\Omega}$  is a  $30 \times 30$  matrix satisfying,

$$\bar{\Omega} \bar{L} \bar{\Omega}^T = \bar{L}. \quad (2.18)$$

In order to get a convenient parametrization of  $\bar{\Omega}$  it is easier to work with the diagonal form of  $\bar{L}$ . The orthogonal matrix  $U$  that diagonalizes  $\bar{L}$  is given by,

$$U = \begin{pmatrix} I_{28} & & \\ & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \quad (2.19)$$

We have,

$$U \bar{L} U^T \equiv \bar{L}_d = \begin{pmatrix} -I_{22} & & & \\ & I_6 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}. \quad (2.20)$$

Then  $U \bar{\Omega} U^T$  preserves  $\bar{L}_d$ .

We can apply this  $O(7,23)$  transformation on a known time independent classical solution to generate new classical solutions of the equations of motion. We shall restrict ourselves to solutions characterized by fixed asymptotic configuration of various fields, representing asymptotically flat space time. For definiteness, we shall look for solutions with the following asymptotic forms for various fields:

$$M_{as} = I_{28}, \quad \Phi_{as} = 0, \quad (A_\mu^{(a)})_{as} = 0, \quad (G_{\mu\nu})_{as} = \eta_{\mu\nu}, \quad (B_{\mu\nu})_{as} = 0. \quad (2.21)$$

This gives,

$$\bar{M}_{as} = \begin{pmatrix} I_{28} & & \\ & -1 & \\ & & -1 \end{pmatrix}. \quad (2.22)$$

Given a solution with any other asymptotically constant field configuration, we can bring it to the form (2.21) by a combination of the  $O(6,22)$  transformation, the transformation  $\Phi \rightarrow \Phi + \text{constant}$ , the general coordinate transformation involving linear change in the coordinates, and the freedom of shifting  $A_\mu^{(a)}$  and  $B_{\mu\nu}$  by constants. Thus we do not suffer from any loss of generality by restricting to field configurations with asymptotic behaviour given in eq.(2.21).

As in ref.[13], we begin with the Kerr solution:

$$\begin{aligned}
ds^2 &\equiv G_{\mu\nu}dx^\mu dx^\nu \\
&= -\frac{\rho^2 + a^2 \cos^2 \theta - 2m\rho}{\rho^2 + a^2 \cos^2 \theta} dt^2 + \frac{\rho^2 + a^2 \cos^2 \theta}{\rho^2 + a^2 - 2m\rho} d\rho^2 + (\rho^2 + a^2 \cos^2 \theta) d\theta^2 \\
&\quad + \frac{\sin^2 \theta}{\rho^2 + a^2 \cos^2 \theta} [(\rho^2 + a^2)(\rho^2 + a^2 \cos^2 \theta) + 2m\rho a^2 \sin^2 \theta] d\phi^2 \\
&\quad - \frac{4m\rho a \sin^2 \theta}{\rho^2 + a^2 \cos^2 \theta} dt d\phi \\
\Phi &= 0, \quad B_{\mu\nu} = 0, \quad A_\mu^{(a)} = 0, \quad M = I_{28}.
\end{aligned} \tag{2.23}$$

Here  $t$ ,  $\rho$ ,  $\theta$ ,  $\phi$  denote the space-time coordinates. This is guaranteed to be a solution of the equations of motion derived from action (2.3) since when the  $\Phi$ ,  $B_{\mu\nu}$  and  $A_\mu^{(a)}$  fields are set to zero, and  $M$  is set to the identity matrix, the equations of motion derived from the action (2.3) become identical to Einstein's equation in matter free space. Using eqs.(2.8)-(2.12) we get,

$$\begin{aligned}
\bar{A}_\phi^{(\bar{a})} &= \delta_{\bar{a},29} \frac{m\rho a \sin^2 \theta}{\rho^2 + a^2 \cos^2 \theta - 2m\rho}, \quad \bar{A}_\theta^{(\bar{a})} = 0, \quad \bar{A}_\rho^{(\bar{a})} = 0, \\
\bar{M} &= \begin{pmatrix} I_{28} & & \\ & -f^{-1} & \\ & & -f \end{pmatrix}, \quad f = \frac{\rho^2 + a^2 \cos^2 \theta - 2m\rho}{\rho^2 + a^2 \cos^2 \theta}, \\
\bar{G}_{ij} dx^i dx^j &= (\rho^2 + a^2 \cos^2 \theta) \left[ \frac{1}{\rho^2 + a^2 - 2m\rho} d\rho^2 + d\theta^2 + \frac{\rho^2 + a^2 - 2m\rho}{\rho^2 + a^2 \cos^2 \theta - 2m\rho} \sin^2 \theta d\phi^2 \right], \\
\bar{B}_{ij} &= 0, \quad \bar{\Phi} = -\frac{1}{2} \ln f.
\end{aligned} \tag{2.24}$$

We can now generate new solutions by performing an  $O(7,23)$  transformation on this solution. Since, however, we want to keep the asymptotic forms of various field configurations fixed, we only use a subgroup of the  $O(7,23)$  transformations which preserve (2.22). This leaves us with an  $O(6,1) \times O(22,1)$  subgroup of the full  $O(7,23)$  group. If we describe the transformation by the matrix  $U\bar{\Omega}U^T$  instead of  $\bar{\Omega}$ , then the  $O(6,1)$  subgroup acts on the 23rd - 28th, and the 30th index of the matrix  $U\bar{M}U^T$ , whereas the  $O(22,1)$  subgroup acts on the 1st - 22nd, and the 29th index of the matrix  $U\bar{M}U^T$ . Not every element of this  $O(6,1) \times O(22,1)$  subgroup generates a new solution however. It is clear from eq.(2.24) that an  $O(22) \times O(6)$  subgroup, which acts on the 1st - 22nd and the 23rd - 28th indices respectively, leaves the solution invariant. Thus the transformations which generate inequivalent solutions, preserving the asymptotic field configuration, can be parametrized by the coset

