

Two Charge System Revisited: Small Black Holes or Horizonless Solutions?

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Abstract

A two charge system in string theory preserving eight supercharges can be described as a small black hole that has zero entropy in the supergravity approximation, but classical higher derivative corrections produce a finite entropy in accordance with the prediction of microstate counting. On the other hand for the same system one can construct smooth horizonless classical solutions whose geometric quantization describes the individual microstates which contribute to the entropy. In this note we point out that there is no duality frame in which the system admits both these classical descriptions. Thus in a given duality frame horizonless classical solutions and small black holes are not alternate descriptions of the same system; their contributions must be added to get a duality invariant result for the macroscopic degeneracy. We discuss the significance of this observation for the macroscopic computation of the degeneracy of BPS states for a general system. We also discuss the relationship between the degeneracy and the index computation and address the puzzle regarding absence of small black holes in toroidally compactified type II string theory.

Contents

1	Introduction and Summary	2
2	Small Black Hole or Smooth Solutions?	4
3	Degeneracy or Index?	12
4	How Does a BPS State Lose Information About Unprotected Quantities?	19
A	Duality Transformation from Heterotic to Type II Description	21

1 Introduction and Summary

BPS states in string theory carrying only two independent charges have played an important role in our understanding of black holes and in particular the effect of higher derivative corrections to the black hole entropy. An example of this is an elementary heterotic string carrying $-n$ units of momentum and w units of winding charge along a circle [1, 2]. From the counting of the states of a fundamental heterotic string one knows that the degeneracy of these states grows as

$$d(n, w) \sim \exp(4\pi\sqrt{nw}), \quad (1.1)$$

for large n, w . Thus it would be natural to ask whether this degeneracy can be reproduced by computing the Bekenstein-Hawking entropy of the BPS black hole carrying the same charges [3, 4, 5, 6]. A straightforward analysis shows that in the supergravity limit such a black hole has zero area event horizon and hence is singular. However it was shown in [7, 8, 9] that upon taking into account the effect of higher derivative corrections to the action the black hole could develop a finite area event horizon; and if this happens then a scaling argument can be used to show that the entropy of such a black hole must have the form $C\sqrt{nw}$ for some numerical constant C . Later the value of C was calculated in vacua with four non-compact space-time dimensions and found to be 4π , in precise agreement with the microscopic result [10, 11, 12, 13, 14]. Explicit computation of C involved taking into account the effect of a class of higher derivative corrections in tree level heterotic string theory using the techniques developed in [15, 16, 17, 18, 19, 20, 21].

In a related development it was discovered that the microstates describing the states of the fundamental heterotic string can be represented as solutions to supergravity equations

of motion. In the heterotic description these solutions have source terms corresponding to fundamental heterotic string [22,23], but if we use a dual description as type IIB string theory then these solutions are completely non-singular [24,25,26,27,28]. The geometric quantization of these solutions leads to a degeneracy of states that agrees with the microscopic result (1.1) [29]. This seems to suggest an alternative description of the microstates as smooth classical solutions without horizon.¹

The goal of this paper is to reexamine the two charge system carefully and explore the relationship between these two different *classical* description of the microstates: as small black hole and as smooth classical solutions. In this set up we find that in any given duality frame the two charge system may have representation either as a small black hole or as smooth classical solutions, but not both. Thus in a fixed duality frame black holes and smooth solutions are not alternate descriptions of the same states; only one of these exists as the possible classical description of the microstates. In particular this would imply that black holes capture information about those states which *do not have representations as smooth horizonless classical solutions*. This result is natural given that a smooth classical solution, by definition, takes into account the gravitational backreaction and yet does not form a horizon. Thus it would be surprising if they can also be represented as black holes in the same duality frame.²

Note our emphasis on the word ‘classical’: we are looking for solutions to classical equations of motion in string theory without external source terms, i.e. we include the effect of higher derivative terms in the action but not string loop corrections. Since under a duality transformation the higher derivative corrections and string loop corrections mix; our analysis will not be manifestly duality invariant. The same microscopic configuration may appear as small black hole in one duality frame and as smooth solutions in another duality frame.

Even though we have emphasized on working with classical solutions, this does not mean that we are forced to ignore quantum effects. Given a (family of) smooth horizonless classical solution(s), we can take into account quantum effects via geometric quantization [39]. On the other hand quantum corrections to a black hole entropy can be computed via Euclidean path integral in which the Wick rotated solution appears as a saddle point. What we do not allow are (black hole) solutions which come into existence only after inclusion of quantum corrections

¹Such smooth classical solutions are sometimes referred to as fuzzballs, but according to the original definition fuzzballs include more general configurations, *e.g.* classical solutions with sources (like the fundamental string source) or quantum states [30,31,32,33,34]. In view of this we shall continue to use the phrase ‘smooth classical solutions’. I wish to thank Samir Mathur for a discussion on this point. For reviews on various aspects of the fuzzball proposal see [35,36,37].

²Similar remarks can be found *e.g.* in [38].

to the effective action *computed in the space-time background without black holes*. We shall elaborate on this in §2 by showing that if we work in a duality frame where classically we have only smooth solutions and no small black holes, and then try to include (wrongly) the quantum corrections to the effective action *computed in the vacuum*, then in the new description based on the quantum effective action we may get a small black hole solution but the smooth solutions will now appear to be singular.

Typically on the microscopic side we compute an appropriate index instead of the degeneracy, but on the macroscopic side the Wald entropy [40] and its quantum generalization proposed in [41, 42] measures the absolute degeneracy. Thus one might wonder whether it is appropriate to compare the two. In §3 we discuss this issue following [42], and show that in some cases we can use the result for degeneracy on the macroscopic side to compute the index and then compare with the microscopic answer. In this context we also discuss the puzzle related to the absence of small black holes in type II string theory representing excited states of the fundamental type II string.

In section 4 we explore the possibility of comparing quantities on the macroscopic and the microscopic side which are not protected by the index theorem. Even though we do not have a proof that this is impossible, we suggest a mechanism that could wipe out the information on such quantities.

