How Do Black Holes Predict the Sign of the Fourier Coefficients of Siegel Modular Forms?

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Abstract

Single centered supersymmetric black holes in four dimensions have spherically symmetric horizon and hence carry zero angular momentum. This leads to a specific sign of the helicity trace index associated with these black holes. Since the latter are given by the Fourier expansion coefficients of appropriate meromorphic modular forms of $Sp(2, \mathbb{Z})$ or its subgroup, we are led to a specific prediction for the signs of a subset of these Fourier coefficients which represent contributions from single centered black holes only. We explicitly test these predictions for the modular forms which compute the index of quarter BPS black holes in heterotic string theory on T^6, as well as in $\mathbb{Z}_N$ CHL models for $N = 2, 3, 5, 7$.

Contents

1 Introduction

Classical single centered black holes in four dimensions are spherically symmetric and hence carry zero angular momentum. Since the black hole breaks part of the supersymmetry of the theory, supersymmetric excitations around the black hole include a set of fermion zero modes, and hence quantization of these fermion zero modes impart certain angular momentum on the black hole. However these fermion zero modes live outside the horizon, and the horizon of the black hole continues to remain spherically symmetric as a consequence of supersymmetry. Given the folklore that black holes describe average properties of an ensemble one might tend to conclude that spherical symmetry implies zero average angular momentum carried by the black hole, – with the individual members of the ensemble carrying different angular momentum. However using AdS_2/CFT_1 correspondence it has been argued in [1] that a spherically symmetric horizon implies that the black hole represents a microcanonical ensemble of states *all of which carry zero angular momentum.* Thus the only source of angular momentum carried by the black hole is from the fermion zero modes associated with broken supersymmetry. This in turn implies that the helicity trace index of the black hole, defined as [2, 3]

$$B_{2n} = \frac{1}{(2n)!} Tr((-1)^F (2h)^{2n}) \, , \tag{1.1}$$

is given by $(-1)^n \, d_{hor}$ where d_{hor} is the degeneracy of the ensemble represented by the horizon of the black hole. Here F denotes fermion number, h denotes the third component of the angular momentum carried by the black hole in its rest frame, the trace is taken over all states carrying a given set of charges, and $4n$ is the number of supersymmetries broken by the black hole, which is equal to the number of fermion zero modes on the black hole. The result quoted above follows from the fact that quantization of each pair of fermion zero modes produces a pair of states with $h = \pm\frac{1}{4}$ and hence $Tr\{(-1)^F(2h)\} = Tr\{e^{2\pi i h}(2h)\} = i$. Thus $2n$ pairs of fermion zero modes will give a contribution to B_{2n} of the form $i^{2n} = (-1)^n$. The factor of $1/(2n)!$ in the definition of B_{2n} cancels against a combinatoric factor that appears when we write $2h$ as the sum of the contribution from individual pairs of fermion zero modes and carry out a binomial expansion of $(2h)^{2n}$, picking up the term that contains one factor of $2h$ for each pair of fermion zero modes. Once the trace of the fermion zero modes has been performed, we just need to evaluate $Tr(-1)^F$ over the rest of the degrees of freedom, and the horizon contribution to this is the same as the degeneracy d_{hor} since $(-1)^F = 1$ for all the states represented by the horizon [1, 4].

We shall focus on quarter BPS black holes in $\mathcal{N} = 4$ supersymmetric string theories which break 12 out of 16 supersymmetries and hence the relevant index is B_6. The analysis given above predicts that $B_6 = -d_{hor}$. d_{hor} can be calculated in principle using quantum entropy function formalism [5], but for our argument the only relevant fact about d_{hor} will be that being a degeneracy it must be positive. This in turn implies that B_6 must be negative [1].

There are several effects which could potentially destroy this prediction.

1. For given set of charges and a given point in the moduli space of the theory the index may receive contribution not only from single centered black holes but also multi-centered black holes. Since multi-centered black holes can carry angular momentum from the fields living outside the black hole horizons [6–10] there is no longer any guarantee that the contribution to B_6 from these black holes will be negative. This problem can however be easily avoided by working in a chamber of the moduli space bounded by the walls of marginal stability that contains the attractor point. In this chamber only single centered black holes contribute to the index [11, 12] and our prediction for the sign of B_6 holds.[1] We shall refer to this chamber of the moduli space as the attractor chamber.

