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QUANTIZATION OF DYON CHARGE AND ELECTRIC-MAGNETIC DUALITY IN STRING THEORY

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ABSTRACT

We analyze the allowed spectrum of electric and magnetic charges carried by dyons in (toroidally compactified) heterotic string theory in four dimensions at arbitrary values of the string coupling constant and θ angle. The spectrum is shown to be invariant under electric-magnetic duality transformation, thereby providing support to the conjecture that this is an exact symmetry in string theory.

It has recently been shown [1] [2] that the equations of motion derived from the low energy effective action in four dimensional string theory are invariant under the electric-magnetic duality transformation [3] [4] that interchanges the electric and magnetic fields, and at the same time interchanges the strong and weak coupling limits of the theory. Using the fact that at least in the four dimensional string theory obtained by toroidal compactification of ten dimensional heterotic string theory, there exists a string like solution in this effective field theory whose zero modes are in one to one correspondence to the dynamical degrees of freedom of the fundamental heterotic string in four dimensions [5] [6], it was argued in ref.[2] that the effective field theory contains all the information about the full string theory, and hence the duality symmetry of the effective field theory might imply duality symmetry of the full string theory under which electrically charged particles get interchanged with magnetically charged particles. Earlier conjectures to this effect in field theory was made in refs.[7], and in string theory in refs.[8]. Similar duality between ten dimensional heterotic string theory and five-brane theory was conjectured in refs.[9][10][11]. Finally, appplications of this duality transformation to generate new classical solutions in the effective field theory were made in refs.[1][2][12].

Our analysis in ref.[2] has been purely classical. In this paper we shall analyze the compatibility of the duality conjecture with the well known quantization condition of the electric and magnetic charges of a dyon [13]. These conditions are known to receive non-trivial modifications in the presence of the theta angle [14]. We shall show that the quantization rules are invariant under duality transformation, i.e. if we start from a given value of the string coupling constant g and θ-angle, and then perform a duality transformation that changes g to g' and θ to θ', then the transformed dyon state is one of the allowed states for string coupling g' and angle θ'. Related work for type IIB superstring theory in ten dimensions was carried out in ref.[15]. Effect of θ-terms on self dual laattice models was studied in refs.[16].

The result of refs.[17][2] may be summarized as follows. The low energy effective field theory describing toroidally compactified heterotic string theory in four dimensions contains the metric $G_{\mu\nu}$, 28 vector fields $A_\mu^{(\alpha)}$ ($1 \le \alpha \le 28$), a complex scalar field $\lambda = \lambda_1 + i\lambda_2$, and a set of scalar fields that can be described by a 28×28 matrix M, satisfying,

$$M^T = M, \qquad M^T L M = L \tag{1}$$

where,

$$L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix} \tag{2}$$

is a 28×28 matrix. I_n denotes the $n \times n$ identity matrix. The equations of motion of the low energy effective field theory follow from the action:

$$\begin{aligned} S = & \frac{1}{32\pi} \int d^4x \sqrt{-\det G}[R - \frac{1}{2(\lambda_2)^2} G^{\mu\nu} \partial_\mu \lambda \partial_\mu \bar{\lambda} - \lambda_2 \vec{F}^T_{\mu\nu}.LML.\vec{F}^{\mu\nu} \\ & + \lambda_1 \vec{F}^T_{\mu\nu}.L.\vec{\tilde{F}}^{\mu\nu} + \frac{1}{8} G^{\mu\nu} Tr(\partial_\mu ML \partial_\nu ML)] \end{aligned} \tag{3}$$

Here the arrow on $\vec{F}_{\mu\nu}$ denotes that it is a 28 dimensional vector:

$$F^{(\alpha)}_{\mu\nu} = \partial_\mu A^{(\alpha)}_\nu - \partial_\nu A^{(\alpha)}_\mu \tag{4}$$

$\tilde{F}^{(\alpha)}_{\mu\nu}$ denotes the dual of $F^{(\alpha)}_{\mu\nu}$. The vector $\vec{F}_{\mu\nu}$ is related to a similar vector defined in ref.[2] by a factor of 2. Also S has been multiplied by an overall factor of $1/32\pi$, which does not affect the classical equations of motion and hence the analysis of ref.[2].[⋆] The equations of motion derived from this action (but not the action itself) are invariant under the following two transformations:

$$\lambda \to \lambda + 1, \quad \vec{F}_{\mu\nu} \to \vec{F}_{\mu\nu}, \quad M \to M, \quad G_{\mu\nu} \to G_{\mu\nu} \tag{5}$$

