A Twist in the Dyon Partition Function[1]

Ashoke Sen

Harish-Chandra Research Institute
Chhatnag Road, Jhusi, Allahabad 211019, India
and
LPTHE, Universite Pierre et Marie Curie, Paris 6
4 Place Jussieu, 75252 Paris Cedex 05, France

E-mail: sen@mri.ernet.in, ashokesen1999@gmail.com

Abstract

In four dimensional string theories with $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supersymmetries one can often define twisted index in a subspace of the moduli space which captures additional information on the partition function than the ones contained in the usual helicity trace index. We compute several such indices in type IIB string theory on $K3 \times T^2$ and T^6, and find that they share many properties with the usual helicity trace index that captures the spectrum of quarter BPS states in $\mathcal{N} = 4$ supersymmetric string theories. In particular the partition function is a modular form of a subgroup of $Sp(2;\ \mathbb{Z})$ and the jumps across the walls of marginal stability are controlled by the residues at the poles of the partition function. However for large charges the logarithm of this index grows as $1/N$ times the entropy of a black hole carrying the same charges where N is the order of the symmetry generator that is used to define the twisted index. We provide a macroscopic explanation of this phenomenon using quantum entropy function formalism. The leading saddle point corresponding to the attractor geometry fails to contribute to the twisted index, but a $\mathbb{Z}_N$ orbifold of the attractor geometry produces the desired contribution.

[1]Dedicated to the memory of Alok Kumar and Jaydeep Majumder.

Contents

1 Introduction and Summary

We now have a good understanding of the spectrum of dyons in $\mathcal{N}=4$ and $\mathcal{N}=8$ supersymmetric string theories [1–33]. For large charges the result for the degeneracy agrees with the macroscopic entropy of a black hole carrying the same charges. We also have a good understanding of how to systematically compute higher derivative corrections [34–38] and quantum corrections [39–42] to the black hole entropy. Some of these corrections have already been used to test the correspondence between the microscopic and black hole entropies beyond the leading order. Eventually one hopes to be able to systematically compute the corrections to the black hole entropy using these techniques, and compare the results with the microscopic answer, thereby testing the correspondence between macroscopic and microscopic entropies to much finer detail. Some attempt to generalize these results to half BPS black holes in the $\mathcal{N}=2$ supersymmetric STU model has also been made in [43, 44].

On the microscopic side one often computes an index rather than the absolute degeneracy, defined so that it receives contribution only from the BPS states in the spectrum. As a result these indices are protected and do not vary continuously as we vary the moduli of the theory.

In four dimensions the standard index is the helicity trace index B_{2n} defined as follows [45,46]

$$B_{2n} = \frac{1}{(2n)!} Tr \left[(-1)^{2h}(2h)^{2n}\right] \, , \tag{1.1}$$

where h is the third component of the angular momentum of a state in the rest frame, and the trace is taken over all states carrying a given set of charges. In order that a given state gives a non-vanishing contribution to this index, the number of supersymmetries broken by the state must be less than or equal to $4n$. This is due to the fact that for every pair of broken supersymmetries we have a pair of fermion zero modes whose quantization gives a bose-fermi degenerate pair of states. As a result $Tr(-1)^{2h}$ will vanish unless we insert a factor of $2h$ which prevents the cancelation between these pair of states, thereby effectively soaking up the pair of fermion zero modes. Thus if we have more than $4n$ broken supersymmetries, and hence more that $4n$ fermion zero modes, then B_{2n} does not contain enough insertions of $2h$ to soak up all the fermion zero modes, and the contribution to the trace from such states vanishes. On the other hand if we have states with precisely $4n$ broken supersymmetries then B_{2n} receives contribution from these states, but not from any other states with more than $4n$ broken supersymmetries. This makes B_{2n} the ideal index for capturing protected information on states with $4n$ broken supersymmetries. Some standard examples are B_2 for half BPS states in $\mathcal{N}=2$ supersymmetric theories, B_4 for half BPS states in $\mathcal{N}=4$ supersymmetric theories, B_6 for quarter BPS states in $\mathcal{N}=4$ supersymmetric theories, B_{14} for 1/8 BPS states in $\mathcal{N}=8$ supersymmetric theories etc. The normalization in the definition of B_{2n} has been adjusted so that the contribution to the trace from the $4n$ fermion zero modes due to broken supersymmetries exactly cancels the denominator factor of $(2n)!$ except for a sign given by $(-1)^n$.

Given that on the macroscopic side the black hole entropy always gives the absolute degeneracy whereas on the microscopic side we compute the helicity trace index, one might wonder whether comparing the two is justified. A resolution of this issue was proposed in [42] where it was shown how using the expression for the degeneracy on the macroscopic side one can compute the helicity trace index. This can then be compared with the microscopic results. This argument will be reviewed in §6.

In this paper we shall study a modified index obtained by twisting the helicity trace index by an appropriate discrete symmetry transformation – both on the microscopic and the macroscopic side – and compare the results. For this we need to restrict the moduli to be on special subspaces of the moduli space where the theory admits extra discrete symmetries generated by an element g, and restrict the charges carried by the dyon to be such that they are invariant under g. In this case we can define a new twisted index as[2]

$$B_{2n}^g \equiv \frac{1}{(2n)!} Tr \left[g \, (-1)^{2h} (2h)^{2n} \right] . \tag{1.2}$$

If all the supersymmetry generators of the theory are invariant under g then our earlier counting holds and we conclude that B_{2n}^g does not receive any contribution from states which breaks more than $4n$ supercharges. However suppose that some of the broken supersymmetry generators are not invariant under g. In that case the corresponding fermion zero modes are also not invariant under g, and the contribution from these modes in $Tr(-1)^{2h} g$ will not vanish. As a result we do not need any factor of $2h$ to soak up this pair of fermion zero modes. If on the other hand we have a pair of fermion zero modes which are invariant under g then we need a factor of $2h$ in the trace to soak up these zero modes. *Thus B_{2n}^g receives non-vanishing contribution from states which break less than or equal to $4n$ g-invariant supersymmetries, – the total number of broken supersymmetries can be more than $4n$.* Conversely, if we have a state that breaks certain number of supersymmetres, and if $4n$ of these broken supersymmetries are invariant under g, then the ideal g-twisted index for capturing information about the spectrum of these states is B_{2n}^g. Since we expect B_{2n}^g to have properties similar to that for B_{2n} (*e.g.* for wall crossing [49–54]), this allows us to introduce an index in extended supersymmetric theories which behaves as an index in a theory with less number of supersymmetries.[3]

The main goal of this paper will be to compute such twisted indices in type II string theory compactified on $\mathcal{M} \times T^2$ where $\mathcal{M}$ can be either T^4 or $K3$. We choose g to be the generator of a geometric $\mathbb{Z}_N$ symmetry acting on $\mathcal{M}$ preserving 16 supersymmetries. For type II string theory on $T^4 \times T^2$ this requires g to commute with 16 out of the 32 unbroken supersymmetries, while for type IIB string theory on $K3 \times T^2$ this will require g to commute with all the 16 unbroken supersymmetries. Examples of such $\mathbb{Z}_N$ transformations have been discussed in [59, 60], and the dyon spectrum on orbifolds of the original theory by these symmetries (accompanied by a translation along a circle) have been analyzed in [9–11]. These (without

[2]Black holes carrying discrete gauge charges have been discussed earlier in [47, 48].

[3]The use of such indices is not new. In particular such an index has been analyzed in detail in $\mathcal{N} = 4$ supersymmetric gauge theories where it was found that supersymmetric multi-monopole solutions preserving quarter of the supersymmetries exist only on a subspace of the moduli space [55–58]. As a result these do not contribute to the usual helicity trace index B_6 which should have the same value everywhere in the moduli space (leaving aside possible jumps across walls of marginal stability). Nevertheless one can define appropriate index to capture information about the multi-monopole states in the particular subspace of the moduli space where the solution exists.

the translations along the circle) will be the $\mathbb{Z}_N$ transformations we shall be using in our analysis. However here we do not take the orbifold of the original theory; we simply use g for defining a twisted partition function in the original theory. In this theory we consider dyonic states preserving 4 supersymmetries all of which are g invariant. Such a dyon breaks 12 of the 16 g-invariant supersymmetries, and the relevant g-twisted index is B_6^g. We find explicit expression for this index in the examples described above, and find that the index has properties similar to B_6 in $\mathcal{N} = 4$ supersymmetric string theories, – the helicity trace index used for encoding information on 1/4 BPS states in this theory. In particular:

1. The index is given by the Fourier transform of the inverse of a modular form of a subgroup of $Sp(2, \mathbb{Z})$.

2. The value of the index in different domains separated by the walls of marginal stability is controlled by the same partition function – with the information on the domain being encoded in the choice of contour along which the Fourier integral is to be performed.

3. The jumps across the walls of marginal stability are controlled by the residues at certain poles of the partition function. The resulting expression for the jump follows the same wall crossing formula as the usual B_6 index.

4. The growth of the index for large charges is controlled by another set of poles of the partition function.

There is however an important difference. If we associate an 'entropy' to this index defined by taking its logarithm, we find that for large charges the entropy is given by

$$S_{BH}/N\,, \tag{1.3}$$

where S_{BH} is the entropy of a black hole carrying the same set of charges as the dyon and N is the order of g.

Given this result for the index it is natural to ask if this can be explained from the macroscopic viewpoint. We show that it is indeed possible to provide an explanation using the quantum entropy function formalism [41,42], – a proposal for calculating systematic quantum corrections to the black hole entropy as a path integral of string theory over the near horizon geometry of the black hole. We find that when we follow the same prescription to compute the twisted index B_6^g, the path integral must be carried out over field configurations satisfying

g-twisted boundary condition along the boundary circle of the AdS_2 factor of the near horizon geometry. Since the circle is contractible in the interior of AdS_2, this is not an allowed boundary condition on the fields in the attractor geometry. As a result the saddle point corresponding to the attractor geometry does not contribute to the path integral. However a $\mathbb{Z}_N$ orbifold of the attractor geometry, which has the same asymptotics as the attractor geometry, does contribute and gives a contribution to the path integral whose semiclassical value is $\exp(S_{BH}/N)$. This provides a natural explanation for the microscopic result (1.3).