2. Another source of breakdown of our argument is the possible existence of additional supersymmetry preserving fermionic excitations outside the horizon (hair modes [14, 15]) besides the fermionic zero modes associated with broken supersymmetry. Quantization of these modes would give both $(-1)^F$ odd and $(-1)^F$ even states, and this could turn

[1]In a subspace of the moduli space where a multi-centered configuration can be embedded in an $\mathcal{N} = 2$ supersymmetric string theory, there exist a family of solutions known as scaling solutions [10] which continue to exist even at the attractor point. At a generic point in the moduli space of $\mathcal{N} = 4$ supersymmetric string theory we do not expect these solutions to exist since they cannot be embedded in an $\mathcal{N} = 2$ supersymmetric theory where they have been constructed [13].

a positive contribution to $Tr(-1)^F$ from the horizon into a negaive contribution. This can in principle be avoided by going to a duality frame in which all the charges carried by the black hole correspond to some kind of brane charges rather than momenta along compact circles. Since the hair modes described in [14,15] come from excitations carrying momentum along some compact directions, this type of hair modes can be avoided if the black hole does not carry any net momentum along any of the internal directions.

3. The final source of breakdown of our argument arises from the possibility that in a given charge sector the contribution to the index could come from horizonless smooth solutions besides the black hole. Indeed a wide class of smooth solutions have been constructed in supergravity theories (see *e.g.* [16] and references therein). If such solutions exist then their contribution to the index must be added to that from the black hole [17] and this could potentially change a negative B_6 of the black hole into a positive value. However it is not obvious that these smooth solutions, even if they exist at a generic point in the moduli space, would contribute to the index. Typically in $\mathcal{N}=4$ supersmmetric theories it is difficult to construct classical solutions which contribute to the index except in very special cases. As an example one can mention multi-centered black holes or two centered black holes at least one of whose centers is quarter BPS. These exist as supersymmetric classical solutions in a subspace of the moduli space of the theory where the solution can be embedded in an $\mathcal{N}=2$ supersymmetric theory. But their contribution to B_6 must vanish as can be seen from the fact that one can find a continuous path in the moduli space of $\mathcal{N}=4$ supersymmetric string theory that does not hit any wall of marginal stability and yet reaches a point where these solutions do not exist [18]. Physically the vanishing of the index can be understood as due to the difficulty in aligning the supersymmetries of different parts of the solution [13, 19]. The essential point is that since a quarter BPS solution breaks 12 out of 16 supersymmetries, each part of the solution aligns its 4 unbroken supersymmetries in a certain way in the space of 16 supersymmetries. In order that the full solution is supersymmetric the supersymmetries of different parts must be compatible, i.e. the four unbroken supersymmetries of different parts must align appropriately inside the space of 16 supersymmetries. This is a stronger requirement in $\mathcal{N}=4$ supersymmetric theory than in $\mathcal{N}=2$ supersymmetric theory since in the latter case the full theory has 8 supersymmetries and hence the four unbroken supersymmetries of different parts need to

be aligned inside the space of 8 supersymmetries.[2] Due to this reason having a classical solution that contributes to the B_6 index in $\mathcal{N} = 4$ supersymmetric theories is more unlikey than in its $\mathcal{N} = 2$ counterpart, and we shall assume that such solutions do not exist for the range of charges for which a single centered black hole solution exists.

So we shall proceed with the assumption that there exists some duality frame in which only single centered black hole solution – whose only hair are the fermion zero modes associated with broken supersymmetry – contributes to B_6 in the attractor chamber. As a result B_6 must be negative. We shall now try to test this prediction using known microscopic results.

2 The result for the index

The index B_6 has been calculated in a wide class of $\mathcal{N} = 4$ supersymmetric string theories for a wide class of charges [20–39] (see [40] for a recent survey of the results). It is convenient to label the charges carried by the state by a pair of (electric, magnetic) charge vectors (Q, P) in a frame where we represent the theory as (an orbifold of) heterotic string theory compactified on T^6. We shall denote by Q^2, P^2 and $Q \cdot P$ the continuous T-duality invariant inner products of Q and P in this duality frame. Then in the $\mathbb{Z}_N$ CHL models [41,42], obtained by taking an appropriate $\mathbb{Z}_N$ quotient of heterotic string theory on T^6, the result for B_6 takes the form:[3]

