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Half-strings, Projectors, and Multiple D-branes in Vacuum String Field Theory

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Abstract

A sliver state is a classical solution of the string field theory of the tachyon vacuum that represents a background with a single D25-brane. We show that the sliver wavefunctional factors into functionals of the left and right halves of the string, and hence can be naturally regarded as a rank-one projector in a space of half-string functionals. By developing an algebraic oscillator approach we are able construct higher rank projectors that describe configurations of multiple D-branes of various dimensionalities and located at arbitrary positions. The results bear remarkable similarities with non-commutative solitons.

Contents

1 Introduction and summary

Vacuum string field theory is a new approach to open string field theory that uses directly the mysterious open string tachyon vacuum for the description of the theory [1, 2]. Among all possible open string backgrounds the tachyon vacuum is particularly natural given its physically expected uniqueness as the endpoint of all processes of tachyon condensation. It is also becoming clear that this vacuum string field theory is structurally much simpler than conventional cubic open string field theory [3].[1] In fact, under the consistent assumption of ghost/matter factorization, equations of motion in the matter sector are simply equations for a projector. A closed form expression for a particular projector has already been obtained using a very special string field called the "sliver" [5, 6, 2]. This sliver state represents a classical solution describing a space-filling D25 brane. Classical solutions representing (single) D-branes of lower dimensions have also been constructed in [2] by using the algebraic description of the sliver as a starting point for generalization. In another work [7] we have shown that the universal definition of the sliver allows one to construct the solution representing the D-brane associated to a deformed boundary conformal field theory.

In this paper we extend the above results to construct multiple D-brane solutions of various dimensions situated at various positions. Conceptually, the starting point for this analysis is the realization that the sliver can be viewed as a rank-one projector in a certain state space. As emphasized in [7], the geometrical picture of the sliver indicates that it is a state originating from a Riemann surface where the left-half and the right-half of the open string are "as far as they can be" from each other. Thus, one can entertain the possibility that the sliver functional factors into the product of a functionals of the left-half and the right half of the open string. We test explicitly this factorization using the representation of the sliver in flat 26 dimensional space-time as a squeezed state in the oscillator basis [6], *i.e.*, an exponential of an oscillator bilinear acting on the Fock vacuum. By rewriting this expression in terms of the coordinates x^L of the left-half of the string and the coordinates x^R of the right-half of the string we confirm that the sliver wave-functional factors into a product of the form $f(x^L)f(x^R)$. Regarding string fields as matrices with row index labeled by x^L and column index labeled by x^R, the $*$-product becomes matrix product, and the sliver is recognized as a rank-one projection operator in a space of half-string functionals.

This makes it clear how to construct higher rank projectors. For example, in order to get a rank two projector, we construct a new half-string functional g that is orthogonal to f, and simply take the sum $f(x^L)f(x^R) + g(x^L)g(x^R)$. Since $f(x^L)f(x^R)$ and $g(x^L)g(x^R)$

[1] String field actions with purely ghost kinetic operators arose previously in ref.[4].

are each rank-one projectors, and they project onto orthogonal subspaces, their sum is a rank-two projector. This is readily generalized to build rank-N projectors. While single D-brane solutions can be identified as rank-one projection operators, a solution representing N D-branes corresponds to a projection operator of rank N. This situation is analogous to the one that arises in the study of non-commutative solitons [8, 9, 10, 11, 12]. We note in passing that in contrast to the sliver, the "identity" string field $\mathcal{I}$, which acts as the identity of the $*$-product, represents a projector into the full half-string Hilbert space and thus, in some sense, represents a configuration of infinite number of D-branes.

The half-string approach to open string field theory, arising from observations in Witten's original paper [3], was developed in [13, 14, 15], and the possible relevance of this formalism to the sliver was anticipated in [6]. The straightforward splitting of the string coordinates into those of the left-half and the right-half of the string holds only for zero momentum string states. For string fields carrying momentum there is additional dependence on the zero mode of the coordinate or equivalently the string mid-point x^M, which cannot be thought of as belonging to either the left-half or the right-half of the string. Thus the above method is not directly applicable to generalize the construction of a D-p-brane solution ($p < 25$) [2] into multiple D-p-brane solutions. However we show that it is possible to construct projectors by taking the string field of the form $f(x^L; x^M)f(x^R; x^M)$ with a certain functional form for f. It is very likely that the lower dimensional D-brane states constructed in ref.[2] are of this type. We can then generalize this to construct higher rank projectors following the same principle as before, but we do not pursue this line of argument any further.

Motivated by the somewhat formal nature of functional manipulations, and by the fact that our evidence for left/right factorization of the sliver is only numerical, we develop a self-contained algebraic approach that incorporates all the desired features in a completely unambiguous way. Going back to the D-25-brane case, we use the oscillator basis to build left/right projectors. These projectors can be directly generalized to lower dimensional brane solutions, with no need of referring to the functional integral half-string language, because the (infinite dimensional) matrices that appear in the construction of the D-25-brane solution and lower dimensional D-brane solutions have very similar structure [6]. This allows us to construct multiple D-brane solutions of lower dimensions. Finally, we can also generalize this method to construct multiple D-brane solutions of various dimensions, situated at various positions.

The paper is organized as follows. In section 2 we give a brief review of the results of refs.[1, 2]. In section 3 we review the construction of the sliver state as an exponential of bilinears in matter oscillators acting on the Fock vacuum. We also list various identities satisfied by the infinite dimensional matrices appearing in the construction of the sliver, and discuss how to compute $*$-product of states built on the sliver by the action of matter oscillators. These identities play a crucial role in the later verification that we have constructed higher rank projectors.

Section 4 is devoted to the functional approach based on half-strings. As in [14] we give the transformations between the Fourier modes of the full string coordinates and those of the left and right-half of the string. This is used in section 4.2 to write the sliver functional in terms of the Fourier modes of the half-strings, and to show numerically that it factorizes into a product $f(x^L)f(x^R)$ with f a gaussian. We construct projectors $g(x^L)g(x^R)$ orthogonal to the sliver by taking g orthogonal to f, and show how to build higher rank projectors. We also discuss briefly how to generalize this procedure to D-branes of lower dimension, but do not study this in detail.

The intuition developed in section 4 allows us to give in section 5 a self-contained rigorous treatment within the algebraic oscillator approach. We construct higher rank projectors in section 5.1 and confirm explicitly that their superposition satisfies the string equations of motion and has the right tension to describe multiple D-brane states. Section 5.2 contains a short discussion on the issue of Lorentz invariance of our solutions, and also the origin of the Chan-Paton factors. In section 5.3 we construct multiple D-p-brane solutions for $p < 25$. As pointed out before, while the idea of left/right factorization of the functional could be pursued for lower branes as well along the lines of section 4.4, there is no need for this since the algebraic analysis of section 5.1 can be generalized in a straightforward manner. Finally in sections 5.4 and 5.5 we combine all the techniques developed earlier to construct multiple D-brane solutions with D-branes of various dimensions and situated at various positions. A brief discussion of the results is given in section 6.

We conclude this introduction with the note that results similar to the ones discussed in this paper have been obtained independently by Gross and Taylor [16].

2 Review of vacuum string field theory

In this section we shall briefly describe the results of refs.[1, 2]. In these papers we proposed a form of the string field theory action around the open bosonic string tachyon vacuum and discussed classical solutions describing D-branes of various dimensions. In order to write concretely this theory (which is formally manifestly background independent [7]) we choose to use the state space $\mathcal{H}$ of the combined matter-ghost boundary conformal field theory (BCFT) describing the D25-brane. The string field Ψ is a state of ghost number one in $\mathcal{H}$ and the string field action is given by:

$$\mathcal{S}(\Psi) \equiv -\frac{1}{g_0^2}\left[\frac{1}{2}\langle\,\Psi\,,\,\mathcal{Q}\,\Psi\rangle + \frac{1}{3}\langle\,\Psi\,,\,\Psi * \Psi\rangle\right], \tag{2.1}$$

where g_0 is the open string coupling constant, $\mathcal{Q}$ is an operator made purely of ghost fields, $\langle\, ,\, \rangle$ denotes the BPZ inner product, and $*$ denotes the usual $*$-product of the string fields [3]. $\mathcal{Q}$ satisfies the requirements:

$$\begin{aligned} &\mathcal{Q}^2 = 0, \\ &\mathcal{Q}(A * B) = (\mathcal{Q}A) * B + (-1)^A A * (\mathcal{Q}B)\,, \\ &\langle\, \mathcal{Q}A, B\,\rangle = -(-)^A \langle A, \mathcal{Q}B \rangle\,. \end{aligned} \tag{2.2}$$

The action (2.1) is then invariant under the gauge transformation:

$$\delta\Psi = \mathcal{Q}\Lambda + \Psi * \Lambda - \Lambda * \Psi\,, \tag{2.3}$$

for any ghost number zero state Λ in $\mathcal{H}$. Ref.[1] contains candidate operators $\mathcal{Q}$ satisfying these constraints; for our analysis we shall not need to make a specific choice of $\mathcal{Q}$.

The equations of motion are

$$\mathcal{Q}\Psi + \Psi * \Psi = 0\,. \tag{2.4}$$

In ref.[2] we made the ansatz that all D-p-brane solutions in this theory have the factorized form:

$$\Psi = \Psi_g \otimes \Psi_m\,, \tag{2.5}$$

where Ψ_g denotes a state obtained by acting with the ghost oscillators on the SL(2,R) invariant vacuum of the ghost BCFT, and Ψ_m is a state obtained by acting with matter operators on the SL(2,R) invariant vacuum of the matter BCFT. Let us denote by $*^g$ and $*^m$ the star product in the ghost and matter sector respectively. Eq.(2.4) then factorizes as

$$\mathcal{Q}\Psi_g = -\Psi_g *^g \Psi_g\,, \tag{2.6}$$

and

$$\Psi_m = \Psi_m *^m \Psi_m\,. \tag{2.7}$$

We further assumed that the ghost part Ψ_g is universal for all D-p-brane solutions. Under this assumption the ratio of energies associated with two different D-brane solutions, with matter parts Ψ_m' and Ψ_m respectively, is given by:

$$\frac{\langle \Psi_m' | \Psi_m' \rangle_m}{\langle \Psi_m | \Psi_m \rangle_m}\,, \tag{2.8}$$

with $\langle | \rangle_m$ denoting BPZ inner product in the matter BCFT. Thus the ghost part drops out of this calculation.

In ref.[2] we constructed the matter part of the solution for different D-p-branes, and verified that we get the correct ratio of tensions of D-p-branes using eq.(2.8). The matter part of the D-25-brane solution was given by the sliver state $|\Xi\rangle$ which will play a central role in the analysis of this paper. The algebraic construction of $|\Xi\rangle$ will be reviewed in section 3 (a detailed geometrical discussion of the sliver can be found in [7]).

In the rest of the paper we shall deal with the matter part of the string state. In order to avoid cluttering up the various formulæ we shall find it convenient to drop the subscript and the superscript m from various states, inner products and $*$-products; but it should be understood that all operations refer to the matter sector only.

