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Symmetries, Conserved Charges and (Black) Holes in Two Dimensional String Theory

Ashoke Sen

Harish-Chandra Research Institute
Chhatnag Road, Jhusi, Allahabad 211019, INDIA

E-mail: ashoke.sen@cern.ch, sen@mri.ernet.in

Abstract

Two dimensional string theory is known to have an infinite dimensional symmetry, both in the continuum formalism as well as in the matrix model formalism. We develop a systematic procedure for computing the conserved charges associated with these symmetries for any configuration of D-branes in the continuum description. We express these conserved charges in terms of the boundary state associated with the D-brane, and also in terms of the asymptotic field configurations produced by this D-brane. Comparison of the conserved charges computed in the continuum description with those computed in the matrix model description facilitates identification of the states between these two formalisms. Using this we put constraints on the continuum description of the hole states in the matrix model, and matrix model description of the black holes solutions of the continuum theory. We also discuss possible generalization of the construction of the conserved charges to the case of D-branes in critical string theory.

Contents

1 Introduction and Summary

Recent investigation in two dimensional string theory[1, 2, 3, 4, 5] has shown that they can provide us with a useful arena for studying various general properties of string theory, most notably the relationship between the open and closed string description of unstable D-brane systems[6, 7, 8, 9, 10, 11]. (See [12, 13] for aspects of open-closed string duality for stable D-branes in this string theory.). The feature that makes this theory most useful is that it has two different formulations. The first one, known as the continuum description[14, 15] (see also [16]), follows the usual formulation of string theory based on a world-sheet action containing a matter part with central charge 26 and a ghost part with central charge -26. The matter part in turn consists of a free time-like scalar field X^0 of central charge 1 and a Liouville scalar field φ with an exponentially growing potential. A linear dilaton background along the Liouville direction makes the Liouville theory have total central charge 25. In this formalism the theory can be studied using the usual string perturbation theory based on genus expansion. The other formulation of the theory, known as the matrix model, is based on discretizing the world sheet path integral and taking an appropriate double scaling limit[17, 18, 19]. This in turn can be shown to be equivalent to a theory of free non-interacting fermions, each moving under an inverted harmonic oscillator potential. The vacuum of the theory is a state in which all levels below a certain fixed energy are filled and all level above this energy are empty. In this formulation we can easily analyze the system to all orders in perturbation theory.

The usual closed string states of the continuum string theory are related to the matrix model states by bosonization of the fermion field followed by a non-local field redefinition[20, 21, 22]. While early work on this subject focussed on the comparison of the properties of closed strings in the two formulations, the recent surge of interest in this subject arises from the study of D-branes in the two descriptions of the theory. The continuum version of the theory admits an unstable D0-brane configuration with an open string tachyon on its world-volume[23]. Following the general method developed in [24, 25] one can construct an exact classical solution describing the open string tachyon rolling away from the maximum of the potential. By studying the closed string description of this process in the continuum string theory following [26, 27, 28], and comparing this with the single fermion excitation in the matrix model using the known relation between the states of the matrix model and the closed string states in the continuum description, it was concluded in [2] that the rolling tachyon configuration on a single D0-brane in the continuum theory describes precisely single fermion excitations in the matrix model.

Despite this new understanding of the relationship between the matrix model and continuum description of two dimensional string theories, several questions remain unanswered. In particular we still do not have a complete map between the known states of the continuum theory and known states of the matrix model. For example the matrix model, besides containing fermionic excitations, also contains hole like excitations where we remove a fermion from an energy level below the fermi level. A completely convincing description of these states in the continuum theory is still missing (although some candidates have been proposed in [5, 29]). On the other hand the continuum version of this theory admits black hole solutions[30, 31]. Although there are some proposals for a representation of the Euclidean black hole in the matrix model[32, 33], a satisfactory description of these black hole states in the Lorenzian version of the matrix model is still lacking.

Both the continuum version and the matrix model version of the theory are known to have infinite number of global symmetries[21, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43], and hence associated with them there must be infinite number of conserved charges. Thus if we can find the precise relation between the conserved charges in the continuum description and those in the matrix model description, then comparison of these conserved charges could provide us useful guidelines for making suitable identification between the states in the continuum description and states in the matrix model description. The construction of the conserved charges in the matrix model description follows straightforward application of Noether's method. Thus the main issue is to construct these conserved charges in the continuum description and relate them to the conserved charges in the matrix model description. This program was initiated in a previous paper[44] where it was shown that requirement of BRST invariance constrains the time evolution of certain components of the boundary state describing a D-brane, and hence leads to certain conserved charges. By evaluating these conserved charges for the rolling tachyon configuration, and comparing them with the conserved charges of the same configuration in the matrix model description, we found the relationship between these two sets of conserved charges. However in this analysis the construction of the conserved charges on the continuum side was somewhat ad hoc, in the sense that it required assuming a certain structure of the boundary state, and within that structure requirement of BRST invariance fixed the time dependence of certain terms of the boundary state. A general procedure for constructing the conserved charges for a general boundary state was not given. Also the conserved charges constructed this way did not get related to any specific symmetry of the theory.

The main goal of this paper will be to develop a systematic method for computing the charge carried by a D-brane associated with a given a global symmetry transformation in any string theory, and then apply this to two dimensional string theory. In particular, we derive a general expression for the conserved charges carried by any D-brane system in terms of its boundary state. Comparison of the charges carried by a D0-brane in the matrix model and the continuum description of the two dimensional string theory then allows us to find a precise relation between the conserved charges in the two descriptions. Using these relations we can put constraints on what kind of D-brane describes the hole states of the matrix model, and also what kind of matrix model configuration describes the black hole states of the continuum string theory.

The paper is organized as follows. In section 2 we review how a rigid closed string gauge transformation, for which the field independent part of the transformation vanishes, generates global symmetries of the open string field theory[45]. Associated with this global symmetry we can associate a conserved charge. We develop an algorithm for computing this conserved charge for any D-brane system in terms of the boundary state describing the D-brane. The final formula for the conserved charge is given in eq.(2.26). The simplest global symmetry of this kind is time translation, and the associated conserved charge gives the energy of the D-brane.

In section 3 we review the structure of rigid closed string gauge transformations in the two dimensional string theory. These are obtained from the elements of relative BRST cohomology in the ghost number one sector, and have been classified in refs.[46, 36, 47, 48, 42]. There are infinite number of such rigid gauge transformations, labelled by SU(2) quantum numbers (j, m) with $j \geq 1$, $m = -(j-1), -(j-2), \ldots (j-1)$. Thus associated with these transformations there are infinite number of conserved charges $Q_{j,m}$. Using the result of section 2 we explicitly write down in eqs.(3.31), (3.32) the expression for $Q_{j,m}$ carried by a D-brane of two dimensional string theory in terms of the boundary state of the D-brane.

