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PICTURE CHANGING OPERATORS IN CLOSED FERMIONIC STRING FIELD THEORY

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ABSTRACT

We discuss appropriate arrangement of picture changing operators required to construct gauge invariant interaction vertices involving Neveu-Schwarz states in heterotic and closed superstring field theory. The operators required for this purpose are shown to satisfy a set of descent equations.

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A closed bosonic string field theory based on non-polynomial interaction has been constructed recently [1 – 4] . In this paper we shall discuss the construction of gauge invariant field theory for heterotic and type II superstring theories based on the same principles. However, our analysis will be confined only to the Neveu-Schwarz sector of the theory; the construction of a similar field theory including Ramond sector states involves extra complications due to problems involving zero modes of various fields. Although the field theory for fermionic strings is not complete without Ramond sector, we can get some insight into the theory just by looking at the field theory involving the Neveu-Schwarz states. In particular, since for the heterotic string theory, all the bosonic fields come from the Neveu-Schwarz sector, the field theory involving Neveu-Schwarz states can be used to study the space of classical solutions, as well as the geometry of the configuration space of the theory.

We shall, for convenience, restrict our discussion to heterotic string theory only, the analysis for superstring theory proceeds in a similar manner. Let \mathcal{H} denote the Hilbert space of states in the Neveu-Schwarz sector in the -1 picture. These are the states created by the matter operators, and the ghost oscillators $b_n, c_n, \beta_n, \gamma_n$ acting on the state $e^{-\phi(0)}|0\rangle \equiv |\Omega\rangle$, where ϕ is the bosonized ghost, related to β, γ through the relations $\beta = e^{-\phi}\partial\xi, \gamma = e^{\phi}\eta$. Here η and ξ are fermionic fields of dimensions 1 and 0 respectively. $|0\rangle$ denotes the $SL(2,C)$ invariant vacuum in the combined matter ghost theory. As in the case of bosonic string theory, a general off-shell string state $|\Psi\rangle$ is taken to be a GSO projected state in \mathcal{H} annihilated by $c_0^- \equiv (c_0 - \bar{c}_0)/\sqrt{2}$ and $L_0^- \equiv (L_0 - \bar{L}_0)/\sqrt{2}$, and created from $|\Omega\rangle$ by an operator of total ghost number 3. Following the construction of bosonic string field theory, we shall look for an action of the heterotic string field theory of the form:

$$S(\Psi) = \frac{1}{2}\langle\Psi|Q_B b_0^-|\Psi\rangle + \sum_{N=3}^{\infty} \frac{g^{N-2}}{N!} \{ \Psi^N \} \quad (1)$$

where $\{A_1 \dots A_N\}$ and $[A_1 \dots A_N]$ are multilinear maps from N -fold tensor product of \mathcal{H} to C (the space of complex numbers) and \mathcal{H} respectively, satisfying relations

identical to the corresponding relations in bosonic string field theory:

$$\{A_1 \dots A_N\} \equiv (-1)^{n_1+1} \langle A_1 | [A_2 \dots A_N] \rangle \quad (2)$$

$$\{A_1 \dots A_N\} = (-1)^{(n_i+1)(n_{i+1}+1)} \{A_1 \dots A_{i-1} A_{i+1} A_i A_{i+2} \dots A_N\} \quad (3)$$

$$\{A_1 \dots (L_0^- A_i) \dots A_N\} = 0 = \{A_1 \dots (b_0^- A_i) \dots A_N\} \quad (4)$$

$$\begin{aligned} (-1)^{n_1} \{(Q_B A_1) A_2 \dots A_N\} &= \sum_{i=2}^N (-1)^{\sum_{j=2}^{i-1} (n_j+1)} \{A_1 \dots (Q_B A_i) \dots A_N\} \\ &- \sum_{\substack{m, n \geq 3 \\ m+n=N+2}} \sum_{\{j_k\}, \{i_l\}} (-1)^{\sigma(\{i_l\}, \{j_k\})} \{A_1 A_{j_1} \dots A_{j_{m-2}} c_0^- [A_{i_1} \dots A_{i_{n-1}}]\} \end{aligned} \quad (5)$$

Here n_i denotes the ghost number of the state $|A_i\rangle$. The sum over $\{i_l\}, \{j_k\}$ in eq.(5) runs over all possible divisions of the set of integers $2, \dots, N$ into the sets $\{i_l\}$ and $\{j_k\}$. $(-1)^{\sigma(\{i_l\}, \{j_k\})}$ is a factor of ± 1 which is computed as follows. Starting from the ordering Q_B, A_2, \dots, A_N , bring them into the order $A_{j_1}, \dots, A_{j_{m-2}}, Q_B, A_{i_1}, \dots, A_{i_{n-1}}$, using the rule that Q_B is anticommuting, and A_i is commuting (anti-commuting) if n_i is odd (even). The sign picked up during this rearrangement is $(-1)^{\sigma(\{i_l\}, \{j_k\})}$. Using eqs.(2)-(5) and the nilpotence of the BRST charge Q_B , one can show that $S(\Psi)$ given in eq.(1) is invariant under a gauge transformation of the form:

$$b_0^- \delta |\Psi\rangle = Q_B b_0^- |\Lambda\rangle + \sum_{N=3}^{\infty} \frac{g^{N-2}}{(N-2)!} [\Psi^{N-2} \Lambda] \quad (6)$$

where $|\Lambda\rangle$ is an arbitrary GSO projected state in \mathcal{H} created from $|\Omega\rangle$ by an operator of ghost number 2, and annihilated by c_0^- and L_0^- .