3 The oscillator representation of the sliver

In this section we discuss the oscillator representation of the sliver. Section 3.1 contains a description of the sliver and some useful identities involving the infinite dimensional matrices that arise in this description. Section 3.2 contains some results on $*$-products of states built on the sliver which will be useful for later analysis.

3.1 Description of the sliver

Here we wish to review the construction of the matter part of the sliver in the oscillator representation and consider the basic algebraic properties that guarantee that the multiplication of two slivers gives a sliver. In fact we follow the discussion of Kostelecky and Potting [6] who gave the first algebraic construction of a state that would star multiply to itself in the matter sector. This discussion was simplified in [2] where due attention was also paid to normalization factors that guarantee that the states satisfy precisely the projector equation (2.7). Here we summarize the relevant parts of this analysis which will be used in the present paper. In doing so we also identify some infinite dimensional matrices with the properties of projection operators. These matrices will be useful in the construction of multiple D-brane solutions in section 5.

In order to star multiply two states $|A\rangle$ and $|B\rangle$ we must calculate

$$|A\rangle * |B\rangle_3 = {}_1\langle A|_2\langle B|V_3\rangle_{123}\,, \tag{3.1}$$

where $|\,\rangle_r$ denotes a state in the r-th string Hilbert space, and $|\,\rangle_{123}$ denotes a state in the product of the Hilbert space of three strings. The key ingredient here is the three-string vertex $|V_3\rangle_{123}$. While the vertex has nontrivial momentum dependence, if the states A and B are at zero momentum, the star product gives a zero momentum state that can be calculated using

$$|V_3\rangle_{123} = \exp\Big(-\frac{1}{2}\sum_{r,s} a^{(r)\dagger}\cdot V^{rs}\cdot a^{(s)\dagger}\Big)|0\rangle_{123}\,, \tag{3.2}$$

and the rule $\langle 0|0\rangle = 1$. Here the V^{rs}, with $r, s = 1, 2, 3$, are infinite matrices V^{rs}_{mn} $(m, n = 1, \cdots\infty)$ satisfying the cyclicity condition $V^{rs} = V^{r+1,s+1}$ and the symmetry condition $(V^{rs})^T = V^{sr}$. These properties imply that out of the nine matrices, three: V^{11}, V^{12} and V^{21}, can be used to obtain all others. $a^{(r)\mu\dagger}_m$ $(0 \leq \mu \leq 25)$ denote oscillators in the r-th string Hilbert space. For simplicity, the Lorentz and the oscillator indices, and the Minkowski matrix $\eta_{\mu\nu}$ used to contract the Lorentz indices, have all been suppressed in eq.(3.2). We shall follow this convention throughout the paper.

One now introduces

$$M^{rs} \equiv CV^{rs}\,, \quad \text{with} \quad C_{mn} = (-1)^m\delta_{mn}, \quad m, n \geq 1\,. \tag{3.3}$$

These matrices can be shown to satisfy the following properties:

$$\begin{aligned} &CV^{rs} = V^{sr}C\,, \quad (V^{rs})^T = V^{sr}\,, \\ &(M^{rs})^T = M^{rs}\,, \quad CM^{rs}C = M^{sr}\,, \quad [M^{rs}, M^{r's'}] = 0\,. \end{aligned} \tag{3.4}$$

In particular note that all the M matrices commute with each other. Defining $X \equiv M^{11}$, the three relevant matrices are X, M^{12} and M^{21}. Explicit formulae exist that allow their explicit computation [17, 18, 19, 20].[2]

They can be shown in general grounds to satisfy the following useful relations:

$$\begin{aligned} &X + M^{12} + M^{21} = 1\,, \\ &M^{12}M^{21} = X^2 - X\,, \\ &(M^{12})^2 + (M^{21})^2 = 1 - X^2\,, \\ &(M^{12})^3 + (M^{21})^3 = 2X^3 - 3X^2 + 1 = (1-X)^2(1+2X)\,, \end{aligned} \tag{3.5}$$

which gives

$$(M^{12} - M^{21})^2 = (1-X)(1+3X)\,. \tag{3.6}$$

The state in the matter Hilbert space that multiplies to itself turns out to take the form [6, 2]

$$|\Psi\rangle = \mathcal{N}^{26}\exp\Big(-\frac{1}{2}\,a^\dagger\cdot S\cdot a^\dagger\Big)|0\rangle\,, \quad \mathcal{N} = \{\det(1-X)\det(1+T)\}^{1/2}\,, \quad S = CT \tag{3.7}$$

[2]Our convention (3.1) for describing the star product differs slightly from that of ref.[17], the net effect of which is that the explicit expression for the matrix V^{rs} listed in appendix A of ref.[2], version 1, is actually the expression for the matrix V^{sr}. Since all the explicit computations performed of ref.[2] involved V^{sr} and V^{rs} symmetrically, this does not affect any of the calculations in that paper.

where the matrix T satisfies $CTC = T$ and the equation

$$XT^2 - (1+X)T + X = 0\,, \tag{3.8}$$

which gives

$$T = (2X)^{-1}(1 + X - \sqrt{(1+3X)(1-X)}). \tag{3.9}$$

In taking the square root we pick that branch which, for small X, goes as $(1+X)$.

In [2] we identified this state as the sliver by computing numerically the matrix S using the equation above and comparing the state obtained this way with the matter part of the sliver $|\Xi\rangle$, which using the techniques of ref.[21], can be expressed as

$$|\Xi\rangle = \widehat{\mathcal{N}}^{26} \exp\Big(-\frac{1}{2}\, a^\dagger \cdot \widehat{S} \cdot a^\dagger\Big)|0\rangle\,, \tag{3.10}$$

$$\widehat{S}_{mn} = -\frac{1}{\sqrt{mn}} \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \frac{1}{z^n\, w^m} \frac{df(z)}{dz} \frac{df(w)}{dw} \frac{1}{(f(z)-f(w))^2}\,, \tag{3.11}$$

with $f(z) = \tan^{-1}(z)$. We found close agreement between the numerical values of S_{mn} and exact answers for $\widehat{S}_{mn}$. This gave convincing evidence that $|\Psi\rangle = |\Xi\rangle$.

We record, in passing, the following additional relations between T and X following from the form of the solution and the equation that relates them ((3.8), (3.9)):

$$\frac{1-TX}{1-X} = \frac{1}{1-T}\,, \quad \frac{1-T}{1+T} = \sqrt{\frac{1-X}{1+3X}}\,, \quad \frac{X}{1-X} = \frac{T}{(1-T)^2}\,. \tag{3.12}$$

We conclude this subsection by constructing a pair of projectors that will be very useful in explicit computations of star products. We define the matrices

$$\begin{aligned} \rho_1 &= \frac{1}{(1+T)(1-X)}\Big[M^{12}(1-TX) + T(M^{21})^2\Big]\,, \\ \rho_2 &= \frac{1}{(1+T)(1-X)}\Big[M^{21}(1-TX) + T(M^{12})^2\Big]\,. \end{aligned} \tag{3.13}$$

One readily verifies that they satisfy the following properties:

$$\rho_1^T = \rho_1\,, \quad \rho_2^T = \rho_2\,, \quad C\rho_1 C = \rho_2\,, \tag{3.14}$$

and more importantly

$$\begin{aligned} \rho_1 + \rho_2 &= 1\,, \\ \rho_1 - \rho_2 &= \frac{M^{12} - M^{21}}{\sqrt{(1-X)(1+3X)}}\,. \end{aligned} \tag{3.15}$$

From eq.(3.6) we see that the square of the second right hand side is the unit matrix. Thus $(\rho_1 - \rho_2)^2 = 1$, and this together with the squared version of the first equation gives

$$\rho_1 \rho_2 = 0\,. \tag{3.16}$$

This equation is also easily verified directly. Multiplying the first equation in (3.15) by ρ_1 and alternatively by ρ_2 we get

$$\rho_1 \rho_1 = \rho_1\,, \quad \rho_2 \rho_2 = \rho_2\,. \tag{3.17}$$

This shows that ρ_1 and ρ_2 are projection operators into orthogonal subspaces, and the C exchanges these two subspaces. We will see later on that ρ_1 and ρ_2 project into the oscillators of the right and the left half of the string respectively. Finally we record the identities

$$\begin{aligned} M^{21}\rho_1 + M^{12}\rho_2 \quad &= X(T-1) \\ (M^{12})^2\rho_1 + (M^{21})^2\rho_2 \quad &= (1-TX)^2\,. \end{aligned} \tag{3.18}$$

3.2 Computing $*$-products of states

In this subsection we will give the methods needed to compute the star product of states involving the (matter) sliver and matter oscillators acting on the sliver. The methods are equally applicable (we give an example) to the case where the products involve a factor where matter oscillators act on the identity string field.

As seen earlier, the matter part of the sliver state is given by

$$|\Xi\rangle = \mathcal{N}^{26} \exp\Big(-\frac{1}{2} a^\dagger \cdot S \cdot a^\dagger\Big)|0\rangle\,. \tag{3.19}$$

Coherent states are defined by letting exponentials of the creation operator act on the vacuum. Treating the sliver as the vacuum we introduce coherent like states of the form

$$|\Xi_\beta\rangle = \exp\Big(\sum_{n=1}^{\infty} (-)^{n+1} \beta_{\mu n} a_n^{\mu\dagger}\Big)|\Xi\rangle = \exp(-a^\dagger \cdot C\beta)|\Xi\rangle\,. \tag{3.20}$$

As built, the states satisfy a simple BPZ conjugation property:

$$\langle \Xi_\beta| = \langle \Xi| \exp\Big(\sum_{n=1}^{\infty} \beta_{n\mu} a_n^\mu\Big) = \langle \Xi| \exp(\beta \cdot a)\,. \tag{3.21}$$

We compute the $*$ product of two such states using the procedure discussed in refs.[6, 2]. We begin by writing out the product using two by two matrices encoding the oscillators of strings one and two:

$$
\begin{aligned}
\Big(|\Xi_{\beta_1}\rangle * |\Xi_{\beta_2}\rangle\Big)_{(3)} &= \Big(\exp(-a^\dagger\cdot C\beta_1)|\Xi\rangle * \exp(-a^\dagger\cdot C\beta_2)|\Xi\rangle\Big)_{(3)} \\
&= {}_{(1)}\langle\Xi|\exp(\beta_1\cdot a_{(1)})\ {}_{(2)}\langle\Xi|\exp(\beta_2\cdot a_{(2)})|V_{123}\rangle \\
&= \langle 0_{12}|\exp\Big(\beta\cdot a - \frac{1}{2}a\cdot\Sigma\cdot a\Big)\exp\Big(-\frac{1}{2}a^\dagger\cdot\mathcal{V}\cdot a^\dagger - \chi^T\cdot a^\dagger\Big)|0_{12}\rangle \\
&\qquad \cdot\exp\Big(-\frac{1}{2}a^\dagger_{(3)}\cdot V^{11}\cdot a^\dagger_{(3)}\Big)|0_3\rangle\ ,
\end{aligned}
\tag{3.22}
$$

where $a = (a_{(1)}, a_{(2)})$, and

$$
\Sigma = \begin{pmatrix} S & 0 \\ 0 & S\end{pmatrix},\quad \mathcal{V} = \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22}\end{pmatrix},\quad \beta = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix},\quad \chi^T = (a^\dagger_{(3)}V^{12}\,,\, a^\dagger_{(3)}V^{21})\,. \tag{3.23}
$$