The rest of the paper is organised as follows. In §2 we consider the simple example of type II string theory on $K3 \times T^2$ and compute the index B_6^g for a $\mathbb{Z}_2$ transformation g that acts geometrically on $K3$ and commutes with all the symmetries. The result is expressed as a triple Fourier integral of a 'partition function'. In §3 we study various properties of this partition function by relating it to a threshold integral [7, 8, 61–63]. In particular we show that it transforms as a modular form under a subgroup of $Sp(2, \mathbb{Z})$. We also determine the location of its zeroes and poles. In §4 we use these properties to derive properties of the index. In particular we prove the S-duality invariance of the index and show that the jump in this index across a wall of marginal stability is controlled by the residues at the poles of the partition function. We also determine the behaviour of the index for large charges, and find that the 'entropy', defined as the logarithm of the index, is half of the entropy of a black hole carrying the same charges. In §5 we generalize these results to type IIB string theory on $\mathcal{M} \times T^2$ where $\mathcal{M} = K3$ or T^4, with g chosen as a $\mathbb{Z}_N$ transformation in $\mathcal{M}$ which preserves 16 of the supersymmetries of the theory. In this case the 'entropy' associated with the index grows as $1/N$ times the entropy of the black hole. In §6 we provide a macroscopic explanation of this phenomenon using quantum entropy function formalism. We end in §7 by discussing some other possible applications of this twisted index.

2 Computation of a $\mathbb{Z}_2$ Twisted Index in type II String Theory on $K3 \times T^2$

In this section we shall consider the spectrum of 1/8 BPS dyons in type IIB string theory on $K3 \times T^2$, identify g as a specific geometric $\mathbb{Z}_2$ symmetry of $K3$ used in [6, 9, 18] that preserves the covariantly constant spinors of $K3$ and leaves invariant 14 of the 22 2-cycles of $K3$, and calculate the index B_6^g for these dyons. This of course forces the moduli of $K3$ to lie in a subspace of the full moduli space admitting this symmetry. We shall denote the a and b

cycles of T^2 by S^1 and $\widetilde{S}^1$, and focus on a specific class of dyons consisting of one D5-brane wrapped along $K3 \times S^1$, (Q_1+1) D1-branes wrapped along S^1 and one Kaluza-Klein (KK) monopole associated with the circle $\widetilde{S}^1$, carrying $-n$ units of momentum along S^1 and J units of momentum along $\widetilde{S}^1$.[4] A duality map involving an S-duality of the ten dimensional type IIB string theory, followed by a T-duality along $\widetilde{S}^1$ and finally a string string duality transformation that relates type IIA string theory on $K3$ to heterotic string theory of T^4, brings this state to a specific state in heterotic string theory. Under this duality map the charges Q_1, J and the single D5-brane wrapping along $K3 \times S^1$ become components of the magnetic charge P, the charges n and the single KK-monopole charge associated with $\widetilde{S}^1$ become components of the electric charge Q, and the transformation g gets mapped to a specific symmetry of the heterotic string theory that exchanges the two E_8 factors of the gauge group. Thus it acts only on the left-moving modes of the world-sheet and preserves all the supersymmetries. The duality group of the theory is $SO(6,22;\ \mathbb{Z})_T \times SL(2,\ \mathbb{Z})_S$ where the subscripts T and S denote that in this heterotic frame they appear as T- and S-duality symmetries respectively. If we denote by Q^2, P^2 and $Q \cdot P$ the $SO(6,22)$ invariant bilinears in the charges then for this particular charge vector Q^2, P^2 and $Q \cdot P$ are given by the relations [9, 18]

$$Q^2 = 2n, \quad P^2 = 2Q_1, \quad Q \cdot P = J\,. \tag{2.1}$$

We define the partition function $Z(\rho, \sigma, v)$ via the relation:

$$Z(\rho,\sigma,v) = \sum_{n,Q_1,J} B_6^g(n,Q_1,J)\,(-1)^J e^{2\pi i Q_1\sigma + 2\pi i n\rho + 2\pi i J v}\,, \tag{2.2}$$

where $B_6^g(n, Q_1, J)$ is the contribution to B_6^g from states carrying quantum numbers (n, Q_1, J). Inverting this relation we get

$$-B_6^g(n,Q_1,J) = (-1)^{J+1} \int_0^1 d\rho \int_0^1 d\sigma \int_0^1 dv\, e^{-2\pi i Q_1\sigma - 2\pi i n\rho - 2\pi i J v}\, Z(\rho,\sigma,v)\,. \tag{2.3}$$

The computation of the $Z(\rho, \sigma, v)$ proceeds as in [9] where the B_6 index of dyons was calculated in type IIB string theory on $K3 \times T^2$, modded out by a $\mathbb{Z}_2$ transformation that involved the same symmetry g accompanied by half unit of translation along S^1. We shall follow the notations of [18] where these results were reviewed. The main difference between our analysis and the one given in [9] will be that 1) here we do not remove any mode since we

[4] The analysis can be generalized to the case of multiple (Q_5) D5-branes following [9]. In any case the final result depends only on the combination $Q_5(Q_1 - Q_5)$.

are not considering an orbifold, and 2) instead of correlating the momentum along S^1 with the g quantum number as in [9], here we use the g quantum number of a mode as the weight of its contribution to the index. However as far as the counting of the modes and their bahaviour under g transformation are concerned, we can directly use the results of [9]. As in the case of [9] we shall express Z as a product of several independent pieces:

1. Partition function for the excitation modes of the KK monopole.
2. Partition function for the D1-D5 center of mass motion in the KK monopole background. This can be further divided into the contribution from the zero modes and the contribution from the non-zero modes.
3. Partition function associated with the motion of the D1-branes relative to the D5-brane.

The world volume degrees of freedom of the three component systems listed above, and the quantum numbers carried by them, including their transformation properties under g, can be found in [9, 18]. As in [9, 18], we shall work in the convention where the four g-invariant unbroken supersymmetries of the system act on the right-moving modes, i.e. modes carrying positive momentum along S^1. In this convention the requirement of unbroken supersymmetry forces all the right-moving modes into their ground states.

In the process of computing the index we must make sure that all the g-invariant fermion zero modes are absorbed by the factors of $2h$ inserted into the trace. Since in the present example g commutes with all the supersymmetries of the theory, all the fermion zero modes associated with the broken supersymmetries will be g even. In particular the KK monopole world-volume breaks 8 of the 16 supersymmetries, producing 8 fermion zero modes. These eight g-even fermion zero modes are soaked up by 4 factors of $2h$ in B^g_6. The D1-D5 system in the KK monopole background breaks four more supersymmetries. This exhausts all the fermion zero modes associated with broken supersymmetry generators as long as we consider only BPS excitations, i.e. excitations carrying negative momentum along S^1.

We begin by computing the contribution to the partition function from the KK monopole. It follows from the results reviewed in [18] that the world-volume of the KK monopole has 16 left-moving bosonic oscillators even under g and 8 left-moving bosonic oscillators odd under g. None of these excitations carry any D1-brane charge or momentum along $\widetilde{S}^1$. Furthermore the ground state of the Kaluza-Klein monopole carries -1 unit of left-moving momentum, i.e. one unit of momenum along S^1. Thus the net contribution to the partition function from these modes is given by

$$Z_{KK} = e^{-2\pi i\rho} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n\rho}\right)^{-16} \prod_{n=1}^{\infty} \left(1 + e^{2\pi i n\rho}\right)^{-8} . \tag{2.4}$$

There are also eight right-moving fermion zero modes which are absorbed by four factors of $2h$ inserted into the trace.

Next we turn to the contribution from the D1-D5 center of mass motion in the background of the KK monopole. Again the various modes and their quantum numbers can be read out from the results reviewed in [18]. In particular all of these modes are g-invariant. There are four fermion zero modes associated with broken supersymmetry; they are absorbed by two factors of $2h$ inserted into the helicity trace. The dynamics of the rest of the zero modes is described by an interacting supersymmetric quantum mechanics with Taub-NUT target space, and using the results of [64, 65] one finds that the corresponding contribution to the partition function is given by $-e^{-2\pi iv}/(1-e^{-2\pi iv})^2$ [18]. On the other hand the non-zero mode oscillators consist of four g-invariant left-moving bosonic modes carrying ± 1 units of momentum along $\widetilde{S}^1$, and four g-invariant left-moving fermionic modes carrying no momentum along $\widetilde{S}^1$.[5] Thus the net contribution to the partition function from the zero modes and the non-zero mode oscillators associated with the D1-D5-brane center of mass motion is given by

$$Z_{cm} = -e^{-2\pi iv} \left(1 - e^{-2\pi iv}\right)^{-2} \prod_{n=1}^{\infty} \left\{ (1 - e^{2\pi i n\rho})^4 \left(1 - e^{2\pi i n\rho + 2\pi iv}\right)^{-2} \left(1 - e^{2\pi i n\rho - 2\pi iv}\right)^{-2} \right\} . \tag{2.5}$$

Finally we turn to the contribution from the motion of the D1-branes relative to the D5-brane. For this we first need to introduce some auxiliary quantities. The low energy dynamics of a single D1-brane inside the D5-brane, wound once along S^1, is described by a superconformal field theory with target space $K3$. We define [18]

$$F^{(r,s)}(\tau, z) \equiv \frac{1}{2} Tr_{RR;g^r} \left(g^s (-1)^{J_L + J_R} e^{2\pi i\tau L_0} e^{-2\pi i\bar{\tau}\bar{L}_0} e^{2\pi i J_L z} \right) , \qquad r, s = 0, 1 , \tag{2.6}$$

where Tr denotes trace over all the g^r twisted Ramond-Ramond (RR) sector states in this CFT, and $J_L/2$ and $J_R/2$ denote the generators of the $U(1)_L \times U(1)_R$ subgroup of the $SU(2)_L \times SU(2)_R$ R-symmetry group of this conformal field theory. In our convention the current associated with J_L is holomorphic and the one associated with J_R is anti-holomorphic. Explicit