$$(O(6,1) \times O(22,1)) / (O(6) \times O(22)). \tag{2.25}$$

This is a 28 dimensional space. We now need to find a convenient representative of a generic element of this coset in the  $O(6,1) \times O(22,1)$  group. This is done as follows. We take  $\bar{\Omega}$  to be of the form:

$$\bar{\Omega} = \bar{\Omega}_2 \bar{\Omega}_1, \tag{2.26}$$

where,

$$U\bar{\Omega}_1U^T = \begin{pmatrix} I_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & \cosh \alpha & 0 & 0 & \sinh \alpha & 0 \\ 0 & 0 & I_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cosh \beta & 0 & \sinh \beta \\ 0 & \sinh \alpha & 0 & 0 & \cosh \alpha & 0 \\ 0 & 0 & 0 & \sinh \beta & 0 & \cosh \beta \end{pmatrix}, \quad (2.27)$$

and,

$$\bar{\Omega}_2 = \begin{pmatrix} R_{22}(\vec{n}) & & \\ & R_6(\vec{p}) & \\ & & I_2 \end{pmatrix}, \quad (2.28)$$

where  $R_N(\vec{k})$  denotes any  $N$ -dimensional rotation matrix that rotates an  $N$ -dimensional column vector with only  $N$ -th component non-zero and equal to 1 to an arbitrary  $N$ -dimensional unit vector  $\vec{k}$ . Here  $\vec{n}$  and  $\vec{p}$  are arbitrary 22 and 6 dimensional unit vectors respectively. Thus  $\bar{\Omega}$  given by eqs.(2.26)-(2.28) is parametrized by 28 parameters  $\alpha$ ,  $\beta$ ,  $\vec{n}$  and  $\vec{p}$ .

It is now a straightforward algebraic exercise to apply the transformation generated by  $\bar{\Omega}$  to the field configuration given in eqs.(2.23), (2.24) and extract the expressions for various fields in the transformed solution. The result is,

$$\begin{aligned} ds^2 &\equiv G_{\mu\nu}dx^\mu dx^\nu \\ &= (\rho^2 + a^2 \cos^2 \theta) \left\{ -\Delta^{-1}(\rho^2 + a^2 \cos^2 \theta - 2m\rho)dt^2 + (\rho^2 + a^2 - 2m\rho)^{-1}d\rho^2 + d\theta^2 \right. \\ &\quad \left. + \Delta^{-1} \sin^2 \theta [\Delta + a^2 \sin^2 \theta (\rho^2 + a^2 \cos^2 \theta + 2m\rho \cosh \alpha \cosh \beta)] d\phi^2 \right. \\ &\quad \left. - 2\Delta^{-1}m\rho a \sin^2 \theta (\cosh \alpha + \cosh \beta) dt d\phi \right\}, \end{aligned} \quad (2.29)$$

where,

$$\Delta = (\rho^2 + a^2 \cos^2 \theta)^2 + 2m\rho(\rho^2 + a^2 \cos^2 \theta)(\cosh \alpha \cosh \beta - 1) + m^2\rho^2(\cosh \alpha - \cosh \beta)^2, \quad (2.30)$$

$$\Phi = \frac{1}{2} \ln \frac{(\rho^2 + a^2 \cos^2 \theta)^2}{\Delta}, \quad (2.31)$$

$$\begin{aligned} A_t^{(a)} &= -\frac{n^{(a)}}{\sqrt{2}} \Delta^{-1} m\rho \sinh \alpha \{(\rho^2 + a^2 \cos^2 \theta) \cosh \beta + m\rho(\cosh \alpha - \cosh \beta)\} \quad \text{for } 1 \leq a \leq 22, \\ &= -\frac{p^{(a-22)}}{\sqrt{2}} \Delta^{-1} m\rho \sinh \beta \{(\rho^2 + a^2 \cos^2 \theta) \cosh \alpha + m\rho(\cosh \beta - \cosh \alpha)\} \quad \text{for } a \geq 23, \end{aligned} \quad (2.32)$$

$$\begin{aligned} A_\phi^{(a)} &= \frac{n^{(a)}}{\sqrt{2}} \Delta^{-1} m\rho a \sinh \alpha \sin^2 \theta \{\rho^2 + a^2 \cos^2 \theta + m\rho \cosh \beta (\cosh \alpha - \cosh \beta)\} \quad 1 \leq a \leq 22, \\ &= \frac{p^{(a-22)}}{\sqrt{2}} \Delta^{-1} m\rho a \sinh \beta \sin^2 \theta \{\rho^2 + a^2 \cos^2 \theta + m\rho \cosh \alpha (\cosh \beta - \cosh \alpha)\} \quad a \geq 23, \end{aligned} \quad (2.33)$$

$$B_{t\phi} = \Delta^{-1} m \rho a \sin^2 \theta (\cosh \alpha - \cosh \beta) \{ \rho^2 + a^2 \cos^2 \theta + m \rho (\cosh \alpha \cosh \beta - 1) \} \quad (2.34)$$

$$M = I_{28} + \begin{pmatrix} Pnn^T & Qnp^T \\ Qpn^T & Ppp^T \end{pmatrix}, \quad (2.35)$$

where,

$$\begin{aligned} P &= 2\Delta^{-1} m^2 \rho^2 \sinh^2 \alpha \sinh^2 \beta, \\ Q &= -2\Delta^{-1} m \rho \sinh \alpha \sinh \beta \{ \rho^2 + a^2 \cos^2 \theta + m \rho (\cosh \alpha \cosh \beta - 1) \}. \end{aligned} \quad (2.36)$$