$$B_6(\vec{Q}, \vec{P}) = \frac{1}{N}\,(-1)^{Q\cdot P} \int_{\mathcal{C}} d\rho\, d\sigma\, dv\, e^{-\pi i(N\rho Q^2 + \sigma P^2/N + 2vQ\cdot P)}\, \frac{1}{\widetilde{\Phi}(\rho, \sigma, v)}\,, \tag{2.1}$$

where for any given N, $\widetilde{\Phi}(\rho, \sigma, v)$ is a known function, transforming as a modular form of certain weight under a subgroup of $Sp(2, \mathbb{Z})$ [43–52], and $\mathcal{C}$ is a three real dimensional subspace of the three complex dimensional space labelled by $(\rho = \rho_1 + i\rho_2, \sigma = \sigma_1 + i\sigma_2, v = v_1 + iv_2)$. Eq.(2.1) encompasses the $N = 1$ case that describes heterotic string theory on T^6. The contour $\mathcal{C}$ takes the form:

$$\begin{aligned} &\rho_2 = M_1, \quad \sigma_2 = M_2, \quad v_2 = -M_3, \\ &0 \le \rho_1 \le 1, \quad 0 \le \sigma_1 \le N, \quad 0 \le v_1 \le 1\,, \end{aligned} \tag{2.2}$$

[2]This argument can be made more precise in terms of alignment of central charges, – the central charge in an $\mathcal{N} = 2$ supersymmetric theory is a two dimensional real vector while in an $\mathcal{N} = 4$ supersymmetric theory it is a six dimensional real vector. Clearly it is easier to align several two dimensional vectors compared to several six dimensional vectors [13].

[3]Note that this result does not hold for all dyons but a subset of dyons belonging to specific duality orbits in these theories.

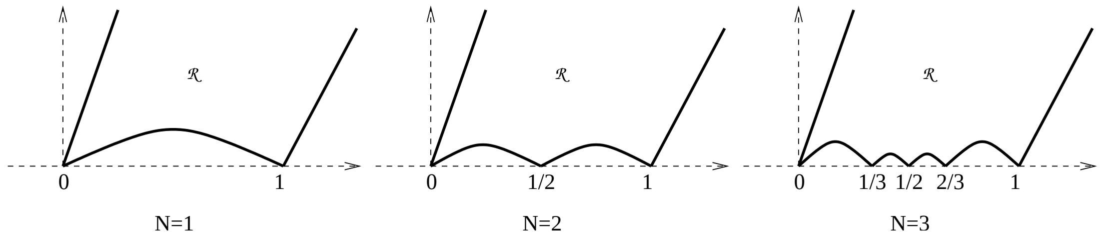

Figure 1: A schematic diagram representing the chamber $\mathcal{R}$ in the upper half τ plane, bounded by the walls of marginal stability, for $\mathbb{Z}_N$ orbifolds of heterotic string theory on T^6 for $N = 1, 2, 3$. The shapes of the circles and the slopes of the straight lines bordering the chamber depend on the charges and other asymptotic moduli, but the vertices are universal.

where M_1, M_2 and M_3 are large but fixed real numbers. The choice of (M_1, M_2, M_3) is governed by the chamber in the moduli space in which we want to compute the index [53, 54] – there being a one to one correspondence between the chambers in the moduli space separated by walls of marginal stability and the domains in the (M_1, M_2, M_3) space separated by poles. The jump in B_6 across a wall of marginal stability is given by the residue of the integrand at the pole that separates the corresponding domains, and is in accordance with the wall crossing formula [11, 55].