Thus, in order to construct a gauge invariant action, we need to construct multilinear maps $\{A_1 \dots A_N\}$ from \mathcal{H}^N to C , satisfying eqs.(2)-(5). We look for

an expression similar to the one in bosonic string theory [3] [5]:

$$\{A_1 \dots A_N\} = - \int_{R^{(N)}} d^{2N-6} \tau \langle f_1^{(N)} \circ (b_0^- P A_1)(0) \dots f_N^{(N)} \circ (b_0^- P A_N)(0) K_N \rangle \quad (7)$$

Here P is the projection operator $\delta_{L_0 \bar{L}_0}$. $f_i^{(N)}$ is a conformal map that maps the unit circle into the i th external string of the N -string diagram associated with the N -string vertex. $f_i^{(N)} \circ (b_0^- P A_i)$ denotes the conformal transform of the field $(b_0^- P A_i)$ under the map $f_i^{(N)}$. τ^i are the modular parameters characterizing the N -string diagram. The region of integration $R^{(N)}$, and the N -string diagram is chosen in such a way that at the boundary $\partial R^{(N)}$ of $R^{(N)}$, the N -string diagram is identical to an m string diagram and an $n(= N + 2 - m)$ string diagram glued by a tube of zero length and twist θ . One particular example of such N -string diagrams is provided by polyhedra with N -faces, the perimeter of each face being equal to 2π . In this case τ_i correspond to independent parameters labelling the lengths of each side of the polyhedron. The region of integration $R^{(N)}$ over τ is such that it includes all such polyhedra with the restriction that the length of any closed curve on the polyhedron constructed out of the edges should be greater than or equal to 2π . (Such polyhedra were called regular polyhedra in ref.[2]).

Finally we turn to the description of the operator K_N appearing in eq.(7). For bosonic string field theory K_N was given by $\prod_{i=1}^{2N-6} (\eta_i | B)$, where

$$(\eta_i | B) = \int d^2 z (\eta_{i\bar{z}}{}^z b(z) + \bar{\eta}_{iz}{}^{\bar{z}} \bar{b}(\bar{z})) \quad (8)$$

$\eta_{i\bar{z}}{}^z, \bar{\eta}_{iz}{}^{\bar{z}}$ are the beltrami differentials, which tell us how the components $g^{zz}, g^{\bar{z}\bar{z}}$ of the metric, induced on the sphere by the N -string diagram, changes as we change τ^i [3]. In the case of fermionic string theories, such a choice of K_N will give vanishing answer for $\{A_1 \dots A_N\}$ due to ghost number non-conservation. This is remedied by introducing appropriate factors of picture changing operators [6 – 8]

in the definition of K_N [9]. We shall make the following choice of K_N :

$$K_N = \sum_{r=0}^{2N-6} C_N^{(r)} \wedge \Phi_N^{(2N-6-r)} \quad (9)$$

where $C_N^{(r)}$ is an r -form (in the $2N - 6$ dimensional moduli space spanned by τ^i) operator, given by,

$$(C_N^{(r)})_{i_1 \dots i_r} = \prod_{k=1}^r (\eta_{i_k} | B) \quad (10)$$

$\Phi_N^{(r)}$, on the other hand, is an r form operator satisfying the ‘descent’ equation:

$$d^{(E)} \Phi_N^{(r)} = [Q_B, \Phi_N^{(r+1)}] \quad (11)$$

where $d^{(E)}$ denotes derivative with respect to τ^i , acting on the explicit τ dependent factors in the expressions for $\Phi_N^{(r)}$, and $[\]$ denotes a commutator (anti-commutator) for odd (even) r in eq.(11). $\Phi_N^{(0)}$ takes the form:

$$\Phi_N^{(0)} = \sum_{\alpha} A^{(\alpha)}(\tau^1, \dots, \tau^{2N-6}) X(w_1^{(\alpha)}(\tau)) \dots X(w_{N-2}^{(\alpha)}(\tau)) \quad (12)$$

where $X(z) = \{Q_B, \xi(z)\}$ is the picture changing operator [6]. The sum over α in eq.(12) runs over a finite set of values. For each value of α we have a set of points $w_1^{(\alpha)}(\tau), \dots, w_{N-2}^{(\alpha)}(\tau)$ on the N -string diagram, and an weight factor $A^{(\alpha)}(\tau)$, satisfying the normalization condition $\sum_{\alpha} A^{(\alpha)}(\tau) = 1$. $\Phi_N^{(r)}$'s, on the other hand, are constructed as linear combinations of products of X 's, $\partial\xi$'s and $(\xi(P) - \xi(Q))$'s.*