In section 4 we evaluate these conserved charges for a specific D-brane system, – namely the rolling tachyon configuration on the unstable D0-brane of two dimensional string theory. These configurations are labelled by a single parameter λ, and we find in eq.(4.38) explicit expression for $Q_{j,m}$ as a function of the parameter λ.

In [44] we had calculated the discrete state closed string background produced by the rolling tachyon configuration in the weak coupling region of large negative φ. We recall these results in section 5 (eq.(5.7)), and point out that there are some additional contributions to this closed string background due to some subtleties which were overlooked in [44]. Eq.(5.14) gives a typical example of such additional contributions.

The matrix model description of the theory also contains an infinite set of conserved charges $W_{l,m}$ with $2m \in Z$, $(l-m) \in Z$, $l \geq |m|$. In section 6 we compare the conserved charges of the matrix model description with the conserved charges of the continuum string theory. By evaluating the conserved charges $W_{l,m}$ of single fermion excitations in the matrix model, and comparing the λ and time dependence of these charges with those of $Q_{j,m}$ carried by a single D0-brane in the continuum description, we determine in eq.(6.6) the relation between the conserved charges of the matrix model and those in the continuum description of the theory. Although these relations are derived by evaluating the charges in the two descriptions of the D0-brane, once the relationship is determined, it must hold for any other system.

In section 7 we try to find a continuum description of the hole states of the matrix model with the help of the conserved charges of the theory. From the description of the conserved charges in the fermionic formulation of the matrix model it is easy to evaluate these conserved charges for a given hole state. These are given in eq.(7.1). Using the known relation between the conserved charges in the matrix model and those ($Q_{j,m}$'s) in the continuum theory, we calculate in eq.(7.2) the expected $Q_{j,m}$ for the hole states. Whatever be the continuum description of the hole must carry the same charges. On the other hand our analysis of section 3 gives an explicit expression for the $Q_{j,m}$'s in terms of the boundary state of the D-brane. This gives constraints on the boundary state describing a hole state, but does not determine it completely. The proposals made in [5, 29] satisfy these constraints; however we argue that they do not reproduce the expected closed string profile of a hole state. We propose a possible candidate for these hols states based on some qualitative arguments, but there is no definite conclusion yet.

Earlier work on the relation between symmetries of the matrix model and those in the continuum description was carried out in the limit of zero potential for the Liouville field, and was based on the comparison of the symmetry algebras in the two descriptions. In order to compare our results to the earlier results, we study in section 8 the limit of our results as we take the potential for the Liouville field to zero. We show first of all that this limit exists and gives a well defined relation (8.11) between the conserved charges in the continuum theory and those in the matrix model. Furthermore, up to normalization factors these relations are consistent with the earlier results obtained by working directly with the Liouville theory without any potential term. We also study the limit of the discrete state closed string background produced by the D0-brane boundary state as we take the Liouville potential to zero keeping the energy of the D0-brane fixed. We find that the closed string field configuration approaches a finite value (8.14) in this limit.

The analysis of section 3 expresses the conserved charges carried by a D-brane in terms of its boundary state. This relation is specific to D-branes since only D-branes are described by boundary states. In section 9 we manipulate the results of section 3 to rewrite the conserved charge in terms of the asymptotic values of certain components of closed string fields produced by the brane for large negative φ. The final formula, given in (9.10), is the analog of the Gauss law for electrodynamics relating the electric charge to asymptotic electric field, or the ADM formula in gravity relating the mass of a system to the asymptotic gravitational field. These relations between conserved charges and asymptotic closed string fields are expected to hold for any system in two dimensional string theory, even if the system cannot be regarded as a collection of D-branes.

In section 10 we use this result to determine the conserved charges carried by the black hole solution of the two dimensional string theory. For simplicity the analysis of this section is carried out for vanishing potential for the Liouville field, which in the matrix model description corresponds to the fermi level coinciding with the maximum of the potential. Although the black hole was initially constructed as a solution of the effective field theory[30] or as an exact conformal field theory[31], it is possible to represent it as a solution in string field theory by using an iterative procedure for solving the equations of motion of string field theory[49]. Using this we can find the asymptotic closed string field configuration associated with the black hole and hence the charges $Q_{j,m}$ carried by the black hole. We find that the black hole carries only the conserved charge $Q_{1,0}$; all other charges $Q_{j,m}$ for $(j, m) \neq (1, 0)$ vanish. Using the known relation (6.6) between $Q_{j,m}$ and the conserved charges in the matrix model description of the system we can then constrain the possible configurations in the matrix model which are compatible with these conserved charges. In particular we find that the matrix model description of the black hole must consist of a large number of low energy fermion-hole pairs instead of a finite number of finite energy fermions and holes.

This description of the black hole poses an apparent puzzle. Since in the matrix model description the fermions are non-interacting, and since the black hole background differs from the usual vacuum in terms of creation of a large number of low energy fermion-hole pairs, a classical D0-brane carrying finite energy should not be able to recognize the difference between a black hole and the ordinary vacuum. Can this be true in two dimensional string theory? While a complete answer to this question requires studying the D0-brane motion in these backgrounds to all orders in α', we show that at least in the approximation where we take the D0-brane world-line action to be of the Dirac-Born-Infeld form, the classical D0-brane cannot distinguish the black hole from the usual linear dilaton background. This is shown by demonstrating that there is a coordinate transformation that converts the effective metric seen by the D0-brane in the black hole background to the effective metric seen by the D0-brane to the linear dilaton background. This coordinate transformation acts only on the space coordinate and leaves the time coordinate unchanged. This is consistent with the fact that both for the black hole and the usual flat background with a linear dilaton field, the time coordinate should be identified with the time coordinate of the matrix model.

Although the main emphasis of the paper has been on the construction of the conserved charges and their interpretation in the two dimensional string theory, we can try to generalize the construction to critical string theory by replacing the primary vertex operators in the Liouville field theory by appropriate primary vertex operators in the critical string theory, and by replacing the Liouville Virasoro generators by the total Virasoro generators associated with all the 25 space-like coordinate fields in the critical string theory. There are however various subtle issues in this approach. There are discussed in section 11.

Finally the appendices contain some technical results which are required for the explicit construction of conserved charges and their normalization in two dimensional string theory.

2 Symmetries to Conserved Charges in Open String Theory

In this section we briefly outline the general procedure for obtaining the conserved charge in classical open string theory associated with a specific global symmetry. We shall focus on the global symmetries associated with rigid gauge transformations in closed string theory[45]. An example of this is space-time translation symmetry, which can be thought of as a rigid general coordinate transformation.