Note that the solution is characterized by non-trivial dilaton as well as other scalar moduli fields  $M$ . From eqs.(2.29) and (2.31) we can also find an expression for the canonical Einstein metric  $g_{\mu\nu} \equiv e^{-\Phi} G_{\mu\nu}$ :

$$\begin{aligned} ds_E^2 &\equiv g_{\mu\nu} dx^\mu dx^\nu = e^{-\Phi} ds^2 \\ &= \Delta^{\frac{1}{2}} \left\{ -\Delta^{-1} (\rho^2 + a^2 \cos^2 \theta - 2m\rho) dt^2 + (\rho^2 + a^2 - 2m\rho)^{-1} d\rho^2 + d\theta^2 \right. \\ &\quad \left. + \Delta^{-1} \sin^2 \theta [\Delta + a^2 \sin^2 \theta (\rho^2 + a^2 \cos^2 \theta + 2m\rho \cosh \alpha \cosh \beta)] d\phi^2 \right. \\ &\quad \left. - 2\Delta^{-1} m \rho a \sin^2 \theta (\cosh \alpha + \cosh \beta) dt d\phi \right\}. \end{aligned} \quad (2.37)$$

One can easily verify that the new solution given in eqs.(2.29)-(2.37) describes a black hole with mass  $M$ , angular momentum  $J$ , electric charge  $Q^{(a)}$  and magnetic dipole moment  $\mu^{(a)}$  given by,

$$M = \frac{1}{2} m (1 + \cosh \alpha \cosh \beta), \quad (2.38)$$

$$J = \frac{1}{2} m a (\cosh \alpha + \cosh \beta), \quad (2.39)$$

$$\begin{aligned} Q^{(a)} &= \frac{m}{\sqrt{2}} \sinh \alpha \cosh \beta n^{(a)} \quad \text{for } 1 \leq a \leq 22 \\ &= \frac{m}{\sqrt{2}} \sinh \beta \cosh \alpha p^{(a-22)} \quad \text{for } 23 \leq a \leq 28, \end{aligned} \quad (2.40)$$

$$\begin{aligned} \mu^{(a)} &= \frac{1}{\sqrt{2}} m a \sinh \alpha n^{(a)} \quad \text{for } 1 \leq a \leq 22 \\ &= \frac{1}{\sqrt{2}} m a \sinh \beta p^{(a-22)} \quad \text{for } 23 \leq a \leq 28. \end{aligned} \quad (2.41)$$

From eqs.(2.40) and (2.41) we see that for generic values of the parameters  $\alpha$ ,  $\beta$ ,  $\vec{n}$  and  $\vec{p}$ , the 28- dimensional vectors  $\vec{\mu}$  and  $\vec{Q}$  are not parallel to each other. The special cases for which  $\vec{\mu}$  and  $\vec{Q}$  are parallel are i)  $\beta = 0$ , ii)  $\alpha = 0$ , and iii)  $\alpha = \beta$ . The black hole solution in case (i) corresponds to the rotating charged black hole solution discussed in ref.[13], whereas the case  $\alpha = \beta$  corresponds to the Kaluza-Klein black hole discussed in refs.[14, 15]. To check that the results (2.40) and (2.41) are consistent with the ones derived in refs.[13, 15], note that the



gyromagnetic ratio, defined as  $2\mu M/QJ$ , is equal to 2 in case (i), and is equal to  $(1 + \text{sech}^2\alpha)$  in case (iii), which varies between 1 and 2 as  $\alpha$  varies between  $\infty$  and 0.

Even though  $\vec{\mu}$  and  $\vec{Q}$  are not parallel in general, and hence we cannot define an overall gyromagnetic ratio, we can define two separate gyromagnetic ratios in the left and the right hand sector as follows. Let us define

$$Q_{L,R}^{(a)} = \frac{1}{2}(I_{28} \mp L)_{ab}Q^{(b)}, \quad \mu_{L,R}^{(a)} = \frac{1}{2}(I_{28} \mp L)_{ab}\mu^{(b)}. \quad (2.42)$$

Then  $\vec{Q}_L$  ( $\vec{\mu}_L$ ) has its last six components zero, whereas  $\vec{Q}_R$  ( $\vec{\mu}_R$ ) has its first 22 components zero. In other words,  $\vec{Q}_L$  ( $\vec{\mu}_L$ ) and  $\vec{Q}_R$  ( $\vec{\mu}_R$ ) can be regarded as 22 and 6 dimensional vectors respectively. Then from eqs.(2.38) - (2.41) we get,

$$\vec{\mu}_L = \frac{1}{2}g_L \frac{J}{M} \vec{Q}_L, \quad \vec{\mu}_R = \frac{1}{2}g_R \frac{J}{M} \vec{Q}_R, \quad (2.43)$$

where,

$$g_L = 2 \frac{1 + \cosh \alpha \cosh \beta}{\cosh \alpha + \cosh \beta} \frac{1}{\cosh \beta}, \quad g_R = 2 \frac{1 + \cosh \alpha \cosh \beta}{\cosh \alpha + \cosh \beta} \frac{1}{\cosh \alpha}. \quad (2.44)$$

Various geometrical properties of the solution can be studied using the canonical metric given in eq.(2.37). There are two horizons corresponding to the surfaces,

$$\rho^2 - 2m\rho + a^2 = 0, \quad (2.45)$$

which gives the location of the horizons at,

$$\rho = m \pm \sqrt{m^2 - a^2} \equiv \rho_H^\pm. \quad (2.46)$$

Note that the horizons disappear for  $a > m$ , leaving behind naked singularity. The limit  $a \rightarrow m$  is known as the extremal limit.