Explicit evaluation continues by using the equation

$$
\begin{aligned}
&\langle 0|\exp\Big(\beta_i a_i - \frac{1}{2}P_{ij}a_i a_j\Big)\exp\Big(-\chi_i a_i^\dagger - \frac{1}{2}Q_{ij}a_i^\dagger a_j^\dagger\Big)|0\rangle \\
&= \det(K)^{-1/2}\exp\Big(-\chi^T K^{-1}\beta - \frac{1}{2}\beta^T Q\, K^{-1}\beta - \frac{1}{2}\chi^T K^{-1}P\chi\Big),\ K\equiv 1-PQ\,.
\end{aligned}
\tag{3.24}
$$

At this time we realize that since $|\Xi\rangle * |\Xi\rangle = |\Xi\rangle$ the result of the product is a sliver with exponentials acting on it; the exponentials that contain β. This gives

$$
|\Xi_{\beta_1}\rangle * |\Xi_{\beta_2}\rangle = \exp\Big(-\chi^T\mathcal{K}^{-1}\beta - \frac{1}{2}\beta^T\mathcal{V}\mathcal{K}^{-1}\beta\Big)|\Xi\rangle\,,\quad \mathcal{K} = (1-\Sigma\mathcal{V})\,. \tag{3.25}
$$

The expression for $\mathcal{K}^{-1}$, needed above is simple to obtain given that all the relevant submatrices commute. One finds that

$$
\mathcal{K}^{-1} = (1-\Sigma\mathcal{V})^{-1} = \frac{1}{(1+T)(1-X)}\begin{pmatrix} 1-TX & TM^{12} \\ TM^{21} & 1-TX\end{pmatrix}\,. \tag{3.26}
$$

We now recognize that the projectors ρ_1 and ρ_2 defined in (3.13) make an appearance in the oscillator term of (3.25)

$$
\begin{aligned}
-\chi^T\mathcal{K}^{-1}\beta &= -a^\dagger\cdot C\,(M^{12},M^{21})\mathcal{K}^{-1}\beta = -a^\dagger\cdot C\,(\rho_1,\rho_2)\beta \\
&= -a^\dagger C\,\cdot(\rho_1\,\beta_1 + \rho_2\,\beta_2)\,.
\end{aligned}
\tag{3.27}
$$

We can verify that

$$
\begin{aligned}
\mathcal{C}(\beta_1,\beta_2) &\equiv \frac{1}{2}(\beta_1,\beta_2)\,\mathcal{V}\mathcal{K}^{-1}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \\
&= \frac{1}{2}(\beta_1,\beta_2)\frac{1}{(1+T)(1-X)}\begin{pmatrix} V^{11}(1-T) & V^{12} \\ V^{21} & V^{11}(1-T)\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}\,.
\end{aligned}
\tag{3.28}
$$

Since the matrix in between is symmetric we have

$$\mathcal{C}(\beta_1,\beta_2)=\mathcal{C}(\beta_2,\beta_1)\,. \tag{3.29}$$

Using (3.27) and (3.28) we finally have:

$$|\Xi_{\beta_1}\rangle * |\Xi_{\beta_2}\rangle = \exp\Big(-\mathcal{C}(\beta_1,\beta_2)\Big)|\Xi_{\rho_1\beta_1+\rho_2\beta_2}\rangle\,. \tag{3.30}$$

This is a useful relation that allows one to compute $*$-products of slivers acted by oscillators by simple differentiation. In particular, using eq.(3.20) we get

$$\begin{aligned}(a^{\mu_1\dagger}_{m_1}\cdots a^{\mu_k\dagger}_{m_k}|\Xi\rangle)*(a^{\nu_1\dagger}_{n_1}\cdots a^{\nu_l\dagger}_{n_l}|\Xi\rangle) \;&=\; (-1)^{\sum_{i=1}^{k}(m_i+1)+\sum_{j=1}^{l}(n_j+1)} \\ &\Big(\frac{\partial}{\partial\beta_{1m_1\mu_1}}\cdots\frac{\partial}{\partial\beta_{1m_k\mu_k}}\frac{\partial}{\partial\beta_{2n_1\nu_1}}\cdots\frac{\partial}{\partial\beta_{2n_l\nu_l}}(|\Xi_{\beta_1}\rangle*|\Xi_{\beta_2}\rangle)\Big)_{\beta_1=\beta_2=0}\,.\end{aligned} \tag{3.31}$$

Since $\rho_1+\rho_2=1$, for $\beta_1=\beta_2$ eq.(3.30) reduces to

$$|\Xi_{\beta}\rangle * |\Xi_{\beta}\rangle = \exp\Big(-\mathcal{C}(\beta,\beta)\Big)|\Xi_{\beta}\rangle\,. \tag{3.32}$$

Using the definition of $\mathcal{C}$ in (3.28) and equations (3.8) and (3.12) one can show that $\mathcal{C}(\beta,\beta)$ simplifies down to

$$\mathcal{C}(\beta,\beta)=\frac{1}{2}\,\beta C(1-T)^{-1}\beta\,. \tag{3.33}$$

It follows from (3.32) that by adjusting the normalization of the Ξ_β state

$$P_\beta \equiv \exp\Big(\mathcal{C}(\beta,\beta)\Big)|\Xi_\beta\rangle\,, \tag{3.34}$$

we obtain projectors

$$P_\beta * P_\beta = P_\beta\,. \tag{3.35}$$

Using eq.(3.24) one can also check that

$$\langle P_\beta|P_\beta\rangle = \langle\Xi|\Xi\rangle\,. \tag{3.36}$$

3.3 The identity string field and BPZ inner products

The same methods can be used to multiply coherent states on the sliver times coherent states on the identity string field.[3] If we restrict ourselves to the matter sector the identity takes the form

$$|\mathcal{I}\rangle = [\det(1-X)]^{13}\exp(-\frac{1}{2}a^\dagger\cdot C\cdot a^\dagger)\,, \tag{3.37}$$

[3]For some remarkable properties of the identity field, see ref.[22].

where the determinant prefactor has been included to guarantee that $\mathcal{I} * \mathcal{I} = \mathcal{I}$. One can also explicitly verify that

$$\mathcal{I} * \Xi = \Xi * \mathcal{I} = \Xi \,, \tag{3.38}$$

and, more generally,

$$\mathcal{I} * \Xi_\beta = \Xi_\beta * \mathcal{I} = \Xi_\beta \,. \tag{3.39}$$

This allows us to simplify the computation of the BPZ norm of any projector P satisfying both $P * \mathcal{I} = \mathcal{I} * P = P$ and $P * P = P$ with exact unit normalization. For this we have

$$\langle P|P\rangle = \langle \mathcal{I} * P|P\rangle = \langle \mathcal{I}|P * P\rangle = \langle \mathcal{I}|P\rangle \,. \tag{3.40}$$

where we have used the property $\langle A*B, C\rangle = \langle A, B*C\rangle$ of the full BPZ product restricted to the matter BPZ product $\langle\cdot|\cdot\rangle$. The above relation is not formal, we have verified directly that $\langle \mathcal{I}|\Xi\rangle = \langle \Xi|\Xi\rangle$ and that $\langle \mathcal{I}|P_\beta\rangle = \langle P_\beta|P_\beta\rangle = \langle \Xi|\Xi\rangle$, as seen in (3.36).

In open string field theory one defines a trace as $\mathrm{Tr}(\Psi) = \langle \mathcal{I}|\Psi\rangle$, this is sometimes called 'integration', and denoted as $\int \Psi$. In this notation we would have that equation (3.40) reads $\mathrm{Tr}(P) = \langle P|P\rangle$. While the sliver will be interpreted as a rank one projector, and thus one would expect $\mathrm{Tr}(\Xi) = 1$, the conformal anomaly in the matter BCFT does not allow us to have both $\Xi * \Xi = \Xi$ and $\langle \Xi|\Xi\rangle = 1$. We normalized the matter sliver so that the projection condition is satisfied exactly. In this case, however, $\langle \Xi|\Xi\rangle$ does not equal one, and in fact is numerically seen to vanish in the infinite level limit. We expect that when we fully understand the ghost sector of the theory and calculate the value of the string field theory action for a D-brane solution, we shall get a finite answer for the action.

In order to consider star products involving the identity and the sliver, both with oscillators acting on them, we introduce

$$|\mathcal{I}_\beta\rangle = \exp\Big(\sum_{n=1}^{\infty} (-)^{n+1} \beta_n a_n^\dagger\Big)|\mathcal{I}\rangle = \exp(-a^\dagger \cdot C\beta)|\mathcal{I}\rangle \,. \tag{3.41}$$

This time we find

$$\begin{aligned} |\Xi_\alpha\rangle * |\mathcal{I}_\beta\rangle &= \exp\Big(-\frac{1}{2}\beta C\,\frac{X(1-T)}{(1-X)}\,\beta - \alpha C\frac{M^{12}}{1-X}\beta\Big)|\Xi_{\alpha+(1+T)\rho_2\beta}\rangle \\ |\mathcal{I}_\beta\rangle * |\Xi_\alpha\rangle &= \exp\Big(-\frac{1}{2}\beta C\,\frac{X(1-T)}{(1-X)}\,\beta - \alpha C\frac{M^{21}}{1-X}\beta\Big)|\Xi_{\alpha+(1+T)\rho_1\beta}\rangle \,. \end{aligned} \tag{3.42}$$

4 Sliver wavefunctional, half-strings and projectors

In this section we shall examine the representation of string fields as functionals of half strings. This viewpoint is possible at least for the case of zero momentum string fields.

It leads to the realization that the sliver functional factors into functionals of the left and right halves of the string, allowing its interpretation as a rank-one projector in the space of half-string functionals. We construct higher rank projectors – these are solution of the equations of motion (2.7) representing multiple D25-branes.

4.1 Zero momentum string field as a matrix

The string field equation in the matter sector is given by

$$\Psi * \Psi = \Psi \,. \tag{4.1}$$

Thus if we can regard the string field as an operator acting on some vector space where $*$ has the interpretation of product of operators, then Ψ_m is a projection operator in this vector space. Furthermore, in analogy with the results in non-commutative solitons [8], we expect that in order to describe a single D-brane, Ψ_m should be a projection operator into a single state in this vector space.