We shall carry out the discussion in the context of string field theory. We begin with some version of covariant open-closed string field theory[45] formulated for a given D-brane in a given space-time background. However our analysis will be quite general and we shall not restrict ourselves to any specific form of the action. Let us denote by $\{\Phi_\alpha\}$ the closed string degrees of freedom and by $\{\Psi_r\}$ the open string degrees of freedom, with the indices α and r, besides containing discrete labels, also carrying information about momenta of the fields along non-compact space-time directions. Then the open-closed string field theory action has the form[45]:

$$\frac{1}{g_s^2} S_0(\Phi) + \frac{1}{g_s} S_1(\Phi, \Psi) + \mathcal{O}(g_s^0)\,, \tag{2.1}$$

where g_s denotes string coupling constant. The order g_s^{-2} and g_s^{-1} terms get contributions respectively from the sphere and disk correlation functions of the world-sheet theory. Let D denote the dimension of space-time in which the closed string theory lives. Then a typical closed string gauge transformation is parametrized by some arbitrary function $\epsilon(p)$ of D dimensional momentum p. The infinitesimal gauge transformation laws take the form:

$$\begin{aligned} \delta\Phi_\alpha &= \sum_{n=0}^\infty g_s^n\, \delta\Phi_\alpha^{(n)} = \int d^D p\ \epsilon(p)\ \Big[h_\alpha^{(0)}(\Phi, p) + g_s h_\alpha^{(1)}(\Phi, \Psi, p) + \mathcal{O}(g_s^2)\Big] \\ \delta\Psi_r &= \sum_{n=0}^\infty g_s^n\, \delta\Psi_r^{(n)} = \int d^D p\ \epsilon(p)\ \Big[f_r^{(0)}(\Phi, \Psi, p) + \mathcal{O}(g_s)\Big]\,, \end{aligned} \tag{2.2}$$

for suitable function $h_\alpha^{(n)}$ and $f_r^{(n)}$. The contributions to $h_\alpha^{(0)}$ come from sphere correlation functions, whereas the contributions to $h_\alpha^{(1)}$ and $f_r^{(0)}$ come from disk correlation functions. Note that the leading contribution to the action and the leading contribution to $\delta\Phi_\alpha$ do not depend on the open string fields Ψ_r. We call this action and gauge transformation laws tree level closed string action and gauge transformation laws respectively. On the other hand

$$S_{open}(\Psi) \equiv \frac{1}{g_s} S_1(\Phi = 0, \Psi) \tag{2.3}$$

is called the tree level open string field theory action in the $\Phi = 0$ closed string background.[1] Invariance of the full action (2.1) under the gauge transformation laws (2.2) gives:

$$\frac{\delta S_0(\Phi)}{\delta \Phi_\alpha} \delta \Phi_\alpha^{(0)} = 0\,, \tag{2.4}$$

$$\frac{\delta S_1(\Phi,\Psi)}{\delta \Phi_\alpha} \delta \Phi_\alpha^{(0)} + \frac{\delta S_1(\Phi,\Psi)}{\delta \Psi_r} \delta \Psi_r^{(0)} + \frac{\delta S_0(\Phi)}{\delta \Phi_\alpha} \delta \Phi_\alpha^{(1)} = 0 \tag{2.5}$$

etc. We shall choose the string field variables such that $\Phi = 0$ is trivially a solution of the classical closed string field equations. Thus $\delta S_0/\delta\Phi_\alpha$ vanishes at $\Phi = 0$. Putting $\Phi = 0$ in eq.(2.5) and using (2.2), (2.3) we get

$$\left[\frac{\delta S_1(\Phi,\Psi)}{\delta \Phi_\alpha}\bigg|_{\Phi=0} h_\alpha^{(0)}(\Phi = 0, p) + \frac{\delta S_{open}(\Psi)}{\delta \Psi_r} f_r^{(0)}(\Phi = 0, \Psi, p) \right] = 0\,. \tag{2.6}$$

In general $h_\alpha^{(0)}(\Phi = 0, p)$ is non-zero. However suppose for some special value of the momentum p it vanishes:

$$h_\alpha^{(0)}(\Phi = 0, p = c) = 0\,. \tag{2.7}$$

Physically it means that the field independent term in the tree level closed string gauge transformation law vanishes. In other words we have a rigid gauge transformation that leaves the $\Phi = 0$ background unchanged. Putting $p = c$ in (2.6) we now get

$$\frac{\delta S_{open}(\Psi)}{\delta \Psi_r} f_r^{(0)}(\Phi = 0, \Psi, p = c) = 0\,. \tag{2.8}$$

This describes a global symmetry of the tree level open string field theory with infinitesimal transformation law

$$\delta \Psi_r = \epsilon\, f_r^{(0)}(\Phi = 0, \Psi, p = c)\,. \tag{2.9}$$

Thus we see that associated with every rigid closed string gauge transformation that leaves the $\Phi = 0$ background unchanged, we have a global symmetry of the classical open string field theory[45].

Given a global symmetry, there should be a conserved charge associated with this symmetry. One possible way to find this charge will be to use Noether method. The symmetry transformations are non-local, but a general Noether method for finding the conserved charge associated with a non-local symmetry transformation in a non-local theory was outlined in [51]. We can follow the same method for finding the expression for the conserved charge associated with these global symmetries of open string theory.

[1] A family of open closed string field theory was constructed in [45], and Witten's open string field theory[50] appears as the open string sector of a special member of this family.

However this procedure only gives the difference between the conserved charge carried by a given open string field configuration, and that carried by the $\Psi = 0$ configuration representing the original D-brane on which we have formulated the open string field theory. In particular if we set the open string field Ψ to zero, then the expression for the conserved charge vanishes. Our main interest on the other hand will be in the expression for the conserved charge that the original D-brane carries. For this we need to use a different method which we shall describe now.