2 Small Black Hole or Smooth Solutions?

In this section we shall explore under what condition tree level higher derivative corrections in a string theory can modify the near horizon geometry of a two charge system and produce a finite entropy in agreement with the microscopic result. The main tool in our analysis will be a limited version of the scaling argument used in [7]. We begin with the observation that the classical action / equations of motion of type IIA/IIB string theory has a scaling symmetry under which the dilaton ϕ gets shifted by a constant $\ln \lambda^{-1}$, all other Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector fields remain invariant and the Ramond-Ramond (RR) sector fields are multiplied by λ . The effect of this scaling is to multiply the action by λ^2 . The same symmetry also exists in classical heterotic string theory except that in this case there are no RR sector fields. Since magnetic charge is directly related to the magnetic field, under this symmetry the magnetic charges \vec{q}_{NSNS}^{mag} for NSNS sector fields remain invariant, while the magnetic charges \vec{q}_{RR}^{mag} for the RR sector fields scale by λ . On the other hand since the electric charge is related

to the derivative of the action with respect to the electric field, the electric charges \vec{q}_{NSNS}^{el} associated with NSNS sector fields scale by λ^2 and the electric charges \vec{q}_{RR}^{el} associated with the RR sector fields scale by λ . Finally the black hole entropy, being proportional to the overall multiplicative factor in the Lagrangian density, gets scaled by λ^2 . This leads to the relation³

$$S_{BH}(\lambda \vec{q}_{RR}, \lambda^2 \vec{q}_{NSNS}^{el}, \vec{q}_{NSNS}^{mag}) = \lambda^2 S_{BH}(\vec{q}_{RR}, \vec{q}_{NSNS}^{el}, \vec{q}_{NSNS}^{mag}), \quad (2.1)$$

in the classical theory. Here \vec{q}_{RR} stands for both electric and magnetic RR charges. In practical terms eq.(2.1) implies that if we have a classical black hole solution with charges $(\vec{q}_{RR}, \vec{q}_{NSNS}^{el}, \vec{q}_{NSNS}^{mag})$ and entropy S_{BH} , then by a simple scaling we can generate another classical solution with charges $(\lambda \vec{q}_{RR}, \lambda^2 \vec{q}_{NSNS}^{el}, \vec{q}_{NSNS}^{mag})$ and entropy $\lambda^2 S_{BH}$.

We shall use this relation to analyze the entropy of two charge black holes in different descriptions. The microscopic system consists of a fundamental heterotic string in heterotic string theory compactified on $\mathcal{M} \times S^1$ for some compact manifold \mathcal{M} , carrying w units of winding and $-n$ units of momentum along S^1 . As pointed out in §1, the microscopic entropy of this system is given by $4\pi\sqrt{nw}$ for large nw . Our goal is to explore the possibility of realizing this system as a (small) black hole solution in a classical string theory.

First consider the heterotic description where both n and w correspond to electric NSNS charges. Thus the scaling relation (2.1) takes the form

$$S_{BH}(\lambda^2 n, \lambda^2 w) = \lambda^2 S_{BH}(n, w). \quad (2.2)$$

The microscopic result $4\pi\sqrt{nw}$ is consistent with this scaling symmetry [7]. This shows that in this case tree level higher derivative corrections could modify the singular near horizon geometry to correctly reproduce the entropy. Furthermore explicit computation with supersymmetrized curvature squared terms in heterotic string theory vacua with four non-compact space-time dimensions correctly reproduces the overall coefficient 4π [10, 11, 12, 13, 14]. The same result for the entropy is obtained if we include the Gauss-Bonnet term in the effective

³Note that this relation does not require any assumption about the structure of the near horizon geometry, and holds as long as the α' -corrected solution has a finite Wald entropy. It does however assume that the entropy for a given set of charges is independent of the asymptotic value of the dilaton and the moduli arising in the RR sector, so that the shift of the dilaton and scaling of the RR fields at infinity do not have any effect on the entropy. For small black holes in heterotic string theory, the independence of the near horizon geometry of the asymptotic values of the dilaton and other moduli was explicitly proven in [7]. If RR field strengths are present at infinity, as in the case of type IIB on $AdS_5 \times S^5$ with RR 5-form background, then our argument also requires that the black hole entropy depends only on a combination of the asymptotic RR field strength and the dilaton that remains invariant under the scaling described here. In the $AdS_5 \times S^5$ example such a combination corresponds to the 't Hooft coupling of the dual $\mathcal{N} = 4$ super Yang-Mills theory.

action [43]. General arguments based on AdS_3/CFT_2 correspondence shows that this result is exact to all orders in the derivative expansion [44].⁴ Explicit construction of small black hole solutions in vacua with five or more non-compact space-time dimensions has not been performed with the same degree of certainty. If we include the Gauss-Bonnet term in the five dimensional action then it does give a non-singular five dimensional black hole horizon with the correct entropy [43], but it is not clear why we are allowed to ignore the other terms in the action. Indeed, some recent discussion on the subtleties of constructing such solutions in five dimensions can be found in [49]. However in heterotic string vacua with five non-compact space-time dimensions one can explicitly construct small black ring solutions describing fundamental heterotic string with angular momentum $J \gg \sqrt{nw}$ by exploiting the fact that the local geometry near the core of a five dimensional small black ring is identical to that of a four dimensional small black hole with zero angular momentum [50, 51, 52]. These reproduce correctly the microscopic entropy

$$4\pi\sqrt{nw - J}. \quad (2.3)$$

More precisely the entropy of the small black ring is given by $4\pi\sqrt{nw - QJ}$ where Q is a ‘dipole charge’ describing fundamental heterotic string winding number along the ring. For given n, w, J the maximum contribution to the entropy comes from solutions with $Q = 1$. These black rings are in fact the right objects to compare with the smooth solutions to be described shortly, since the latter also describe five dimensional configurations carrying angular momentum. See footnote 5 for a more detailed discussion.

Let us now examine if there are smooth horizonless classical solutions carrying charges (n, w) . There are indeed classical solutions in heterotic string theory describing these microstates [22, 23], but they are known to require source terms and hence are not genuine solutions to the equations of motion of classical string field theory. Thus we conclude that in this description the macroscopic entropy of the system arises from small black holes / black rings, and not smooth classical solutions.

Now for $\mathcal{M} = T^4$, i.e. for heterotic string theory compactified on a five dimensional torus $T^4 \times S^1$, the same system has a dual description as a two charge D1-D5 system in type IIB

⁴While for systems carrying both electric and magnetic charges the near horizon geometry can be analyzed in a straightforward manner, it is more involved for a system carrying only electric charges as is the situation here. See [45, 46, 47, 48] for detailed discussions on this. It would be fair to say that while all approaches lead to the same value $4\pi\sqrt{nw}$ for the entropy of the small black hole, the detailed structure of the near horizon geometry still remains a mystery. Given that the world-sheet theory at the horizon is strongly coupled, this is to be expected since a field redefinition involving higher derivative terms can completely change the form of the near horizon geometry without affecting the entropy.