For large charges the contribution from single centered black holes is the dominant contribution in all chambers [13, 18] and hence the argument presented in §1 will imply that B_6 is negative in all the chambers [1]. This has been explicitly verified by analyzing the behaviour of (2.1) for large charges [33]. Our goal is to verify the prediction for the sign of B_6 for finite charges, and for this we must work in the attractor chamber. There are several approaches we can follow. For a given (Q, P) we can determine the values of (M_1, M_2, M_3) when the moduli are at the attractor point, – a general algorithm for finding this has been given in [11]. One can also try to first define a generating function for single centered black holes starting from (2.1) and use it to extract the B_6 indices for single centered black holes [56]. We shall follow a third approach which we find most practical. Fig. 1 shows the shapes of the some of the walls of marginal stability in the heterotic axion-dilaton moduli space labelled by the complex field τ taking values in the upper half plane, for fixed values of the other moduli [53]. We shall denote by $\mathcal{R}$ a specific chamber that lies just to the right of the wall that connects 0 to $i\infty$ in the τ plane, determine the constraints on the charges that makes the attractor point lie inside the chamber $\mathcal{R}$, and verify that B_6 in $\mathcal{R}$ is negative for all these charges. Since for heterotic string theory on T^6, and for $\mathbb{Z}_2$ and $\mathbb{Z}_3$ CHL models, every chamber can be mapped to $\mathcal{R}$ by an S-duality transformation [53], this would prove that for all single centered black holes B_6 is negative, provided the charges carried by the black hole fall on the duality orbit for which (2.1) holds. Whether there exist duality transformations mapping every chamber to $\mathcal{R}$ is not known for the $\mathbb{Z}_5$ and $\mathbb{Z}_7$ CHL models. Nevertheless the negativity of B_6 for single centered dyons in $\mathcal{R}$ is a necessary condition which can be tested even in these models.

The choice of (M_1, M_2, M_3) corresponding to the chamber $\mathcal{R}$ is [30, 53]:

$$M_1, M_2 >> 0, \quad M_3 << 0, \quad |M_3| << M_1, M_2 \,. \tag{2.3}$$

In practical terms this means that to extract B_6 in this chamber we first expand $1/\widetilde{\Phi}$ in powers of $e^{2\pi i\rho}$ and $e^{2\pi i\sigma}$ and then expand each term in this expansion in powers of $e^{-2\pi i v}$. This is best done using the product representation of $\widetilde{\Phi}$. For the $\mathbb{Z}_N$ CHL models with $N = 1, 2, 3, 5, 7$ we have [26, 28, 30]

$$\begin{aligned}\widetilde{\Phi}(\rho,\sigma,v)^{-1} &= e^{-2\pi i(\rho+\sigma/N+v)} \\ &\times \prod_{b=0}^{1}\prod_{r=0}^{N-1}\prod_{\substack{k\in\mathbb{Z}+\frac{r}{N}, l\in\mathbb{Z}, j\in 2\mathbb{Z}+b\\ k,l\geq 0, j<0 \text{ for } k=l=0}} \left(1-\exp\left(2\pi i(k\sigma+l\rho+jv)\right)\right)^{-\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c_b^{(r,s)}(4kl-j^2)}\,,\end{aligned} \tag{2.4}$$

where the coefficients $c_b^{(r,s)}(u)$ are defined as follows [26]. First we define[4]

$$\begin{aligned}F^{(0,0)}(\tau,z) &= \frac{8}{N}A(\tau,z)\,,\\ F^{(0,s)}(\tau,z) &= \frac{8}{N(N+1)}\,A(\tau,z) - \frac{2}{N+1}\,B(\tau,z)\,E_N(\tau) \qquad \text{for } 1\leq s\leq (N-1)\,,\\ F^{(r,rk)}(\tau,z) &= \frac{8}{N(N+1)}\,A(\tau,z) + \frac{2}{N(N+1)}\,E_N\left(\frac{\tau+k}{N}\right)\,B(\tau,z)\,,\\ &\qquad \text{for } 1\leq r\leq (N-1),\ 0\leq k\leq (N-1)\,,\end{aligned} \tag{2.5}$$

where

$$A(\tau,z) = \left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right]\,, \tag{2.6}$$

[4] A different but equivalent description of the functions $F^{(r,s)}$ can be derived from the general result of [38].

$$B(\tau,z)=\eta(\tau)^{-6}\vartheta_1(\tau,z)^2\,, \tag{2.7}$$

and

$$E_N(\tau)=\frac{12i}{\pi(N-1)}\,\partial_\tau\left[\ln\eta(\tau)-\ln\eta(N\tau)\right]=1+\frac{24}{N-1}\sum_{\substack{n_1,n_2\geq 1\\ n_1\neq 0 \bmod N}} n_1 e^{2\pi i n_1 n_2\tau}\,. \tag{2.8}$$

Then $c_b^{(r,s)}(u)$ is defined via the expansion:

$$F^{(r,s)}(\tau,z)\equiv\sum_{b=0}^{1}\sum_{j\in 2\mathbb{Z}+b,n\in\mathbb{Z}/N} c_b^{(r,s)}(4n-j^2)e^{2\pi i n\tau+2\pi i j z}\,. \tag{2.9}$$