A possible operator interpretation of the string field was suggested in Witten's original paper [3], and was further developed in ref.[13, 14, 15]. In this picture the string field is viewed as a matrix where the role of the row index and the column index are taken by the left-half and the right-half of the string respectively. In order to make this more concrete, let us consider the standard mode expansion of the open string coordinate [17]:

$$X^\mu(\sigma) = x_0^\mu + \sqrt{2}\sum_{n=1}^\infty x_n^\mu \cos(n\sigma)\,, \quad \text{for} \quad 0 \le \sigma \le \pi\,. \tag{4.2}$$

Now let us introduce coordinates $X^{L\mu}$ and $X^{R\mu}$ for the left and the right half of the string as follows[4]:

$$\begin{aligned} X^{L\mu}(\sigma) &= X^\mu(\sigma/2) - X^\mu(\pi/2)\,, && \text{for} \quad 0 \le \sigma \le \pi\,. \\ X^{R\mu}(\sigma) &= X^\mu(\pi-\sigma/2) - X^\mu(\pi/2), && \text{for} \quad 0 \le \sigma \le \pi\,. \end{aligned} \tag{4.3}$$

$X^{L\mu}(\sigma)$ and $X^{R\mu}(\sigma)$ satisfy the usual Neumann boundary condition at $\sigma = 0$ and a Dirichlet boundary condition at $\sigma = \pi$. Thus they have expansions of the form:

$$\begin{aligned} X^{L\mu}(\sigma) &= \sqrt{2}\sum_{n=1}^\infty x_n^{L\mu} \cos((n-\frac{1}{2})\sigma), \\ X^{R\mu}(\sigma) &= \sqrt{2}\sum_{n=1}^\infty x_n^{R\mu} \cos((n-\frac{1}{2})\sigma)\,. \end{aligned} \tag{4.4}$$

[4]Here the half strings are parameterized both from $\sigma = 0$ to $\sigma = \pi$, as opposed to the parameterization of [14] where the half strings are parameterized from 0 to $\pi/2$.

Comparing (4.2) and (4.4) we get an expression for the full open string modes in terms of the modes of the left-half and the modes of the right-half:

$$x_n^\mu = A^+_{nm}\, x_m^{L\mu} + A^-_{nm}\, x_m^{R\mu}, \qquad m, n \geq 1\,, \tag{4.5}$$

where the matrices $A^\pm$ are

$$A^\pm_{nm} = \pm\frac{1}{2}\delta_{n,2m-1} + \frac{1}{2\pi}\epsilon(n,m)\Big(\frac{1}{2m+n-1} + \frac{1}{2m-n-1}\Big)\,, \tag{4.6}$$

and

$$\epsilon(n,m) = (1+(-1)^n)(-1)^{m+\frac{1}{2}n-1}\,. \tag{4.7}$$

Alternatively we can write the left-half modes and right half modes in terms of the full string modes

$$\begin{aligned} x_m^{L\mu} &= \widetilde{A}^+_{mn}\, x_n^\mu, \\ x_m^{R\mu} &= \widetilde{A}^-_{mn}\, x_n^\mu, \qquad m, n \geq 1\,, \end{aligned} \tag{4.8}$$

where one finds

$$\widetilde{A}^\pm_{mn} = 2A^\pm_{nm} - \frac{1}{\pi}\,\epsilon(n,m)\,\Big(\frac{2}{2m-1}\Big)\,. \tag{4.9}$$

Note that

$$A^+ = CA^-, \quad \widetilde{A}^+ = \widetilde{A}^- C\,, \tag{4.10}$$

where $C_{mn} = (-1)^n\delta_{mn}$ is the twist operator. Note also that the relationship between x_n^μ and $(x_n^{L\mu},\, x_n^{R\mu})$ does not involve the zero mode x_0^μ of X^μ. Finally, we observe that by their definition, the various matrices must satisfy the relations

$$\begin{aligned} A^+\widetilde{A}^+ + A^-\widetilde{A}^- &= 1\,, \\ \widetilde{A}^\pm A^\mp &= 0\,, \\ \widetilde{A}^\pm A^\pm &= 1\,. \end{aligned} \tag{4.11}$$

By the last equation, the $\widetilde{A}^\pm$ matrices are left inverses of the $A^\pm$ matrices, but not right inverses, as it follows from the first of the equations.

A general string field configuration can be regarded as a functional of $X^\mu(\sigma)$, or equivalently a function of the infinite set of coordinates x_n^μ. Now suppose we have a translationally invariant string field configuration. In this case it is independent of x_0^μ and we can regard this as a function $\psi(\{x_n^{L\mu}\}, \{x_n^{R\mu}\})$ of the collection of modes of the left and the right half of the string. (The sliver is an example of such a state). We will use vector notation to represent these collections of modes

$$\begin{aligned} x^L &\equiv= \{x_n^{L\mu}\,|\, n = 1, \cdots\infty;\ \mu = 0, \cdots 25\}\,, \\ x^R &\equiv= \{x_n^{R\mu}\,|\, n = 1, \cdots\infty;\ \mu = 0, \cdots 25\}\,. \end{aligned} \tag{4.12}$$

We can also regard the function $\psi(x^L, x^R)$ as an infinite dimensional matrix, with the row index labeled by the modes in x^L and the column index labeled by the modes in x^R. The reality condition on the string field is the hermiticity of this matrix:

$$\psi^*(x^L, x^R) = \psi(x^R, x^L)\,, \tag{4.13}$$

where the * as a superscript denotes complex conjugation. Twist symmetry, on the other hand, exchanges the left and right half-strings, so it acts as transposition of the matrix: twist even (odd) string fields correspond to symmetric (antisymmetric) matrices in half-string space. Furthermore, given two such functions $\psi(x^L, x^R)$ and $\chi(x^L, x^R)$, their $*$ product is given by [3]

$$(\psi * \chi)\,(x^L, x^R) = \int [\mathrm{d}y]\ \psi(x^L, y)\,\chi(y, x^R)\,. \tag{4.14}$$

Thus in this notation the $*$-product becomes a generalized matrix product. It is clear that the vector space on which these matrices act is the space of functionals of the half-string coordinates x^L (or x^R). A projection operator P into a one dimensional subspace of the half string Hilbert space, spanned by some appropriately normalized functional f, will correspond to a functional of the form:

$$\psi_P(x^L, x^R) = f(x^L) f^*(x^R)\,. \tag{4.15}$$

The two factors in this expression are related by conjugation in order to satisfy condition (4.13). The condition $\psi_P * \psi_P = \psi_P$ requires that

$$\int [\mathrm{d}y] f^*(y) f(y) = 1\,. \tag{4.16}$$

By the formal properties of the original open string field theory construction one has $\langle A, B\rangle = \int A * B$ where $\int$ has the interpretation of a trace, namely identification of the left and right halves of the string, together with an integration over the string-midpoint coordinate $x^{M\,\mu} = X^\mu(\pi/2)$. Applied to a projector P with associated wavefunction $\psi_P(x^L, x^R) = f^*(x^L) f(x^R)$, and focusing only in the matter sector we would find

$$\begin{aligned} \langle P, P\rangle &= \int P * P = \int P = V \int [\mathrm{d}x^L][\mathrm{d}x^R]\delta(x^L - x^R)\psi_P(x^L, x^R) \\ &= V \int [\mathrm{d}y] f^*(y) f(y) = V\,, \end{aligned} \tag{4.17}$$

where V is the space-time volume coming from integration over the string midpoint $x^{M\,\mu}$. This shows that (formally) rank-one projectors are expected to have BPZ normalization V. In our case, due to conformal anomalies, while the matter sliver squares precisely as a projector, its BPZ norm approaches zero as the level is increased [2]. The above argument applies to string fields at zero momentum, thus the alternate projector constructed in ref.[2] representing lower dimensional D-branes need not have the same BPZ norm as the sliver. Note that the above discussion is the functional counterpart of the algebraic discussion in subsection 3.3.

4.2 The left-right factorization of the sliver wavefunctional

The first question we would like to address is: is the sliver a projection operator into a one dimensional subspace in the sense just described? In order to answer this question we need to express the sliver wave-function as a function of $x_n^{L\mu}$, $x_n^{R\mu}$ and then see if it factorizes in the sense of eq.(4.15). We start with the definition of the sliver written in the harmonic oscillator basis:

$$|\Xi\rangle = \mathcal{N}^{26} \exp(-\frac{1}{2}\eta_{\mu\nu}\, a_m^{\mu\dagger}\, S_{mn}\, a_n^{\nu\dagger})|0\rangle\,. \tag{4.18}$$

We now note the relation between the operators[5] $\hat{x}_n^\mu$ and $a_m^{\mu\dagger}$, a_m^μ:

$$\hat{x}_n^\mu = \frac{i}{2}\sqrt{\frac{2}{n}}\,(a_n^\mu - a_n^{\mu\dagger})\,, \quad \rightarrow \quad \hat{x} = \frac{i}{2}\,E\cdot(a-a^\dagger)\,, \quad E_{nm} = \delta_{nm}\sqrt{\frac{2}{n}}\,, \tag{4.19}$$

which we also write in compact matrix notation using the matrix E defined above. Using this one can express the position eigenstate in the harmonic oscillator basis:

$$\langle\vec{x}| = K_0^{26}\,\langle 0|\exp\Big(-x\cdot E^{-2}\cdot x + 2i\,a\cdot E^{-1}\cdot x + \frac{1}{2}a\cdot a\Big)\,, \tag{4.20}$$

where K_0 is a normalization constant whose value we shall not need to know. Thus the sliver wave-function, expressed in the position basis, is given by

$$\psi_\Xi(\vec{x}) = \langle\vec{x}\,|\Xi\rangle = \widetilde{\mathcal{N}}^{26}\exp(-\frac{1}{2}\,x\cdot V\cdot x)\,. \tag{4.21}$$

The evaluation of this contraction is done with the general formula (3.24) in section 3.2. One finds that the normalization factor is $\widetilde{\mathcal{N}} = \mathcal{N}\,K_0(\det(1+S))^{-1/2}$ and that the matrix V above is given as

$$V = 2E^{-2} - 4E^{-1}S(1+S)^{-1}E^{-1} \quad \rightarrow \quad V_{mn} = n\delta_{mn} - 2\sqrt{mn}(S(1+S)^{-1})_{mn}\,. \tag{4.22}$$

We can now rewrite ψ_Ξ as a function of x^L and x^R using eq.(4.5). This gives:

$$\psi_\Xi(x^L,x^R) = \widetilde{\mathcal{N}}^{26}\exp\Big(-\frac{1}{2}x^L\cdot K\cdot x^L - \frac{1}{2}x^R\cdot K\cdot x^R - x^L\cdot L\cdot x^R\Big)\,, \tag{4.23}$$

[5] We shall denote by $\hat{x}_n^\mu$ the coordinate operators and by x_n^μ the eigenvalues of these operators.

where

$$K = A^{+T} V A^{+} = A^{-T} V A^{-} , \qquad L = A^{+T} V A^{-} \,. \tag{4.24}$$

The equality of the two forms for K follows from eq.(4.10) and the relation $CSC = S$. The superscript T denotes transposition.

From this it is clear that in order that the sliver wave-function factorizes in the sense described in eq.(4.15), the matrix L defined above must vanish. While we have not attempted an analytic proof of this, we have checked using level truncation that this indeed appears to be the case. In particular, as the level is increased, the elements of the matrix L become much smaller than typical elements of the matrix K. If L vanishes, then the sliver indeed has the form given in eq.(4.15) with

$$f(x^L) = \widetilde{\mathcal{N}}^{13} \exp(-\frac{1}{2}\, x^L \cdot K \cdot x^L) \,. \tag{4.25}$$

In this form we also see that the functional f is actually real. This is expected since the sliver is twist even, and it must then correspond to a symmetric matrix in half-string space.