[5] At the center of the KK monople the quantum number labeling $\widetilde{S}^1$ momentum can be identified as the $U(1)_L$ generator of the $SU(2)_R \times SU(2)_L$ rotation group in the tangent space of the KK monopole.

computation gives [7]

$$F^{(0,0)}(\tau,z) = 4\left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right],$$
$$F^{(0,1)}(\tau,z) = 4\,\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2}, \quad F^{(1,0)}(\tau,z) = 4\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}, \quad F^{(1,1)}(\tau,z) = 4\,\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2}, \tag{2.7}$$

where ϑ_i are the Jacobi theta functions. (2.7) can be rewritten as

$$F^{(r,s)}(\tau,z) = h_0^{(r,s)}(\tau)\,\vartheta_3(2\tau,2z) + h_1^{(r,s)}(\tau)\,\vartheta_2(2\tau,2z) \tag{2.8}$$

where

$$\begin{aligned}
h_0^{(0,0)}(\tau) &= 8\,\frac{\vartheta_3(2\tau,0)^3}{\vartheta_3(\tau,0)^2\vartheta_4(\tau,0)^2} + 2\,\frac{1}{\vartheta_3(2\tau,0)} \\
h_1^{(0,0)}(\tau) &= -8\,\frac{\vartheta_2(2\tau,0)^3}{\vartheta_3(\tau,0)^2\vartheta_4(\tau,0)^2} + 2\,\frac{1}{\vartheta_2(2\tau,0)} \\
h_0^{(0,1)}(\tau) &= 2\,\frac{1}{\vartheta_3(2\tau,0)}, \qquad h_1^{(0,1)}(\tau) = 2\,\frac{1}{\vartheta_2(2\tau,0)}, \\
h_0^{(1,0)}(\tau) &= 4\frac{\vartheta_3(2\tau,0)}{\vartheta_4(\tau,0)^2}, \qquad h_1^{(1,0)}(\tau) = -4\frac{\vartheta_2(2\tau,0)}{\vartheta_4(\tau,0)^2}, \\
h_0^{(1,1)}(\tau) &= 4\frac{\vartheta_3(2\tau,0)}{\vartheta_3(\tau,0)^2}, \qquad h_1^{(1,1)}(\tau) = 4\frac{\vartheta_2(2\tau,0)}{\vartheta_3(\tau,0)^2}\,.
\end{aligned} \tag{2.9}$$

We now define the coefficients $c_b^{(r,s)}(u)$ through the expansions

$$h_b^{(r,s)}(\tau) = \sum_{n\in\frac{1}{2}\mathbf{Z}-\frac{1}{4}b^2} c_b^{(r,s)}(4n)q^n, \qquad b=0,1\,. \tag{2.10}$$

Substituting (2.10) into (2.8) and using the Fourier expansions of $\vartheta_3(2\tau,2z)$, $\vartheta_2(2\tau,2z)$ we get

$$F^{(r,s)}(\tau,z) = \sum_{b=0}^{1}\sum_{j\in 2\mathbf{Z}+b,\,n\in\frac{1}{2}\mathbf{Z}} c_b^{(r,s)}(4n-j^2)e^{2\pi in\tau+2\pi ijz}\,. \tag{2.11}$$

Consider now the motion of a single D1-brane, wound w times along S^1, inside a D5-brane. The dynamics of this system is described by a superconformal field theory with target space $K3$, but since the D1-brane has length w times the period of S^1, one unit of left-moving momentum along S^1 will appear as w units of left-moving momentum (L_0) on the D1-brane.

Also in the background of the Kaluza-Klein monopole the quantum numbers (J_L, J_R) can be identified respectively with the $\widetilde{S}^1$ momentum J and the fermion number F of the four dimensional theory [3, 5, 66]. We denote by $(-1)^j\, n(w,l,j;k)$ the total number of bosonic minus fermionic states of this D1-brane, carrying g quantum number $(-1)^k$, momentum $-l$ along S^1 and momentum j along $\widetilde{S}^1$. Then it follows from (2.6), (2.11) that [9, 18][6]

$$n(w,l,j;k) = \sum_{s=0}^{1} (-1)^{sk}\, c_b^{(0,s)}(4lw - j^2)\,, \quad l, w, j \in \mathbb{Z}\,, \quad b = j \bmod 2, \quad l \geq 0, \quad w \geq 1\,. \tag{2.12}$$

Using this and the techniques of [67] we can compute the contribution to the partition function from the general motion of the D1-branes inside a D5-brane, where we have excitations involving multiple D1-branes carrying different values of w, l and j. This is given by

$$\begin{aligned} Z_{D1D5} &= e^{-2\pi i\sigma} \prod_{w=1}^{\infty}\prod_{l=0}^{\infty}\prod_{j\in\mathbb{Z}}\prod_{k=0}^{1} \left(1 - (-1)^k\, e^{2\pi i(w\sigma + l\rho + jv)}\right)^{-n(w,l,j;k)} \\ &= e^{-2\pi i\sigma} \prod_{b=0}^{1}\prod_{w=1}^{\infty}\prod_{l=0}^{\infty}\prod_{j\in 2\mathbb{Z}+b}\prod_{k=0}^{1} \left(1 - (-1)^k\, e^{2\pi i(w\sigma + l\rho + jv)}\right)^{-\sum_{s=0}^{1}(-1)^{sk}\, c_b^{(0,s)}(4lw - j^2)}\,. \end{aligned} \tag{2.13}$$

The extra factor of $e^{-2\pi i\sigma}$ reflects that the total number of D1-branes is $Q_1 + 1$ and not Q_1. This extra shift in Q_1 by 1 accounts for the fact that a single D5-brane wrapped on $K3$ carries -1 unit of D1-brane charge. Using the results

$$\begin{aligned} &c_0^{(0,0)}(0) = 10\,, \qquad c_1^{(0,0)}(-1) = 1\,, \qquad c_0^{(0,1)}(0) = 2\,, \qquad c_1^{(0,1)}(-1) = 1\,, \\ &c_0^{(1,0)}(0) = 4, \qquad c_1^{(1,0)}(-1) = 0, \qquad c_0^{(1,1)}(0) = 4, \qquad c_1^{(1,1)}(-1) = 0\,, \end{aligned} \tag{2.14}$$

we can express the total partition function as

$$\begin{aligned} Z(\rho,\sigma,v) &= Z_{KK} Z_{cm} Z_{D1D5} = 1/\Phi(\rho,\sigma,v)\,, \\ \Phi(\rho,\sigma,v) &= e^{2\pi i(\rho+\sigma+v)} \prod_{w=0}^{\infty}\prod_{l=0}^{\infty}\prod_{\substack{j\in\mathbb{Z} \\ j<0\, for\, w=l=0}}\prod_{k=0}^{1} \left(1 - (-1)^k\, e^{2\pi i(w\sigma + l\rho + jv)}\right)^{\sum_{s=0}^{1}(-1)^{sk}\, c_b^{(0,s)}(4lw - j^2)}\,, \\ & \qquad b = j \bmod 2\,. \end{aligned} \tag{2.15}$$

The $w = 0$ term in the above product is obtained from the $Z_{KK} Z_{cm}$ term computed from (2.4), (2.5).

[6]The main difference between our results and those reviewed in [18] is that in [18] the g transformation law was correlated with the momentum along S^1. In our case they are independent data.

3 Properties of the Dyon Partition Function

Various symmetries of the dyon partition function are best studied by relating the function Φ defined in (2.15) to a threshold integral [7, 9, 10, 61–63]. Most of the results that we shall be using can be found in appendices C and D of [18] (the function Φ was called $\widehat{\Phi}$ in [18]). We begin by defining:

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}, \tag{3.1}$$

and

$$\begin{aligned} \frac{1}{2}p_R^2 &= \frac{1}{4\det Im\Omega}|-m_1\rho+m_2+n_1\sigma+n_2(\sigma\rho-v^2)+jv|^2, \\ \frac{1}{2}p_L^2 &= \frac{1}{2}p_R^2+m_1n_1+m_2n_2+\frac{1}{4}j^2\,. \end{aligned} \tag{3.2}$$

We now consider the integral [18]:

$$\widehat{\mathcal{I}}(\rho,\sigma,v) = \sum_{b=0}^{1}\sum_{r,s=0}^{1}\widehat{\mathcal{I}}_{r,s,b}\,, \tag{3.3}$$

where

$$\widehat{\mathcal{I}}_{r,s,b} = \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\left[\sum_{\substack{m_1,n_1\in\mathbb{Z},m_2\in\mathbb{Z}/2\\ n_2\in 2\mathbb{Z}+r,j\in 2\mathbb{Z}+b}} q^{p_L^2/2}\bar{q}^{p_R^2/2}(-1)^{2m_2 s}h_b^{(r,s)}(\tau)-\delta_{b,0}\,\delta_{r,0}\,c_0^{(0,s)}(0)\right], \qquad q\equiv e^{2\pi i\tau}\,. \tag{3.4}$$

$\mathcal{F}$ denotes the fundamental region of $SL(2,\ \mathbb{Z})$ in the upper half plane. The subtraction terms proportional to $c_0^{(0,s)}(0)$ have been chosen so that the integrand vanishes faster than $1/\tau_2$ in the $\tau\to i\infty$ limit, rendering the integral finite. Following the procedure described in [7] one can show that [18][7]

$$\widehat{\mathcal{I}}(\rho,\sigma,v) = -2\ln\left[(\det\,\mathrm{Im}\Omega)^k\right]-2\ln\Phi(\rho,\sigma,v)-2\ln\overline{\Phi(\rho,\sigma,v)}+\text{constant} \tag{3.5}$$

where bar denotes complex conjugation,

$$k=\frac{1}{2}\sum_{s=0}^{1}c_0^{(0,s)}(0)=6\,, \tag{3.6}$$

[7]Note that although the definition of Φ contains only the coefficients $c_b^{(0,s)}$, $\widehat{\mathcal{I}}$ given in (3.3), (3.4) contains $c_b^{(r,s)}$ for all (r,s). This is due to the fact that the manipulations leading to (3.5) makes use of the modular property of $h_b^{(r,s)}(\tau)$ and such modular transformations give $h_b^{(0,s)}(\tau)$ in terms of $h_b^{(r',s')}(\tau')$ for all (r',s').

and Φ is the same function that appears in the expression (2.15) for the partition function.