Besides the descent equation (11), $\Phi_N^{(r)}$ are also required to satisfy the following boundary conditions. As has been indicated before, the boundary $\partial R^{(N)}$ of $R^{(N)}$ consists of several pieces; each piece corresponds to gluing two vertices with less

* Appearance of extra factors proportional to $\partial\xi$ when the locations of the picture changing operators are moduli dependent, was discussed in ref.[10].

number of external states (m and $n = N + 2 - m$, say) along one of the external strings from each vertex with a certain twist θ . We demand that on such a component of $\partial R^{(N)}$,

$$\Phi_N^{(r)} \Big|_{\partial R^{(N)}} = \sum_{s=0}^r \Phi_m^{(r-s)} \wedge \Phi_n^{(s)} \quad (13)$$

Here $\Phi_N^{(r)} \Big|_{\partial R^{(N)}}$ denotes the component of $\Phi_N^{(r)}$ tangential to $\partial R^{(N)}$. Finally, $\Phi_N^{(r)}$ should be invariant under permutation of the external strings. Besides these restrictions, the choice of $\Phi_N^{(r)}$ (i.e. the quantities $A^{(\alpha)}$, $w_i^{(\alpha)}$, and the corresponding quantities appearing in the expressions for $\Phi_N^{(r)}$ for $r \geq 1$) is completely arbitrary.[†]

We shall now show that the quantities $\{A_1 \dots A_N\}$ constructed this way are non-zero in general, and satisfy eqs.(3)-(5) with $[A_1 \dots A_{N-1}]$ defined through eq.(2). To see that $\{A_1 \dots A_N\}$ is non-zero in general, we only need to note that the contribution to the right hand side of eq.(9) from the $r = 2N - 6$ term is of the form

$$\prod_{i=0}^{2N-6} (\eta_i | B) \sum_{\alpha} A^{(\alpha)}(\tau) X(w_1^{(\alpha)}) \dots X(w_{N-2}^{(\alpha)}) \quad (14)$$

Besides providing the appropriate factors of $(\eta_i | B)$ as in the case of bosonic string theory, we now also have the correct number of picture changing operators. Thus, if each $|A_i\rangle$ is created by a ghost number 3 operator acting on $|\Omega\rangle$, at least the contribution from the $r = 2N - 6$ term in eq.(9) to $\{A_1 \dots A_N\}$ will be non-zero. As we shall now see, the other terms are necessary for $\{A_1 \dots A_N\}$ to satisfy eq.(5).

Verification of eq.(2)-(4) with the definition of $\{A_1 \dots A_N\}$ given in eq.(7) is straightforward, so we turn to the verification of eq.(5). Using eq.(7), both sides of eq.(5) may be expressed as integrals (over τ^i) of appropriate correlation functions in the conformal field theory. In the left hand side of eq.(5), we may express

[†] Although we shall try to choose $\Phi_N^{(r)}$ in such a way that they are continuous inside $R^{(N)}$, this is not a necessary constraint, since, as can be seen from eq.(11), a discontinuity of $\Phi_N^{(r)}$ inside $R^{(N)}$ may be compensated by a δ -function singularity in $\Phi_N^{(r+1)}$.

$(Q_B A_1)$ as the contour integral of the BRST current around the location of the operator $b_0^- A_1$. We may now deform the BRST contour and shrink it to a point, in the process picking up residues from the locations of various other operators. The residues from the locations of the operators $b_0^- A_i$ ($i \geq 2$) give rise to the first set of terms on the right hand side of eq.(5). We are now left with terms proportional to $[Q_B, K_N]$. Residues from the terms proportional to $b(z), \bar{b}(\bar{z})$ in $(\eta_i|B)$ generates a factor of $\int d^2z(\eta_{i\bar{z}} z T(z) + \bar{\eta}_{iz} \bar{z} \bar{T}(\bar{z})) \equiv T_i^{(1)}$. Thus we may write,

$$[Q_B, C_N^{(r)}] = T^{(1)} \wedge C_N^{(r-1)} \quad (15)$$

where $C_N^{(r)}$ has been defined in eq.(10). Since insertion of a $T_i^{(1)}$ inside a correlation function generates the τ^i derivative of the correlation function, with the derivative acting on the implicit τ^i dependence of the correlation function due to the dependence of the string diagram on τ , we see that the terms proportional to $[Q_B, C_N^{(r)}]$ in $[Q_B, K_N]$ give rise to a term of the form:

$$\int_{R^{(N)}} d^{2N-6} \tau d^{(I)} \mathcal{B} \quad (16)$$

where,

$$\mathcal{B} = (-1)^{n_1} \left\langle \prod_{i=1}^N (f_i^{(N)} \circ b_0^- A_i(0)) \sum_{r=1}^{2N-6} C_N^{(r-1)} \wedge \Phi_N^{(2N-6-r)} \right\rangle \quad (17)$$

and $d^{(I)}$ denotes the τ derivative acting on the implicit τ dependence of the correlation function but not on the explicit τ dependence appearing in $\Phi_N^{(r)}$. On the other hand, terms proportional to $[Q_B, \Phi_N^{(r)}]$ in $[Q_B, K_N]$ may be analyzed using the descent equation (11), and gives an answer similar to eq.(16) with $d^{(I)}$ replaced by $d^{(E)}$. Thus if $d \equiv d^{(I)} + d^{(E)}$ denotes the total τ derivative, the contribution from the term proportional to $[Q_B, K_N]$ that appears in the analysis of the left

hand side of eq.(5) may be written as,

$$\int_{R^{(N)}} d^{2N-6} \tau d\mathcal{B} = \int_{\partial R^{(N)}} d^{2N-7} \tau \mathcal{B} \quad (18)$$

Let us now consider a specific component of the boundary $\partial R^{(N)}$ of $R^{(N)}$ that corresponds to gluing of two lower order string diagrams of m and $n = N + 2 - m$ external states, with a twist θ . Let $\tau_{(1)}$ and $\tau_{(2)}$ be the modular parameters describing these lower order string diagrams. Then,

$$d^{2N-7} \tau \Big|_{\partial R^{(N)}} = d^{2m-6} \tau_{(1)} d^{2n-6} \tau_{(2)} d\theta \quad (19)$$

$$\left(C_N^{(r)} \right)_{\theta a_1 \dots a_{r-1}} \Big|_{\partial R^{(N)}} = (\eta_\theta | B) \left(\sum_{s=0}^{r-1} C_m^{(s)} \wedge C_n^{(r-s-1)} \right)_{a_1 \dots a_{r-1}} \quad (20)$$

Using the boundary conditions given in eqs.(13) and (20) it is easy to see that,

$$\begin{aligned} & \left(\sum_{r=1}^{2N-6} C_N^{(r-1)} \wedge \Phi_N^{(2N-6-r)} \right)_{\theta \tau_{(1)}^1 \dots \tau_{(1)}^{2m-6} \tau_{(2)}^1 \dots \tau_{(2)}^{2n-6}} \Big|_{\partial R^{(N)}} \\ &= (\eta_\theta | B) \left(\sum_{r=0}^{2m-6} C_m^{(r)} \wedge \Phi_m^{(2m-6-r)} \right)_{\tau_{(1)}^1 \dots \tau_{(1)}^{2m-6}} \left(\sum_{s=0}^{2n-6} C_n^{(s)} \wedge \Phi_n^{(2n-6-s)} \right)_{\tau_{(2)}^1 \dots \tau_{(2)}^{2n-6}} \end{aligned} \quad (21)$$

Standard manipulations identical to the one for the bosonic string theory can now be used to show that the contribution to the right hand side of eq.(18) is identical to the second set of terms to the right hand side of eq.(5). This completes the proof of eq.(5).

We shall now indicate the basic steps involved in the calculation of Feynman amplitudes in this field theory. The external states are taken to be physical states of the form $b_0^- |A_i\rangle = c\bar{c}V_i(0)|\Omega\rangle \equiv c\bar{c}\hat{V}_i(0)|0\rangle$ where V_i is a dimension $(1/2, 1)$ superconformal primary field in the matter sector. Since the contribution from

various Feynman diagrams can be brought into the form of the contribution from elementary N -point vertex given in eq.(7) using standard techniques [11], we shall analyze the contribution to $A(1, \dots, N)$ from the elementary N -point vertex only. In analyzing this contribution we shall use the familiar expression for $\eta_{i\bar{z}}^z, \bar{\eta}_{i\bar{z}}^{\bar{z}}$ in terms of quasi conformal deformations [12] $v^z, \bar{v}^{\bar{z}}$, and write,

$$(\eta_i|B) = \oint_C \left(dz \frac{\delta v^z}{\delta \tau^i} b(z) + d\bar{z} \frac{\delta \bar{v}^{\bar{z}}}{\delta \tau^i} \bar{b}(\bar{z}) \right) \quad (22)$$