The basic procedure can be understood in analogy with the computation of the energy momentum tensor of a D-brane. Defining the energy-momentum tensor in the open string field theory through the Noether prescription gives correctly the difference in the energy-momentum tensor between two open string configurations[51], but this does not give the energy-momentum tensor of the D-brane itself. The latter can be calculated by examining the coupling of the metric to the D-brane world-volume. In a similar spirit one would expect that the information about all other conserved charges carried by the D-brane, which are associated with rigid gauge transformations that leave the closed string background $\Phi = 0$ invariant, should also be calculable by examining the coupling of the various closed string modes to the D-brane. In fact the relevant information is already contained in eq.(2.6). Using eq.(2.7) and assuming that $h^{(0)}_\alpha(\Phi = 0, p)$ is analytic at $p = c$, we can write

$$h^{(0)}_\alpha(\Phi = 0, p) = (p_\mu - c_\mu)\, \widehat{h}^\mu_\alpha(p)\,, \tag{2.10}$$

for some $\widehat{h}^\mu_\alpha$. If we define

$$\mathcal{G}_\alpha(\Psi) = \frac{\delta S_1(\Phi, \Psi)}{\delta \Phi_\alpha}\bigg|_{\Phi=0}\,, \tag{2.11}$$

then eq.(2.6) can be rewritten as

$$(p_\mu - c_\mu)\ \mathcal{G}_\alpha(\Psi)\ \widehat{h}^\mu_\alpha(p) + \frac{\delta S_{open}(\Psi)}{\delta \Psi_r}\ f^{(0)}_r(\Phi = 0, \Psi, p) = 0\,. \tag{2.12}$$

Now if the fields Ψ_s satisfy their equations of motion then $\delta S_{open}(\Psi)/\delta\Psi_r = 0$. In this case we have

$$(p_\mu - c_\mu)\, \mathcal{G}_\alpha(\Psi)\, \widehat{h}^\mu_\alpha(p) = 0\,. \tag{2.13}$$

If we define

$$F^\mu(x) = \int d^D p\, e^{-ip.x}\, \mathcal{G}_\alpha(\Psi) \widehat{h}^\mu_\alpha(p) \tag{2.14}$$

then (2.13) may be rewritten as

$$\partial_\mu \left(e^{ic.x} F^\mu(x) \right) = 0\,. \tag{2.15}$$

This gives the conserved charge

$$\int d^{D-1}x \, e^{ic.x} F^0(x) \, . \tag{2.16}$$

We shall now make this construction more explicit by working with specific representation of closed string fields. We shall restrict our analysis to a closed string background in which the time direction has associated with it a world-sheet conformal field theory (CFT) of a free scalar field X^0 which does not couple to any other world-sheet field. In this case we can work with the Euclidean continuation of the theory obtained by the replacement $x^0 \to -ix$. We can represent the closed string field by a state $|\Phi\rangle$ of ghost number two in the bulk CFT on a cylinder, satisfying

$$(b_0 - \bar{b}_0)|\Phi\rangle = 0, \qquad (L_0 - \bar{L}_0)|\Phi\rangle = 0 \, , \tag{2.17}$$

where b_n, $\bar{b}_n$, c_n, $\bar{c}_n$ denote the usual ghost oscillators, and L_n, $\bar{L}_n$ denote the total Virasoro generators of the world-sheet theory of matter and ghost fields. Closed string gauge transformations in this theory are generated by ghost number one states $|\Lambda\rangle$ of the CFT on a cylinder, satisfying

$$(b_0 - \bar{b}_0)|\Lambda\rangle = 0, \qquad (L_0 - \bar{L}_0)|\Lambda\rangle = 0 \, . \tag{2.18}$$

The effect of the infinitesimal gauge transformations on the closed string fields is given by:

$$\delta|\Phi\rangle = (Q_B + \bar{Q}_B)|\Lambda\rangle + \mathcal{O}(\Phi) \, , \tag{2.19}$$

where Q_B and $\bar{Q}_B$ are the holomorphic and antiholomorphic components of the BRST charge in closed string theory. Thus for a $|\Lambda\rangle$ satisfying

$$(Q_B + \bar{Q}_B)|\Lambda\rangle = 0 \, , \tag{2.20}$$

the infinitesimal gauge transformation of $|\Phi\rangle$ vanishes at $|\Phi\rangle = 0$. As a result this gauge transformation leaves the $|\Phi\rangle = 0$ background unchanged. By our previous argument, this must generate a symmetry of the pure open string field theory living on a D-brane, and give rise to a conserved charge in this theory.[2]

Our goal will be to construct expressions for these conserved charges explicitly for any D-brane system living in this closed string background. For simplicity we shall evaluate

[2]Of course we can trivially find solutions to (2.20) by choosing $|\Lambda\rangle = (Q_B + \bar{Q}_B)|s\rangle$ for some ghost number zero state $|s\rangle$ satisfying conditions similar to those in (2.17), (2.18). But one can show that these generate gauge symmetries in the open string field theory, and do not give rise to any additional global symmetry[45]. We shall see later that in this case the associated conserved charge vanishes identically.

the charge in trivial open string background $\Psi = 0$, – this will evaluate the charge carried by the specific D-brane used in the construction of the open closed string field theory without any further open string excitations on the brane. Let $|\Lambda(p)\rangle$ denote a family of closed string gauge transformation parameters labelled by X momentum p, such that $|\Lambda(p = c)\rangle = |\Lambda\rangle$ for some special momentum c.[3] Then $(Q_B + \bar{Q}_B)|\Lambda(p)\rangle$ vanishes at $p = c$ and we can write

$$(Q_B + \bar{Q}_B)|\Lambda(p)\rangle = (p - c)|\phi(p)\rangle\,, \tag{2.21}$$

where $|\phi(p)\rangle$ is some ghost number two state. Now since $(Q_B + \bar{Q}_B)$ is nilpotent, we see from eq.(2.21) that $|\phi(p)\rangle$ is BRST invariant for all $p \neq c$, and hence by analytic continuation BRST invariant also for $p = c$. Furthermore it has the property that for any $p \neq c$ it is BRST trivial, but for $p = c$ it could be a non-trivial element of the BRST cohomology in the ghost number two sector. We shall see later that we can get non-trivial conserved charges only if $|\phi(p = c)\rangle$ is not BRST trivial.

Let us denote by $|\mathcal{B}\rangle$ the boundary state associated with the D-brane on which we have formulated the open string theory. Then the full string field theory action contains a coupling:

$$\langle\mathcal{B}|(c_0 - \bar{c}_0)|\Phi\rangle\,. \tag{2.22}$$

Invariance of this term under the infinitesimal gauge transformation (2.19) generated by the family of gauge transformation parameters $|\Lambda(p)\rangle$ requires:

$$\langle\mathcal{B}|(c_0 - \bar{c}_0)\,(Q_B + \bar{Q}_B)|\Lambda(p)\rangle = 0\,, \tag{2.23}$$

For ordinary D-branes eq.(2.23) follows from the BRST invariance of $\langle\mathcal{B}|$ and the analogs of (2.17), (2.18):

$$\langle\mathcal{B}|(Q_B + \bar{Q}_B) = 0, \quad \langle\mathcal{B}|(b_0 - \bar{b}_0) = 0, \quad \langle\mathcal{B}|(L_0 - \bar{L}_0) = 0\,. \tag{2.24}$$

This allows us to replace $(c_0 - \bar{c}_0)(Q_B + \bar{Q}_B)$ by $\{(c_0 - \bar{c}_0), (Q_B + \bar{Q}_B)\}$ in (2.23). This does not have any zero mode of $(c_0 - \bar{c}_0)$ and hence the matrix element vanishes.