Due to the relation (3.5), symmetries of Φ can be determined from the symmetries of $\widehat{\mathcal{I}}$ [7].[8] Consider in particular $O(3,2;\ \mathbb{Z})$ transformation on the variables (ρ, σ, v) and (m_1, n_1, m_2, n_2, j) defined as follows:

$$\begin{pmatrix} m_1' \\ m_2' \\ n_1' \\ n_2' \\ j' \end{pmatrix} = S \begin{pmatrix} m_1 \\ m_2 \\ n_1 \\ n_2 \\ j \end{pmatrix}, \qquad \begin{pmatrix} \sigma' \\ \rho'\sigma' - v'^2 \\ -\rho' \\ 1 \\ 2v' \end{pmatrix} = \lambda\, S \begin{pmatrix} \sigma \\ \rho\sigma - v^2 \\ -\rho \\ 1 \\ 2v \end{pmatrix} \tag{3.7}$$

where S is a 5×5 matrix with integer entries, satisfying

$$S^T L S = L, \qquad L = \begin{pmatrix} 0 & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \tag{3.8}$$

and λ is a number to be adjusted so that the fourth element of the vector on the left hand side of (3.7) is 1. I_n denotes $n \times n$ identity matrix. One can easily check that p_R^2 and p_L^2 are invariant under these transformations. Thus as long as the transformation (3.7) preserves the restriction on (m_1, n_1, m_2, n_2, j) in the sum in (3.4), and preserves m_2 mod 1, this transformation is a symmetry of $\widehat{\mathcal{I}}$, and hence of $(\det Im\Omega)^k \Phi\bar{\Phi}$. From the known modular transformation properties of $\det Im\Omega$ it then follows that Φ transforms as a modular form of weight k under these transformations. Since $O(3,2;\ \mathbb{Z})$ is isomorphic to the modular group $Sp(2;\ \mathbb{Z})$ of genus two Riemann surfaces we see that Φ is a modular form of a subgroup of $Sp(2,\ \mathbb{Z})$. The generators of this subgroup have been given explicitly in [6, 18].

A special subgroup of the symmetry group is generated by the following transformations:

$$\begin{aligned} & (m_1, n_1, m_2, n_2, j) \;\to\; (-n_1, -m_1, m_2, n_2, -j), \\ \text{and} \quad & (m_1, n_1, m_2, n_2, j) \;\to\; (m_1 - n_1 - j, n_1, m_2, n_2, j + 2n_1)\,. \end{aligned} \tag{3.9}$$

Via (3.7) these induce the symmetries

$$\Phi(\rho, \sigma, v) = \Phi(\sigma, \rho, -v), \qquad \text{and} \qquad \Phi(\rho, \sigma, v) = \Phi(\rho, \sigma + \rho - 2v, v - \rho)\,. \tag{3.10}$$

As we shall see in §4.1, these generate the S-duality symmetry of the partition function Φ^{-1}.

[8] Under some transformations $\widehat{\mathcal{I}}$ may remain invariant but Φ may pick up a phase, but one can show that for the symmetries which will be relevant for our discussion this does not happen.

Eq.(3.5) also allows us to determine the zeroes and poles of Φ, since they correspond to logarithmically divergent contribution to the threshold integral $\widehat{\mathcal{I}}$. A detailed analysis can be found in [9,18]; here we summarize the main results. Analyzing (3.4) one can show that $\widehat{\mathcal{I}}$ can get logarithmically divergent contributions when p_R^2 vanishes and $p_L^2 = \frac{1}{4}$ for one of the terms in the sum, i.e. near the point

$$-m_1\rho + m_2 + n_1\sigma + n_2(\sigma\rho - v^2) + jv = 0\,, \qquad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}\,, \tag{3.11}$$

where (m_1, n_1, m_2, n_2, j) takes one of the values which appear in the sum in (3.4). Near such a point $\Phi(\rho, \sigma, v)$ behaves as [18]

$$\begin{aligned}\Phi(\rho,\sigma,v) \quad &\sim \quad (-m_1\rho + m_2 + n_1\sigma + n_2(\sigma\rho - v^2) + jv)^{\sum_{s=0}^{1}(-1)^{2\,m_2 s} c_1^{(r,s)}(-1)}\,, \\ & r = n_2 \mod 2, \quad m_1, n_1, n_2 \in \mathbb{Z}, \quad m_2 \in \mathbb{Z}/2\,. \end{aligned} \tag{3.12}$$

Now from (2.14) we see that $c_1^{(1,s)}(-1) = 0$ for all s. Thus there are no poles and zeroes of Φ for odd n_2. On the other hand for n_2 even we have $r = 0$ in (3.12), and from (2.14) we get

$$\sum_{s=0}^{1} (-1)^{2\,m_2\,s} c_1^{(0,s)}(-1) = 1 + (-1)^{2\,m_2}\,. \tag{3.13}$$

Thus Φ has a second order zero for even values of $2m_2$ in (3.12) and no poles.

4 Properties of the Index

In this section we shall make use of the various properties of the dyon partition function derived in §3 to derive properties of the index B_6^g. In particular we shall study the properties of B_6^g under S-duality, study the jump in B_6^g under wall crossing and study the asymptotic growth of B_6^g for large charges. Our starting point will be (2.3), but using (2.1) we shall express it as:

$$-B_6^g(Q,P) = (-1)^{Q\cdot P+1} \int_C d\rho\, d\sigma\, dv\, e^{-2\pi i\left(\sigma\frac{P^2}{2} + \rho\frac{Q^2}{2} + vQ\cdot P\right)} \frac{1}{\Phi(\rho,\sigma,v)}\,. \tag{4.1}$$

The integral has been written as an integral over a 'contour' C which, according to (2.3), lies along the real (ρ, σ, v) axes. Naively in (2.3) we can fix $Im(\rho)$, $Im(\sigma)$ and $Im(v)$ to any values we like; however in order that (2.2) converges we need to choose the imaginary parts of (ρ, σ, v) to lie in certain domains in $\mathbb{R}^3$. As is well understood by now, the choice of the imaginary parts of (ρ, σ, v) depend on the point in the moduli space of the theory we are at [13,14,16,17] since the spectrum – and hence the convergence property of (2.2) – changes discontinuously as the moduli cross the walls of marginal stability. Nevertheless one finds that the function $Z(\rho,\sigma,v)$ defined through (2.2) is the same – or more precisely the analytic continuation of the same meromorphic function – irrespective of where in the moduli space we compute it. The logic leading to this conclusion will be reviewed at the end of §4.1. This is equivalent to the statement that B_6^g in different domains in the moduli space are given by (4.1) with the same Φ, but the choice of the contour C is different in different domains in the moduli space, dictated by the convergence of the sum in (2.2). A convenient prescription for the choice of contour as a function of the moduli is [17]

$$\begin{aligned} Im(\sigma) &= \Lambda\left(\frac{|\tau|^2}{\tau_2}+\frac{Q_R^2}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right), \qquad 0\le Re(\sigma)\le 1\,, \\ Im(\rho) &= \Lambda\left(\frac{1}{\tau_2}+\frac{P_R^2}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right), \qquad 0\le Re(\rho)\le 1\,, \\ Im(v) &= -\Lambda\left(\frac{\tau_1}{\tau_2}+\frac{Q_R\cdot P_R}{\sqrt{Q_R^2P_R^2-(Q_R\cdot P_R)^2}}\right), \qquad 0\le Re(v)\le 1\,, \end{aligned} \tag{4.2}$$

where Λ is a large positive number,

$$Q_R^2 = Q^2+Q^TMQ, \quad P_R^2 = P^2+P^TMP, \quad Q_R\cdot P_R = Q\cdot P+Q^TMP\,, \tag{4.3}$$

$\tau\equiv\tau_1+i\tau_2$ denotes the asymptotic value of the axion-dilaton moduli which belong to the gravity multiplet and M is the asymptotic value of the symmetric $O(6,22)$ matrix valued moduli field of the matter multiplet, – in the heterotic description these represent the moduli of T^6 and Wilson lines along T^6. The choice (4.2) of course is not unique since we can deform the contour without changing the result for the index as long as we do not cross a pole of the partition function. However (4.2) gives a useful bookkeeping device for associating domains in the moduli space to domains in the $(Im(\rho), Im(\sigma), Im(v))$ space. Under small deformations of the moduli (τ, M), (4.2) induces small deformations of the contour C. While generically the value of the integral does not change under such small deformations, the result does change if the contour (4.2) crosses a zero of Φ. Physically these jumps are associated with the jumps in the index across walls of marginal stability [13, 14].

Using T-duality symmetry of the theory one can argue that any other charge vector that can be related to the D1-D5-KK monopole system considered here via a g-invariant T-duality transformation will have B_6^g given by (4.1). The choice of contour given in (4.2) is manifestly invariant under T-duality transformation.