where the contour C encloses all the points z_i , as well as the locations of all the operators appearing in $\hat{\Phi}_N^{(r)}$. δv^z ($\delta \bar{v}^{\bar{z}}$) is analytic (anti-analytic) inside the contour C but not outside C . We can now deform C and shrink it to a point, picking up residues from the locations of various operators in this process. In the case of bosonic string theory, the only possible residues are picked up from the locations of the vertex operators $b_0^- A_i = c\bar{c}\hat{V}_i$. This removes the $c\bar{c}$ factors from $N - 3$ of the vertex operators and generates appropriate measure factors that converts the integration over τ^i to integration over the locations of the $(N - 3)$ vertices. In the present case, however, the integration contour can pick up residues from the locations of the picture changing operators inside $\Phi_N^{(r)}$ also. For any r form operator \mathcal{O} , let us define an $r + 1$ form operator $\delta\mathcal{O}$ as,

$$(\delta\mathcal{O})_{i_1 \dots i_{r+1}} = A \left(\oint_C \left(dz \frac{\delta v^z}{\delta \tau^{i_1}} b(z) + d\bar{z} \frac{\delta \bar{v}^{\bar{z}}}{\delta \tau^{i_1}} \bar{b}(\bar{z}) \right) \mathcal{O}_{i_2 \dots i_{r+1}} \right) \quad (23)$$

where A denotes antisymmetrization in the indices i_1, \dots, i_{r+1} , and C denotes a contour enclosing the locations of all the operators in \mathcal{O} . Using eqs.(7)-(10) the contribution to $A(1, \dots, N)$ from the elementary N -point vertex may be written as,

$$g^{N-2} \int_{R^{(N)}} d^{2N-6}\tau \sum_{r=0}^{2N-6} \sum_{s=0}^r \frac{1}{s!(r-s)!} \left\langle \delta^s \left(\prod_{k=1}^N c\bar{c}\hat{V}_k(z_k, \bar{z}_k) \right) \wedge \delta^{r-s} \Phi_N^{(2N-6-r)} \right\rangle \quad (24)$$

where δ^s denotes s successive operations of δ and $z_i = f_i^N(0)$.

So far our description has been independent of the choice of coordinates of the moduli space. We can now simplify our analysis by choosing the moduli parameters τ^i to be identical to the coordinates x_i ($1 \leq i \leq 2N - 6$) defined as follows:

$$x_i = z_{\frac{i+1}{2}} \quad \text{for } i \text{ odd}; \quad x_i = \bar{z}_{\frac{i}{2}} \quad \text{for } i \text{ even} \quad (25)$$

In that case, for odd i ,

$$\begin{aligned} \frac{\delta v^z}{\delta \tau^i} &= 1 \quad \text{at } z = z_{\frac{i+1}{2}}; & \frac{\delta v^z}{\delta \tau^i} &= 0 \quad \text{at } z = z_j \text{ for } j \neq \frac{i+1}{2} \\ \frac{\delta \bar{v}^{\bar{z}}}{\delta \tau^i} &= 0 \quad \text{at } z = z_j \text{ for all } j \end{aligned} \quad (26)$$

and, for even i ,

$$\begin{aligned} \frac{\delta v^z}{\delta \tau^i} &= 0 \quad \text{at } z = z_j \text{ for all } j \\ \frac{\delta \bar{v}^{\bar{z}}}{\delta \tau^i} &= 1 \quad \text{at } z = z_{\frac{i}{2}}; & \frac{\delta \bar{v}^{\bar{z}}}{\delta \tau^i} &= 0 \quad \text{at } z = z_j \text{ for } j \neq \frac{i}{2} \end{aligned} \quad (27)$$

Let us now define,

$$\hat{\Phi}_N^{(s)} = \sum_{r=0}^s \frac{1}{r!} \delta^r \Phi_N^{(s-r)} \quad (28)$$

and,

$$L_N^{(s)} = \frac{1}{s!} \delta^s \left(\prod_{k=1}^N c \bar{c} \hat{V}_k \right) \quad (29)$$

so that,

$$\left(L_N^{(2N-6-r)} \right)_{i_1 \dots i_{2N-6-r}} = \frac{\delta}{\delta C_{i_1}} \dots \frac{\delta}{\delta C_{i_{2N-6-r}}} \prod_{i=1}^N c \bar{c} \hat{V}_i(z_i, \bar{z}_i) \quad (30)$$

where,

$$C_j = c(z_{\frac{j+1}{2}}) \quad \text{for } j \text{ odd}; \quad C_j = \bar{c}(\bar{z}_{\frac{j}{2}}) \quad \text{for } j \text{ even} \quad (31)$$

Then eq.(24) takes the form:

$$A(1, \dots, N) = g^{N-2} \int d^{2N-6} x_i \sum_{r=0}^{2N-6} \langle L_N^{(r)} \wedge \hat{\Phi}_N^{(2N-6-r)} \rangle \quad (32)$$

By techniques identical to those used in the case of bosonic string field theory one can show that the integral over x_i runs over the full moduli space when we add the contribution from all the Feynman diagrams.