string theory on $K3 \times \widehat{S}^1$ with w denoting the number of D5-branes wrapped on $K3 \times \widehat{S}^1$ and n denoting the number of D1-branes wrapped on \widehat{S}^1 . In this case both n and w denote RR charges and the scaling relation (2.1) takes the form

$$S_{BH}(\lambda n, \lambda w) = \lambda^2 S_{BH}(n, w). \quad (2.4)$$

This is incompatible with the microscopic result $4\pi\sqrt{nw}$, showing that the classical string theory corrections cannot reproduce the statistical entropy of the system. In fact, had there been a classical black hole solution it would carry far larger entropy than the microscopic result. Thus the only consistent scenario is that a (small) black hole / black ring solution carrying these charges does not exist in classical type IIB string theory. On the other hand in this case the microstates of the D1-D5 system can be described as smooth, horizonless, source free solutions to the classical equations of motion [26, 27, 28, 53] and furthermore the geometric quantization of these solutions leads to a degeneracy in agreement with the microscopic result [29]. Thus we conclude that in this duality frame the two charge system can be described as smooth solutions but not as small black holes or black rings.⁵

The reader may find our insistence on using classical action (instead of the quantum effective action) a bit disturbing, so we shall now explain the difficulties which arise if we try to use quantum corrected effective action *computed in the vacuum*. This discussion will be closely related to the one in [55] with a slightly different emphasis. It is undoubtedly true that the quantum effective action of type IIB string theory on $K3 \times \widehat{S}^1$ has a Gauss-Bonnet term which maps to the tree level Gauss-Bonnet term of the heterotic theory on T^5 after a duality transformation. Thus we can construct a small black hole / ring solution of the equations of motion derived from the quantum effective action of the type IIB string theory on T^5 , since the corresponding solution exists in the heterotic theory. However the difficulty with this is the following. A simple analysis of the duality map tells us that the coefficient of this term in the

⁵For a distant observer the smooth solutions constructed in [27, 28] look like a ring [54] carrying a ‘dipole charge’ of Kaluza-Klein (KK) monopole associated with the \widehat{S}^1 compactification. Under the series of duality transformations which take a D1-D5 system to the fundamental heterotic string carrying momentum and winding along S^1 , the KK monopole becomes a fundamental heterotic string extending along the non-compact directions. This is precisely the structure of the black ring solution on the heterotic side, carrying fundamental string winding charge along the ring, besides the usual momentum and winding charges along S^1 . Thus the duality between the solutions in the type IIB and the heterotic description is also visible locally in five dimensional space-time as long as we stay far from the core of the ring. In general however one should be careful in comparing the classical profiles / quantum wave-functions of BPS states in different descriptions since this is not protected against quantum corrections. This has been discussed briefly in §4.

type IIB theory is proportional to the inverse power of the radius of \widehat{S}^1 .⁶ Thus it diverges in the limit when the radius of \widehat{S}^1 goes to zero, reflecting the fact that in actual computation this term arises after integrating out the modes carrying momentum and winding along \widehat{S}^1 [56]. Thus if we choose to work with this quantum effective action we have to forget about the original ten dimensional description and use an effective five dimensional description. However the ten dimensional description plays a non-trivial role in constructing the smooth classical solutions describing the D1-D5 system in type IIB string theory. In particular the \widehat{S}^1 circle is contractible in the full ten dimensional solution [28]; hence the smooth solutions will not appear as smooth solutions in the effective five dimensional description. This will of course be the wrong approach to this problem, and the correct approach will be to begin with the full ten dimensional smooth solution and study quantum corrections to that solution. We expect these to be small since the ten dimensional geometry is perfectly regular [55]. This shows that from the point of view of type IIB string theory it will not be correct to describe these states as small black holes / black rings by including the quantum corrections to the five dimensional effective action *computed in the vacuum*. On the other hand in the heterotic description we are perfectly justified in starting with the tree level effective action and constructing small black hole / black ring solutions corresponding to these states, particularly since the string coupling remains small at the core of the solution. The extra circle that is used to smoothen out the solutions on the type IIB side is not visible in the perturbative heterotic string theory.

Let us now turn to another example where we choose the compact space to be $T^5 \times S^1$ on the heterotic side. This theory has a dual description as type IIB string theory compactified on $K3 \times \widetilde{S}^1 \times S^1$ where the configuration under study is mapped to a system containing w KK monopoles associated with the circle \widetilde{S}^1 and $-n$ units of momentum along S^1 . First let us analyze the possibility of the existence of a small black hole describing this system. Since in this case w denotes an NSNS sector magnetic charge and n denotes an NSNS sector electric charge, the scaling relation (2.1) gives

$$S_{BH}(\lambda^2 n, w) = \lambda^2 S_{BH}(n, w). \quad (2.5)$$

This is incompatible with the microscopic result $4\pi\sqrt{nw}$, showing that there cannot be any classical black hole solution associated with these charges.

⁶More generally, it has been shown in appendix A that all the tree level higher derivative terms in the heterotic theory, when written in the type IIB variables, have the property that the coefficients of these terms have inverse powers of the radius of \widehat{S}^1 .

However one can argue that in this case the entropy can again be reproduced by quantization of smooth classical solutions [57, 58, 59]. At a generic point in the moduli space a system of w KK monopoles is described by a smooth classical solution, – the w -centered Taub-NUT space.⁷ This has $3w$ normalizable zero modes associated with the transverse motion of the w KK monopoles, leading to deformations parametrized by $3w$ massless scalar fields on the $(1+1)$ -dimensional world-volume theory of this system spanned by the time coordinate t and the coordinate y along S^1 .⁸ Furthermore in the background of w KK monopoles there are w normalizable self-dual harmonic 2-forms ω_m ($m = 1, \dots, w$) [60]. Thus every 2-form field C of the six dimensional theory describing type IIB string theory on K3 will give rise to w massless scalar fields ϕ_m on the $(1+1)$ dimensional world-volume spanned by (y, t) coordinates via the decomposition

$$C = \sum_{m=1}^w \phi_m(y, t) \omega_m. \quad (2.6)$$

Furthermore if the field strength dC is (anti-)self-dual, the corresponding scalar fields ϕ_m are (right-) left-moving on S^1 . Thus the 21 self-dual and 5 anti-self-dual 2-form fields in type IIB string theory on K3 – arising from the NSNS and RR 2-form fields of the ten dimensional type IIB string theory and the reduction of the RR 4-form field on the 2-cycles of K3 – give rise to deformations parametrized by $21w$ left-moving and $5w$ right-moving scalars along S^1 . Thus we have altogether $24w$ left-moving and $8w$ right-moving scalars along S^1 . There are also fermionic deformations parametrized by $8w$ right-moving fermion fields along S^1 , with each KK monopole contributing 8 right-moving fermions. If we want these deformations to preserve the supersymmetries of the undeformed background as is needed for counting BPS states, we must keep the right-movers along S^1 in their ground state but allow for arbitrary excitation of the left-movers. Since the left-movers describe a $(1+1)$ -dimensional conformal field theory with total central charge $24w$, the standard application of Cardy formula tells us that upon quantization of these classical deformations, the degeneracy of states carrying $-n$ units of momentum along S^1 grows as $\exp(4\pi\sqrt{nw})$, in agreement with the microscopic

⁷Again if we insist on using the quantum effective action we shall get a curvature squared term which can generate a small black hole solution with the right entropy, but the coefficient of this term blows up as the size of \tilde{S}^1 goes to zero, indicating that these terms are generated by integrating out the winding modes along \tilde{S}^1 . Hence the inclusion of such a term in the quantum effective action forces us to use a description in terms of a five dimensional effective theory. In this description we lose the KK monopole as a smooth solution since the circle \tilde{S}^1 shrinks to zero size at the core of the KK monopole.