4.3 Building orthogonal projectors

Given that the sliver describes a projection operator into a one dimensional subspace, the following question arises naturally : is it possible to construct a projection operator into an orthogonal one dimensional subspace? If we can construct such a projection operator χ, then we shall have

$$\Xi * \chi = \chi * \Xi = 0, \qquad \chi * \chi = \chi \,, \tag{4.26}$$

and $\Xi + \chi$ will satisfy the equation of motion (4.1) and represent a configuration of two D-25-branes.

From eq.(4.15) it is clear how to construct such an orthogonal projection operator. We simply need to choose a function g satisfying the same normalization condition (4.16) as f and orthonormal to f:

$$\begin{aligned} &\int [\mathrm{d}y]\, f^*(y) g(y) = 0 \,, \\ &\int [\mathrm{d}y]\, g^*(y) g(y) = \int [\mathrm{d}y]\, f^*(y) f(y) \, = 1 \,, \end{aligned} \tag{4.27}$$

and then define

$$\psi_\chi(x^L, x^R) = g(x^L) g^*(x^R) \,. \tag{4.28}$$

There are many ways to construct such a function g, but one simple class of such functions is obtained by choosing:

$$g(x^L) = \lambda_{\mu n} x_n^{L\mu} \, f(x^L) \, \equiv \lambda \cdot x^L \, f(x^L) \, , \tag{4.29}$$

where λ is a constant vector. Since $f(-x) = f(x)$, the function $g(x)$ is orthogonal to $f(x)$. For convenience we shall choose λ to be real. Making use of (4.25) the normalization condition (4.27) for g requires:

$$\frac{1}{2} \lambda \cdot K^{-1} \cdot \lambda = 1 \, . \tag{4.30}$$

Additional orthogonal projectors are readily obtained. Given another function $h(x^L)$ of the form

$$h(x^L) = \lambda' \cdot x^L \, f(x^L) \, , \tag{4.31}$$

with real λ', we find another projector orthogonal to the sliver and to χ if

$$\lambda \cdot K^{-1} \cdot \lambda' = 0, \qquad \frac{1}{2} \, \lambda' \cdot K^{-1} \lambda' = 1. \tag{4.32}$$

Since we can choose infinite set of mutually orthonormal vectors of this kind, we can construct infinite number of projection operators into mutually orthogonal subspaces, each of dimension one. By superposing N of these projection operators we get a solution describing N D-branes.

Note that these projectors are not manifestly Lorentz invariant although, as we shall discuss in 5.2, they may well be invariant under a combined Lorentz and gauge transformation, which is all that is necessary. On the other hand, we could also construct manifestly Lorentz invariant projectors by taking $g(x^L)$ to be of the form $C_{mn} x_m^{L\mu} x_n^{L\nu} \eta_{\mu\nu} + D$ with appropriately chosen constants C_{mn} and D.

One could continuously interpolate between the sliver Ξ and the state χ defined above via a family of rank one projection operators χ_θ characterized by an angle θ. This corresponds to choosing[6]

$$g_\theta(x^L) = (\sin\theta \, \lambda \cdot x^L + \cos\theta) \, f(x^L) \, . \tag{4.33}$$

It is instructive to re-express the string state χ, given by eqs.(4.28) and (4.29) in the harmonic oscillator basis. We have,

$$|\chi\rangle = (\lambda \cdot \hat{x}^L) \, (\lambda \cdot \hat{x}^R) |\Xi\rangle \, . \tag{4.34}$$

Using eqs.(4.8) and (4.19) we can rewrite this as

$$|\chi\rangle = -\frac{1}{4} \, (\lambda \cdot \widetilde{A}^+ E(a - a^\dagger)) \, (\lambda \cdot \widetilde{A}^- E(a - a^\dagger)) |\Xi\rangle \, . \tag{4.35}$$

[6]Note that we could have used a general $U(2)$ rotation. This will give rise to a complex $g(x^L)$.

We would like to express this in terms of only creation operators acting on the sliver state $|\Xi\rangle$. For this we use the identity

$$a\,|\Xi\rangle = -Sa^\dagger\,|\Xi\rangle\,. \tag{4.36}$$

Using this we can rewrite eq.(4.35) as

$$|\chi\rangle = \Big(-\,\xi\cdot a^\dagger\;\widetilde{\xi}\cdot a^\dagger + \kappa\Big)|\Xi\rangle\,, \tag{4.37}$$

where $\widetilde{\xi}_{\mu n} = \xi_{\mu m}C_{mn}$, and

$$\begin{aligned} \xi &= \frac{1}{2}\,\lambda\cdot\widetilde{A}^{+}E(1+S)\,, \\ \kappa &= \frac{1}{4}\,\lambda\cdot\widetilde{A}^{+}E(1+S)E\widetilde{A}^{-T}\cdot\lambda = \xi\cdot(1+S)^{-1}\cdot\widetilde{\xi}\,. \end{aligned} \tag{4.38}$$

The interpolating state χ_θ can also be expressed in terms of oscillators following an identical procedure. The result is

$$|\chi_\theta\rangle = \Big(-\sin^2\theta\,\xi\cdot a^\dagger\;\widetilde{\xi}\cdot a^\dagger - i\sin\theta\cos\theta(\xi+\widetilde{\xi})\cdot a^\dagger + \cos^2\theta + \kappa\sin^2\theta\Big)|\Xi\rangle\,. \tag{4.39}$$

Before we conclude this section, we would like to emphasize that neither the sliver nor the state χ can be thought of as a projector into a one dimensional subspace of the *full string Hilbert space* even at zero momentum. If we think of the zero momentum string field as a matrix, then the full string Hilbert space is the space of all matrices. The sliver (or χ) is represented by a projection operator of the form $P_{ij} = u_i u_j$, and acting on a general matrix M_{ij}, gives $(PM)_{ik} = u_i u_j M_{jk} \equiv u_i v_k$. The subset S of matrices of the form $u_i v_j$ for fixed vector u but arbitrary vector v is closed under matrix multiplication; furthermore multiplication by any element of S from the left takes any matrix to this subset S. Thus this subset of matrices can be thought of as an ideal of the algebra associated with the full string Hilbert space, and the operator P_{ij} projects any element of the algebra into this ideal. In fact in this case the set of matrices in S is also closed under addition, and hence S can also be regarded as a subspace of the full string Hilbert space.

4.4 Lower dimensional D-branes

The above method is not directly applicable to the construction of lower dimensional D-brane solutions since the string wave-function now has additional dependence on the zero mode coordinates x_0^α, or equivalently the string mid-point coordinates $\{X^\alpha(\pi/2)\} \equiv x^M$ where X^α denotes directions transverse to the D-brane. The string field is now a function of x^L, x^R and x^M, and the star product of two string fields ψ and χ is given by

$$(\psi * \chi)\,(x^L, x^R; x^M) = \int [\mathrm{d}y]\;\psi(x^L, y; x^M)\,\chi(y, x^R; x^M)\,. \tag{4.40}$$

Note that on the right hand side of eq.(4.40) the integration is carried out over the modes of the half string only *without the mid-point*. In analogy with the discussion in the previous section we could try to get a solution of the equation $\psi_P * \psi_P = \psi_P$ by taking ψ_P of the form:

$$\psi_P(x^L, x^R) = f(x^L; x^M) f^*(x^R; x^M)\,, \tag{4.41}$$

with

$$f(x^L; x^M) = \mathcal{N}' \exp\big(- K'_{mn}({x^L}_m - c_m x^M)({x^L}_n - c_n x^M)\big)\,. \tag{4.42}$$

Here K'_{mn} is some appropriate matrix, c_n is an appropriate vector, and $\mathcal{N}'$ is a suitable normalization constant such that

$$\int [\mathrm{d}y] f^*(y; x^M) f(y; x^M) = 1\,. \tag{4.43}$$

In writing down the above formulæ we have suppressed the Lorentz indices, but they can be easily put back. We expect that for a suitable choice of K and c_m, the state ψ_P defined through eqs.(4.41) and (4.42) coincides with the D-p-brane solution constructed in ref.[2]. Following the methods of previous subsections, we can easily construct other projection operators orthogonal to ψ_P. We shall not pursue this line of argument any further, however, and we now turn to the more rigorous algebraic formulation of the problem.

5 Multiple D-brane solutions – Algebraic approach

In this section we give an oscillator construction of multiple brane solutions. While inspired by the considerations of the previous section, the analysis is self-contained and purely within the algebraic oscillator approach. Not only is the algebraic approach less formal than the functional presentation of the previous section, but it also allows straightforward generalization to the case of lower dimensional branes, as well as to the case of non-coincident branes.

5.1 Algebraic construction of higher rank projectors

Since the operators $\widetilde{A}^+$, $\widetilde{A}^-$ and S are known explicitly, eqs.(4.37), (4.38) give an explicit expression for a string state which squares to itself and whose $*$-product with the sliver vanishes. Since the treatment of star products as delta functionals that glue half strings in path integrals could conceivably be somewhat formal, and also the demonstration that the sliver wave-functional factorizes was based on numerical study, in this section we will examine the problem algebraically using the oscillator representation of star products. We shall give an explicit construction of the state $|\chi\rangle$ without any reference to the matrices $\widetilde{A}^+, \widetilde{A}^-$. For this we take a trial state of the same form as in eq.(4.37):

$$|\chi\rangle = \Big(-\xi \cdot a^\dagger \; \widetilde{\xi} \cdot a^\dagger + \kappa \Big) |\Xi\rangle \,. \tag{5.1}$$

Here $\widetilde{\xi} \equiv C\xi$, ξ is taken to be an arbitrary vector to be determined, and κ is a constant to be determined. We shall actually constrain ξ to satisfy

$$\rho_1 \xi = 0, \qquad \rho_2 \xi = \xi \,, \tag{5.2}$$

where the ρ_i are the projector operators defined in (3.13). We believe that ξ, as defined in eq.(4.38), automatically satisfies eq.(5.2) for any λ. This is the case if:

$$\rho_1 \,(1+S)\, E\, \widetilde{A}^{+T} = 0 \,. \tag{5.3}$$

We do not have a proof of this equation, but numerical evidence indicates it is indeed true. At any rate, for the analysis of this section we do not use any of the results of the previous subsection. So we will simply proceed by taking ξ to be an independent vector satisfying eq.(5.2). All the results that follow are consistent with the results of the previous subsection if we assume eq.(5.3) to be correct.