⋆ This factor can always be absorbed into a rescaling of λ and $G_{\mu\nu}$.

and,

$$
\begin{aligned}
&\lambda \to \lambda' = -\frac{1}{\lambda}, \qquad \vec{F}_{\mu\nu} \to \vec{F}'_{\mu\nu} = -\lambda_2 M L \vec{\tilde{F}}_{\mu\nu} - \lambda_1 \vec{F}_{\mu\nu} \\
&M \to M, \qquad G_{\mu\nu} \to G_{\mu\nu}
\end{aligned}
\tag{6}
$$

Together these generate an SL(2,$\mathbf{Z}$) symmmetry. Although the equations of motion are invariant under $\lambda \to \lambda + c$ for any real number c, this symmetry is broken down to $\lambda \to \lambda + 1$ by instanton corrections [1].

Since it helps us to fix some of the as yet undetermined normalizations in the theory, we shall now show how to obtain the above result. Let us consider a specific embedding of one of the U(1) gauge fields (say $A^{(28)}_\mu$) into an SU(2) subgroup of one of the E_8 (or SO(32)/$\mathbf{Z}_2$) groups of the heterotic string theory. Using the freedom of scaling λ by a constant c and $\vec{F}$ by $1/\sqrt{c}$ which leaves the action invariant, we can always ensure that $A^{(28)}_\mu$ is related to the third component $B_{3\mu}$ of the SU(2) gauge fields as $B_{3\mu} = A^{(28)}_\mu/\sqrt{2}$.† The term proportional to $F^{(28)}_{\mu\nu}\tilde{F}^{(28)\mu\nu}$ in the action then becomes part of an SU(2) invariant term,

$$
\frac{1}{16\pi}\int d^4x \sqrt{-\det G}\ \lambda_1 \sum_{a=1}^{3} H_{a\mu\nu}\tilde{H}^{\mu\nu}_a \tag{7}
$$

where $H_{a\mu\nu} = \partial_\mu B_{a\nu} - \partial_\nu B_{a\mu} + \epsilon^{abc} B_{b\mu} B_{c\nu}$. Using the relation,

$$
\int d^4x \sqrt{-\det G} \sum_{a=1}^{3} H_{a\mu\nu}\tilde{H}^{\mu\nu}_a = 32\pi^2 \tag{8}
$$

for a single instanton configuration, we see that S changes by 2π times an integer under $\lambda_1 \to \lambda_1 + 1$, and hence e^{iS} remains invariant under this transformation. Symmetry under a general transformation of the form $\lambda \to \lambda + c$ is broken by the instanton corrections.

† The factor of $\sqrt{2}$ also provides a normalization such that the electric charge vector, defined later, takes value on an even, self-dual lattice.

Let us now denote by $\lambda^{(0)} = \lambda_1^{(0)} + i\lambda_2^{(0)}$ the asymptotic value of λ, and by $M^{(0)}$ the asymptotic value of the matrix M. From the form of the action we see that $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$ are related to the string coupling constant g and the θ angle by the relations:

$$\lambda_2^{(0)} = \frac{8\pi}{g^2}, \qquad \lambda_1^{(0)} = \frac{\theta}{2\pi} \tag{9}$$

Finally, for a given state, we define the 28 dimensional electric and magnetic charge vectors $\vec{Q}_e$ and $\vec{Q}_m$ in terms of the asymptotic form of $\vec{F}_{\mu\nu}$ as follows:

$$F_{0r}^{(\alpha)} \simeq \frac{Q_e^{(\alpha)}}{r^2}, \qquad \tilde{F}_{0r}^{(\alpha)} \simeq \frac{Q_m^{(\alpha)}}{r^2} \tag{10}$$

From eqs.(6) and (10) we see that under a duality transformation,

$$\begin{aligned}
&\lambda_1^{(0)} \to \lambda_1^{(0)\prime} = -\frac{\lambda_1^{(0)}}{|\lambda^{(0)}|^2}, \qquad \lambda_2^{(0)} \to \lambda_2^{(0)\prime} = \frac{\lambda_2^{(0)}}{|\lambda^{(0)}|^2} \\
&\vec{Q}_e \to \vec{Q}_e' = -\lambda_2^{(0)} M^{(0)} L \vec{Q}_m - \lambda_1^{(0)} \vec{Q}_e \\
&\vec{Q}_m \to \vec{Q}_m' = \lambda_2^{(0)} M^{(0)} L \vec{Q}_e - \lambda_1^{(0)} \vec{Q}_m
\end{aligned} \tag{11}$$