Using (2.21), eq.(2.23) becomes:

$$(p - c)\langle\mathcal{B}|(c_0 - \bar{c}_0)|\phi(p)\rangle = 0\,. \tag{2.25}$$

If we define:

$$F(x) = \int \frac{dp}{2\pi}\, e^{-ipx}\langle\mathcal{B}|(c_0 - \bar{c}_0)|\phi(p)\rangle\,, \tag{2.26}$$

[3]We are assuming that the other momentum components have already been set equal to the specific values for which $(Q_B + \bar{Q}_B)|\Lambda(p)\rangle$ vanishes.

then (2.25) may be rewritten as

$$\partial_x \left(e^{icx} F(x) \right) = 0 \,. \tag{2.27}$$

Replacing x by ix^0 we now get:

$$\partial_0 \left(e^{-cx^0} F(ix^0) \right) = 0 \,. \tag{2.28}$$

Thus $e^{-cx^0} F(ix^0)$ is a conserved charge. This gives a general procedure for constructing the conserved charge carried by a D-brane corresponding to a specific rigid gauge transformation in closed string theory. The suggestion that the BRST invariance of $\langle \mathcal{B}|$ carries information about conserved charges has been made earlier in [52].

Given an element $|\Lambda\rangle$ of the BRST cohomology with a specific momentum c, there are clearly infinite number of families $|\Lambda(p)\rangle$ with the property that $|\Lambda(p = c)\rangle = |\Lambda\rangle$. On physical grounds the conserved charge associated with the symmetry generated by $|\Lambda\rangle$ should not depend on the choice of the family. We shall now prove this explicitly by demonstrating that if two families of gauge transformations parameters $|\Lambda^{(1)}(p)\rangle$ and $|\Lambda^{(2)}(p)\rangle$ approach the same value at $p = c$, then they give rise to the same conserved charge. In this case, we may write

$$|\Lambda^{(1)}(p)\rangle - |\Lambda^{(2)}(p)\rangle = (p - c)|\Lambda^{(0)}(p)\rangle \,, \tag{2.29}$$

for some $|\Lambda^{(0)}(p)\rangle$, so that the difference between $|\Lambda^{(1)}(p)\rangle$ and $|\Lambda^{(2)}(p)\rangle$ vanishes at $p = c$. Eqs.(2.21) and (2.29) now give:

$$|\phi^{(1)}(p)\rangle - |\phi^{(2)}(p)\rangle = (Q_B + \bar{Q}_B)|\Lambda^{(0)}(p)\rangle \,, \tag{2.30}$$

where $|\phi^{(i)}(p)\rangle$ is related to $|\Lambda^{(i)}(p)\rangle$ as in eq.(2.21). If $F^{(1)}(x)$ and $F^{(2)}(x)$ denote the corresponding conserved charges as defined in (2.26), then we have

$$F^{(1)}(x) - F^{(2)}(x) = \int \frac{dp}{2\pi} \, e^{-ipx} \langle \mathcal{B}|(c_0 - \bar{c}_0)(Q_B + \bar{Q}_B)|\Lambda^{(0)}(p)\rangle \,. \tag{2.31}$$

Since $\langle \mathcal{B}|$ is annihilated by $(Q_B + \bar{Q}_B)$, we can replace the $(c_0 - \bar{c_0})(Q_B + \bar{Q}_B)$ term by $\{(c_0 - \bar{c_0}), (Q_B + \bar{Q}_B)\}$. This in turn does not contain any zero mode $(c_0 - \bar{c}_0)$. Since both $\langle \mathcal{B}|$ and $|\Lambda^{(0)}(p)\rangle$ are annihilated by $(b_0 - \bar{b}_0)$, we see that the right hand side of (2.31) vanishes due to the absence of the zero mode $(c_0 - \bar{c}_0)$ in the matrix element. This in turn establishes that the conserved charges $F^{(1)}$ and $F^{(2)}$ are identical.

It also follows from the above analysis that if $|\Lambda\rangle$ is BRST trivial then the corresponding conserved charge vanishes. To see this let us assume that $|\Lambda\rangle = (Q_B + \bar{Q}_B)|\chi\rangle$ for some ghost number zero state $|\chi\rangle$. Let $|\chi(p)\rangle$ denote a family of states labelled by the momentum p such that $|\chi(p=c)\rangle = |\chi\rangle$. Then the family $|\widetilde{\Lambda}(p)\rangle \equiv (Q_B+\bar{Q}_B)|\chi(p)\rangle$ has the property that it reduces to $|\Lambda\rangle$ for $p=c$. Thus we can compute the conserved charge associated with this symmetry using this family $|\widetilde{\Lambda}(p)\rangle$. However in this case $(Q_B+\bar{Q}_B)|\widetilde{\Lambda}(p)\rangle$ vanishes for all p, and hence the corresponding state $|\widetilde{\phi}(p)\rangle = (p-m)^{-1}(Q_B+\bar{Q}_B)|\widetilde{\Lambda}(p)\rangle$ also vanishes for all p. Using the definition (2.26) of the conserved charge we see clearly that the corresponding conserved charge also vanishes in this case.

Finally we note from the definition (2.26) of $F(x)$ and the fact that $F(x) \propto e^{-icx}$ due to the conservation law, that the value of F depends on the matrix element $\langle\mathcal{B}|(c_0-\bar{c}_0)|\phi(p)\rangle$ at $p=c$. If $|\phi(p=c)\rangle$ is BRST exact then this matrix element vanishes and we do not get a non-trivial conserved charge.

3 Symmetries and Conserved Charges in Two Dimensional String Theory

In this section we shall use the results of section 2 to construct infinite number of conserved charges in two dimensional bosonic string theory. We begin with a brief review of two dimensional string theory. The world-sheet description of the theory involves a time like scalar field X^0, a Liouville field theory with $c=25$ and the usual ghost fields b, c, $\bar{b}$, $\bar{c}$. In the $\alpha'=1$ unit that we shall be using, the Liouville theory is described by a single scalar field φ with exponential potential in the world sheet action:[4]

$$s_{liouville} = \int d^2z \left(\frac{1}{2\pi}\,\partial_z\varphi\partial_{\bar{z}}\varphi + \mu\, e^{2\varphi}\right), \tag{3.1}$$

where μ is a parameter which we shall set to unity by a shift of the field φ. There is also a linear dilaton field background along the Liouville direction

$$\Phi_D = 2\varphi\,, \tag{3.2}$$

which makes the total central charge associated with the Liouville direction to be 25. For large negative φ, the potential term $e^{2\varphi}$ is negligible, and φ behaves as a free scalar field with background charge. This is also the region where the effective string coupling constant $e^{\Phi_D} = e^{2\varphi}$ is small. We shall call this region the weak coupling region.