4.1 S-duality Invariance

S-duality transformations act on the charges and moduli as

$$\begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} Q' \\ P' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}, \quad \tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \quad M \to M\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\ \mathbb{Z})\,. \tag{4.4}$$

We want to show that (4.1), with the choice of contour given in (4.3), is invariant under this transformation. Let us define

$$\begin{aligned} \sigma' &= a^2\sigma + b^2\rho - 2abv\,, \\ \rho' &= c^2\sigma + d^2\rho - 2cdv\,, \\ v' &= -ac\sigma - bd\rho + (ad+bc)v\,. \end{aligned} \tag{4.5}$$

One can easily verify that

$$e^{-\pi i(\rho\, Q^2 + \sigma\, P^2 + 2v\, Q\cdot P)} = e^{-\pi i(\rho' Q'^2 + \sigma' P'^2 + 2v' Q'\cdot P')}\,, \tag{4.6}$$

and

$$d\rho\, d\sigma\, dv = d\rho'\, d\sigma'\, dv'\,. \tag{4.7}$$

Furthermore, using the results $Q^2, P^2 \in 2\ \mathbb{Z}$ and $Q\cdot P \in\ \mathbb{Z}$ one finds that $(-1)^{Q\cdot P} = (-1)^{Q'\cdot P'}$. Using these relations we can rewrite (4.1) as

$$B_6^g(Q,P;\tau,M) = (-1)^{Q'\cdot P'+1} \int_C d\rho' d\sigma'\, dv'\, e^{-2\pi i\left(\sigma' \frac{P'^2}{2} + \rho' \frac{Q'^2}{2} + v' Q'\cdot P'\right)} \frac{1}{\Phi(\rho,\sigma,v)}\,. \tag{4.8}$$

Note that we have now explicitly indicated the dependence of B_6^g on the moduli τ, M via the choice of the contour (4.2). One can now further observe that

- The choice of the contour C given in (4.2) is S-duality covariant under simultaneous S-duality transformation on (τ, Q, P) given in (4.4) and of (ρ, σ, v) given in (4.6). Thus we can replace C by C' in (4.8) with the understanding that C' corresponds to the contour where all the variables are replaced by primed variables in (4.2).

- We have the relation

  $$\Phi(\rho,\sigma,v) = \Phi(\rho',\sigma',v')\,. \tag{4.9}$$

  This follows from the fact that (3.10) corresponds to invariance of Φ under (4.5) for

  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{4.10}$$

  and that the two matrices in (4.10) generate the whole S-duality group.

Using these two results we can rewrite (4.8) as

$$
\begin{aligned}
-B_6^g(Q,P;\tau,M) &= (-1)^{Q'\cdot P'+1}\int_{C'} d\rho' d\sigma'\, dv'\, e^{-2\pi i\left(\sigma'\frac{P'^2}{2}+\rho'\frac{Q'^2}{2}+v'Q'\cdot P'\right)}\,\frac{1}{\Phi(\rho',\sigma',v')} \\
&= -B_6^g(Q',P';\tau',M)\,. \qquad (4.11)
\end{aligned}
$$

This finishes the proof of S-duality invariance of B_6^g.

In practice we derive the expression (4.1), (4.2) for the index only in certain domains in the moduli space where the type IIB string theory is weakly coupled [9, 18] and then extend the result to other domains by *requiring S-duality invariance of the spectrum [13, 14].* Thus our analysis in this section should really be regarded as a proof not of S-duality invariance but of (4.1), (4.2), – i.e. of the statement that the same function Φ can be used to capture the index in different domains in the moduli space just by changing the contour according to (4.2).

4.2 Wall Crossing

If we keep the charges fixed and vary the asymptotic moduli then the integration contour C varies. When it hits a pole of the integrand there is a jump in B_6^g given by the residue of the integrand at the pole. The physical interpretation of this jump is that at these points in the moduli space we have a wall of marginal stability and the jump in the degeneracy is due to the jump in the index across the wall of marginal stability [13,14]. One can show that [13] the poles which are encountered in this process are of the form

$$
\sigma\gamma-\rho\beta+v(\alpha-\delta)=0\,,\qquad \alpha\delta=\beta\gamma,\qquad \alpha+\delta=1,\qquad \alpha,\beta,\gamma,\delta\in\mathbb{Z}\,. \qquad (4.12)
$$

The corresponding wall of marginal stability is associated with the decay

$$
(Q,P)\to(\alpha Q+\beta P,\gamma Q+\delta P)+(\delta Q-\beta P,-\gamma Q+\alpha P)\,. \qquad (4.13)
$$

In fact via S-duality transformation all of these decays can be related to the decay [13]

$$
(Q,P)\to(Q,0)+(0,P)\,, \qquad (4.14)
$$

and the corresponding pole of the partition function is at

$$
v=0\,. \qquad (4.15)
$$

Using (2.15) and the identity

$$
\sum_{b=0}^{1}\sum_{j\in 2\mathbb{Z}+b} c_b^{(r,s)}(4n-j^2)=\delta_{n,0}\left\{c_0^{(r,s)}(0)+2\,c_1^{(r,s)}(-1)\right\}, \qquad (4.16)
$$

we see that for small v

$$\Phi(\rho,\sigma,v) = -4\,\pi^2\,v^2\,g(\rho)\,g(\sigma) + \mathcal{O}(v^4)\,, \tag{4.17}$$

$$g(\rho) \equiv e^{2\pi i\rho} \prod_{n=1}^{\infty} \left(1-e^{2\pi i n\rho}\right)^{16} \left(1+e^{2\pi i n\rho}\right)^{8} . \tag{4.18}$$

Thus the jump in the index, given by the residue at the pole of the integrand in (4.1) at $v=0$, is given by[9]

$$\Delta B_6^g = (-1)^{Q\cdot P+1}(Q\cdot P)\,f(Q)\,f(P)\,, \tag{4.19}$$

where

$$f(Q) = \int_0^1 d\rho\, e^{-i\pi\rho Q^2}\,\frac{1}{g(\rho)}\,. \tag{4.20}$$

It is easy to see that $1/g(\rho)$ is precisely the partition function that computes the B_4^g index of the dyons carrying charges $(Q,0)$ (i.e. the KK monopole carrying momentum along S^1) or $(0,P)$ (i.e. the D1-D5 system carrying momentum along $\widetilde{S}^1$) and preserving half of the g-invariant supersymmetries. For example $1/g(\rho)$ is precisely the partition function of the Kaluza-Klein monopole given in (2.4) after removing the contribution from the fermion zero modes. Thus (4.19) can be rewritten as

$$\Delta B_6^g = (-1)^{Q\cdot P+1}\,(Q\cdot P)\,B_4^g((Q,0))\,B_4^g((0,P))\,, \tag{4.21}$$

in agreement with the wall crossing formula for the index B_6 [68].

4.3 Asymptotic Growth

We shall now study the asymptotic growth of the index B_6^g for large charges. As was shown in [1, 2, 6, 9] and reviewed in [18], when Q^2, P^2 and $Q\cdot P$ are all large and of the same order the asymptotic growth of the index, given by (2.3), is controlled by the pole of the partition function, i.e. zeroes of Φ at

$$-m_1\rho + m_2 + n_1\sigma + n_2(\sigma\rho - v^2) + jv = 0\,, \qquad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}\,, \tag{4.22}$$

for $|n_2| > 0$. Furthermore the contribution from a pole of this type is of order

$$\exp\left[\pi\sqrt{Q^2P^2-(Q\cdot P)^2}/|n_2| + \cdots\right]\,, \tag{4.23}$$

[9]The sign of ΔB_6^g of course depends on in which direction the contour crosses the pole, which in turn is determined by the direction in which the moduli cross the wall of marginal stability.

where $\cdots$ denotes terms which are of order unity or suppressed by powers of the charges. Thus the leading contribution to the entropy comes from the poles with lowest possible non-zero value of $|n_2|$. Since from the analysis below (3.12) it follows that Φ has no zeroes for odd n_2 we see that the leading contribution to B_6^g comes from the pole(s) with $|n_2| = 2$. Thus for large charges we have

$$\ln |B_6^g| = \frac{\pi}{2}\sqrt{Q^2P^2 - (Q\cdot P)^2} + \cdots . \tag{4.24}$$

Since the entropy of a BPS black hole carrying these charges is given by $\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}$ [69, 70], we see that $\ln |B_6^g|$ is half of the entropy of a black hole carrying the same charges (and also half of the 'entropy' computed from the usual helicity trace index B_6). We shall provide a macroscopic explanation of this phenomenon in §6.

5 Generalization to $\mathbb{Z}_N$ Twisted Index

The above results can be easily generalized to the case where the theory under consideration is type IIB string theory on $\mathcal{M} \times T^2$ where $\mathcal{M}$ can be either T^4 or $K3$, and the $\mathbb{Z}_2$ symmetry generator g is replaced by a $\mathbb{Z}_N$ symmetry generator g that has a geometric action on $K3$ or T^4, and commutes with an $\mathcal{N} = 4$ subalgebra of the full supersymmetry algebra. This implies that for $\mathcal{M} = T^4$ g commutes with half of the 32 supersymmetries, while for $\mathcal{M} = K3$ g must commute with all the 16 supersymmetries.[10] The $\mathbb{Z}_N$ transformations we shall be using can be found in [11] and references therein. However unlike in [11], we are not computing the spectrum in a new theory obtained by taking a $\mathbb{Z}_N$ orbifold of the original theory. Our interest is to compute the g twisted index B_6^g in the original theory, i.e. in type IIB string theory on $T^4 \times T^2$ or $K3 \times T^2$.