⁸We shall take the deformations to be independent of the coordinates of K3 in order to preserve the BPS condition.

answer. Thus we see that in this case the macroscopic entropy arises from quantization of smooth deformations of the w KK-monopole solution. Therefore it is just as well that in this duality frame there are no small black holes accounting for the entropy of this system.

The above argument has been somewhat abstract, but at least for $w = 1$ all the left-moving modes can be constructed explicitly as horizonless, supersymmetric, finite deformations of the original Taub-NUT geometry, taking into account the effect of gravitational backreaction. For this we need to take the family of deformations of BMPV black hole in Taub-NUT space constructed in [61] and simply replace the BMPV metric by flat metric. This ensures that quantization of these deformations can generate not only low lying states associated with small fluctuations but highly excited states which contribute to the degeneracy. We expect that similar construction will be possible even for the $w \neq 1$ case – *e.g.* by taking a \mathbb{Z}_w orbifold of the deformed $w = 1$ solutions and then switching on additional deformations involving twisted sector NSNS and RR fields – but in this case the geometric quantization of the solutions may be more complicated as the moduli space of multiple KK monopoles has singularities at the points where the monopoles coincide.

An alert reader could wonder whether it is possible to get around the scaling argument by considering small black rings or other black objects which carry, besides the usual gauge charges, also dipole charges [62, 63, 64, 65]. In this case the scaling argument will require us to also scale the dipole charges, and we may be able to violate the scaling relations by working in sectors with fixed dipole charges. This however does not allow us to get around the no go results. To illustrate this let us focus on the particular example of the D1-D5 system in type IIB string theory on $K3 \times \widehat{S}^1$, and assume, for the sake of argument, that there is a small black ring solution at tree level that carries, besides the D5-brane charge w and D1-brane charge n , some electric dipole charge p in the NSNS sector. Then under the scaling p scales to $\lambda^2 p$ and the scaling relation (2.1) takes the form

$$S_{BH}(\lambda n, \lambda w, \lambda^2 p) = \lambda^2 S_{BH}(n, w, p). \quad (2.7)$$

Thus a relation of the form $S_{BH} = 4\pi\sqrt{nw p}$ will be consistent with (2.7) and one could argue that the small black ring with $p = 1$ produces the desired entropy. While this is correct, the point is that there is no reason why we should only include in our analysis small black rings with $p = 1$. In particular the scaling argument tells us that if in the classical theory there exists a small black ring solution with $p = 1$ and entropy $4\pi\sqrt{nw}$ for a given n and w , then there also exists a small black ring with $p = \lambda^2$, D5-brane charge $w' = \lambda w$ and D1-brane

charge $n' = \lambda n$ with entropy $4\pi\lambda^2\sqrt{nw} = 4\pi\lambda\sqrt{n'w'}$. For large λ this is larger than the required answer $4\pi\sqrt{n'w'}$, leading to a contradiction with the microscopic results. Thus we would conclude that such small black ring solutions do not exist in classical type IIB string theory on $K3 \times \widehat{S}^1$.

What general lesson can we draw from this analysis? One lesson is that in any given duality frame, in computing the degeneracy of states using a macroscopic description we must include the contribution from both (small) black holes *and* smooth classical solutions. In some description the contribution may come solely from smooth solutions and in some other description it may come solely from (small) black holes and only after adding their contributions together we can recover the complete duality invariant answer. This is also consistent with the proposal of [42] for a general formula for the macroscopic expression for the degeneracy of BPS states carrying a given set of charge quantum numbers \vec{q} (including angular momentum):

$$d_{macro}(\vec{q}) = \sum_s \sum_{\substack{\{\vec{q}_i\}, \vec{q}_{hair} \\ \sum_{i=1}^s \vec{q}_i + \vec{q}_{hair} = \vec{q}}} \left\{ \prod_{i=1}^s d_{hor}(\vec{q}_i) \right\} d_{hair}(\vec{q}_{hair}; \{\vec{q}_i\}). \quad (2.8)$$

The s -th term on the right hand side of (2.8) represents the contribution to the degeneracy from an s -centered black hole configuration.⁹ $d_{hor}(\vec{q}_i)$ is the degeneracy associated with the horizon of a single centered black hole (or any other black object) carrying charge \vec{q}_i . It is given by the exponential of the Wald entropy in the classical limit but more generally by the path integral of string fields over an Euclidean space with the asymptotic boundary conditions set by the attractor geometry [41]. $d_{hair}(\vec{q}_{hair}; \{\vec{q}_i\})$ is the degeneracy associated with the hair [66, 61] – smooth deformations of the black hole solution with support outside the horizon(s) – carrying total charge \vec{q}_{hair} , of an s -centered black hole whose horizons carry charges $\vec{q}_1, \vec{q}_2, \dots, \vec{q}_s$. From this viewpoint the degeneracy of states obtained by quantizing smooth solutions without horizon will be represented by the $s = 0$ terms in eq.(2.8), and will have to be included in addition to the contributions from black holes to get the complete macroscopic result for the degeneracy. This can then be compared with the microscopic result.

If instead of small black holes we consider large black holes then eq.(2.8) would lead to the following conclusion. Since large black holes have finite area event horizon in the supergravity limit, they exist in all duality frames, although the contribution to the horizon entropy d_{hor} , obtained from path integral over string fields in the near horizon geometry, could be different in

⁹In general this sum should include sum over all BPS black objects, including those which might be localized in one or more internal directions. For brevity we shall refer to all of them as black holes.

different duality frames after we take into account higher derivative and quantum corrections. This would mean in particular that the right hand side of (2.8) has a non-vanishing contribution from the $s = 1$ term in all duality frames. Thus in no duality frame d_{macro} will be described just by the $s = 0$ term, i.e. by smooth solutions without horizon. If such solutions do exist, then their contribution must be *added* to the contribution from black holes represented by the $s \geq 1$ terms in the sum.