We first discuss some of the algebraic implications of the constraints imposed in (5.2). Since $C\rho_1 C = \rho_2$, eq.(5.2) gives

$$\rho_2 \widetilde{\xi} = 0, \qquad \rho_1 \widetilde{\xi} = \widetilde{\xi} \,. \tag{5.4}$$

Since ρ_1 and ρ_2 are represented by symmetric matrices, the transposed versions of eqns. (5.2) and (5.4) also hold. Moreover, being projectors into complementary orthogonal subspaces: $\xi^T \widetilde{\xi} = \xi^T (\rho_1 + \rho_2) \widetilde{\xi} = 0$. In fact, since ρ_1 and ρ_2 commute with M^{rs}, X and T, as follows from eqs.(3.4), (3.9) and (3.13), we have the stronger relation:

$$\xi^T f(M^{rs}, X, T)\, \widetilde{\xi} = 0 \,, \tag{5.5}$$

that holds for any function f. Finally, we record the relations:

$$\begin{aligned} M^{12}\xi &= -X(1-T)\xi, \\ M^{21}\xi &= (1-TX)\xi \,. \end{aligned} \tag{5.6}$$

The first follows directly from the first equation in (3.18). The second follows from the first and the relation $X + M^{12} + M^{21} = 1$.

We now begin the computation in earnest. This will have three steps:

(1) We require $\chi * \Xi = 0$ and use this to fix κ.
(2) We will see that indeed $\chi * \chi = \chi$ if ξ is suitably normalized.
(3) We will show that $\langle\chi|\chi\rangle = \langle\Xi|\Xi\rangle$ and that $\langle\chi|\Xi\rangle = 0$.

We begin with step (1). The product $\chi * \Xi$ (and $\Xi * \chi$) requires multiplying a sliver times a sliver with two oscillators acting on it. As explained in section 3.2, one can obtain this result by applying the differential operator $\frac{\partial^2}{\partial\beta_{1m\mu}\partial\beta_{1n\nu}}$ on both sides of the master equation eq.(3.30), and then setting β_1 and β_2 to zero. One finds

$$|\chi * \Xi\rangle = |\Xi * \chi\rangle = (\kappa + \xi^T(\mathcal{V}\mathcal{K}^{-1})_{11}\widetilde{\xi})|\Xi\rangle\,, \tag{5.7}$$

where the matrix $\mathcal{V}\mathcal{K}^{-1}$, playing an important role in this kind of calculation has been computed in (3.28). Since we require $\chi * \Xi = 0$, we need to set

$$\kappa = -\xi^T(\mathcal{V}\mathcal{K}^{-1})_{11}\widetilde{\xi} = -\xi^T T(1-T^2)^{-1}\xi\,, \tag{5.8}$$

where the last equation in (3.12) was used to simplify the expression for $(\mathcal{V}\mathcal{K}^{-1})_{11}$. On the other hand, using the relation $(1+S)^{-1} = (1-S)(1-S^2)^{-1} = (1-CT)(1-T^2)^{-1}$, and using eq.(5.5) we can show that eq.(4.38) leads to precisely the same equation for κ. Thus we see that eq.(5.8) agrees with the second equation in (4.38). This concludes step one of the calculation.

In step two we calculate $\chi * \chi$ by differentiating both sides of eq.(3.30) with respect to $\beta_{1m\mu}$ and $\beta_{2m\mu}$ appropriate number of times, and then setting β_1 and β_2 to zero. After using eq.(5.5), we get the result for the $*$-product to be:

$$\begin{aligned}\chi * \chi \;&=\; -(\xi^T(\mathcal{V}\mathcal{K}^{-1})_{12}\widetilde{\xi})\,\xi\cdot a^\dagger\,\widetilde{\xi}\cdot a^\dagger|\Xi\rangle \\ &\quad +\Big((\xi^T(\mathcal{V}\mathcal{K}^{-1})_{11}\widetilde{\xi})(\xi^T(\mathcal{V}\mathcal{K}^{-1})_{22}\widetilde{\xi}) + (\xi^T(\mathcal{V}\mathcal{K}^{-1})_{12}\widetilde{\xi})(\widetilde{\xi}^T(\mathcal{V}\mathcal{K}^{-1})_{12}\xi) - \kappa^2\Big)|\Xi\rangle\,.\end{aligned} \tag{5.9}$$

Using (3.28), the last equation in (3.12), (5.6), and (5.8) one finds that

$$\widetilde{\xi}^T(\mathcal{V}\mathcal{K}^{-1})_{12}\xi = -\xi^T T(1-T^2)^{-1}\xi = \kappa\,. \tag{5.10}$$

Furthermore $(\mathcal{V}\mathcal{K}^{-1})_{11} = (\mathcal{V}\mathcal{K}^{-1})_{22}$. Using this and eqs.(5.8), (5.10), we see that eq.(5.9) can be written as

$$\chi * \chi = (\xi^T(\mathcal{V}\mathcal{K}^{-1})_{12}\widetilde{\xi})\Big(-\xi\cdot a^\dagger\,\widetilde{\xi}\cdot a^\dagger + \kappa\Big)|\Xi\rangle\,. \tag{5.11}$$

If we now normalize ξ such that

$$\xi^T\,(\mathcal{V}\mathcal{K}^{-1})_{12}\,\widetilde{\xi} = 1\,, \tag{5.12}$$

then eq.(5.11) reduces to the desired equation:

$$\chi * \chi = \chi\,. \tag{5.13}$$

The normalization condition eq.(5.12) can be simplified using eq.(5.6) and the first equation in (3.12) to obtain:

$$\xi^T (1-T^2)^{-1}\xi = 1\,. \tag{5.14}$$

This concludes step two of the procedure.

We shall now show that the new solution χ also represents a single D-25-brane. For this we shall calculate the tension associated with this solution and try to verify that it agrees with the tension of the brane described by the sliver. Since the tension of the brane associated to a given state is proportional to the BPZ norm of the state [2], all we need to show is that $\langle\chi|\chi\rangle$ is equal to $\langle\Xi|\Xi\rangle$. This is a straightforward calculation using the result

$$\begin{aligned} &\langle 0|\exp\Big(-\frac{1}{2}a\cdot Sa + \lambda\cdot a\Big)\exp\Big(-\frac{1}{2}a\cdot Sa^\dagger + \beta\cdot a\Big)|0\rangle \\ = \;& \det(1-S^2)^{-1}\exp\Big(\beta^T\cdot(1-S^2)^{-1}\cdot\lambda - \frac{1}{2}\beta^T\cdot S(1-S^2)^{-1}\cdot\beta \\ &\qquad -\frac{1}{2}\lambda^T\cdot S(1-S^2)^{-1}\cdot\lambda\Big)\,, \end{aligned} \tag{5.15}$$

which follows from the more general overlap (3.24). One then differentiates both sides of this equation with respect to components of λ and β to calculate the required correlator. The final result, after using eqs.(5.8), (5.14), is

$$\langle\chi|\chi\rangle = \langle\Xi|\Xi\rangle\,. \tag{5.16}$$

Thus the solution described by χ has the same tension as the solution described by Ξ. Similar calculation also yields:

$$\langle\Xi|\chi\rangle = 0\,. \tag{5.17}$$

Thus the BPZ norm of $|\Xi\rangle + |\chi\rangle$ is $2\langle\Xi|\Xi\rangle$. This shows that $|\Xi\rangle + |\chi\rangle$ represents a configuration with twice the tension of a single D-25-brane. This concludes the third and last step of the main calculation.[7]

The new projector χ is not unrelated to the sliver Ξ. One can prove directly that χ is obtained from Ξ by a rotation in the $*$-algebra. For this let us consider the interpolating state given in eq.(4.39):

$$|\chi_\theta\rangle = \Big(-\sin^2\theta\,\xi\cdot a^\dagger\,\widetilde{\xi}\cdot a^\dagger - i\sin\theta\cos\theta(\xi+\widetilde{\xi})\cdot a^\dagger + \cos^2\theta + \kappa\sin^2\theta\Big)|\Xi\rangle\,, \tag{5.18}$$

[7] The computations of BPZ inner products could have been done also by using $\langle\chi|\chi\rangle = \langle\mathcal{I}|\chi\rangle$, as discussed in subsection 3.3.

with ξ now interpreted to be an arbitrary vector satisfying eqs.(5.2), (5.14), and κ given by eq.(5.8). It is straightforward to show using the techniques described earlier in this subsection that $\chi_\theta * \chi_\theta = \chi_\theta$. Furthermore, one can show that

$$\frac{d\chi_\theta}{d\theta} = (\chi_\theta * \Lambda - \Lambda * \chi_\theta)\,, \tag{5.19}$$

where

$$|\Lambda\rangle = i(\xi - \widetilde{\xi}).a^\dagger|\Xi\rangle\,. \tag{5.20}$$

Thus changing θ induces a rotation of χ_θ by the generator Λ. We shall call such transformations $*$-rotation. Since $\chi_{\pi/2} = \chi$ and $\chi_0 = \Xi$, we get

$$\chi = e^{-\Lambda\pi/2} * \Xi * e^{\Lambda\pi/2}\,, \tag{5.21}$$

where in the expansion of the exponential all products must be interpreted as $*$ products. This establishes that χ is indeed a $*$-rotation of the sliver. We would like to believe that this corresponds to a gauge transformation in the full string field theory, but we cannot settle this point without knowing the form of the kinetic operator $\mathcal{Q}$. Nevertheless we have argued in refs.[2, 7] that under some suitable assumptions $*$-rotations can indeed be regarded as gauge transformations.

Consider now another projector χ' built just as χ but using a vector ξ':

$$|\chi'\rangle = \Big(-\xi'\cdot a^\dagger\,\widetilde{\xi}'\cdot a^\dagger + \kappa'\Big)|\Xi\rangle\,, \tag{5.22}$$

with

$$\rho_1\xi' = 0, \qquad \rho_2\xi' = \xi'\,, \tag{5.23}$$

κ' given as

$$\kappa' = -\xi'^T T(1-T^2)^{-1}\xi'\,, \tag{5.24}$$

and normalization fixed by

$$\xi'^T(1-T^2)^{-1}\xi' = 1\,. \tag{5.25}$$

Thus χ' is a projector orthogonal to Ξ. We now want to find the condition under which χ' projects into a subspace orthogonal to χ as well, *i.e.* the condition under which $\chi * \chi'$ vanishes. We can compute $\chi * \chi'$ in a manner identical to the one used in computing $\chi * \chi$ and find that it vanishes if:

$$\xi^T(1-T^2)^{-1}\xi' = 0\,. \tag{5.26}$$

Since this equation is symmetric in ξ and ξ', it is clear that $\chi' * \chi$ also vanishes when eq.(5.26) is satisfied. Given eqs.(5.25) and (5.26) we also have:

$$\langle\chi'|\chi'\rangle = 1, \quad \langle\chi|\chi'\rangle = \langle\Xi|\chi'\rangle = 0\,. \tag{5.27}$$

Thus $|\Xi\rangle + |\chi\rangle + |\chi'\rangle$ describes a solution with three D-25-branes. This procedure can be continued indefinitely to generate solutions with arbitrary number of D-25-branes.