The dyon system we consider is identical to the one described in §2, with the only difference that for $\mathcal{M} = T^4$ we denote the number of D1-branes by Q_1. The method of analysis is also identical to that in §2 and all the technical results needed for the computation can be found in [11, 18]. The only extra complication for $\mathcal{M} = T^4$ arises from the additional degrees of freedom associated with the Wilson line on the D5-brane along T^4, but their effect can be easily computed since the quantum numbers and the g transformation laws of these additional degrees of freedom have been given in [11, 18]. We shall describe only the final results. First

[10] Following the same set of duality transformations as in the $\mathbb{Z}_2$ example described earlier, one can map this theory to type II or heterotic string theory on T^6, with the $\mathbb{Z}_N$ acting only on the left-moving fields on the world-sheet.

we define

$$F^{(r,s)}(\tau,z) \equiv \frac{1}{N} Tr_{RR;g^r}\left(g^s(-1)^{J_L+J_R} e^{2\pi i\tau L_0} e^{-2\pi i\bar{\tau}\bar{L}_0} e^{2\pi i J_L z}\right), \qquad 0\le r,s\le N-1, \quad r,s\in \mathbb{Z}, \tag{5.1}$$

where Tr denotes trace over all the g^r twisted RR sector states in the (4,4) superconformal field theory with target space $\mathcal{M}$, and $J_L/2$ and $J_R/2$ denote the generators of the $U(1)_L\times U(1)_R$ subgroup of the $SU(2)_L\times SU(2)_R$ R-symmetry group of this conformal field theory. For $\mathcal{M}=T^4$ the computation of $F^{(r,s)}$ is straightforward since we have a free conformal field theory. For $\mathcal{M}=K3$ we can calculate $F^{(r,s)}$ by working at special points in the moduli space *e.g.* at the orbifold points or the Gepner points. For prime values of N explicit expression for $F^{(r,s)}$ can be found in [7, 10, 18]. On general grounds $F^{(r,s)}(\tau,z)$ can be shown to have an expansion of the form

$$F^{(r,s)}(\tau,z) = \sum_{b=0}^{1} \sum_{\substack{j\in 2\mathbb{Z}+b,\, n\in\mathbb{Z}/N\\ 4n-j^2\ge -b^2}} c_b^{(r,s)}(4n-j^2)\, e^{2\pi i n\tau+2\pi i jz}\,. \tag{5.2}$$

This defines the coefficients $c_b^{(r,s)}(u)$. We also define

$$Q_{r,s} = N\,\left(c_0^{(r,s)}(0) + 2\, c_1^{(r,s)}(-1)\right)\,. \tag{5.3}$$

Some useful relations are

$$c_1^{(0,s)}(-1) = \begin{cases} \frac{2}{N} & \text{for} \quad \mathcal{M}=K3 \\ \frac{1}{N}\left(2-e^{2\pi i s/N}-e^{-2\pi i s/N}\right) & \text{for} \quad \mathcal{M}=T^4 \end{cases}. \tag{5.4}$$

The index B_6^g is then given by

$$-\,B_6^g(Q,P) = (-1)^{Q\cdot P+1}\int_0^1 d\rho \int_0^1 d\sigma \int_0^1 dv\, e^{-\pi i\sigma\, P^2-\pi i\rho\, Q^2-2\pi i v\, Q\cdot P}\, \frac{1}{\Phi(\rho,\sigma,v)}\,, \tag{5.5}$$

where

$$\begin{aligned}\Phi(\rho,\sigma,v) \;&=\; C^3\, e^{2\pi i\widehat{\alpha}(\rho+\sigma+v)} \\ &\prod_{b=0}^{1}\prod_{r=0}^{N-1} \prod_{\substack{(k',l)\in\mathbb{Z},\, j\in 2\mathbb{Z}+b\\ k',l\ge 0,\, j<0 \text{ for } k'=l=0}} \left\{1-e^{2\pi i r/N}\, e^{2\pi i(k'\sigma+l\rho+jv)}\right\}^{\sum_{s=0}^{N-1} e^{-2\pi i rs/N} c_b^{(0,s)}(4k'l-j^2)}\,,\end{aligned} \tag{5.6}$$

$$\widehat{\alpha} = \begin{cases} 1 & \text{for} \quad \mathcal{M}=K3 \\ 0 & \text{for} \quad \mathcal{M}=T^4 \end{cases}, \qquad C = \begin{cases} 1 & \text{for} \quad \mathcal{M}=K3 \\ \left(1-e^{2\pi i/N}\right)^{-1}\left(1-e^{-2\pi i/N}\right)^{-1} & \text{for} \quad \mathcal{M}=T^4 \end{cases}. \tag{5.7}$$

The factor of C^3 in Φ comes from the quantization of the g non-invariant fermion zero modes carrying no J quantum number.

The threshold integral that can be used to derive various properties of Φ is given by (see appendix C of [18])

$$\widehat{\mathcal{I}}(\rho,\sigma,v) = \sum_{r,s=0}^{N-1}\sum_{b=0}^{1}\widehat{\mathcal{I}}_{r,s,b}\,, \tag{5.8}$$

where

$$\widehat{\mathcal{I}}_{r,s,b} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\left[\sum_{\substack{m_1,n_1\in\mathbb{Z},m_2\in\mathbb{Z}/N\\ n_2\in N\mathbb{Z}+r,j\in 2\mathbb{Z}+b}} q^{p_L^2/2}\bar{q}^{p_R^2/2}e^{2\pi i m_2 s}h_b^{(r,s)}(\tau) - \delta_{b,0}\delta_{r,0}c_0^{(0,s)}(0)\right]\,, \quad q\equiv e^{2\pi i\tau}\,, \tag{5.9}$$

$$\begin{aligned}\frac{1}{2}p_R^2 &= \frac{1}{4\det Im\Omega}|-m_1\rho+m_2+n_1\sigma+n_2(\sigma\rho-v^2)+jv|^2,\\ \frac{1}{2}p_L^2 &= \frac{1}{2}p_R^2+m_1n_1+m_2n_2+\frac{1}{4}j^2\,,\end{aligned} \tag{5.10}$$

$$\Omega = \begin{pmatrix}\rho & v\\ v & \sigma\end{pmatrix}\,, \tag{5.11}$$

and $\mathcal{F}$ denotes the fundamental region of $SL(2,\ \mathbb{Z})$ in the upper half plane. $\widehat{\mathcal{I}}$ is related to Φ by the relation

$$\widehat{\mathcal{I}}(\rho,\sigma,v) = -2\ln\left[(\det\ \mathrm{Im}\Omega)^k\right] - 2\ln\Phi(\rho,\sigma,v) - 2\ln\overline{\Phi(\rho,\sigma,v)} + \text{constant} \tag{5.12}$$

where

$$k = \frac{1}{2}\sum_{s=0}^{N-1} c_0^{(0,s)}(0)\,. \tag{5.13}$$

By making the rearrangement

$$\begin{aligned} &(m_1,n_1,m_2,n_2,j) &\to& \ (-n_1,-m_1,m_2,n_2,-j),\\ \text{and}\quad &(m_1,n_1,m_2,n_2,j) &\to& \ (m_1-n_1-j,n_1,m_2,n_2,j+2n_1)\,,\end{aligned} \tag{5.14}$$

one can prove the invariance of $\widehat{\mathcal{I}}$ and of Φ under the transformations

$$(\rho,\sigma,v)\to(\sigma,\rho,-v), \qquad \text{and} \qquad (\rho,\sigma,v)\to(\rho,\sigma+\rho-2v,v-\rho)\,. \tag{5.15}$$

As in the case of $\mathbb{Z}_2$ twisted index in type IIB string theory on $K3\times T^2$, the symmetry (5.15) can be used to prove the S-duality invariance of the partition function. More generally

Φ transforms as a modular form of weight k under a subgroup of $O(3,2;\ \mathbb{Z})$, cconsisting of $O(3,2)$ matrices which, acting as in (3.7), preserves the restrictions on $(\vec{m},\vec{n},j)$ in the sum in (5.9), and preserves m_2 mod 1 so that the $e^{2\pi i m_2 s}$ factor in (5.9) also remains invariant. The generators of this subgroup of $Sp(2,\ \mathbb{Z})$ have been given explicitly in [11, 18].

The zeroes and poles of Φ can also be found from $\widehat{\mathcal{I}}$ following the analysis of [9, 11] reviewed in [18] (appendix D). The result is that the zeroes and poles of Φ are of the following form:

$$\begin{aligned}
&\Phi \sim \big(n_2(\sigma\rho - v^2) + jv + n_1\sigma - \rho m_1 + m_2\big)^{\sum_{s=0}^{N-1} e^{2\pi i m_2 s}\, c_1^{(r,s)}(-1)}\,, \\
&m_1, n_1, n_2 \in\ \mathbb{Z}, \quad j \in 2\ \mathbb{Z}+1, \quad m_2 \in\ \mathbb{Z}/N, \quad r = n_2 \bmod N, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}\,.
\end{aligned} \tag{5.16}$$

For $N \geq 5$, we also have additional zeroes/poles of the type

$$\begin{aligned}
&\Phi \sim \big(n_2(\sigma\rho - v^2) + jv + n_1\sigma - \rho m_1 + m_2\big)^{\sum_{s=0}^{N-1} e^{2\pi i m_2 s}\, c_1^{(r,s)}(-1+\frac{4p}{N})}\,, \\
&m_1, n_1, n_2 \in\ \mathbb{Z}, \quad j \in 2\ \mathbb{Z}+1, \quad m_2 \in\ \mathbb{Z}/N, \quad r = n_2 \bmod N, \\
&m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4} - \frac{p}{N}, \quad p \in\ \mathbb{Z}, \quad 1 \leq p < \frac{N}{4}\,.
\end{aligned} \tag{5.17}$$

It was argued in [18] (appendix D) that the exponents in (5.16) and (5.17) are always negative or zero for $r \neq 0$ mod N, hence they correspond to poles of Φ.[11] Since our main interest is in determining the poles of the partition function which come from the zeroes of Φ, we can ignore the contribution from the $r \neq 0$ terms. For $r = 0$ we must have $n_2 = 0$ mod N. In this case the constraints on (m_i, n_i, j) forces p to vanish in (5.17), reducing it to the case described in (5.16). Finally using the identities reviewed in [18]

$$\begin{aligned}
\mathcal{M} = K3 \quad &: \quad \sum_{s=0}^{N-1} e^{2\pi i l s/N}\, c_1^{(0,s)}(-1) = \begin{cases} 2 \text{ for } l \in N\ \mathbb{Z} \\ 0 \text{ otherwise} \end{cases}\,, \\
\mathcal{M} = T^4 \quad &: \quad \sum_{s=0}^{N-1} e^{2\pi i l s/N}\, c_1^{(0,s)}(-1) = \begin{cases} 2 \text{ for } l \in N\ \mathbb{Z} \\ -1 \text{ for } l \in N\ \mathbb{Z} \pm 1 \\ 0 \text{ otherwise} \end{cases}\,,
\end{aligned} \tag{5.18}$$

we see that only zeroes of Φ, both for $K3$ and T^4, arise from the choice $m_2 \in\ \mathbb{Z}$ in (5.16) and are of the form

$$\Phi \quad \sim \quad \big(n_2(\sigma\rho - v^2) + jv + n_1\sigma - \rho m_1 + m_2\big)^2$$