3 Degeneracy or Index?

In comparing the spectrum of BPS states in different descriptions one often makes use of an appropriate index instead of absolute degeneracy since the former is protected against quantum corrections. As discussed in [42], one can also write down a formula analogous to (2.8) that involves the index instead of the degeneracy. For simplicity we shall first confine our discussion to theories with four non-compact space-time dimensions, and then briefly describe its generalization to higher dimensions. In heterotic string theory on T^6 the relevant index for half BPS states is the fourth helicity trace B_4 , where in general B_{2k} is defined as [67, 68]

$$B_{2k} = (-1)^k Tr [(-1)^{2h} (2h)^{2k}] / (2k)! . \quad (3.1)$$

Here h denotes the helicity of the state (or component of angular momentum along some specific direction in the rest frame) in a fixed charge sector. For $k = 0$ this will be the Witten index. The purpose of inserting the $(2h)^{2k}$ factor in the trace is to soak up the contribution from the fermion zero modes associated with $4k$ broken supersymmetry generators; without this factor the trace will vanish. The generalization of (2.8) for B_4 takes the form:

$$B_{4;macro}(\vec{q}) = \sum_s \sum_{\substack{\{\vec{q}_i\}, \vec{q}_{hair} \\ \sum_{i=1}^s \vec{q}_i + \vec{q}_{hair} = \vec{q}}} \left\{ \prod_{i=1}^s B_{0;hor}(\vec{q}_i) \right\} B_{4;hair}(\vec{q}_{hair}; \{\vec{q}_i\}) , \quad (3.2)$$

where $B_{0;hor}(\vec{q}_{hor})$ denotes the zeroth helicity trace – i.e. the Witten index – of the horizon degrees of freedom and $B_{4;hair}(\vec{q}_{hair}; \{\vec{q}_i\})$ denotes the fourth helicity trace of the hair degrees of freedom in given charge sectors. Note that in this case the charge vector \vec{q} no longer contains angular momentum, since we have already summed over angular momenta for computing the helicity trace. In arriving at (3.2) we have used the fact that the fermion zero modes associated with broken supersymmetry are typically part of the hair degrees of freedom and hence we need

to pick from the $(2h)^4$ factor in B_4 the $(2h_{hair})^4$ term in order to avoid vanishing of the trace over the fermion zero modes. Now in a given charge sector $B_{0;hor}(\vec{q}_{hor})$ is given by

$$B_{0;hor}(\vec{q}_{hor}) = \sum_{h_{hor}} (-1)^{2h_{hor}} d_{hor}(\vec{q}_{hor}, h_{hor}), \quad (3.3)$$

with h_{hor} denoting the angular momentum associated with the horizon. However in four dimensions supersymmetric black hole horizons carry no angular momentum. Hence only the $h_{hor} = 0$ term will contribute and we shall have $B_{0;hor}(\vec{q}_{hor}) = d_{hor}(\vec{q}_{hor}, h_{hor} = 0)$. Thus we can replace (3.2) by

$$B_{4;macro}(\vec{q}) = \sum_s \sum_{\substack{\{\vec{q}_i\}, \vec{q}_{hair} \\ \sum_{i=1}^s \vec{q}_i + \vec{q}_{hair} = \vec{q}}} \left\{ \prod_{i=1}^s d_{hor}(\vec{q}_i, h_i = 0) \right\} B_{4;hair}(\vec{q}_{hair}; \{\vec{q}_i\}). \quad (3.4)$$

Eq.(3.4) again shows that in any given duality frame the contribution from the smooth solutions represented by the $s = 0$ term and small black holes represented by the $s = 1$ term must be added. In fact in this case there are no contributions to the index from the $s \geq 2$ terms, i.e. multi-centered black holes. Hence the sum of the $s = 0$ and $s = 1$ terms will have to be compared with the index computed on the microscopic side.

For fundamental heterotic string states carrying quantum numbers (n, w) the index computed in the microscopic description grows in the same way as the degeneracy, i.e. as $\exp(4\pi\sqrt{nw})$, up to factors involving (inverse) powers of nw . On the macroscopic side there are no smooth solutions in the heterotic description, but there are small black holes for which $d_{hor}(n, w, h = 0)$ grows as $\exp(4\pi\sqrt{nw})$. If the only hair degrees of freedom are the fermionic zero modes associated with the eight broken supersymmetries, then $\vec{q}_{hair} = 0$ and $B_{4;hair} = 1$. This gives $B_{4;macro}(n, w) = d_{hor}(n, w, h = 0) \sim \exp(4\pi\sqrt{nw})$. Even if the hair modes include some additional degrees of freedom, typically their contribution to $\ln B_{4;macro}$ is small compared to the horizon contribution and we still get $B_{4;macro}(n, w) \sim \exp(4\pi\sqrt{nw})$, *as long as there are no additional fermion zero modes among the hair degrees of freedom*. Assuming that the last condition holds, we see that in the heterotic description we get a contribution to $B_{4;macro}$ of order $\exp(4\pi\sqrt{nw})$ from the single black hole sector in agreement with the microscopic result. In the type II description where the contribution to the macroscopic entropy comes from the smooth solutions – e.g. as KK monopole carrying momentum along its world-volume – geometric quantization directly gives the degeneracy for any angular momentum, and hence also the index. Thus as long as the classical modes are in one to one correspondence with the

oscillation modes of the fundamental heterotic string, we shall automatically get the correct degeneracy and index from geometric quantization of these modes.

Dealing with the index rather than the absolute degeneracy also throws some light on the difficulties associated with small black holes in type II string theory on T^6 , representing elementary type II string states. If we consider type IIA/IIB string theory on T^6 and take a fundamental type IIA/IIB string carrying winding charge w and momentum $-n$ along one of the circles, then the microscopic degeneracy of these states grows as $\exp(2\sqrt{2}\pi\sqrt{nw})$. Thus one might ask if these can be associated with the entropy of a small black hole. Even though the scaling analysis (2.2) would tell us that the classical small black hole could carry entropy proportional to \sqrt{nw} , explicit analysis has failed to find such a small black hole essentially due to the absence of curvature squared corrections in tree level type II string theory.¹⁰ We shall now argue that this could be a consequence of the (generalization of the) index formula (3.4); in fact if we had found only spherically symmetric small black holes in type II string theory on T^6 , as in the case of heterotic string theory on T^6 , and if the only fermion zero modes associated with this solution were the ones associated with the broken supersymmetry generators, it would have led to a contradiction. The relevant states are quarter-BPS and the relevant index is the twelfth helicity trace B_{12} . On the microscopic side these states are obtained by keeping the right-moving oscillators in their ground states, but considering excited states of the left-moving oscillators. Up to an overall normalization the microscopic contribution to B_{12} is given by the coefficient of $q^{nw} v^4$ in the expansion of

$$\prod_{k=1}^{\infty} (1 - q^k e^{iv})^{-1} (1 - q^k e^{-iv})^{-1} (1 - q^k)^{-6} (1 - q^k e^{iv/2})^4 (1 - q^k e^{-iv/2})^4. \quad (3.5)$$