5.2 Lorentz invariance and Chan-Paton factors

Although the sliver is a Lorentz invariant state, the state χ defined in eq.(5.1) is apparently not Lorentz invariant since it involves the vector $\xi_{\mu m}$. A general Lorentz transformation takes this state $|\chi\rangle$ to another state $|\chi'\rangle$ of the form:

$$|\chi'\rangle = \Big(- \xi' \cdot a^\dagger \, \widetilde{\xi}' \cdot a^\dagger + \kappa \Big) |\Xi\rangle \, , \tag{5.28}$$

where $\xi'_{\mu m}$ is the Lorentz transform of $\xi_{\mu m}$ and $\widetilde{\xi}' = C\xi'$. Since ρ_1 commutes with a Lorentz transformation, we have $\rho_1 \xi' = 0$. It is easy to see that the new state is a $*$-rotation of the original state χ. One way to show this is to rotate χ' to Ξ using the analog of eq.(5.21) with primed variables and then rotate Ξ back to χ using eq.(5.21). Thus although χ is not invariant under a Lorentz transformation by itself, a combination of Lorentz transformation and a $*$-rotation leaves the state invariant.

A similar analysis can be used to show that the multiple D-25-brane solution constructed earlier is invariant under a Lorentz transformation accompanied by a $*$-rotation. This $*$-rotation can be constructed according to the following algorithm. Under a general Lorentz transformation the set of vectors $\xi^{(i)}$ used to construct the N D-brane solution gets rotated into another set of vectors $\xi'^{(i)}$. The initial set of vectors $\xi^{(i)}$ as well as the final set of vectors $\xi'^{(i)}$ each define a set of N orthonormal vectors in half-string state space. In general, given two sets of N orthonormal vectors in a vector space there is a rotation taking one set to the other which can be written as the exponential of a generator. We can therefore construct the generator that rotates the final set of N vectors into the initial set in the half string state space. Given this generator we can explicitly represent it as a state Λ in the full string state space via the correspondance between matrices in half-string space and string fields. This Λ is the generator of the desired $*$-rotation.

This, however, does not prove the Lorentz invariance of these solutions since at present $*$-rotation in the matter sector only appears to be a symmetry of the equations of motion for the restricted class of field configurations of the form (2.5). If, as discussed in refs.[2, 7], $*$-rotation can be regarded as a gauge transformation in the full string field theory, then the above analysis would establish Lorentz invariance of our solutions.

A solution describing N coincident D-25-branes corresponds to a projection operator into an N-dimensional subspace of the half-string state space. If $*$-rotation really describes gauge transformation, then the $U(N)$ symmetry acting on this N-dimensional subspace of the half-string state space is a gauge symmetry. This can be identified as the $U(N)$ gauge symmetry of N coincident D-25-branes, $-$ in the same spirit as in the description of D-branes as non-commutative solitons [10].

5.3 Multiple D-p-brane solutions for $p < 25$

In the previous subsections we have described methods for constructing space-time independent solutions of the matter part of the field equation $\Psi * \Psi = \Psi$ which have vanishing $*$-product with the sliver. Taking the superposition of such a solution and the sliver we get a solution representing two D-branes. In this subsection we shall discuss similar construction for the D-branes of lower dimension.

The explicit solution of the field equations representing D-p-branes of all $p \le 25$ have been given in ref.[2]. These solutions have explicit dependence on the zero mode x_0^μ of the coordinate fields transverse to the D-brane, and hence the method of subsection 4.2 relying on the factorization of space-time independent string functionals is not directly applicable.[8] The algebraic method of subsection 5.1, however, is still applicable because the solutions associated with directions transverse to the D-brane have a similar structure to the solutions associated with directions tangential to the D-brane. Thus, for example, if $x^{\bar\mu}$ denote directions tangential to the D-p-brane ($0 \le \mu \le p$) and x^α denote directions transverse to the D-brane ($p+1 \le \alpha \le 25$), then the solution representing the D-p-brane has the form:

$$|\Xi_p\rangle = \mathcal{N}^{p+1} \exp\Big(-\frac{1}{2}\eta_{\bar\mu\bar\nu} S_{mn} a_m^{\bar\mu\dagger} a_n^{\bar\nu\dagger}\Big)|0\rangle \otimes (\mathcal{N}')^{25-p} \exp\Big(-\frac{1}{2} S'_{mn} a_m^{\alpha\dagger} a_n^{\alpha\dagger}\Big)|\Omega\rangle\,, \qquad (5.29)$$

where in the second exponential the sums over m and n run from 0 to ∞, $\mathcal{N}'$ is an appropriate normalization constant determined in ref.[2], and S' is given by an equation identical to the one for S (see eqs.(3.3), (3.4), (3.8), and (3.9)) with all matrices M^{rs}, V^{rs}, X, C and T replaced by the corresponding primed matrices. The primed matrices carry indices running from 0 to ∞ in contrast with the unprimed matrices whose indices run from 1 to ∞. But otherwise the primed matrices satisfy the same equations as the unprimed matrices. Indeed, all the equations in section 3.1 are valid with unprimed matrices replaced by primed matrices. In particular we can define ρ_1' and ρ_2' in a manner analogous to eq.(3.13). The equations in section 3.2 also generalize to the case when $|\Xi\rangle$ is replaced by $|\Xi_p\rangle$ and $\beta \cdot a^\dagger$ is interpreted as $(\beta_{n\bar\mu} a_n^{\bar\mu\dagger} + \beta'_{n\alpha} a_n^{\alpha\dagger})$. We now choose vectors $\xi_{\bar\mu m}$, $\xi'_{\alpha m}$ such that

$$\rho_1 \xi_{\bar\mu} = 0, \qquad \rho_1' \xi'_\alpha = 0\,. \qquad (5.30)$$

We also define

$$\widetilde\xi_{\bar\mu} = C\xi_{\bar\mu}, \qquad \widetilde\xi'_\alpha = C'\xi'_\alpha, \qquad \kappa' = -\xi^T T(1-T^2)^{-1}\xi - \xi'^T T'(1-T'^2)^{-1}\xi'\,, \qquad (5.31)$$

and normalize ξ, ξ' such that

$$\xi^T(1-T^2)^{-1}\xi + \xi'^T(1-T'^2)^{-1}\xi' = 1\,. \qquad (5.32)$$

[8] The setup of subsection 4.4 may apply.

In that case following the procedure identical to that of subsection 5.1 we can show that the state:

$$|\chi_p\rangle = \Big(- (\xi_{\bar\mu} \cdot a^{\bar\mu\dagger} + \xi'_\alpha \cdot a^{\alpha\dagger})(\widetilde{\xi}_{\bar\nu} \cdot a^{\bar\nu\dagger} + \widetilde{\xi}'_\alpha \cdot a^{\alpha\dagger}) + \kappa' \Big)|\Xi_p\rangle \,, \tag{5.33}$$

satisfies:

$$\chi_p * \Xi_p = \Xi_p * \chi_p = 0 \,, \tag{5.34}$$

and

$$\chi_p * \chi_p = \chi_p \,. \tag{5.35}$$

Thus $\chi_p + \Xi_p$ will describe a configuration with two D-p-branes. This construction can be generalized easily following the procedure of subsection 5.1 to multiple D-p-brane solutions.

5.4 Parallel separated D-branes

At this stage it is natural to ask if we could superpose two (or more) parallel D-p-brane solutions which are separated in the transverse direction. For this we first need to construct the single D-p-brane state which is translated in the transverse direction along some vector $\vec{s}$ (say). This is easily constructed by noting that the generator of translation in the transverse direction is the momentum operator $\hat{p}_\alpha$ along the x^α direction. This is related to a_0^α, $a_0^{\alpha\dagger}$ through the relations:

$$\widehat{p}^\alpha = \frac{1}{\sqrt{b}}(a_0^\alpha + a_0^{\alpha\dagger}) \,, \tag{5.36}$$

where b is a constant introduced in ref.[2]. Then the shifted D-brane solution is given by

$$|\Xi_p^{(\vec{s})}\rangle = \exp(is_\alpha \widehat{p}^\alpha)|\Xi_p\rangle \,. \tag{5.37}$$

Since translation is a symmetry of the action, this state must square to itself. This is instructive to check explicitly. It suffices to work along a single specific direction, and to focus on the portion of the state in (5.29) that involves that direction. This gives

$$\begin{aligned} &\exp(is\widehat{p}) \exp\Big(-\frac{1}{2} a^\dagger \cdot S' \cdot a^\dagger\Big)|\Omega\rangle \\ &= \exp(-\frac{s^2}{2b}) \exp(\frac{is}{\sqrt{b}} a_0^\dagger) \exp(\frac{is}{\sqrt{b}} a_0) \exp\Big(-\frac{1}{2} a^\dagger \cdot S' \cdot a^\dagger\Big)|\Omega\rangle \\ &= \exp\Big(-\frac{s^2}{2b}(1 - S'_{00})\Big) \exp(\frac{is}{\sqrt{b}}(1 - S')_{0m} a_m^\dagger) \exp\Big(-\frac{1}{2} a^\dagger \cdot S' \cdot a^\dagger\Big)|\Omega\rangle \,, \end{aligned} \tag{5.38}$$

where use was made of the Baker-Campbell-Hausdorf relation. We now claim that the final state is actually of the form of the states $P_\beta \equiv \exp\Big(\mathcal{C}(\beta,\beta)\Big)|\Xi_\beta\rangle$ introduced in section 3.2 (see (3.34)). One identifies

$$\beta_m = -\frac{is}{\sqrt{b}}((1-S')C')_{0m} = -\frac{is}{\sqrt{b}}(1-T')_{0m}\,, \tag{5.39}$$

where in the last step we have used $S' = C'T' = T'C'$ to recognize that $S'_{0m} = T'_{0m}$. A similar replacement can be done in the first two exponentials of (5.38). It is now a simple computation to verify that the first exponential on the right hand side of (5.38) correctly arises as $\exp(\mathcal{C}(\beta,\beta))$ as required by the identification. Since $P_\beta * P_\beta = P_\beta$ this gives the desired confirmation.

We note, however, that the $*$-product of $\Xi_p^{(\vec{s})}$ with Ξ_p does not vanish manifestly.[9] Hence we do not attempt to represent two displaced D-branes as $\Xi_p + \Xi_p^{(\vec{s})}$. Let us consider, however, a different projector built as in (5.33), with $\xi_{\bar{\mu}m} = 0$ but $\xi'^{\alpha}_m \neq 0$:

$$|\chi_p\rangle = (-\xi'\cdot a^\dagger\,\widetilde{\xi}'\cdot a^\dagger + \kappa')|\Xi_p\rangle, \qquad \widetilde{\xi}' = C'\xi', \qquad \rho'_1\xi' = 0\,,$$

$$\kappa' = -\xi'^T T'(1-T'^2)^{-1}\xi'\,, \qquad \xi'^T(1-T'^2)^{-1}\xi' = 1\,. \tag{5.40}$$

If we define $\eta' = \rho'_2\beta$, $\widetilde{\eta}' = \rho'_1\beta$, with β given in eq.(5.39), then the relation $C'\beta = \beta$, which follows from eq.(5.39) leads to $\widetilde{\eta}' = C'\eta'$, as the notation suggests. It is then easy to verify that[10]

$$\Xi_p^{(\vec{s})} * \chi_p = \chi_p * \Xi_p^{(\vec{s})} = 0\,, \tag{5.41}$$

if

$$\xi'^T(1-T'^2)^{-1}\eta' = 0\,. \tag{5.42}$$

Thus in this case $\chi_p + \Xi_p^{(\vec{s})}$ is a solution of the equations of motion, and describes parallel separated D-branes.