Using the relations $\{Q_B, b(z)\} = T(z)$, $\{Q_B, \bar{b}(\bar{z})\} = \bar{T}(\bar{z})$, the fact that the insertion of $\int d^2z (\eta_{i\bar{z}}^z T(z) + \bar{\eta}_{i\bar{z}}^{\bar{z}} \bar{T}(\bar{z}))$ in a correlation function generates the (implicit) derivative of the correlation function with respect to τ^i , and the descent equation (11) we get,

$$[Q_B, \delta^r \Phi_N^{(l)}] = r d^{(I)} (\delta^{r-1} \Phi_N^{(l)}) + d^{(E)} (\delta^r \Phi_N^{(l-1)}) \quad (33)$$

Using eq.(33) one can easily verify that $\hat{\Phi}_N^{(r)}$ defined in eq.(28) satisfies the descent equation:

$$d \hat{\Phi}_N^{(r)} = [Q_B, \hat{\Phi}_N^{(r+1)}] \quad (34)$$

where d denotes total derivative with respect to the x_i 's.

We now turn to the question of comparing the amplitude given in eq.(32) with the one calculated from the first quantized formalism. In order to do so we shall first prove the following

Lemma: If we have two sets of operators $\hat{\Phi}_N^{(r)}$ and $\hat{\Psi}_N^{(r)}$, both satisfying the descent equations given in eq.(34), and, if,

$$\hat{\Phi}_N^{(0)} - \hat{\Psi}_N^{(0)} = \{Q_B, \chi_N^{(0)}\} \quad (35)$$

for some $\chi_N^{(0)}$, then,

$$\int d^{2N-6} x_i \sum_{r=0}^{2N-6} \langle L_N^{(r)} \wedge (\hat{\Phi}_N^{(2N-6-r)} - \hat{\Psi}_N^{(2N-6-r)}) \rangle = 0 \quad (36)$$

The lemma is proved in the following way. From eq.(35) and the descent equation for $\hat{\Phi}_N^{(r)}$, $\hat{\Psi}_N^{(r)}$, we get,

$$\{Q_B, \hat{\Phi}_N^{(1)} - \hat{\Psi}_N^{(1)}\} = \{Q_B, d\chi_N^{(0)}\} \quad (37)$$

This gives,

$$\hat{\Phi}_N^{(1)} - \hat{\Psi}_N^{(1)} = d\chi_N^{(0)} + [Q_B, \chi_N^{(1)}] \quad (38)$$

for some $\chi_N^{(1)}$. Repeating this, we get the general equation,

$$\hat{\Phi}_N^{(r)} - \hat{\Psi}_N^{(r)} = d\chi_N^{(r-1)} + [Q_B, \chi_N^{(r)}] \quad (39)$$

Finally, using the definition of $L_N^{(r)}$ and the commutation relations,

$$\begin{aligned} [Q_B, \hat{V}_i(z_i, \bar{z}_i)] &= \frac{\partial}{\partial z_i}(c\hat{V}_i(z_i, \bar{z}_i)) + \frac{\partial}{\partial \bar{z}_i}(\bar{c}\hat{V}_i(z_i, \bar{z}_i)) \\ \{Q_B, c\hat{V}_i(z_i, \bar{z}_i)\} &= \frac{\partial}{\partial \bar{z}_i}(\bar{c}\hat{V}_i), \quad \{Q_B, \bar{c}\hat{V}_i\} = \frac{\partial}{\partial z_i}(c\hat{V}_i), \quad [Q_B, c\bar{c}\hat{V}_i] = 0 \end{aligned} \quad (40)$$

we get,

$$[Q_B, L_n^{(r)}] = dL_n^{(r-1)} \quad (41)$$

Using eqs.(39), (41), and the fact that the expectation value of a BRST exact operator vanishes, we can bring the left hand side of eq.(36) into the form:

$$\int d^{2N-6}x_i d\left(\sum_{r=0}^{2N-7} L_N^{(r)} \wedge \chi_N^{(2N-7-r)}(-1)^r\right) \quad (42)$$

which vanishes after integration over x_i . This completes the proof of the lemma.

Let us now choose,

$$\hat{\Psi}_N^{(0)} = X(z_3)X(z_4)\dots X(z_N) \quad (43)$$

Using the fact that $\hat{\Phi}_N^{(0)}$ has the form $\sum_{\alpha} \tilde{A}^{(\alpha)}(\{x_i\})X(w_1^{(\alpha)})\dots X(w_{N-2}^{(\alpha)})$ with $\sum_{\alpha} \tilde{A}^{(\alpha)} = 1$, we see that $\hat{\Phi}_N^{(0)} - \hat{\Psi}_N^{(0)}$ is BRST exact. Thus by the above lemma,

we can replace $\hat{\Phi}_N^{(r)}$ by $\hat{\Psi}_N^{(r)}$ in the expression for $A(1, \dots, N)$ given in eq.(32). A set of $\hat{\Psi}_N^{(r)}$ satisfying the descent equation are given by,

$$\begin{aligned} (\hat{\Psi}_N^{(r)})_{i_1 \dots i_r} &= \prod_{k=1}^r \partial \xi(z_{i_{\frac{k+1}{2}}}) \prod_{\substack{j=3 \\ j \neq \frac{i_k+1}{2}}}^N X(z_j) \text{ for } i_k \text{ odd and } 3 \leq \frac{i_k+1}{2} \leq N-3 \text{ for all } k \\ &= 0 \text{ otherwise} \end{aligned} \tag{44}$$