Physically (3.5) represents the partition function of left-moving scalars and Green-Schwarz fermions on the fundamental string world-sheet, with v being the variable conjugate to the helicity. The 8 left-handed and 8 right-handed zero modes of the Green-Schwarz fermions are soaked up by the eight factors of h in the helicity trace. The remaining four factors of h are needed to soak up the bose-fermi degeneracy among the states created by the non-zero mode oscillators, reflecting the fact that not all the zero modes associated with the broken supersymmetries appear as zero modes in the world-sheet theory, – this is the reason why we need to compute the coefficient of the v^4 term. Collecting the order v^4 term in the expansion

¹⁰A proposal for the near horizon theory of these black holes has been suggested in [46], but there are no explicit solutions.

of (3.5) gives

$$\sum_{k=1}^{\infty} q^k (1 + 4q^k + q^{2k})(1 - q^k)^{-4}, \quad (3.6)$$

up to an overall normalization. If $b(N)$ denotes the coefficient of q^N for large N , then one can easily verify [69], using the modular properties of (3.5), that for large N , $\ln b(N)$ grows as $3 \ln N$. Thus we have

$$\ln B_{12;micro}(n, w) \simeq 3 \ln(nw). \quad (3.7)$$

Now suppose that we have a small spherically symmetric quarter-BPS black hole solution, carrying charge quantum numbers of elementary string in type II string theory on T^6 , and suppose further that there are no other classical quarter-BPS black hole solutions with non-zero angular momentum, carrying the same charge quantum numbers. Then the argument of [42] would lead to the relation

$$B_{12;macro}(\vec{q}) = \sum_s \sum_{\substack{\{\vec{q}_i\}, \vec{q}_{hair} \\ \sum_{i=1}^s \vec{q}_i + \vec{q}_{hair} = \vec{q}}} \left\{ \prod_{i=1}^s d_{hor}(\vec{q}_i, h_i = 0) \right\} B_{12;hair}(\vec{q}_{hair}; \{\vec{q}_i\}). \quad (3.8)$$

For states carrying quantum numbers of the fundamental string there are no contributions from the $s \geq 2$ terms. On the other hand we do not expect to have smooth classical solutions describing fundamental type II string, and so the $s = 0$ term also does not contribute. Thus we focus on the contribution from the $s = 1$ term, i.e. single centered black hole solutions. If the only fermion zero modes associated with the hair are those associated with the twelve broken supersymmetry generators, then B_{12} associated with the hair carrying $\vec{q}_{hair} = 0$ is non-zero and the helicity trace (3.8) receives a contribution proportional to $d_{hor}(n, w, h = 0)$. Now based on general scaling arguments given earlier we shall have [9]

$$\ln d_{hor}(n, w, h = 0) = C \sqrt{nw}, \quad (3.9)$$

for some constant C . This would give

$$\ln B_{12;macro}(n, w) \simeq C \sqrt{nw}. \quad (3.10)$$

This is in clear contradiction to (3.7). There seem to be two natural possibilities: 1) there is no small black hole describing the fundamental type II string; 2) there is a spherically symmetric small black hole solution, but the hair degrees of freedom associated with this black hole carry additional fermionic zero modes besides the ones associated with the broken

supersymmetry generators. In this case quantization of these additional fermion zero modes will make $B_{12,hair}$ vanish and hence this small black hole will not contribute to $B_{12,macro}$.

We can gain further insight by taking an asymmetric orbifold of type IIA/IIB string theory on T^6 by $(-1)^{F_L}$ accompanied by a half shift along one of the circles S^1 . This gives an $\mathcal{N} = 4$ supersymmetric string theory for which the helicity trace B_4 associated with the elementary string states grow as $\exp(2\sqrt{2}\pi\sqrt{nw})$ [70]. Thus supersymmetric small black holes must show up on the macroscopic side. Furthermore, the scaling argument of §2 tells us that in order that the logarithm of the index computed from this black hole is of order $C\sqrt{nw}$ it must arise at string tree level [7,9]. As a result it must also exist in type II string theory on T^6 , since the tree level effective action is not affected by compactification. The microscopic contribution to the helicity trace in the orbifold theory however has the strange property that it oscillates between positive and negative values depending on whether the momentum along S^1 is even or odd. Such a behaviour of the index cannot be reproduced by a spherically symmetric extremal black hole whose only hair degrees of freedom are the fermion zero modes associated with the broken supersymmetry generators; such a system will always give a positive contribution $d_{hor}(n, w; h = 0)$ to the index. Instead we need to look for new classes of supersymmetric extremal small black holes carrying non-zero angular momentum correlated with the momentum along S^1 .

Combining all these informations we shall now propose a possible scenario, but we do not claim that this is the only possible resolution. In this scenario type II string theory on T^6 has a quarter BPS small black hole carrying the same charges as the elementary string states, but besides the twelve fermion zero modes associated with broken supersymmetry generators, the world-volume degrees of freedom of the black hole also contain a massless (1+1) dimensional fermion field on S^1 arising out of R-NS sector fields.¹¹ Thus these modes are odd under $(-1)^{F_L}$. Due to the additional zero modes from this fermion field this small black hole will not contribute to the index B_{12} in type IIB string theory on T^6 . However since these zero modes are projected out in the orbifold theory, the black hole in the orbifold theory will have non-vanishing index. Furthermore even though the zero modes are projected out, the fermionic modes carrying odd momentum along S^1 survive the projection since they are even under the combined operation of a half shift along S^1 and $(-1)^{F_L}$. Since in order to get a state with odd momentum along

¹¹For black holes whose near horizon geometry contains an AdS_2 factor, we expect a clean separation between the hair and the horizon degrees of freedom due to the infinite AdS_2 throat. However it is not necessary that the proposed small black hole of type IIB string theory should have an AdS_2 factor, and hence there may not be a separation between the hair and horizon degrees of freedom. For this reason it may not be meaningful to ask whether this fermionic field lives on the hair or inside the horizon.

S^1 we need to excite odd number of modes of this fermion, they will be fermionic states and hence will give a negative contribution to the index. This would automatically explain why in the orbifold theory $B_{4;macro}$ oscillates between positive and negative values as we change the momentum along S^1 .

This unusual property of these black holes, – namely presence of additional fermionic modes on the world-volume, – could be the reason that these are not seen as solutions to the effective action containing just the ‘F-terms’; additional higher derivative D-terms may be needed for their existence. Such terms are undoubtedly present in the effective action of type II string theory compactified on T^6 , but we have much less control on these terms to be able to analyze small black hole solutions in their presence.