We shall see in the next subsection how a simple procedure allows us to obtain more general D-brane configurations.

5.5 Intersecting D-branes

In this subsection we shall discuss the construction of multiple D-brane solutions of different dimensions at different positions.

[9] Although $\Xi_p^{(\vec{s})} * \Xi_p$ is not manifestly zero, we believe that on general grounds [7] such products do vanish. In this case, this could happen as a result of the vanishing of $\exp(-\mathcal{C}(0,\beta))$ for the particular β in equation (5.39). This is discussed further in [7].

[10] This can be done by differentiating $\Xi_{\beta_1} * \Xi_{\beta_2}$ with respect to components of β_1 or β_2.

It is clear from our earlier discussion that in order to construct multiple brane solutions, we need to construct states Ψ_1, Ψ_2, $\ldots$ Ψ_n satisfying

$$\Psi_i * \Psi_j = \delta_{ij} \Psi_i \,, \tag{5.43}$$

and then simply consider the solution

$$\Psi = \sum_i \Psi_i \,. \tag{5.44}$$

Thus the question is: in general how do we construct solutions Ψ_i representing D-branes of different types and still satisfying eq.(5.43)?

In order to carry out this construction we need to assume that all the D-branes that we want to superpose have one tangential (or transverse) direction in common. For definiteness we shall take this to be the time direction x^0. We shall now take the Ψ_i's to be states of the factorized form:

$$|\Psi_i\rangle = |\chi^{(i)}\rangle_0 \otimes |\psi^{(i)}\rangle_{space} \,. \tag{5.45}$$

Here $|\,\rangle_0$ denotes that we are describing a state in the Hilbert space of the BCFT associated with X^0 and $|\,\rangle_{space}$ denotes that we are describing a state in the Hilbert space of the BCFT associated with the spatial coordinates. Since for such states the $*$-product factorizes into the $*$-product in the time part and the space part, we have

$$|\Psi_i * \Psi_j\rangle = |\chi^{(i)} * \chi^{(j)}\rangle_0 \otimes |\psi^{(i)} * \psi^{(j)}\rangle_{space} \,. \tag{5.46}$$

The idea now is to choose the space part $|\psi^{(i)}\rangle_{space}$ to be the space part of arbitrary D-p-brane solutions (with possible shifts and different values of p) described earlier in this section, or even the BCFT deformations discussed in ref.[7], but to choose the $|\chi^{(i)}\rangle_0$ such that

$$|\chi^{(i)} * \chi^{(j)}\rangle_0 = \delta_{ij} |\chi^{(i)}\rangle_0 \,. \tag{5.47}$$

This is achieved by choosing

$$|\chi^{(i)}\rangle_0 = -(-\xi^{(i)} \cdot a^{0\dagger}\ \widetilde{\xi}^{(i)} \cdot a^{0\dagger} + \kappa^{(i)}) |\Xi\rangle_0, \tag{5.48}$$

with $\rho_1 \xi = 0$, and

$$\kappa^{(i)} = -\eta_{00}\, \xi^{(i)T} T (1-T^2)^{-1} \xi^{(i)}, \qquad \eta_{00}\, \xi^{(i)T} (1-T^2)^{-1} \xi^{(j)} = -\delta_{ij} \,. \tag{5.49}$$

The overall minus sign on the right hand side of eq.(5.48) compared to (5.1) is due to the choice of extra minus sign in the normalization condition given in the second equation in (5.49) as compared to (5.14). This in turn is required in order to find solutions to the normalization condition. This choice of Ψ_i satisfies eq.(5.43), and hence Ψ defined in eq.(5.44) satisfies the equation of motion. This has the interpretation of superposition of D-branes of different dimensions at different positions.

Before we end we would like to note that not only could we have chosen the $|\psi^{(i)}\rangle_{space}$ to be the space part of the solutions for arbitrary D-p-branes at arbitrary positions, but with arbitrary orientations as well. We simply take the space part of a known D-p-brane solution and apply a rotation to construct such a $|\psi^{(i)}\rangle_{space}$. Superposition of states of the form $|\chi^{(i)}\rangle_0 \otimes |\psi^{(i)}\rangle_{space}$ will then give rise to configurations of D-branes at different angles. A slight generalization of this allows us to construct solutions describing superpositions of moving D-branes with arbitrary velocities and arbitrary orientation, as long as all the D-branes share one common space-like tangential coordinate, or one common space-like transverse coordinate which is orthogonal to the velocities of all the branes. Let us call this direction x^1. We can then use this direction to construct a set of mutually orthogonal projectors which will now play the role of $|\chi^{(i)}\rangle$.[11] The $|\psi^{(i)}\rangle$'s are generated from the original D-brane solution Ξ_p by first removing the factor associated with the direction x^1, and then applying combinations of translations, boosts and rotations in directions perpendicular to x^1. Superposition of solutions of the type $|\chi^{(i)}\rangle \otimes |\psi^{(i)}\rangle$ will then give rise to a solution describing moving D-branes with different velocities and orientations. We believe that the restriction of having one common tangential or transverse direction is only a technical difficulty, and a resolution based on a different line of argument is presented in [7].

6 Discussion

We now offer some brief remarks on our results and discuss some of the open questions.

- The first point we would like to make is that the present work gives credence to the idea that half-string functionals do play a role in open string theory. At least for zero momentum string fields, as explained here, it is on the space of half-string functionals that the sliver is a rank-one projection operator. We do believe that the left/right factorization of the sliver is an exact result, and it would be interesting to find an explicit proof using the relevant matrices.

- We have seen that the sliver is a rank-one projector, and we have learned how to construct higher rank projectors. The identity string field, on the other hand is

[11] When x^1 is a common transverse coordinate orthogonal to the velocities and the branes are displaced along it we use the construction at the end of section 5.4 to produce the required orthogonal factors.

an infinite rank projector. Since rank-N projectors are associated to configurations with N D-branes, one would be led to believe that the identity string field is a classical solution of vacuum string field theory representing a background with an infinite number of D-branes. While no doubt technical complications related to normalization and regularization would be encountered in discussing concretely such background, it is interesting to note that a background with infinite number of D-branes is natural for a general K-theory analysis of D-brane states [23].

- We have shown that various rank-one projectors are equivalent under $*$-rotation; a similarity transformation generated by a state built with oscillators acting on the sliver. We would expect $*$-rotations to be gauge symmetries of vacuum string field theory, but this issue seems difficult to settle without explicit knowledge of the kinetic operator $\mathcal{Q}$ of vacuum string field theory. For more discussion (but not a resolution) of this question, the reader may consult [7].

- In the study of C^* algebras and von Neumann algebras projectors play a central role in elucidating their structure. We may be optimistic that having finally found how to construct (some) projectors in the star algebra of open strings, a more concrete understanding of the gauge algebra of open string theory, perhaps based on operator algebras[12], may be possible to attain in the near future. This would be expected to have significant impact on our thinking about string theory.

Vacuum string field theory appears to be rather promising. Multiple configurations of branes appear to be as simple in this string field theory as they are in non-commutative field theory. In turn, non-commutative solitons are in many ways simpler than ordinary field theory solitons. All in all we are in the surprising position where we realize that in string field theory some non-perturbative physics – such as that related to multiple D-brane configurations – could be argued to emerge more simply than the analogous phenomena does in ordinary field theory.

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[12]For some readable introductory comments on the possible uses of C^* algebras in K-theory see [24], section 4.

References

[1] L. Rastelli, A. Sen and B. Zwiebach, *String field theory around the tachyon vacuum*, hep-th/0012251.

[2] L. Rastelli, A. Sen and B. Zwiebach, *Classical solutions in string field theory around the tachyon vacuum,* hep-th/0102112.

[3] E. Witten, *Noncommutative Geometry And String Field Theory,* Nucl. Phys. **B268**, 253 (1986).

[4] G. T. Horowitz, J. Morrow-Jones, S. P. Martin and R. P. Woodard, *New Exact Solutions For The Purely Cubic Bosonic String Field Theory,* Phys. Rev. Lett. **60**, 261 (1988).

[5] L. Rastelli and B. Zwiebach, *Tachyon potentials, star products and universality,* hep-th/0006240.

[6] A. Kostelecky and R. Potting, *Analytical construction of a nonperturbative vacuum for the open bosonic string,* hep-th/0008252.

[7] L. Rastelli, A. Sen and B. Zwiebach, *Small and large deformations of D-brane solutions in vacuum string field theory*, to appear.

[8] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, JHEP**0005**, 020 (2000) [hep-th/0003160].

[9] K. Dasgupta, S. Mukhi and G. Rajesh, *Noncommutative tachyons,* JHEP **0006**, 022 (2000) [hep-th/0005006].

[10] J. A. Harvey, P. Kraus, F. Larsen and E. J. Martinec, *D-branes and strings as noncommutative solitons,* hep-th/0005031.

[11] M. Schnabl, *String field theory at large B-field and noncommutative geometry*, JHEP**0011**, 031 (2000) [hep-th/0010034].

[12] J. A. Harvey, P. Kraus and F. Larsen, *Exact noncommutative solitons*, JHEP**0012**, 024 (2000) [hep-th/0010060].

[13] C. Hong-Mo and T. Sheung Tsun, Phys. Rev. **D 35**, 2474 (1987); Phys. Rev. **D 39**, 555 (1989).

[14] J. Bordes, H. Chan, L. Nellen and S. T. Tsou, *Half string oscillator approach to string field theory,* Nucl. Phys. B **351** (1991) 441.

[15] A. Abdurrahman and J. Bordes, "The relationship between the comma theory and Witten's string field theory. I," Phys. Rev. D **58**, 086003 (1998).

[16] D. J. Gross and W. Taylor, *Split string field theory I*, hep-th/0105059.

[17] D. J. Gross and A. Jevicki, *Operator Formulation Of Interacting String Field Theory*, Nucl. Phys. **B283**, 1 (1987).

[18] D. J. Gross and A. Jevicki, *Operator Formulation Of Interacting String Field Theory. 2,* Nucl. Phys. **B287**, 225 (1987).

[19] E. Cremmer, A. Schwimmer and C. Thorn, *The vertex function in Witten's formulation of string field theory,* Phys. Lett. **B179** (1986) 57.

[20] S. Samuel, *The Physical and Ghost Vertices In Witten's String Field Theory,* Phys. Lett. **B181**, 255 (1986).

[21] A. LeClair, M. E. Peskin and C. R. Preitschopf, *String Field Theory On The Conformal Plane. 1. Kinematical Principles,* Nucl. Phys. **B317**, 411 (1989); *String Field Theory On The Conformal Plane. 2. Generalized Gluing,* Nucl. Phys. **B317**, 464 (1989).

[22] I. Ellwood, B. Feng, Y.-H. He, and N. Moeller, *The identity string field and the tachyon vacuum*, hep-th/0105024.

[23] E. Witten, *Overview of K-theory applied to strings,* Int. J. Mod. Phys. A **16**, 693 (2001) [hep-th/0007175].

[24] J. A. Harvey, *Komaba lectures on noncommutative solitons and D-branes,* hep-th/0102076.