Let us now study the spectrum $(\vec{Q}_m, \vec{Q}_e)$ of magnetic and electric charges in this theory. We start from the states with zero magnetic charge. With the normalization convention that we have adopted, the charge spectrum of such states is given by [18] [19],

$$(\vec{Q}_m, \vec{Q}_e) = (0, \frac{1}{\lambda_2^{(0)}} \vec{\alpha}) \tag{12}$$

where $\vec{\alpha}$ is a lattice vector belonging to a 28 dimensional self-dual, even, Lorentzian lattice with metric L. Let us denote this lattice by P. Then, for $\vec{\alpha}, \vec{\beta} \in P$, we have,

$$\vec{\alpha}^T.L.\vec{\beta} = integer, \qquad \vec{\alpha}^T.L.\vec{\alpha} = even\ integer \tag{13}$$

Let us now consider a general dyon state $(\vec{Q}_m, \vec{Q}_e)$. A consistent spectrum of $\vec{Q}_m$ is obtained by demanding that the Dirac string attached to the magnetic charge is not visible to the particle of charge $(0, \vec{\alpha}/\lambda_2^{(0)})$. With the normalization convention we have adopted, this requires,

$$\lambda_2^{(0)} \vec{Q}_m . L M^{(0)} L . \frac{1}{\lambda_2^{(0)}} \vec{\alpha} = integer \tag{14}$$

Using eqs.(1), (2), and the self duality of the lattice P we see that the most general solution of eq.(14) is of the form:

$$\vec{Q}_m = M^{(0)} L \vec{\beta}, \qquad \vec{\beta} \in P \tag{15}$$

We now try to determine the allowed values of $\vec{Q}_e$ for the value of $\vec{Q}_m$ given in eq.(15). Naively one might have expected that these are given by $\vec{\alpha}/\lambda_2^{(0)}$ with $\vec{\alpha} \in P$ as in eq.(12). However we know that in the presence of a theta angle the allowed electric charges of a dyon are shifted by an amount proportional to the magnetic charge [14]. The shift in this case can be computed following the procedure of ref.[14] and is given by $\lambda_1^{(0)} \vec{\beta}/\lambda_2^{(0)}$. Thus the spectrum of electric and magnetic charges carried by the dyon is given by,

$$(\vec{Q}_m, \vec{Q}_e) = \Big(M^{(0)} L \vec{\beta}, \frac{1}{\lambda_2^{(0)}} (\vec{\alpha} + \lambda_1^{(0)} \vec{\beta}) \Big), \qquad \vec{\alpha}, \vec{\beta} \in P \tag{16}$$

We can now easily perform a duality transformation and compute $(\vec{Q}'_m, \vec{Q}'_e)$ using eq.(11). The result is,

$$(\vec{Q}'_m, \vec{Q}'_e) = \Big(M^{(0)} L \vec{\alpha}, \frac{1}{{\lambda_2^{(0)}}'} (-\vec{\beta} + {\lambda_1^{(0)}}' \vec{\alpha}) \Big) \tag{17}$$

This shows that the spectrum given by eq.(16) is invariant under the duality transformation, since the transformed spectrum (17) has the same form as the original spectrum (16), with $\vec{\alpha}$, $\vec{\beta}$ transforming as,

$$\vec{\alpha} \to \vec{\alpha}' = -\vec{\beta}, \quad \vec{\beta} \to \vec{\beta}' = \vec{\alpha} \tag{18}$$

Note also that under the other generator of the SL(2,**Z**) transformation, $\lambda_1^{(0)} \to \lambda_1^{(0)\prime} = \lambda_1^{(0)} + 1$ with all other quantities remaining fixed,

$$(\vec{Q}'_m, \vec{Q}'_e) = (\vec{Q}_m, \vec{Q}_e) = \Big(M^{(0)} L \vec{\beta}, \frac{1}{\lambda_2^{(0)\prime}} (\vec{\alpha} - \vec{\beta} + \lambda_1^{(0)\prime} \vec{\beta}) \Big) \tag{19}$$

Thus the spectrum again retains its form, with the transformations,

$$\vec{\alpha} \to \vec{\alpha}' = \vec{\alpha} - \vec{\beta}, \quad \vec{\beta} \to \vec{\beta}' = \vec{\beta} \tag{20}$$

This establishes that the electric and the magnetic charge spectrum of dyons is invariant under the full SL(2, **Z**) group of transformations. This, in turn, shows that the laws of quantization of dyon charge are consistent with the idea that electric-magnetic duality is an exact symmetry of four dimensional string theory.

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