The area of the outer event horizon, which is proportional to the Bekenstein entropy of the black hole, is given by,

$$A = \int d\theta d\phi \sqrt{g_{\theta\theta}g_{\phi\phi}} \Big|_{\rho=\rho_H^+} = 4\pi m(\cosh \alpha + \cosh \beta) (m + \sqrt{m^2 - a^2}). \quad (2.47)$$

The surface gravity of the black hole, calculated at  $\theta = 0$ , is given by,

$$\kappa = \lim_{\rho \rightarrow \rho_H^+} \sqrt{g^{\rho\rho}} \partial_\rho \sqrt{-g_{tt}} \Big|_{\theta=0} = \frac{\sqrt{m^2 - a^2}}{m(\cosh \alpha + \cosh \beta)(m + \sqrt{m^2 - a^2})}. \quad (2.48)$$

$\kappa/2\pi$  can be interpreted as the Hawking temperature of the black hole. Finally, the angular velocity  $\Omega$  at the horizon is found by demanding that the vector  $\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}$  is null at the horizon. This gives,

$$g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2 = 0. \quad (2.49)$$

The solution to the above equation is

$$\Omega = \frac{a}{m(\cosh \alpha + \cosh \beta)(m + \sqrt{m^2 - a^2})}. \quad (2.50)$$

From eqs.(2.47), (2.48) we see that in the extremal limit  $a \rightarrow m$ ,

$$A \rightarrow 4\pi m^2(\cosh \alpha + \cosh \beta) = 8\pi|J|, \quad \kappa \rightarrow 0. \quad (2.51)$$

For non-rotating black holes ( $a = 0$ ) special care is needed to study the extremal limit. From eqs.(2.37), (2.30) we see that in this case,

$$ds_E^2 = -\Delta^{-\frac{1}{2}}(\rho^2 - 2m\rho)dt^2 + \Delta^{\frac{1}{2}}(\rho^2 - 2m\rho)^{-1}d\rho^2 + \Delta^{\frac{1}{2}}(d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.52)$$

where,

$$\Delta = \rho^2\{\rho^2 + 2m\rho(\cosh \alpha \cosh \beta - 1) + m^2(\cosh \alpha - \cosh \beta)^2\}. \quad (2.53)$$

The metric describes a black hole with horizon at  $\rho = 2m$  and singularity at  $\rho = 0$ . The extremal limit corresponds to the case when the horizon approaches the singularity, *i.e.*  $m \rightarrow 0$  keeping the physical mass  $M$  defined in eq.(2.38) fixed. We shall now study this limit in three separate cases.

Case I:  $\alpha > \beta$

In this case we consider the limit

$$m \rightarrow 0, \quad \alpha \rightarrow \infty, \quad m \cosh \alpha \equiv m_0 \text{ finite}, \quad \beta \text{ finite}. \quad (2.54)$$

Then eqs.(2.38) and (2.40) take the form:

$$M = \frac{m_0}{2} \cosh \beta, \quad \vec{Q}_L = \frac{m_0}{\sqrt{2}} \cosh \beta \vec{n}, \quad \vec{Q}_R = \frac{m_0}{\sqrt{2}} \sinh \beta \vec{p}. \quad (2.55)$$

$\vec{Q}_L$  and  $\vec{Q}_R$  are 22 and 6 dimensional vectors respectively, and have been defined below eq.(2.42). From this we get the following relations between  $M$ ,  $\vec{Q}_L$  and  $\vec{Q}_R$ :

$$M^2 = \frac{1}{2}\vec{Q}_L^2, \quad \vec{Q}_R = \sqrt{2}M \tanh \beta \vec{p}. \quad (2.56)$$

Thus here  $\vec{Q}_R^2 < \vec{Q}_L^2$ . From eqs.(2.47) and (2.48) we see that in this limit,

$$A \rightarrow 0, \quad \kappa \rightarrow \frac{1}{4M} \cosh \beta. \quad (2.57)$$

Case II:  $\alpha < \beta$

In this case we consider the limit

$$m \rightarrow 0, \quad \beta \rightarrow \infty, \quad m \cosh \beta \equiv m_0 \text{ finite}, \quad \alpha \text{ finite}. \quad (2.58)$$

In this limit, eqs.(2.38) and (2.40) take the form:

$$M = \frac{m_0}{2} \cosh \alpha, \quad \vec{Q}_L = \frac{m_0}{\sqrt{2}} \sinh \alpha \vec{n}, \quad \vec{Q}_R = \frac{m_0}{\sqrt{2}} \cosh \alpha \vec{p}. \quad (2.59)$$

Thus now,

$$M^2 = \frac{1}{2} \vec{Q}_R^2, \quad \vec{Q}_L = \sqrt{2} M \tanh \alpha \vec{n}. \quad (2.60)$$

In this limit  $\vec{Q}_L^2 < \vec{Q}_R^2$ . Also,

$$A \rightarrow 0, \quad \kappa \rightarrow \frac{1}{4M} \cosh \alpha. \quad (2.61)$$

Case III:  $\alpha = \beta$ .

In this case we consider the limit:

$$m \rightarrow 0, \quad \alpha = \beta \rightarrow \infty, \quad m \cosh^2 \alpha \equiv m_0 \text{ finite}. \quad (2.62)$$

In this limit,

$$M = \frac{m_0}{2}, \quad \vec{Q}_L = \frac{m_0}{\sqrt{2}} \vec{n}, \quad \vec{Q}_R = \frac{m_0}{\sqrt{2}} \vec{p}. \quad (2.63)$$

Thus,

$$M^2 = \frac{1}{2} \vec{Q}_L^2 = \frac{1}{2} \vec{Q}_R^2. \quad (2.64)$$

Also in this limit,

$$A \rightarrow 0, \quad \kappa \rightarrow \infty. \quad (2.65)$$

In the normalization convention that we have been using, the Bogomol'nyi bound that follows from the space-time supersymmetry of the theory, is given by  $M^2 \geq (\vec{Q}_R^2/2)$ [24, 25]. Thus we see that the extremal black holes in cases II and III saturate the Bogomol'nyi bound, and hence give rise to supersymmetric solutions, whereas those in case I do not. Also for non-zero  $J$ , eqs.(2.38)-(2.40) shows that  $M^2$  is always larger than  $(\vec{Q}_R^2/2)$ , even in the extremal limit  $a \rightarrow m$ . This shows that the extremal black holes carrying non-zero angular momentum do not correspond to supersymmetric solutions. (Similar observations have been made in ref.[26].) For special cases  $\alpha = \beta$  and  $\alpha = 0$ , the non-rotating solutions constructed here reproduce the solutions discussed in ref.[5].