[11] For prime values of N explicit computation using known values of $c_1^{(r,s)}(u)$ shows that the exponent in (5.16) always vanishes for $r \neq 0$, whereas the exponent in (5.17) is given by $-48/(N^2-1)$ for $N = 5$ and $N = 7$ when $m_2 r = -1/N$ and vanishes otherwise.

$$m_1, n_1, m_2 \in \mathbb{Z}, \quad n_2 \in N\,\mathbb{Z}, \quad j \in 2\,\mathbb{Z}+1, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}\,. \quad (5.19)$$

The knowledge of the zeroes of Φ gives us two important informations. First of all the zeroes of the form described in (4.12), obtained by choosing $n_2 = 0$ in (5.19), give us information on wall crossing for the decay (4.13). Again using S-duality transformation all such decays can be related to the decay given in (4.14) with the corresponding wall at $v = 0$. Using (5.6) and the identity

$$\sum_{b=0}^{1} \sum_{j\in 2\mathbb{Z}+b} c_b^{(r,s)}(4n - j^2) = \delta_{n,0}\,\{c_0^{(r,s)}(0) + 2\,c_1^{(r,s)}(-1)\}\,, \tag{5.20}$$

one finds that near $v = 0$,

$$\Phi(\rho,\sigma,v) = -4\,\pi^2\,v^2\,g(\rho)\,g(\sigma) + \mathcal{O}(v^4)\,, \tag{5.21}$$

where

$$g(\rho) = C^2\,e^{2\pi i\widehat{\alpha}\rho} \prod_{r=0}^{N-1}\prod_{n=1}^{\infty}\left\{1 - e^{2\pi i r/N}\,e^{2\pi i n\rho}\right\}^{\sum_{s=0}^{N-1} e^{-2\pi i r s/N}\left(c_0^{(0,s)}(0)+2c_1^{(0,s)}(-1)\right)}\,. \tag{5.22}$$

The constant C has been defined in (5.7). Using (5.21) we see that the jump in B_6^g across the wall of marginal stability, given by the residue of the integrand in (5.5) at the pole at $v = 0$, is given by

$$(-1)^{Q\cdot P+1}\,Q\cdot P\,f(Q)\,f(P)\,, \tag{5.23}$$

where

$$f(Q) = \int_0^1 d\rho\,e^{-i\pi Q^2\rho}\,(g(\rho))^{-1}\,d\rho\,. \tag{5.24}$$

It is easy to check, *e.g.* by computing the index B_4^g of a Kaluza-Klein monopole carrying momentum along S^1, that $f(Q)$ and $f(P)$ have the interpretation of the index B_4^g for dyons carrying charge $(Q,0)$ and $(0,P)$ respectively. Thus we get

$$\Delta B_6^g = (-1)^{Q\cdot P+1}\,(Q\cdot P)\,B_4^g((Q,0))\,B_4^g((0,P))\,, \tag{5.25}$$

in agreement with the expected wall crossing formula for the index B_6.

The second application of the knowledge of the zeroes of Φ is in the determination of the asymptotic behaviour of B_6^g for large charges. They are controlled by the zeroes of Φ given in (5.19) for $|n_2| > 0$. The constraint $n_2 \in N\,\mathbb{Z}$ in (5.19) implies that the lowest value of $|n_2|$ other than zero for which there is a pole is $|n_2| = N$. The analysis of the asymptotic growth of the index, which is controlled by this pole, then tells us that the index grows as

$$\exp\left(\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}/N\right)\,. \tag{5.26}$$

6 Macroscopic Explanation from Quantum Entropy Function

The near horizon attractor geometry of an extremal black hole in four dimensions contains an $AdS_2 \times S^2$ factor. After euclidean continuation the metric on $AdS_2 \times S^2$ takes the form

$$ds^2 = v_1(d\eta^2 + \sinh^2\eta d\theta^2) + v_2(d\psi^2 + \sin^2\psi d\phi^2)\,, \quad 0 \le \eta < \infty, \quad \theta \equiv \theta + 2\pi, \quad (\psi, \phi) \in S^2\,, \tag{6.1}$$

where v_1 and v_2 are constants whose values are determined by the charges carried by the black hole. Besides the metric the background also has non-vanishing fluxes through various cycles and constant vacuum expectation values of the scalars which we have not written down explicitly. These background fields respect the $SO(3) \times SO(2,1)$ isometry of $AdS_2 \times S^2$.

According to the quantum entropy function proposal [41, 42] (see also [71]) the degeneracy associated with the black hole horizon is given by an appropriate path integral of string theory in the euclidean near horizon geometry of the black hole. More precisely the degeneracy associated with the black hole horizon is given by

$$d_{hor}(\vec{q}) = Z^{finite}\,, \tag{6.2}$$

$$Z \equiv \left\langle \exp[-iq_i \oint d\theta\, A^{(i)}_\theta] \right\rangle_{AdS_2}\,. \tag{6.3}$$

Here $\langle\ \rangle$ denotes the result of path integral over the string fields, q_i are the electric charges describing fluxes of the $U(1)$ gauge fields $A^{(i)}_\mu$ in AdS_2 and $\oint d\theta$ denotes an integral along the boundary of AdS_2. The superscript 'finite' refers to the infrared finite part of the amplitude defined as follows. If we carry out the path integral by putting an infrared cut-off at $\eta = \eta_0$ so that the boundary has a finite length L, then we may express Z computed via (6.3) as [41, 42]

$$Z = e^{CL + \mathcal{O}(L^{-1})}\, Z^{finite}\,. \tag{6.4}$$

for some L independent constant C. As indicated in (6.4), Z^{finite} is obtained by removing the e^{CL} factor from Z and taking the $L \to \infty$ limit. This prescription for computing $d_{hor}(\vec{q})$ follows naturally from AdS_2/CFT_1 correspondence and reduces to the exponential of the Wald entropy in the classical limit [41].

In (6.3) the path integral is to be carried out over all string field configurations whose asymptotic geometry coincides with the attractor geometry. In particular in integrating over the gauge fields we must keep the electric fields fixed at infinity and allow the constant modes of $A_\theta^{(i)}$ to fluctuate [41]. This follows from the fact that the electric field modes represent non-normalizable deformations of AdS_2 whereas the constant $A_\theta^{(i)}$ modes represent normalizable deformations. With this prescription one can argue, via AdS_2/CFT_1 correspondence, that the d_{hor} defined in (6.2), (6.3) measures the degeneracy of a dual quantum mechanical system whose Hilbert space is degenerate and finite dimensional, containing the grounds states of the black hole carrying a fixed set of charges.

Since d_{hor} measures the degeneracy associated with the horizon whereas in the microscopic analysis we calculate an index, one might wonder how we can compare the two quantities. For the helicity trace index B_{2n} this issue was addresed in [42] where the following explanation was offered. The main idea is to try to compute the index B_{2n} on the macroscopic side as well and then compare this with the microscopic result. The first step is to note that the factors of $2h$ inserted into the trace are used in soaking up the fermion zero modes associated with the broken supersymmetries. Typically these modes are always part of the hair degrees of freedom of the black hole, i.e. live outside the horizon. This allows us to relate B_{2n} to $Tr(-1)^{2h}$ associated with the horizon degrees of freedom [42]. In fact if the only hair degrees of freedom are the fermion zero modes associated with broken supersymmetry then, up to a sign, the macroscopic B_{2n} can be shown to be equal to $Tr(-1)^{2h}$ associated with the horizon degrees of freedom. The second step is to note that although from the point of view of the asymptotic observer $2h$ measures angular momentum, in AdS_2 it can be regarded as an electric charge of the $U(1) \subset SU(2)$ gauge field $\mathcal{A}$ associated with the rotational isometry of S^2 [72]. Thus while carrying out the path integral this charge must be fixed, and in fact has zero value since the attractor geometry is spherically symmetric.[12] Thus we have $(-1)^{2h} = 1$ and B_{2n} can be directly related to d_{hor} defined in (6.2), (6.3). Put another way, computation of $Tr(-1)^{2h}$ can be expressed as a path integral of the kind described in (6.3), but with a twisted boundary condition that requires the fields to transform by $(-1)^{2h}$ as θ changes by 2π. This can be regarded as a Wilson line of $\mathcal{A}$ along the boundary circle. But while carrying out the path integral over the gauge fields we are instructed to integrate over the Wilson lines of $\mathcal{A}$ along the boundary circle keeping the electric fields fixed. Thus a background Wilson line along the boundary circle can be removed by a shift in the integration variables and does not affect the

[12]In classical supergravity in four dimensional Minkowski space all supersymmetric black holes are spherically symmetric. One can argue that supersymmetric black holes whose near horizon geometry has an AdS_2 factor must be spherically symmetric even in the full theory. For this we note that due to the presence of the AdS_2 throat, the full symmetry algebra at the horizon contains an $sl(2, R)$ subalgebra besides the supersymmetry generators. The (anti-)commutators of the $sl(2, R)$ and the supersymmetry generators then generate the $su(1, 1|2)$ algebra which contains $su(2)$ as its subalgebra.

final value of the path integral.