In order to understand the structure of BRST cohomology in the two dimensional string theory, it will be convenient to first regard the left and right moving components

[4]The Liouvile theory with $c=25$ actually has a term $\propto \varphi e^{2\varphi}$ in the world-sheet action[53, 54]. As in [1, 2, 3] we shall regard the $c=25$ Liouville theory as the $c\to 25$ limit of theories with $c>25$. For $c>25$, (3.1) (with $e^{2\varphi}$ replaced by an appropriate power of e^{φ}) is the correct form of the action, but μ undergoes an infinite renormalization as we take the $c\to 25$ limit.

of the world-sheet scalar field $X = iX^0$ and the Liouville field φ as independent, construct states in the left- and the right-moving sectors separately, and then combine them matching the momenta in the left- and the right-moving sector to construct proper states of the two dimensional string theory. We begin with the CFT associated with the free scalar field X. Let us denote by X_L and X_R the left and the right-moving components of X. The CFT, besides containing the usual primary states $e^{ikX_L(0)}|0\rangle_X$ and $e^{ikX_R(0)}|0\rangle_X$, contains a set of primaries $|j,m\rangle_L$, $|j,m\rangle_R$ of the form[55]:

$$|j,m\rangle_L = \mathcal{P}^L_{j,m} e^{2imX_L(0)}|0\rangle_X, \qquad |j,m\rangle_R = \mathcal{P}^R_{j,m} e^{2imX_R(0)}|0\rangle_X, \tag{3.3}$$

where $\mathcal{P}^L_{j,m}$ and $\mathcal{P}^R_{j,m}$ are some combination of non-zero mode X_L, X_R oscillators of level $(j^2 - m^2)$, and (j,m) are SU(2) quantum numbers with $-j \le m \le j$.[5] For example, we have $|1,0\rangle_L = \alpha_{-1}|0\rangle_X$, $|1,0\rangle_R = \bar\alpha_{-1}|0\rangle_X$, where α_n, $\bar\alpha_n$ are the usual oscillators of the X-field. $\mathcal{P}^L_{j,\pm j}$ and $\mathcal{P}^R_{j,\pm j}$, being of level 0, must be identity operators. Thus $|j,j\rangle_L = e^{2ijX_L(0)}|0\rangle$, $|j,j\rangle_R = e^{2ijX_R(0)}|0\rangle$.

We shall combine the left and the right-moving modes to define:

$$|j,m\rangle_X = |j,m\rangle_L \times |j,m\rangle_R = \mathcal{P}^L_{j,m}\mathcal{P}^R_{j,m}\, e^{2imX(0)}|0\rangle_X\,. \tag{3.4}$$

In fact this theory contains a more general set of primaries $|j,m\rangle_L \times |j',m\rangle_R$, but we shall not introduce a special symbol to label these states. For later use we shall also define:

$$|j,m,p\rangle_L = \mathcal{P}^L_{j,m}\, e^{ipX_L(0)}|0\rangle_X, \qquad |j,m,p\rangle_R = \mathcal{P}^R_{j,m}\, e^{ipX_R(0)}|0\rangle_X\,, \tag{3.5}$$

for an arbitrary X-momentum p, and

$$|j,m,p\rangle_X = |j,m,p\rangle_L \times |j,m,p\rangle_R = \mathcal{P}^L_{j,m}\mathcal{P}^R_{j,m}\,|p\rangle_X\,, \tag{3.6}$$

where

$$|p\rangle_X = e^{ipX(0)}|0\rangle_X\,. \tag{3.7}$$

We shall normalize $\mathcal{P}^L_{j,m}$, $\mathcal{P}^R_{j,m}$ such that:

$${}_X\langle j,m,p|j',m,p'\rangle_X = {}_X\langle p|p'\rangle_X\,\delta_{jj'} = 2\pi\,\delta(p+p')\,\delta_{jj'}, \tag{3.8}$$

where ${}_X\langle\cdot|\cdot\rangle_X$ denotes BPZ inner product in the CFT of the X-field. The vanishing of this inner product for $j \ne j'$ follows simply from the fact that the two states have different conformal weights.

[5]The underlying SU(2) algebras in the left and right-moving sectors of the world-sheet are generated by $e^{\pm 2iX_L}$, $i\partial X_L$, $e^{\pm 2iX_R}$ and $i\bar\partial X_R$. These generators do not have well defined action on a vertex operator e^{ikX} for non-integer k, and hence do not generate a symmetry of the theory unless X is compactified on a circle of self-dual radius. Nevertheless this algebra is useful in analyzing the properties of the special primary states $|j,m\rangle_X$ even when X represents a non-compact coordinate.

The Liouville field theory contains a set of primary vertex operators V_β of conformal weight (h_β, h_β) with

$$h_\beta = \frac{1}{4}\beta(4-\beta)\,. \tag{3.9}$$

For large negative φ where φ behaves as a free scalar field with background charge, $V_\beta \sim e^{\beta\varphi}$ for $\beta < 2$. Although the world-sheet action is not that of a free field theory, it describes a solvable CFT[56, 57, 58, 59, 60]. There is a reflection symmetry[58, 59] that relates the vertex operator V_β to $V_{4-\beta}$ up to a constant of proportionality. Of these only the vertex operators V_{2+iP} for real P describe δ-function normalizable states. The normalization of these vertex operators will be chosen such that

$$\langle V_{2+iP}(1) V_{2+iP'}(0)\rangle_{liouville} = 2\pi\left[\delta(P+P') - \left(\frac{\Gamma(iP)}{\Gamma(-iP)}\right)^2 \delta(P-P')\right]. \tag{3.10}$$

The second term in this expression is required by the reflection symmetry[58, 59] $V_{2+iP} \equiv -\left(\frac{\Gamma(-iP)}{\Gamma(iP)}\right)^2 V_{2-iP}$. With this normalization

$$V_{2+iP} \;\sim\; e^{(2+iP)\varphi} - \left(\frac{\Gamma(iP)}{\Gamma(-iP)}\right)^2 e^{(2-iP)\varphi}\,, \tag{3.11}$$

for large negative φ. The second term reflects the effect of the exponentially growing potential for large positive φ.

This normalization differs from the implicit normalization assumed in [44] where the second delta function in (3.10) was absent, and we pretended that V_{2+iP} and V_{2-iP} are independent vertex operators. As long as we work with appropriate linear combinations of V_{2+iP} and V_{2-iP} which obey the reflection symmetry, this procedure gives the correct result. The closest analogy of this in ordinary quantum mechanics is that while studying a free particle on a half line with Neumann (Dirichlet) boundary condition on the wave-function at the origin, we can study the theory on the full line with basis states e^{ikx} and at the end restrict the field configurations to be even (odd) under reflection around the origin. In contrast in this paper we use the convention that V_{2+iP} itself obeys the reflection symmetry dictated by the CFT. This is analogous to using $2\cos x$ $(2\sin x)$ as basis functions for free particle on a half line with Neumann (Dirichlet) boundary condition on the wave-function at the origin. Any basis independent relation *e.g.* eq.(5.7) will not be affected by this difference in the choice of basis.