For terms in (2.4) with either l or k non-zero, the procedure of expansion is straightforward; we simply expand the $\left(1-e^{2\pi i(k\sigma+l\rho+jv)}\right)^{-\sum_{s=0}^{N-1}e^{-2\pi i s l/N}c_b^{(r,s)}(4kl-j^2)}$ term in a power series in $e^{2\pi i(k\sigma+l\rho+jv)}$. Special care needs to be taken for the the $k=l=0$ term which, together with the $e^{-2\pi i v}$ factor in the front, is given by $e^{-2\pi i v}/(1-e^{-2\pi i v})^2$. The contour prescripton for chamber $\mathcal{R}$, corresponding to the choice of M_i given in (2.3), requires us to expand this factor in powers of $e^{-2\pi i v}$. This gives a completely well defined prescription for expanding $1/\widetilde{\Phi}$ and computing B_6 in the chamber $\mathcal{R}$.

3 Kinematic constraints on the charges

Now that we have described the algorithm for calculating B_6 in the chamber $\mathcal{R}$, the next question we need to ask is: for which charges (Q,P) the attractor point in the moduli space lies inside $\mathcal{R}$? Once we determine these charges, our previous argument will tell us that $B_6(Q,P)$ for these charges, computed inside the chamber $\mathcal{R}$, must be negative. There are various approaches to answer this question, we shall describe one of them.

We begin with the $N=1$ model, i.e. heterotic string theory on T^6. First consider the wall that connects 0 to $i\infty$. For reasons which will become clear soon, we shall assign an orientation to this line which we take to be directed away from 0 and towards the point at $i\infty$. A necessary condition that the attractor point lies inside $\mathcal{R}$ is that it lies to the right of the wall going from 0 to $i\infty$. Now if we denote by M the symmetric $SO(6,22)$ matrix valued moduli of the string theory ($SO(6,22)$ will be replaced by $SO(6,r)$ for some other integer r for CHL models), by L the $O(6,22)$ invariant matrix of signature $(+^6-^{22})$, and by

$$Q_R=\frac{1}{2}(M+L)Q,\qquad P_R=\frac{1}{2}(M+L)P, \tag{3.1}$$

then at the attractor point

$$Q_R^2 = Q^2, \quad P_R^2 = P^2, \quad Q_R.P_R = Q.P\,, \tag{3.2}$$

$$\tau_1 = \frac{Q.P}{P^2}, \quad \tau_2 = \frac{\sqrt{Q^2P^2 - (Q.P)^2}}{P^2}\,. \tag{3.3}$$

On the other hand the wall of marginal stability joining 0 and $i\infty$ is described by the equation [53]

$$\tau_1 + \frac{Q_R.P_R}{\sqrt{Q_R^2 P_R^2 - (Q_R.P_R)^2}}\tau_2 = 0\,. \tag{3.4}$$

If we choose M and τ_2 to be at their attractor values given in (3.2), (3.3) then the value of τ_1 computed from (3.4) is given by $-Q.P/P^2$. Thus in order that the attractor point lies to the right of this wall, we need τ_1 given in (3.3) to be larger than $-Q.P/P^2$, i.e. have $Q.P/P^2 \geq 0$.[5] Since we shall always consider the range in which $Q^2, P^2 > 0, (Q.P)^2 < Q^2P^2$ (non-singular supersymmetric black holes exist only in this range) we must have

$$Q.P \geq 0\,. \tag{3.5}$$

Since the equations for the other walls of $\mathcal{R}$ are also known [53, 57] we can use similar method to determine the condition on the charges which will ensure that the attractor point lies inside $\mathcal{R}$. But we shall now describe a simpler method for determining this using S-duality transformation that acts simultaneously on the charges and the τ-moduli as

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} Q' \\ P' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})\,. \tag{3.6}$$

If we consider the wall from 0 to 1 in the τ-plane then the $SL(2, \mathbb{Z})$ transformation by $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ maps it to a wall from 0 to $i\infty$ in the τ' plane. Now in order that the attractor point corresponding to the charge (Q, P) in the τ plane lies inside $\mathcal{R}$ it must lie to the *left* of the wall from 0 to 1. Thus in the τ' plane the attractor point for (Q', P') must lie to the left of the wall from 0 to $i\infty$. From our previous analysis this requires $Q'.P' \leq 0$. Now from (3.6) we have $(Q' = Q, P' = P - Q)$ and hence the condition $Q'.P' \leq 0$ translates to $Q.P \leq Q^2$. Similarly mapping the wall from 1 to $i\infty$ to the wall from 0 to $i\infty$ by the transformation $\tau' = \tau - 1$ we get the third condition $Q.P \leq P^2$. Together with these three conditions we must add the conditions $Q^2, P^2, \{Q^2P^2 - (Q.P)^2\} > 0$ since classical black hole solutions