### 3 Rotating Dyonic Black Holes

In this section we shall discuss construction of black holes carrying both electric and magnetic charges. One class of dyons that can be constructed trivially are the ones for which the magnetic charge vector  $\vec{Q}_{mag}$  is parallel to  $L\vec{Q}_{el}$ , where  $\vec{Q}_{el}$  is the electric charge vector. This is done with

the help of  $SL(2, \mathbf{R})$  transformations[27, 16, 12, 28]. To see how this works, we first note that the equation of motion for the  $B_{\mu\nu}$  field, derived from the action (2.3), allows us to introduce a field  $\Psi$ , related to  $H_{\mu\nu\rho}$  through the relation:

$$g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} H_{\mu'\nu'\rho'} = -(-\det g)^{-\frac{1}{2}} e^{2\Phi} \epsilon^{\mu\nu\rho\sigma} \partial_\sigma \Psi, \quad g_{\mu\nu} \equiv e^{-\Phi} G_{\mu\nu}. \quad (3.1)$$

Let us now define the complex scalar field:

$$\lambda = \Psi + ie^{-\Phi} \equiv \lambda_1 + i\lambda_2. \quad (3.2)$$

Then the equations of motion and the Bianchi identities of the theory can be shown to be invariant under the transformations:

$$\begin{aligned} \lambda &\rightarrow \frac{a\lambda + b}{c\lambda + d}, & g_{\mu\nu} &\rightarrow g_{\mu\nu}, & M &\rightarrow M, \\ F_{\mu\nu}^{(a)} &\rightarrow (c\lambda_1 + d)F_{\mu\nu}^{(a)} + c\lambda_2(ML)_{ab}\tilde{F}_{\mu\nu}^{(b)}, & ad - bc &= 1, & a, b, c, d &\in \mathbf{R}, \end{aligned} \quad (3.3)$$

where,

$$\tilde{F}_{\mu'\nu'}^{(a)} = \frac{1}{2}(-\det g)^{-\frac{1}{2}} g_{\mu\mu'} g_{\nu\nu'} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^{(a)}. \quad (3.4)$$

Thus given a solution of the classical equations of motion, we can generate a new solution by performing the above  $SL(2, \mathbf{R})$  transformation on the original solution.

In accordance with our earlier spirit we shall look for solutions with fixed asymptotic values of various fields. Let us choose the asymptotic values of  $\Phi$  and  $\Psi$  to be zero. A solution with non-zero asymptotic values of  $\Phi$  and  $\Psi$  can be brought to this form by using the freedom of shifting  $\Phi$  and  $\Psi$  by constants keeping  $G_{\mu\nu}$ ,  $A_\mu^{(a)}$  and  $M$  fixed. Thus we do not suffer from any loss of generality by restricting our solutions this way. This forces us to consider only those  $SL(2, \mathbf{R})$  transformations which leave the point  $\Phi = \Psi = 0$  fixed. These transformations belong to an  $SO(2)$  subgroup of  $SL(2, \mathbf{R})$ , and are represented by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix}. \quad (3.5)$$

We can now apply this transformation to the electrically charged rotating black hole solution given in eqs.(2.29)-(2.37). Since  $g_{\mu\nu}$  does not transform under the  $SL(2, \mathbf{R})$  transformation, the geometry remains identical. The matrix valued scalar field  $M$  also remains the same as given in eqs.(2.35), (2.36). The fields  $\Phi$ ,  $A_\mu^{(a)}$  and  $B_{\mu\nu}$  (or equivalently  $\Psi$ ) change. We shall not write down the transformed solutions explicitly, but only give the asymptotic forms of the gauge field strengths which allow us to extract the electric and the magnetic charges associated with this solution. These are given by,

$$\begin{aligned} F_{\rho t}^{(a)} &\simeq \frac{n^{(a)} m}{\sqrt{2} \rho^2} \sinh \alpha \cosh \beta \cos \gamma \quad \text{for } 1 \leq a \leq 22, \\ &\simeq \frac{p^{(a-22)} m}{\sqrt{2} \rho^2} \sinh \beta \cosh \alpha \cos \gamma \quad \text{for } 23 \leq a \leq 28, \end{aligned} \quad (3.6)$$

$$\begin{aligned}
\tilde{F}_{\rho t}^{(a)} &\simeq -\frac{n^{(a)}}{\sqrt{2}} \frac{m}{\rho^2} \sinh \alpha \cosh \beta \sin \gamma \quad \text{for } 1 \leq a \leq 22, \\
&\simeq \frac{p^{(a-22)}}{\sqrt{2}} \frac{m}{\rho^2} \sinh \beta \cosh \alpha \sin \gamma \quad \text{for } 23 \leq a \leq 28.
\end{aligned} \tag{3.7}$$

This gives,

$$\begin{aligned}
Q_{el}^{(a)} &= \frac{n^{(a)}}{\sqrt{2}} m \sinh \alpha \cosh \beta \cos \gamma \quad \text{for } 1 \leq a \leq 22, \\
&= \frac{p^{(a-22)}}{\sqrt{2}} m \sinh \beta \cosh \alpha \cos \gamma \quad \text{for } 23 \leq a \leq 28,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
Q_{mag}^{(a)} &= -\frac{n^{(a)}}{\sqrt{2}} m \sinh \alpha \cosh \beta \sin \gamma \quad \text{for } 1 \leq a \leq 22, \\
&= \frac{p^{(a-22)}}{\sqrt{2}} m \sinh \beta \cosh \alpha \sin \gamma \quad \text{for } 23 \leq a \leq 28.
\end{aligned} \tag{3.9}$$