For simplifying the notation we shall regard the vertex operator V_β as a product of a left-chiral vertex operator V^L_β and a right-chiral vertex operator V^R_β of dimension $(h_\beta, 0)$ and $(0, h_\beta)$ respectively, although in the final expression only the product $V^L_\beta V^R_\beta$ will appear.

We shall now review some results on the chiral BRST cohomology of the world-sheet theory[46, 47, 48, 36, 42, 5, 61, 62] in ghost number zero and one sectors.[6] As we shall see, these will be the basic building blocks for the construction of non-trivial symmetry generators of the two dimensional string theory under which D-branes are charged. For definiteness we shall describe the results in the left-moving (holomorphic) sector, but identical results hold in the right-moving sector as well. We begin in the ghost number one sector. In this sector we have an infinite number of elements of the BRST cohomology labelled by the SU(2) quantum numbers (j,m) with $-j \le m \le j$, represented by the states

$$\begin{aligned} |Y^L_{j,m}\rangle &= |j,m\rangle_L \otimes V^L_{2(1-j)}(0)|0\rangle_{liouville} \otimes c_1|0\rangle_{ghost} \\ &= \mathcal{P}^L_{j,m} e^{2imX_L(0)}|0\rangle_X \otimes V^L_{2(1-j)}(0)|0\rangle_{liouville} \otimes c_1|0\rangle_{ghost}\,, \end{aligned} \tag{3.12}$$

where $\mathcal{P}^L_{j,m}$ has been defined in (3.3). By construction these states have zero L_0 eigenvalue. For later use we define:

$$|Y^L_{j,m}(p)\rangle = \mathcal{P}^L_{j,m} e^{ipX_L(0)}|0\rangle_X \otimes V^L_{2(1-j)}(0)|0\rangle_{liouville} \otimes c_1|0\rangle_{ghost}\,. \tag{3.13}$$

In the ghost number 0 sector also we have an infinite number of elements of the BRST cohomology labelled by the SU(2) quantum numbers $(j-1,m)$ with $-(j-1) \le m \le j-1$. The representative elements of the BRST cohomology can be chosen to be of the form

$$|\mathcal{O}^L_{j-1,m}\rangle = \mathcal{Q}^L_{j-1,m}|j-1,m\rangle_L \otimes V^L_{2(1-j)}(0)|0\rangle_{liouville} \otimes c_1|0\rangle_{ghost}\,, \tag{3.14}$$

where $\mathcal{Q}^L_{j-1,m}$ is an operator of ghost number -1, level $(2j-1)$ constructed from negative moded ghost oscillators and X and Liouville Virasoro generators[61]. Using (3.3) this can be rewritten as

$$|\mathcal{O}^L_{j-1,m}\rangle = \mathcal{R}^L_{j-1,m} e^{2imX_L(0)}|0\rangle_X \otimes V^L_{2(1-j)}(0)|0\rangle_{liouville} \otimes c_1|0\rangle_{ghost} \tag{3.15}$$

[6]Much of the earlier analysis on the BRST cohomology in two dimensional string theory was done for $\mu = 0$. We shall not try to carry out a complete analysis of the BRST cohomology for $\mu \neq 0$. Instead we shall use the elements of BRST cohomology found in earlier analysis which do not require explicit use of the oscillators of φ, but involve only those states in the Liouville theory which are obtained by the action of the Liouville Virasoro generators on primary operators of the form V_α. These states can be easily defined in the interacting theory. Thus for example states of the form $(c\partial\varphi(0) + \partial c(0))|0\rangle$, which are valid elements of the BRST cohomology for $\mu = 0$, will not be included in our analysis since $\partial\varphi(0)|0\rangle_{liouville}$ cannot be expressed as a combination of Liouville Virasoro generators acting on a primary state $V_\alpha(0)|0\rangle_{liouville}$. On the other hand a state of the form $c\partial^2 c(0)|0\rangle$, which is BRST invariant but was not included as an element of the relative BRST cohomology since it is proportional to $Q_B c\partial\varphi(0)|0\rangle$, will now be regarded as a valid element of the relative BRST cohomology. The other main difference from the $\mu = 0$ case is that for $\mu = 0$ a state was taken to vanish only if it vanishes when expressed in terms of the Liouville oscillators. Since we are not making use of Liouville oscillators, we shall consider a state to vanish if it is a linear combination of null states of the Liouville Virasoro algebra. Since the one point function of null states on a disk vanishes for the D0-brane boundary CFT[23], this prescription does not lead to any internal contradiction.

where $\mathcal{R}^L_{j-1,m} \equiv \mathcal{Q}^L_{j-1,m}\,\mathcal{P}^L_{j-1,m}$ is an operator of ghost number -1 constructed from negative moded X and ghost oscillators and Liouville Virasoro generators.[7] For later use we now define:

$$|\mathcal{O}^L_{j-1,m}(p)\rangle = \mathcal{R}^L_{j-1,m} e^{ipX_L(0)}|0\rangle_X \otimes V^L_{2(1-j)}(0)|0\rangle_{liouville} \otimes c_1|0\rangle_{ghost}\,. \tag{3.16}$$

Note that $|Y^L_{j,m}(p)\rangle$ and $|\mathcal{O}^L_{j-1,m}(p)\rangle$ are both built by the action of X and ghost oscillators and Liouville Virasoro generators on the same Fock vacuum $e^{ipX_L(0)}|0\rangle_X \otimes V^L_{2(1-j)}(0)|0\rangle_{liouville} \otimes c_1|0\rangle_{ghost}$, and satisfy

$$b_0|\mathcal{O}^L_{j-1,m}(p)\rangle = 0, \qquad b_0|Y^L_{j,m}(p)\rangle = 0\,. \tag{3.17}$$

Furthermore, since $|Y^L_{j,m}(p=2m)\rangle$ and $|\mathcal{O}^L_{j-1,m}(p=2m)\rangle$ have zero L_0 eigenvalues, we have

$$L_0|\mathcal{O}^L_{j-1,m}(p)\rangle = \frac{1}{4}\,(p^2-4m^2)\,|\mathcal{O}^L_{j-1,m}(p)\rangle\,, \qquad L_0|Y^L_{j,m}(p)\rangle = \frac{1}{4}\,(p^2-4m^2)\,|Y^L_{j,m}(p)\rangle\,. \tag{3.18}$$