If we try to follow a similar logic for the index B^g_{2n} then the first part of the argument goes through as usual, i.e. the factors of $(2h)^{2n}$ inserted into the trace are absorbed by the fermion zero modes living on the hair. At the end we are left with $Tr(-1)^{2h}g$ associated with the horizon degrees of freedom. This can be expressed as a path integral similar to the one described in (6.2), (6.3), but with a twisted boundary condition on the fields which require the fields to transform by $g(-1)^{2h}$ as θ changes by 2π. Let us call this partition function Z_g. Of these the factor of $(-1)^{2h}$ can be removed by the same argument described above. For the factor of g there are two possibilities. If g can be regarded as a rigid gauge transformation for some U(1) gauge field living on AdS_2 – like $(-1)^{2h}$ as an element of the gauge group associated with the rotational invariance on S^2 – then the effect of the insertion of g is a background value of the Wilson line associated with the gauge group. Since in carrying out the path integral we integrate over the mode representing the Wilson line at the boundary, the effect of g insertion has no effect. Alternatively one can say that since the electric charges associated with all the gauge fields are fixed at the boundary, all states which contribute to d_{hor} have the same value of g and hence the effect of insertion of g into the trace is trivial. On the other hand if g is not an element of a $U(1)$ gauge group then there is no such interpretation. In this case the attractor geometry is not a valid saddle point in the theory, since the boundary circle along which we insert the twist by g is contractible at the center of AdS_2 ($\eta = 0$) and a twist by g will produce a singularity at the center of AdS_2.

This however is not the end of the story. In carrying out the string path integral we are instructed to integrate over all configurations preserving the required boundary condition at $\eta \to \infty$. So we can look for other saddle points. One of the criteria one must use in searching for these saddle points is supersymmetry; if the saddle point breaks too much supersymmetry then integration over the associated fermion zero modes will make the path integral vanish. It was shown in [73] that if we take an orbifold of the original geometry by a transformation that involves equal amount of rotation in AdS_2 and S^2, possibly accompanied by another symmetry that commutes with supersymmetry, then the resulting orbifold, if consistent, preserves the necessary number of supersymmetries so that the contribution from this saddle point to the path integral does not vanish due to integration over the fermion zero modes. This suggests the following procedure for constructing a saddle point that contributes to B^g_6: we take the original attractor geometry geometry and then take an orbifold of this by a $\mathbb{Z}_N$ transformation that combines the action of g with a shift of θ by $2\pi/N$ and a shift of ϕ by $2\pi/N$. It can be shown following [74] that this geometry satisfies the required boundary condition as $\eta \to \infty$. For this we need to carry out a rescaling $\theta \to \theta/N$, $\eta \to \eta + \ln N$ so that in the new coordinate system the AdS_2 part of the metric takes the form:

$$v_1 \left[d\eta^2 + N^{-2} \sinh^2 (\eta + \ln N)\, d\theta^2 \right] = v_1 \left[d\eta^2 + \left\{ \sinh \eta + \frac{1}{2} e^{-\eta} (1 - N^{-2}) \right\}^2 d\theta^2 \right] . \qquad (6.5)$$

Clearly as $\eta \to \infty$ the metric approaches that of AdS_2. The orbifold group now acts as $\theta \to \theta + 2\pi$, $\phi \to \phi + \frac{2\pi}{N}$ together with an action of g. The g action is exactly as required for computing the g twisted index. On the other hand the $2\pi/N$ rotation in ϕ is part of a gauge transformation from the point of view of the theory on AdS_2 and hence, by our previous argument, has no effect on the macroscopic computation of the index, except possibly an overall phase. Put another way, the path integral involves integrating over all values of the Wilson line at ∞, and hence a saddle point corresponding to any specific value of the Wilson lline is an admissible configuration in the path integral. Thus we conclude that this saddle point contributes to Z_g.

It follows from the analysis of [74] that the semiclassical contribution to Z_g^{finite} from this saddle point is given by $\exp(S_{wald}/N)$ where S_{wald} is the Wald entropy of the BPS black hole carrying the same set of charges. The argument is fairly straightforward: if we keep the cut-off of η fixed as we take the orbifold action, then the classical action gets divided by N. On the other hand the length of the boundary also gets divided by N. Thus after subtracting the term proportional to the length of the boundary to extract the finite part of the action, we find that the finite part of the action for the orbifold is $1/N$ times the finite part of the action for the original attractor geometry. Since the latter is given by $S_{wald} = \pi\sqrt{Q^2P^2 - (Q \cdot P)^2}$, we get

$$Z_g^{finite} \sim \exp\left[\frac{\pi}{N} \sqrt{Q^2P^2 - (Q \cdot P)^2} \right] . \qquad (6.6)$$

This is in agreement with the microscopic result given in (5.26).

One possible problem with this construction however is that this saddle point has a $\mathbb{Z}_N$ orbifold singularity on a codimension eight subspace that lies at the center of AdS_2, either at the north or the south pole of S^2 and at the fixed points of g in $\mathcal{M}$. In the absence of flux this is a consistent orbifold of string theory, but it is not clear if the presence of the background flux makes the orbifold inconsistent. If the attractor geometry contains a circle C then one way to avoid the existence of the orbifold singularity is to accompany the $\mathbb{Z}_N$ transformation also by $1/N$ unit of shift along C [42, 75, 76]. The $\mathbb{Z}_N$ now acts freely on the attractor geometry and hence there is no fixed point. By our previous argument the $1/N$ unit of shift along C under $\theta \to \theta + 2\pi$ effectively amounts to switching on a Wilson line at the boundary of AdS_2 and hence is an admissible saddle point. However in this case there is another subtlety, arising from the fact that at the origin of AdS_2 i.e. at $\eta = 0$, the shift along θ is irrelevant. Thus we have an identification under $\phi \to \phi + 2\pi/N$ together with the action of g and $1/N$ unit of shift along C. As a result at the center of AdS_2 any magnetic flux through the three cycle $S^2 \times C$, possibly accompanied by some other cycles of $\mathcal{M} \times T^2$, will get divided by N, and unless the original flux through this cycle is a mutiple of N, the orbifold is not a consistent background of string theory. Thus we must judiciously choose the circle C such that $C \times S^2$ either does not carry any magnetic flux or carries N units of magnetic flux. In the present example S^1 provides us with such a circle, in a spirit close to the one discussed in [75].

This finishes our proof that the macroscopic 'entropy' associated with the index B_6^g grows as $\pi\sqrt{Q^2P^2 - (Q \cdot P)^2}/N$ for large charges, in agreement with the microscopic results.

7 Discussion

In this paper we have computed the twisted helicity trace index for a class of dyons in type II string theory compactified on T^6 or $K3 \times T^2$ and studied their properties. We have also provided a macroscopic explanation for the observed asymptotic growth of this index by identifying the leading saddle point in the path integral which contributes to this index.

One of the special features of an index of this type is that in deriving the various general properties of this index we can focus on supersymmetries which commute with this twist. In particular if the full theory has M supersymmetries, and if only N of these commute with this twist, then the general properties of the index (*e.g.* under wall crossing) will be similar to those of the usual helicity trace index in a theory with N supersymmetries.

We shall now give some more examples of such indices which could have potential application. Consider type IIB / IIA string theory compactified on T^6. The spectrum of elementary strings in this theory contains quarter BPS states obtained by keeping the right-moving world-sheet degrees of freedom in their ground state and exciting only the left-moving modes. Since these states break 24 out of 32 supersymmetries, the relevant index for these states is B_{12}. This can be easily computed in the light-cone gauge Green-Schwarz formulation, where the world-sheet degrees of freedom consist of 8 scalars, 8 left-moving fermions and eight right-moving fermions. However it turns out that due to an extra cancelation between the bosonic and fermionic states in the left-moving sector of the world-sheet, B_{12} grows with charges at a rate much lower than the rate at which the absolute degeneracies grow [77, 78]. Thus the contribution from most of the states cancel due to the cancelation between the contributions from fermionic and bosonic states, – indeed whereas the degeneracies grow exponentially with the charges, the index grows only as a power of the charges.

However consider now the subspace of the full moduli space of the theory where we set all the Ramond-Ramond (RR) moduli to zero. In this subspace the theory has a discrete symmetry denoted as $(-1)^{F_L}$ that changes the signs of all the RR and R-NS sector states. Since elementary string states only carry charges under the $NS-NS$ sector gauge fields, these charges are automatically invariant under $(-1)^{F_L}$. Thus we can choose $g=(-1)^{F_L}$ for defining a new index B^g_{2n}. For elementary string states in the ground state of the right-moving sector but with arbitrary excitations in the left-moving sector of the world-sheet all the 16 supersymmetries in the R-NS sector, and 8 supersymmetries in the NS-R sector are broken. Thus we have 16 g-odd and 8 g-even broken supersymmetries, and these elementary string states contribute to B^g_4. The eight g-invariant fermion zero modes from the right-moving sector are soaked up by the factor of $(2h)^4$, whereas the eight fermion zero modes from the left-moving sector are even under $(-1)^{F_L}(-1)^{2h}=(-1)^{F_R}$ and hence gives a factor of $2^4=16$ in the trace. The rest of the computation involves keeping the right-movers in their ground state and computing the degeneracy of states created by the left-moving oscillators without any weight factor. As a result there is no cancelation, and B^g_4 grows at the same rate as the degeneracy, i.e. exponentially, according to the Cardy formula for a CFT with eight bosons and eight fermions.

The next example involves quarter BPS states in the heterotic string theory on T^6. Since these states break 12 out of 16 supersymmetries the appropriate index is B_6. But we can consider a subspace of the moduli space where T^6 factorizes into a product of T^4 and T^2. In this subspace the theory has an extra discrete symmetry that involves reversing the sign of the four coordinates of T^4. We identify this as our symmetry g. If we consider charge vectors which carry only momentum and winding charges, and KK monopole and H-monopole charges along the two circles of T^2 then these charges are invariant under g. Thus we can define an index B^g_{2n} for these charges. Under the action of g half of the 16 supersymmetries are odd and half are even, but one can show that all the 8 g-odd supersymmetries are broken by the dyon. Thus these dyons have 4 broken supersymmetries which are even under g, and hence contribute to the index B^g_2. We expect this index to have properties similar to that of B_2, – the index that is relevant for capturing the spectrum of half BPS states in $\mathcal{N}=2$ supersymmetric string theories.[13] In particular the wall crossing formula for this index will be controlled by the Kontsevich-Soibelman formula [79, 80].

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