Before concluding this section we shall briefly describe the generalization of this analysis to higher dimensions. In four dimensions the information about the angular momentum carried by the individual states is lost in the index since we have to sum over all states carrying different angular momenta in defining the index. In five and higher dimensions one can do somewhat better. For definiteness we shall focus on the five dimensional theory obtained by compactifying heterotic string theory on T^5 . In this case the rotation group is $SO(4) \simeq SU(2) \times SU(2)$ and we can use a pair of quantum numbers (h_1, h_2) , labelling the J^3 quantum numbers of the two $SU(2)$ ’s, to characterize a state. We can now define an index

$$I_{2k}(\vec{q}, h_1) = \frac{(-1)^k}{(2k)!} \text{Tr}_{\vec{q}, h_1} ((-1)^{2h_1+2h_2} (2h_2)^{2k}). \quad (3.11)$$

Note that the trace is taken over a fixed charge and fixed h_1 sector. In defining the index we have broken the symmetry between the two angular momenta; which one we choose to keep fixed is a matter of convention. The index I_2 for fixed quantum numbers (n, w, h_1) receives contribution only from the half BPS states of the heterotic string. To see this we note that a half BPS state will break eight supersymmetries, and the associated fermion zero modes will contain four modes with $(h_1, h_2) = (0, \pm 1/2)$ and four modes with $(h_1, h_2) = (\pm 1/2, 0)$. The $(2h_2)^2$ factor is needed to soak up the fermion zero modes with quantum numbers $(0, \pm 1/2)$. On the other hand since the trace is taken over a fixed h_1 sector the fermion zero modes carrying quantum numbers $(\pm 1/2, 0)$ do not give a vanishing result. For a non-BPS state the number of fermion zero modes double in each sector and the trace over the eight fermion zero modes carrying quantum numbers $(0, \pm 1/2)$ will make the result vanish.

The analog of the formula (3.2) for the macroscopic index now takes the form:

$$I_{2;macro}(\vec{q}, h_1) = \sum_s \sum_{\substack{\{\vec{q}_i\}, \vec{q}_{hair} \\ \sum_{i=1}^s \vec{q}_i + \vec{q}_{hair} = \vec{q}, \sum_{i=1}^s h_{1i} + h_{1;hair} = h_1}} \left\{ \prod_{i=1}^s I_{0;hor}(\vec{q}_i; h_{1i}) \right\} I_{2;hair}(\vec{q}_{hair}; \{\vec{q}_i\}; h_{1,hair}), \quad (3.12)$$

again taking into account the fact that the fermion zero modes associated with broken supersymmetry generators are part of the hair degrees of freedom. In the $s = 1$ sector of the heterotic description, i.e. the sector containing a single small ‘black object’, as long as the only fermion zero modes associated with the hair degrees of freedom are the ones associated with the broken supersymmetry generators, $I_{2;hair}$ will give a finite contribution. In particular if the only hair degrees of freedom are the fermion zero modes associated with the broken supersymmetry generators then $\vec{q}_{hair} = 0$ and $I_{2;hair} = -1, 2, -1$ for $h_{1,hair} = -\frac{1}{2}, 0, \frac{1}{2}$.¹² This gives

$$I_{2;macro}(\vec{q}, h_1) = 2I_{0;hor}(\vec{q}, h_1) - I_{0;hor}(\vec{q}, h_1 - \frac{1}{2}) - I_{0;hor}(\vec{q}, h_1 + \frac{1}{2}) \simeq -\frac{1}{4} \frac{\partial^2 I_{0;hor}(\vec{q}, h_1)}{\partial h_1^2}. \quad (3.13)$$

On the other hand $I_{0;hor}(n, w, h_1)$ can be equated to $\sum_{h_2} (-1)^{2h_1+2h_2} d_{hor}(n, w, h_1, h_2)$. As discussed in §2, in this case the contribution to $d_{hor}(n, w, h_1, h_2)$ comes from a small black ring rotating in a plane, i.e. carrying $|h_1| = |h_2|$, and the entropy is given by $4\pi\sqrt{nw - |2h_1|}$. This would give

$$I_{0;hor}(n, w, h_1) \sim \exp[4\pi\sqrt{nw - |2h_1|}]. \quad (3.14)$$

When we substitute this in (3.13) the derivatives with respect to h_1 bring down inverse powers of charges, but do not curb the exponential growth. As a result we have

$$I_{2;macro}(n, w, h_1) \sim \exp[4\pi\sqrt{nw - |2h_1|}], \quad (3.15)$$

where \sim denotes equality up to factors involving (negative) powers of n , w and h_1 . As in the case of four dimensional small black holes, this result continues to hold even if the hair modes include some additional degrees of freedom, as long as they do not have any additional fermion zero modes.

On the other hand the microscopic computation of this index can be performed using the known spectrum of elementary heterotic string states. The BPS states correspond to states of

¹²These numbers arise from the quantization of the four fermion zero modes carrying $(h_1, h_2) = (\pm 1/2, 0)$.

the fundamental string for which the right-movers are kept in their ground state. The index $I_2(n, w, h_1)$ is given by the coefficient of $q^{nw} z^{2h_1}$ term in the expansion of:

$$q^{-1} \prod_{k=1}^{\infty} (1 - q^k)^{-20} (1 - q^k z)^{-2} (1 - q^k z^{-1})^{-2} (iz^{1/2} - iz^{-1/2})^2. \quad (3.16)$$

Here the $(1 - q^k)^{-20}$ term is the contribution from the left-moving bosonic oscillators along the internal directions, and the $(1 - q^k z)^{-2} (1 - q^k z^{-1})^{-2}$ term represents contribution from the left-moving bosonic oscillators along the four non-compact transverse directions. The $(iz^{1/2} - iz^{-1/2})^2$ term is the contribution from the four fermion zero modes carrying $(h_1, h_2) = (\pm 1/2, 0)$. Using a slight variation of the method developed in [6] one finds that for large nw and h_1 the index grows as

$$\exp[4\pi \sqrt{nw - |2h_1|}], \quad (3.17)$$

up to factors involving powers of the charges. This is in agreement with the macroscopic result (3.15).

In the type IIB description where these states are described as smooth classical solutions one needs to calculate the index by geometric quantization of the classical solutions with the extra factor of $((-1)^{2h_1+2h_2} (2h_2)^2)$ inserted. Since these classical solutions are in one to one correspondence to classical oscillations of the dual heterotic string, we expect the index to be reproduced correctly after we quantize this system.

4 How Does a BPS State Lose Information About Unprotected Quantities?

In §3 we have discussed comparison of the results of microscopic and macroscopic computation based on protected quantities like the helicity trace indices. One might wonder if we can compare more; e.g. the wave-functions computed in different descriptions, which in the classical limit gives information about the profile of the solutions.