Although the above solution represents a rotating dyonic black hole solution, it does not correspond to a black hole with a general electric and magnetic charge vector. In particular, the electric and the magnetic charge vectors are related as

$$Q_{mag}^{(a)} = \tan \gamma L_{ab} Q_{el}^{(b)}. \tag{3.10}$$

We shall now discuss the construction of a rotating black hole solution carrying independent electric and magnetic charge vectors. The basic strategy that we employ is to make successive use of  $SL(2, R)$  and  $O(7, 23)$  transformations. Since these two sets of transformations do not commute, we would expect to produce solutions carrying charges that are more general than the ones given in eqs.(3.8), (3.9) by using this procedure. However, as pointed out in ref.[11], there is a potential problem with this approach. The magnetically charged solution is characterized by an  $A_{\mu}^{(a)}$  which is not globally defined, but need to be defined using two different coordinate patches. Now, if we perform a general  $O(7, 23)$  transformation on this solution, the component  $G_{ti}$  of the metric mixes with  $A_i^{(a)}$ . Thus in the resulting field configuration, the metric will not be globally defined, but need to be defined in two separate coordinate patches, one around the positive  $z$ -axis, and the other around the negative  $z$ -axis. On the overlap of the two coordinate patches, the two solutions are related by a coordinate transformation of the form:

$$t \rightarrow t + c\phi, \tag{3.11}$$

for some constant  $c$ . This, however is not a globally defined coordinate transformation, since it is not invariant under  $\phi \rightarrow \phi + 2\pi$ . Hence the solution constructed this way would not be an acceptable field configuration. Alternatively, one could use only one coordinate patch, but then the resulting metric will have a singularity either on the positive or on the negative  $z$ -axis.

This shows that we have to be careful about applying SL(2,R) and O(7,23) transformations on a solution successively, but it does not rule out the possibility that suitable combinations of these transformations can be found which give rise to a non-singular metric. We shall now show how this can be done in a systematic manner to produce a 58 parameter black hole solution carrying arbitrary mass, angular momentum, electric charge and magnetic charge. The first point to note is that since for time independent solutions the equations of motion of this theory are the same as those of the ten dimensional heterotic string theory for field configurations independent of seven of the ten dimensions (the time and the six internal coordinates), these equations of motion are expected to have an O(8,24) symmetry[21, 22].<sup>2</sup> To see how this O(8,24) symmetry manifests itself in the present case, let us first note that the equations of motion of the  $\bar{A}_i^{(\bar{a})}$  fields, derived from the action (2.14) gives,

$$\partial_i \left( \sqrt{\det \bar{G}} e^{-\bar{\Phi}} (\bar{M}\bar{L})_{\bar{a}\bar{b}} \bar{G}^{ii'} \bar{G}^{jj'} \bar{F}_{i'j'}^{(\bar{b})} \right) = 0, \quad 1 \leq \bar{a} \leq 30. \quad (3.12)$$

This allows us to introduce a set of fields  $\psi^{\bar{a}}$  through the equations

$$\sqrt{\det \bar{G}} e^{-\bar{\Phi}} (\bar{M}\bar{L})_{\bar{a}\bar{b}} \bar{G}^{ii'} \bar{G}^{jj'} \bar{F}_{i'j'}^{(\bar{b})} = \frac{1}{2} \epsilon^{ijk} \partial_k \psi^{\bar{a}}. \quad (3.13)$$

The bianchi identity for the gauge fields,  $\epsilon^{ijk} \partial_i \bar{F}_{jk}^{(\bar{a})} = 0$  gives,

$$\bar{D}^i (e^{\bar{\Phi}} (\bar{M}\bar{L})_{\bar{a}\bar{b}} \partial_i \psi^{\bar{b}}) = 0, \quad (3.14)$$

where  $\bar{D}_i$  denotes the covariant derivative with respect to the metric  $\bar{G}_{ij}$ . Let us now regard  $\psi$  as a 30 dimensional column vector and define,

$$\mathcal{M} = \begin{pmatrix} \bar{M} - e^{2\bar{\Phi}} \psi \psi^T & e^{2\bar{\Phi}} \psi & \bar{M}\bar{L}\psi - \frac{1}{2} e^{2\bar{\Phi}} \psi (\psi^T \bar{L}\psi) \\ e^{2\bar{\Phi}} \psi^T & -e^{2\bar{\Phi}} & \frac{1}{2} e^{2\bar{\Phi}} \psi^T \bar{L}\psi \\ \psi^T \bar{L}\bar{M} - \frac{1}{2} e^{2\bar{\Phi}} \psi^T (\psi^T \bar{L}\psi) & \frac{1}{2} e^{2\bar{\Phi}} \psi^T \bar{L}\psi & -e^{-2\bar{\Phi}} + \psi^T \bar{L}\bar{M}\bar{L}\psi - \frac{1}{4} e^{2\bar{\Phi}} (\psi^T \bar{L}\psi)^2 \end{pmatrix}, \quad (3.15)$$

$$\mathcal{L} = \begin{pmatrix} \bar{L} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (3.16)$$

and,

$$\bar{g}_{ij} = e^{-2\bar{\Phi}} \bar{G}_{ij}. \quad (3.17)$$

If we restrict ourselves to field configurations for which  $\bar{H}_{ijk} = 0$ , then eq.(3.14) and the other equations of motion derived from the action (2.14) can be shown to be identical to the equations of motion derived from the action:

$$\tilde{S} = \tilde{C} \int d^3x \sqrt{\det \bar{g}} [R_{\bar{g}} + \frac{1}{8} \bar{g}^{ij} Tr(\partial_i \mathcal{M} \mathcal{L} \partial_j \mathcal{M} \mathcal{L})]. \quad (3.18)$$

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<sup>2</sup>Actually, since the Kerr solution is independent of two coordinates,  $t$  and  $\phi$ , there is in fact an infinite parameter Geroch group of transformations that can be used to generate new solutions[17]. However, most of these solutions are singular, and so we restrict ourselves to the O(8,24) group of transformations.