[5] For $Q.P = 0$ a two centered black hole carrying charges $(Q, 0)$ and $(0, P)$ may exist, but its contribution to the index, being proportional to $Q.P$, vanishes. For this reason we have used $\geq$ instead of $>$.

with non-singular event horizon exists only when this condition is satisfied. Thus we would conclude the for heterotic string theory on T^6 the B_6 index in $\mathcal{R}$ must be negative when all of the following conditions are satisfied:

$$Q.P \geq 0, \quad Q.P \leq Q^2, \quad Q.P \leq P^2, \quad Q^2, P^2, \{Q^2P^2 - (Q.P)^2\} > 0\,. \tag{3.7}$$

Similar analysis can be performed for the CHL models obtained by taking the $\mathbb{Z}_N$ orbifold of heterotic string theory on T^6. Let us first consider the case of $N = 2$ for which the region $\mathcal{R}$ is bounded by four walls shown in Fig.1. In this case the S-duality group is $\Gamma_1(2)$. As before the wall connecting 0 and $i\infty$ gives the condition $Q.P \geq 0$. Now the other walls from 0 to 1/2, 1 to 1/2 and 1 to $i\infty$ can all be mapped to the wall from 0 to $i\infty$ with the help of $\Gamma_1(2)$ transfrmations[6]

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \tag{3.8}$$

respectively. This can be used to derive the following conditions on (Q, P) for the attractor point to lie inside the region $\mathcal{R}$:

$$Q.P \geq 0, \quad Q.P \leq 2\,Q^2, \quad Q.P \leq P^2, \quad 3\,Q.P \leq 2\,Q^2 + P^2, \quad Q^2, P^2, \{Q^2P^2 - (Q.P)^2\} > 0\,. \tag{3.9}$$

The same analysis can be repeated for $N = 3$. The walls from 0 to 1/3, 1/2 to 1/3, 1/2 to 2/3, 1 to 2/3 and 1 to $i\infty$ are mapped to the wall from 0 to $i\infty$ via the $\Gamma_1(3)$ transformations

$$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.10}$$

The conditions on (Q, P) for the attractor point to lie inside the region $\mathcal{R}$ is

$$\begin{aligned} &Q.P \geq 0, \quad Q.P \leq 3\,Q^2, \quad Q.P \leq P^2, \quad 5\,Q.P \leq 6\,Q^2 + P^2, \quad 5\,Q.P \leq 3\,Q^2 + 2\,P^2, \\ &7\,Q.P \leq 6\,Q^2 + 2\,P^2, \quad Q^2, P^2, \{Q^2P^2 - (Q.P)^2\} > 0\,. \end{aligned} \tag{3.11}$$

To summarize, our argument of §1 predicts that B_6 computed in the region $\mathcal{R}$ must be negative for $(Q^2, P^2, Q.P)$ satisfying the constraints (3.7) for heterotic string theory on T^6,

[6] Even though we have used $\Gamma_1(N)$ transformations to map the walls bordering $\mathcal{R}$ to the wall connecting 0 and $i\infty$, this is not necessary. The walls are results of kinematical constraints and transform covariantly under any $SL(2, R)$ transformation. Thus given a wall connecting a point $\tau = a$ to $\tau = b$ with $b > a$, we can use the $SL(2, R)$ transformaton $(b-a)^{-1/2}\begin{pmatrix} 1 & -a \\ -1 & b \end{pmatrix}$ to map it to the wall connecting 0 and $i\infty$. The constraint on (Q, P) in order that the point lies above the wall connecting p and q now translates to $(Q - aP).(bP - Q) \leq 0$.