We cannot prove that this is impossible, but there is no known index theorem that protects such information. In what follows we shall suggest a possible mechanism by which a BPS state could lose such information by drawing analogy to chiral fermions – an analogy often cited in motivating the index theorem for BPS states. As is well known, the difference between the number of left and right chiral fermions is protected by an index theorem. We shall now illustrate how a chiral fermion can lose information about its wave-function and hence

various attributes by mixing with non-chiral fermions while preserving the net number of chiral fermions in accordance with index theorem. Let us consider a theory with a single right-chiral fermion ψ_R and N non-chiral fermions (χ_{iL}, χ_{iR}) for $1 \leq i \leq N$ with standard kinetic terms and let us suppose that in the unperturbed theory the mass term has the form

$$M \sum_{i=1}^N \bar{\chi}_{iL} \chi_{iR} + c.c. . \quad (4.1)$$

In this theory ψ_R represents the massless fermion, – the analog of the BPS state. Now add a perturbation of the form

$$\epsilon \sum_{i=1}^N a_i \bar{\chi}_{iL} \psi_R + c.c. , \quad (4.2)$$

where ϵ is a small mass parameter and the a_i 's are finite dimensionless constants. We can make a unitary rotation $\tilde{\chi}_{iL} = U_{ij} \chi_{jL}$, $\tilde{\chi}_{iR} = U_{ij} \chi_{jR}$, $U^\dagger U = 1$, such that

$$\tilde{\chi}_{1L} = |\vec{a}|^{-1} \sum_{i=1}^N a_i^* \chi_{iL} . \quad (4.3)$$

In this basis the mass term takes the form

$$\epsilon |\vec{a}| \tilde{\chi}_{1L} \psi_R + M \sum_{i=1}^N \tilde{\chi}_{iL} \tilde{\chi}_{iR} + c.c. . \quad (4.4)$$

Finally we define

$$\hat{\chi}_R = (\epsilon |\vec{a}| \psi_R + M \tilde{\chi}_{1R}) / \sqrt{\epsilon^2 \vec{a}^2 + M^2}, \quad \hat{\psi}_R = (\epsilon |\vec{a}| \tilde{\chi}_{1R} - M \psi_R) / \sqrt{\epsilon^2 \vec{a}^2 + M^2} \quad (4.5)$$

so that $\hat{\psi}_R$, $\hat{\chi}_R$, $\tilde{\chi}_{1L}$ and $\tilde{\chi}_{iR}$, $\tilde{\chi}_{iL}$ for $2 \leq i \leq N$ have standard kinetic term. In this basis the mass term is given by

$$\sqrt{\epsilon^2 \vec{a}^2 + M^2} \tilde{\chi}_{1L} \hat{\chi}_R + M \sum_{i=2}^N \tilde{\chi}_{iL} \tilde{\chi}_{iR} + c.c. . \quad (4.6)$$

Thus in the final theory $\hat{\psi}_R$ represents the chiral fermion. Since $\vec{a}^2 \sim N$, we see from (4.5) it is made predominantly out of $\tilde{\chi}_{1R}$ if $N\epsilon^2 \gg M^2$. Thus even with small mixing the wave-function of the chiral fermion can change completely if there are a large number of states it could mix with.

To make this simple result look more dramatic we could imagine that the theory under consideration comes from a brane-world model in which all the fermions live on some space filling branes, with ψ_R living on brane A and the χ_i 's living on brane B , and suppose further that the branes A and B are separated by a finite distance along some compact direction. Then in the unperturbed theory the chiral fermion lives on the brane A . But once we switch on the deformation we see that the chiral fermion lives predominantly on the brane B . Thus it chooses to live not on the brane in which the original chiral fermion lived, but where most of the (non-chiral) fermions lived!

If the same mechanism operates on BPS states then this will mean that even for a small deformation the wave-function of a BPS state can change considerably by mixing with those of non-BPS states if there are a large number of non-BPS states carrying the same charges. This is precisely the situation in string theory where for fixed set of charges, and say within one string units of energy, the number of non-BPS states is exponentially larger than the number of BPS states. Thus the information about the wave-function of the BPS state may be wiped out by such mixings. (See [71] for a somewhat related discussion.)

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A Duality Transformation from Heterotic to Type II Description

In this appendix we shall analyze the fate of the tree level effective action of heterotic string theory under the duality map that takes it to the type IIB string theory. We begin with heterotic string theory on T^4 which is dual to type IIA string theory on $K3$ [72, 73, 74, 75, 76]. In the supergravity approximation where we keep only two derivative terms, the two actions are mapped to each other under this duality transformation. The heterotic metric $G^{(H)}$ is

related to the type IIA metric $G^{(A)}$ and the type IIA dilaton ϕ_A by the relation

$$G^{(H)} = e^{-2\phi_A} G^{(A)}. \quad (\text{A.1})$$

Consider now a term in the tree level heterotic string theory carrying $2n + 2$ derivatives. This will carry n extra powers of the inverse metric of heterotic string theory compared to the two derivative terms, and hence, when expressed in the type IIA variables, will carry an extra power of $e^{2n\phi_A}$ compared to the supergravity action. In particular if we consider only the NSNS sector fields of IIA the term in the type IIA action will carry a factor of $e^{2(n-1)\phi_A}$ since the supergravity action carries a factor of $e^{-2\phi_A}$.

Now consider compactifying this theory on a circle S^1 . On the heterotic side the state of interest is the one that carries fundamental string winding charge and momentum along S^1 . On the type IIA side this corresponds to NS 5-branes wrapped on $K3 \times S^1$ and momentum along S^1 . Since both charges are in the NSNS sector we are justified in working with the effective action in the NSNS sector. The effective five dimensional action on the type IIA side, coming from the tree level $2n + 2$ derivative terms in the heterotic string theory, will then contain a factor of

$$e^{2(n-1)\phi_A} R_A, \quad (\text{A.2})$$

where R_A is the radius of S^1 in the type IIA metric. We now make a T-duality transformation along S^1 to go to the type IIB description. The dilaton ϕ_B and the radius R_B of the dual circle \widehat{S}^1 will be related to ϕ_A and R_A via the relations

$$R_A = 1/R_B, \quad e^{\phi_A} = e^{\phi_B}/R_B. \quad (\text{A.3})$$

Using (A.2), (A.3) we see that the effective action in the type IIB string theory, coming from the tree level $2n + 2$ derivative term of the heterotic string theory will contain a factor of

$$e^{2(n-1)\phi_B} (R_B)^{1-2n}. \quad (\text{A.4})$$

In the IIB description the original fundamental heterotic string carrying winding and momentum along S^1 gets mapped to an NS 5-brane wrapped on $K3 \times \widehat{S}^1$ and a fundamental string wrapped on \widehat{S}^1 . A further strong-weak coupling duality transformation of the ten dimensional type IIB string theory will take this to a D1-D5 system, but this does not change the power of R_B in the effective action. Thus from (A.4) we see that any higher derivative ($n \geq 1$) term in the tree level heterotic string effective action, when represented in the type IIB description, carries an inverse power of the radius of the circle measured in the type IIB metric.

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