Using this we get the following expression for $A(1, \dots, N)$:

$$\begin{aligned} A(1, \dots, N) &= g^{N-2} \int \prod_{i=1}^{N-3} d^2 z_i \prod_{i=1}^2 \hat{V}_i(z_i, \bar{z}_i) \prod_{i=N-2}^N (c \bar{c} \hat{V}_i(z_i, \bar{z}_i) X(z_i)) \\ &\quad \prod_{i=3}^{N-3} \left(\hat{V}_i(z_i, \bar{z}_i) (X(z_i) - c(z_i) \partial \xi(z_i)) \right) \end{aligned} \tag{45}$$

where $d^2 z_i \equiv d\bar{z}_i \wedge dz_i$. In the above equation, $\hat{V}_i(z_i, \bar{z}_i)$ corresponds to integrated vertex operator in the -1 picture [6]. Also,

$$\hat{V}_i(z_i, \bar{z}_i) (X(z_i) - c(z_i) \partial \xi(z_i)) = \{Q_B, \hat{V}_i(z_i, \bar{z}_i) \xi(z_i)\} - \partial (\hat{V}_i(z_i, \bar{z}_i) c(z_i) \xi(z_i)) - \bar{\partial} (\hat{V}_i(z_i, \bar{z}_i) \bar{c}(z_i) \xi(z_i)) \tag{46}$$

correspond to integrated vertex operators in the zero picture. Finally, $\hat{V}_i(z_i, \bar{z}_i) c(z_i) \bar{c}(\bar{z}_i) X(z_i)$ correspond to unintegrated vertex operators in the zero picture. Thus we see that the right hand side of eq.(45) has precisely the form expected from the analysis of the first quantized theory.

We now turn to the problem of determining $\Phi_N^{(r)}$ satisfying eqs.(11) and (13). We shall discuss one particular construction, but one should keep in mind that this construction is in no way unique, and there are (infinitely) many other choices possible. In order to prescribe $\Phi_N^{(r)}$ in a way that avoids the divergences associated with collision of picture changing operators [13],^{*} we shall find it more convenient

^{*} Removal of such divergences in open string field theory has been discuss by several authors [14 – 16]

to take the N -string diagram not just a regular polyhedra with N faces, but regular polyhedra with N faces with tubes of a certain fixed length l_0 attached to each of the faces; together with diagrams corresponding to such regular polyhedra with $m_1, m_2, \dots, m_{r+1} = N + 2r - (m_1 + \dots + m_r)$ faces, joined by tubes of length l_1, \dots, l_r ($0 \leq l_i \leq 2l_0$) and twist $\theta_1, \dots, \theta_r$ ($0 \leq \theta_i < 2\pi$). (Such vertices have been used in ref.[17] for a different purpose.) We shall denote by $R_e^{(N)}$ the component of $R^{(N)}$ for which the corresponding string diagram is a regular N -hedron with tubes of length l_0 attached to its faces. $R_c^{(N)}$ will denote the component of $R^{(N)}$ for which the corresponding string diagram is given by two or more such regular polyhedra connected by tubes of length $\leq 2l_0$.

We shall first discuss the construction of $\Phi_N^{(0)}$ for points inside $R_e^{(N)}$. In this case the picture changing operators are taken to be at the mid-points of the edges of the polyhedron. Since the number of edges ($3N - 6$) of an N -hedron is larger than the number of picture changing operators ($N - 2$), there are several possibilities. We average over all configurations, with the weight factor for a given configuration being proportional to $\prod_i f(s_i)$, where s_i is the length of the i th edge, the product over i runs over all the $(N - 2)$ edges containing the picture changing operators, and $f(s_i)$ is a smooth function of s_i satisfying the constraint:

$$f(s_i) = 0 \text{ for } s_i \leq \eta; \quad f(s_i) = 1 \text{ for } s_i \geq 2\eta \quad (47)$$

where η is a small but fixed number. This construction guarantees that the picture changing operators are always inserted at the mid-points of the edges which have length $\geq \eta$, and hence two picture changing operators never collide inside $R_e^{(N)\dagger}$. This completely specifies $\Phi_N^{(0)}$ inside $R_e^{(N)}$.