$(Q^2, P^2)\backslash Q.P$	-2	0	1	2	3	4
(2,2)	-209304	**50064**	**25353**	648	327	0
(2,4)	-2023536	**1127472**	**561576**	**50064**	8376	-648
(4,4)	-16620544	**32861184**	**18458000**	**3859456**	**561576**	12800
(2,6)	-15493728	**16491600**	**8533821**	**1127472**	130329	-15600
(4,6)	-53249700	**632078672**	**392427528**	**110910300**	**18458000**	**1127472**
(6,6)	2857656828	**16193130552**	**11232685725**	**4173501828**	**920577636**	**110910300**

Table 1: Some results for $-B_6$ in heterotic string theory on T^6 for different values of Q^2, P^2 and $Q.P$. The boldfaced entries are for charges which satisfy the constraints (3.7). We have given the results only for $Q^2 \leq P^2$ since the results are symmetric under $Q^2 \leftrightarrow P^2$. Note that some of the entries are the same; this is a consequence of a $\mathbb{Z}_3$ subgroup of S-duality transformation $\tau \to 1 - \tau^{-1}$ which maps $\mathcal{R}$ to $\mathcal{R}$ but changes the charges as $(Q^2, P^2, Q \cdot P) \to (P^2 + Q^2 - 2Q.P, Q^2, Q^2 - Q.P)$.

the constraints (3.9) for the $\mathbb{Z}_2$ CHL model, and the constraints (3.11) for the $\mathbb{Z}_3$ CHL model. Since various mathematical properties of $\widetilde{\Phi}$ have been analyzed in [12, 58–61], it will be interesting to see if these predictions follow from these properties.

For $N > 3$ the number of walls bordering $\mathcal{R}$ becomes infinite [53] and so there are infinite number of constraints. The wall from 0 to $i\infty$ still gives the constraint $Q.P \geq 0$. Thus if we can show, for the range of (Q^2, P^2) for which we carry out the analysis, that B_6 is negative for all $Q.P$ satisfying

$$Q.P \geq 0, \qquad (Q.P)^2 < Q^2 P^2, \qquad Q^2, P^2 > 0, \tag{3.12}$$

then it will imply that B_6 is negative for single centered dyons in this range of charges. Note that this test is sufficient but not necessary; if we find a positive B_6 value for some charges satisfying (3.12) then it may still be consistent with our result if the charges fail to satisfy any of the other conditions associated with the other walls of $\mathcal{R}$.

$(Q^2, P^2)\backslash Q.P$	-2	0	1	2	3	4
(1,2)	-5410	**2164**	**360**	-2	0	0
(1,4)	- 26464	**18944**	**4352**	160	0	0
(2,4)	-124160	**198144**	**67008**	**6912**	64	0
(1,6)	-114524	**125860**	**36024**	**2164**	52	0
(2,6)	-473088	**1580672**	**671744**	**101376**	**4352**	-16
(3,6)	- 779104	**15219528**	**7997655**	**1738664**	**149226**	**2164**

Table 2: Some results for $-B_6$ in the $\mathbb{Z}_2$ CHL model for different values of Q^2, P^2 and $Q.P$. The boldfaced entries are for charges which satisfy the constraints (3.9). We have only given the results for $2Q^2 \leq P^2$, since due to a symmetry of $\widetilde{\Phi}$ the B_6 index has a symmetry under $P^2 \leftrightarrow 2Q^2$ [28].

$(Q^2, P^2)\backslash Q.P$	-2	0	1	2	3	4
(2/3,2)	- 1458	**540**	**27**	0	0	0
(2/3,4)	- 5616	**3294**	**378**	0	0	0
(4/3,4)	-21496	**23008**	**4912**	**136**	0	0
(2/3,6)	- 18900	**16200**	**2646**	54	0	0
(4/3,6)	- 70524	**128706**	**37422**	**2484**	6	0
(2,6)	- 208584	**820404**	**318267**	**37818**	**801**	0

Table 3: Some results for $-B_6$ in the $\mathbb{Z}_3$ CHL model for different values of Q^2, P^2 and $Q.P$. The boldfaced entries are for charges which satisfy the constraints (3.11). We have only given the results for $3Q^2 \leq P^2$, since due to a symmetry of $\widetilde{\Phi}$ the B_6 index has a symmetry under $P^2 \leftrightarrow 3Q^2$ [28].