We shall now study the effect of these transformations on the original Kerr solution (2.23). The metric  $\bar{g}_{ij}$  and matrix  $\mathcal{M}$  for this solution can be easily computed, and the result is,

$$\bar{g}_{ij}dx^i dx^j = (\rho^2 + a^2 \cos^2 \theta - 2m\rho) \left[ \frac{1}{\rho^2 + a^2 - 2m\rho} d\rho^2 + d\theta^2 + \frac{\rho^2 + a^2 - 2m\rho}{\rho^2 + a^2 \cos^2 \theta - 2m\rho} \sin^2 \theta d\phi^2 \right], \quad (3.26)$$

$$\mathcal{M} = \begin{pmatrix} I_{28} & 0 & 0 & 0 & 0 \\ 0 & -f^{-1} & 0 & 0 & -g \\ 0 & 0 & -f - fg^2 & g & 0 \\ 0 & 0 & g & -f^{-1} & 0 \\ 0 & -g & 0 & 0 & -f - fg^2 \end{pmatrix} \equiv \mathcal{M}^{(0)}, \quad (3.27)$$

where,

$$\begin{aligned} f &= (\rho^2 + a^2 \cos^2 \theta - 2m\rho) / (\rho^2 + a^2 \cos^2 \theta) \\ g &= 2ma \cos \theta / (\rho^2 + a^2 \cos^2 \theta - 2m\rho). \end{aligned} \quad (3.28)$$

It is clear that the matrix  $\mathcal{U}\mathcal{M}^{(0)}\mathcal{U}^T$  is left invariant under the  $O(22) \times O(6)$  subgroup of the  $O(22,2) \times O(6,2)$  group, which act on the first 22 and the 23rd-28th indices of the matrix respectively. What is perhaps not so obvious is that this matrix is also left invariant under an  $SO(2)$  subgroup of  $O(22,2) \times O(6,2)$ , represented by the matrix

$$\mathcal{U}\tilde{\Omega}\mathcal{U}^T = \begin{pmatrix} I_{28} & & & & \\ & \cos \alpha & 0 & \sin \alpha & 0 \\ & 0 & \cos \alpha & 0 & \sin \alpha \\ & -\sin \alpha & 0 & \cos \alpha & 0 \\ & 0 & -\sin \alpha & 0 & \cos \alpha \end{pmatrix}. \quad (3.29)$$

This can be verified by explicit computation. Thus the transformations which generate inequivalent solutions are parametrized by the coset:

$$(O(22,2) \times O(6,2)) / (O(22) \times O(6) \times SO(2)). \quad (3.30)$$

This coset is parametrized by 57 parameters. This, together with the original parameters  $m$  and  $a$ , gives a 59 parameter solution. But we now recall that not all of these solutions are non-singular, due to the possible singularity in the  $G_{t\phi}$  component of the metric discussed previously. The presence of such a singularity is signalled by an asymptotic value of  $\partial_\theta G_{t\phi}$  proportional to  $\sin \theta$ , in the same way that the presence of magnetic charge associated with the gauge field  $A_\mu^{(a)}$  is signalled by the presence of an asymptotic value of  $F_{\theta\phi}^{(a)}$  proportional to  $\sin \theta$ . From eqs.(2.8) and (3.13) we see that such an asymptotic form of  $G_{t\phi}$  would induce an asymptotic  $\psi^{30}$  proportional to  $1/\rho$ . Thus in order to get a non-singular metric, we must demand that the coefficient of  $1/\rho$  in the asymptotic expression of  $\psi^{30}$  must vanish. (This is equivalent to demanding that the coefficient  $c$  appearing in eq.(3.11) vanishes.) This gives one constraint among the 59 parameters, thereby reducing the number of independent parameters to 58. Using eqs.(3.15), (3.19), (3.27) and (3.28) we can derive an explicit form of this constraint on the  $O(22,2) \times O(6,2)$  matrix  $\tilde{\Omega}$ :

$$\tilde{\Omega}_{30,29}\tilde{\Omega}_{31,29} + \tilde{\Omega}_{30,31}\tilde{\Omega}_{31,31} - \tilde{\Omega}_{30,30}\tilde{\Omega}_{31,30} - \tilde{\Omega}_{30,32}\tilde{\Omega}_{31,32} = 0. \quad (3.31)$$



Note that 58 is precisely the expected number of parameters required to label the most general static black hole solution consistent with the no hair theorem, since such a black hole will be characterized by mass, angular momentum, 28 electric charges and 28 magnetic charges. Thus although we have not explicitly shown that the solutions constructed this way are free from naked singularities, there is good reason to believe that this is indeed the case.