† Note that we could also have, in principle, chosen the picture changing operators at the vertices, imitating the corresponding construction for open string field theory. But in this case special care (like moving the picture changing operators away from the vertices) is needed to ensure that two picture changing operators do not collide, since for $N \geq 5$, $R_e^{(N)}$ contains configurations where the number of well separated vertices (say, by a distance $\geq \eta$) is less than $N - 2$. Such a situation does not occur if we insert the picture changing operators at the midpoints of the edges, since a regular N -hedron contains at least N edges of finite length.

Let us now turn to a point inside $R_c^{(N)}$ which corresponds to two regular polyhedra joined by a tube of length l . Let h be some fixed length $\leq l_0$. For $l \geq 2h$, we choose $\Phi_N^{(0)}$ to be simply the product of $\Phi_N^{(0)}$ on the two polyhedra. This automatically ensures that $\Phi_N^{(0)}$ satisfies the boundary condition (13) at the boundary $l = 2l_0$ of $R^{(N)}$. For $l \leq h$, we choose the picture changing operators to be on the edges of the two polyhedra in such a way that $\Phi_N^{(0)}$ is independent of l and is identical to its value at $l = 0$, where it is given by the previous construction of $\Phi_N^{(0)}$ for points inside $R_e^{(N)}$. This does not specify $\Phi_N^{(0)}$ completely, since a picture changing operator inserted on the right boundary of the tube or the left boundary of the tube at the same angle generates the same configuration when the tube length is collapsed to zero. We remove this ambiguity by assigning equal weight factor to each of these configurations. Finally, in the region $h \leq l \leq 2h$, $\Phi_N^{(0)}$ is chosen to be a linear combination of the expressions for $\Phi_N^{(0)}$ for $l \leq h$ and $l \geq 2h$ such that the resulting expression smoothly interpolates between the values of $\Phi_N^{(0)}$ for $l \leq h$ and $l \geq 2h$.

This construction can easily be generalized to points inside $R_c^{(N)}$ representing string diagrams where more than two polyhedra are joined together by tubes of length $\leq 2l_0$. Ambiguities similar to the one discussed above arises in the region when some of the tubes are of length $\leq h$, since many different segments of edges belonging to different polyhedra may correspond to the same segment when all tubes of length $\leq h$ is collapsed. Hence if the polyhedron obtained after this collapse has an insertion of picture changing operator on this segment, we need to decide how to distribute it over the various segments before the collapse. One possible consistent way is to distribute it only among the two segments which are at the two extreme ends of the chain of tubes connecting these different segments, with equal weight factor.

This finishes the discussion on the construction of $\Phi_N^{(0)}$. Once $\Phi_N^{(0)}$ is given, $\Phi_N^{(r)}$ may be obtained by solving the descent equations. We shall not discuss the general case here, but as an example, give a specific solution for $\Phi_4^{(r)}$. In this case

$\Phi_4^{(0)}$ may be expressed as

$$\Phi_4^{(0)} = X(P)X(Q) + \sum_{\alpha} f^{(\alpha)}(\tau)X(P^{(\alpha)})(X(Q^{(\alpha)}) - X(R^{(\alpha)})) \quad (48)$$

where the sum over α runs over a finite set. The corresponding solution for $\Phi_4^{(r)}$ is,

$$\begin{aligned} \Phi_4^{(1)} &= (d\xi(P)X(Q) + X(P)d\xi(Q)) + \sum_{\alpha} df^{(\alpha)}X(P^{(\alpha)})(\xi(Q^{(\alpha)}) - \xi(R^{(\alpha)})) \\ &\quad + \sum_{\alpha} f^{(\alpha)}\left(d\xi(P^{(\alpha)})(X(Q^{(\alpha)}) - X(R^{(\alpha)})) + X(P^{(\alpha)})(d\xi(Q^{(\alpha)}) - d\xi(R^{(\alpha)}))\right) \\ \Phi_4^{(2)} &= d\xi(P) \wedge d\xi(Q) - \sum_{\alpha} df^{(\alpha)} \wedge d\xi(P^{(\alpha)})(\xi(Q^{(\alpha)}) - \xi(R^{(\alpha)})) \\ &\quad + \sum_{\alpha} f^{(\alpha)}d\xi(P^{(\alpha)}) \wedge (d\xi(Q^{(\alpha)}) - d\xi(R^{(\alpha)})) \\ \Phi_4^{(r)} &= 0 \quad \text{for } r \geq 3 \end{aligned} \quad (49)$$

where $d\xi(P) = \partial\xi(P)dP$ etc.

To summarize, in this paper we have discussed the construction of a gauge invariant field theory for the Neveu-Schwarz (NS) sector of the heterotic string theory. This construction can easily be generalized to include the NS-NS sector of closed superstring field theory as well. The Ramond sector, however, suffers from extra problems due to the presence of zero modes of various operators, and cannot, at present, be treated by the same method. We hope to come back to this question in the future.

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