$(Q^2, P^2)\backslash Q.P$	-2	0	1	2	3	4
(2/5,2)	- 392	**100**	1	0	0	0
(2/5,4)	- 1120	**460**	**20**	0	0	0
(4/5,4)	- 3200	**2280**	**240**	0	0	0
(2/5,6)	- 2940	**1720**	**125**	0	0	0
(4/5,6)	- 8380	**9180**	**1460**	**20**	0	0
(6/5,6)	- 21660	**39960**	**9345**	**390**	0	0

Table 4: Some results for $-B_6$ in the $\mathbb{Z}_5$ CHL model for different values of Q^2, P^2 and $Q.P$. The boldfaced entries are for charges which satisfy the constraints (3.12). We have only given the results for $5Q^2 \leq P^2$, since due to a symmetry of $\widetilde{\Phi}$ the B_6 index has a symmetry under $P^2 \leftrightarrow 5Q^2$ [28].

$(Q^2, P^2)\backslash Q.P$	-2	0	1	2	3	4
(2/7,2)	- 162	**36**	0	0	0	0
(2/7,4)	- 396	**138**	**3**	0	0	0
(4/7,4)	- 968	**564**	**40**	0	0	0
(2/7,6)	- 918	**444**	**18**	0	0	0
(4/7,6)	- 2244	**1916**	**210**	0	0	0
(6/7,6)	- 5184	**6892**	**1152**	**18**	0	0

Table 5: Some results for $-B_6$ in the $\mathbb{Z}_7$ CHL model for different values of Q^2, P^2 and $Q.P$. The boldfaced entries are for charges which satisfy the constraints (3.12). We have only given the results for $7Q^2 \leq P^2$, since due to a symmetry of $\widetilde{\Phi}$ the B_6 index has a symmetry under $P^2 \leftrightarrow 7Q^2$ [28].

4 Test of positivity of the index

As already mentioned, the negativity of B_6 has been proved explicitly in the limit when all the charges become large keeping the ratios $Q.P/P^2$, Q^2/P^2 fixed [33], not only for heterotic string theory on T^6 but all $\mathcal{N}=4$ supersymmetric string theories where the answer for B_6 is known. In this section we shall try to test this for finite charges. Note that due to the $(-1)^{Q.P}$ factor in (2.1), negativity of B_6 means positive (negative) sign for the Fourier coefficients of $1/\widetilde{\Phi}$ for odd (even) powers of $e^{2\pi i v}$.

We begin with heterotic string theory on T^6. The results for $-B_6$ in $\mathcal{R}$ for a range of values of Q^2, P^2 and $Q.P$ have been shown in table 1. Clearly the entries have positive and negative values. But for charges which satisfy the restrictions given in (3.7) we have represented the entries by bold faced letters, and as we can see, all the bold faced entries are manifestly positive. We have in fact checked that up to all values of Q^2 and P^2 up to 10 and all values of $Q.P$, the positivity of $-B_6$ inside $\mathcal{R}$ holds whenever (3.7) holds.

Similar analysis is possible for $\mathbb{Z}_N$ CHL models. We have checked the positivity of $-B_6$ for several charges in these models and the result is again in accordance with the general prediction from the black hole side. Some of the results are shown in tables 2, 3, 4 and 5. We have in fact tested the required positivity of $-B_6$ for all values of $NQ^2, P^2 \leq 10$ and all allowed values of $Q.P$. We have not gone to very high values of the charges, but it is more important to test this for low charges since we already know that the prediction holds in the large charge limit.

In all the tables we have specifically displayed the results for $Q \cdot P = -2$ sector to emphasize the need for focussing on single centered black holes for the positivity test of $-B_6$. Due to a $v \to -v$ symmetry of $\widetilde{\Phi}$ the index for negative $Q.P$ values in the chamber $\mathcal{R}$ can be related to the index for positive $Q.P$ values in the chamber $\mathcal{L}$ lying to the left of the wall from 0 to $i\infty$. Thus the results for $Q.P = -2$ given in the tables can be reinterpreted as the results for $Q.P = 2$ in the chamber $\mathcal{L}$, and the difference between $Q.P = -2$ and the $Q.P = 2$ entries in the tables can be accounted for by the wall crossing formula across the wall connecting 0 to $i\infty$. As we move from $\mathcal{R}$ to $\mathcal{L}$ across this wall new two centered configurations of a pair of half-BPS states, carrying charges $(Q, 0)$ and $(0, P)$, appear. As can be seen from the tables, the negative contribution to $-B_6$ from these states overwhelm the positive contribution from single centered black holes for low values of the charges.

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