We shall end this section by giving an interpretation of the 59th parameter that takes us out of the class of non-singular solutions. A typical representative  $O(8,24)$  transformation which does not satisfy (3.31), and hence, acting on a Kerr solution, produces a singular solution, is given by,

$$\mathcal{U}\tilde{\Omega}\mathcal{U}^T = \begin{pmatrix} I_{28} & & & & \\ & \cos \alpha & 0 & \sin \alpha & 0 \\ & 0 & 1 & 0 & 0 \\ & -\sin \alpha & 0 & \cos \alpha & 0 \\ & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.32)$$

For simplicity, let us restrict ourselves to studying the effect of this transformation on the Schwarzschild solution ( $a = 0$ ). In this case, the transformed solution is given by,

$$\begin{aligned} G_{\mu\nu}dx^\mu dx^\nu &= -\frac{\rho(\rho - 2m)}{\tilde{\Delta}}(dt + m \sin \alpha \cos \theta d\phi)^2 + \frac{\tilde{\Delta}}{\rho(\rho - 2m)}d\rho^2 + \tilde{\Delta}(d\theta^2 + \sin^2 \theta d\phi^2), \\ \Phi &= 0 \quad A_\mu^{(a)} = 0, \quad M = I_{28}, \quad B_{\mu\nu} = 0, \end{aligned} \quad (3.33)$$

where,

$$\tilde{\Delta} = \rho^2 - 4m\rho \sin^2 \frac{\alpha}{2} + 4m^2 \sin^2 \frac{\alpha}{2}. \quad (3.34)$$

This can be easily recognized as the Taub-NUT solution[29]. The action of the other elements of the  $O(8,24)$  group then produces the Taub-NUT dyon solutions discussed in refs.[26, 30]. For non-zero  $a$ , the solutions generated this way correspond to rotating Taub-NUT dyon solutions given in ref.[18].

Thus we see that the rotating dyonic Taub-NUT solutions can also be generated via  $O(8,24)$  transformation of the original Kerr solution. This has already been observed before[18, 19, 20], and can be traced to the fact that the transformation on the fields represented by the  $O(8,24)$  matrix (3.32) is precisely the Ehlers-Geroch transformation[31] that takes us from the Schwarzschild solution to the Taub-NUT family of solutions. To this effect, we note that the scalar field  $\psi$ <sup>30</sup> defined through eqs.(3.13), (2.8) is simply the generalization of the NUT potential[32], and requiring it to fall off faster than  $1/\rho$  asymptotically corresponds to setting the NUT charge to zero.

## 4 Conclusion

In this paper we have explicitly constructed the general electrically charged rotating black hole solution in heterotic string theory compactified on a six dimensional torus. We have also given

an algorithm to construct the general rotating black hole solution carrying both, electric and magnetic charges. This is the most general black hole solution in this theory consistent with no hair theorem.

We hope that these results can be used to study the relationship between black holes and elementary string states in the spirit of refs.[2, 3]. To this end, we shall conclude this paper by pointing out some amusing coincidences between the properties of black holes and those of elementary strings in the extremal limit. First we compare the gyromagnetic ratios of the solutions found in this paper and those of elementary string states in the limit  $\beta \rightarrow \infty$  with the physical mass  $M$ , the physical angular momentum  $J$ , and the parameter  $\alpha$  fixed. Here  $\alpha$  and  $\beta$  are the parameters appearing in the black hole solution given in section 2. (Note that in this limit the solution develops naked singularity, but we shall ignore this problem. We can take the limit  $J \rightarrow 0$  at the end of the calculation, so that the solution approaches an extremal black hole.) From eq.(2.44) we see that in this limit,

$$g_L \rightarrow 0, \quad g_R \rightarrow 2. \quad (4.1)$$

On the other hand, these gyromagnetic ratios for elementary string states can be computed using a slight generalization of the work of ref.[3]. The answer is,

$$g_L = 2 \frac{S_R}{S_R + S_L}, \quad g_R = 2 \frac{S_L}{S_R + S_L}, \quad (4.2)$$

where  $S_L$  and  $S_R$  denote the contribution to the  $z$  component of the angular momentum from the left and the right moving oscillators respectively. From eqs.(2.38) - (2.40) we see that in the limit  $\beta \rightarrow \infty$ , with  $M$ ,  $J$  and  $\alpha$  fixed,  $M^2 - \vec{Q}_R^2/2 \rightarrow 0$ . Thus these states saturate the Bogomol'nyi bound, which, in turn, implies that the right hand part of this state must be the lowest state in the Neveu-Schwarz (or Ramond) sector consistent with GSO projection[25]. Thus for these states  $|S_R| \leq \hbar$ , and most of the contribution to  $J$  comes from  $S_L$ . This shows that in this limit  $|S_R| \ll |S_L|$ , and  $g_L$  and  $g_R$  given in eq.(4.2) agree with those given in eq.(4.1).

Next we shall compare the area of the stretched horizon and the logarithm of the density of single string states in the extremal limit. We shall restrict ourselves to the non-rotating solution since in this case the extremal limit (2.58) corresponds to saturation of the Bogomol'nyi bound, and various non-renormalization theorems are expected to be valid. As we can see from eq.(2.61), the area of the event horizon vanishes in the limit (2.58). However, if  $A_s$  denotes the area of the surface  $\rho = \rho_H^+ + \eta$ , where  $\eta$  is a fixed length of the order of the Planck length, then using eq.(2.52) we find that it is non-zero and has the value,

$$A_s = 8\pi M\eta / \cosh \alpha = 8\pi\eta \sqrt{M^2 - \frac{\vec{Q}_L^2}{2}}. \quad (4.3)$$

This surface is similar to the stretched horizon discussed in refs.[2, 3]. On the other hand, the density of states of elementary string excitations in the extremal limit  $M^2 \rightarrow \vec{Q}_R^2/2$  can be easily calculated (see, for example, ref.[3]) and is given by,

$$d_S \sim \exp(4\pi\sqrt{n}), \quad (4.4)$$

where  $n$  denotes the total contribution to mass<sup>2</sup> of the state from the left moving oscillators. (Note that in this case there is no appreciable contribution to the density of states from the right moving oscillators.) This, in turn, is proportional to  $M^2 - (\vec{Q}_L^2/2)$ . Thus we see that,

$$A_s \propto \ln d_S. \quad (4.5)$$

This establishes the relationship between the density of string states and the area of the stretched horizon in the extremal limit.

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