Given that $|\mathcal{O}^L_{j-1,m}\rangle = |\mathcal{O}^L_{j-1,m}(p=2m)\rangle$ and $|Y^L_{j,m}\rangle = |Y^L_{j,m}(p=2m)\rangle$ are BRST invariant, we must have[8]

$$Q_B|\mathcal{O}^L_{j-1,m}(p)\rangle = (p-2m)|\eta^L_{(j),m}(p)\rangle\,, \tag{3.19}$$

and

$$Q_B|Y^L_{j,m}(p)\rangle = (p-2m)|\psi^L_{(j),m}(p)\rangle\,, \tag{3.20}$$

for some states $|\eta^L_{(j),m}(p)\rangle$ and $|\psi^L_{(j),m}(p)\rangle$. It follows from eqs.(3.19) and (3.20) and the nilpotence of Q_B that both $|\eta^L_{(j),m}(p)\rangle$ and $|\psi^L_{(j),m}(p)\rangle$ are BRST invariant for any $p \neq 2m$ and hence by analytic continuation also for $p = 2m$. We also see from eqs.(3.19), (3.20) that for $p \neq 2m$, $|\eta^L_{(j),m}(p)\rangle$ and $|\psi^L_{(j),m}(p)\rangle$ are BRST trivial but for $p = 2m$ they can be BRST non-trivial. Finally we note that since $|\mathcal{O}^L_{j-1,m}(p)\rangle$ and $|Y^L_{j,m}(p)\rangle$ have non-vanishing L_0 eigenvalues proportional to (p^2-4m^2), $|\eta^L_{(j),m}(p)\rangle$ and $|\psi^L_{(j),m}(p)\rangle$ defined through (3.19) and (3.20) are not annihilated by b_0 in general.

It has been shown in appendix A that at $p = 2m$,

$$|\eta^L_{(j),m}\rangle \equiv |\eta^L_{(j),m}(p=2m)\rangle = |Y^L_{j,m}\rangle + |\widehat{\eta}^L_{(j),m}\rangle\,, \tag{3.21}$$

[7]Note that while $\mathcal{Q}^L_{j-1,m}$ is built from X Virasoro generators, $\mathcal{R}^L_{j-1,m}$ is built from X oscillators. The distinction is important. While any combination of X Virasoro generators, acting on a state of definite X-momentum, can be expressed in terms of X-oscillators, the reverse is not always true.

[8]For a chiral state we shall use the convention that at $p = 2m$ a state labelled by the subscript ${}_{j,m}$ carries definite SU(2) quantum numbers (j,m), whereas a state labelled be the subscript ${}_{(j),m}$ does not in general have definite SU(2) quantum numbers. After combining the left- and the right-moving sectors to get full closed string states we shall no longer follow this convention.

$$|\psi^L_{(j),m}\rangle \equiv |\psi^L_{(j),m}(p=2m)\rangle = m\, c_0\, |Y^L_{j,m}\rangle + |\tau^L_{j-1,m}\rangle\,, \tag{3.22}$$

where $|\widehat{\eta}^L_{(j),m}\rangle$ is a linear combination of states carrying SU(2) quantum numbers $(j-1,m)$ and $(j-2,m)$, and $|\tau^L_{j-1,m}\rangle$ has SU(2) quantum numbers $(j-1,m)$.

This finishes our discussion of chiral BRST cohomology in ghost numbers zero and one sectors. We shall now combine the left and the right-moving states matching X and φ momenta to construct a family of states $|\Lambda_{j,m}(p)\rangle$ of ghost number 1, satisfying the requirement (2.21). We define:[9]

$$\begin{aligned}|\Lambda_{j,m}(p)\rangle = \frac{1}{2}\Big[|\mathcal{O}^L_{j-1,m}(p)\rangle \times |Y^R_{j,m}(p)\rangle - |Y^L_{j,m}(p)\rangle \times |\mathcal{O}^R_{j-1,m}(p)\rangle\Big]\,, \\ j\ge 1, \quad -(j-1)\le m \le j-1\,. \end{aligned}\tag{3.23}$$

From (3.17), (3.18), their right-moving counterpart, it follows that

$$(b_0-\bar b_0)|\Lambda_{j,m}(p)\rangle = (L_0-\bar L_0)|\Lambda_{j,m}(p)\rangle = 0\,, \tag{3.24}$$

Also using (3.19), (3.20) and their right-moving counterpart, we get:

$$(Q_B+\bar Q_B)|\Lambda_{j,m}(p)\rangle = (p-2m)|\phi_{j,m}(p)\rangle\,, \tag{3.25}$$

where

$$\begin{aligned}|\phi_{j,m}(p)\rangle &= \frac{1}{2}\Big[|\eta^L_{(j),m}(p)\rangle \times |Y^R_{j,m}(p)\rangle + |\mathcal{O}^L_{j-1,m}(p)\rangle \times |\psi^R_{(j),m}(p)\rangle \\ &\quad -|\psi^L_{(j),m}(p)\rangle \times |\mathcal{O}^R_{j-1,m}(p)\rangle + |Y^L_{j,m}(p)\rangle \times |\eta^R_{(j),m}(p)\rangle\Big]\,. \end{aligned}\tag{3.26}$$

Eqs.(3.24), (3.25) now give

$$(b_0-\bar b_0)|\phi_{j,m}(p)\rangle = 0, \qquad (L_0-\bar L_0)|\phi_{j,m}(p)\rangle = 0\,. \tag{3.27}$$

For explicit computation of the conserved charge in section 4 we shall need the form of $|\phi_{j,m}(p=2m)\rangle$. Using eqs.(3.21), (3.22) we get

$$\begin{aligned}|\phi_{j,m}(p=2m)\rangle &= |Y^L_{j,m}\rangle \times |Y^R_{j,m}\rangle + |\omega_{j,m}\rangle \\ &\quad +\frac{1}{2}\Big[|\widehat{\eta}^L_{(j),m}\rangle \times |Y^R_{j,m}\rangle + |Y^L_{j,m}\rangle \times |\widehat{\eta}^R_{(j),m}\rangle\Big] \\ &\quad +\frac{m}{2}\Big[|\mathcal{O}^L_{j-1,m}\rangle \times \bar c_0 |Y^R_{j,m}\rangle - c_0|Y^L_{j,m}\rangle \times |\mathcal{O}^R_{j-1,m}\rangle\Big]\,, \end{aligned}\tag{3.28}$$

[9]Note that these gauge transformation parameters are of order $e^{2(1-j)\varphi}$ for large negative φ and hence grow exponentially in this region for $j>1$. Nevertheless, since the D0-brane is located in the strong coupling region of $\varphi>0$, we expect these transformations to describe sensible symmetries of the open string field theory on the D0-brane.

where

$$\begin{aligned}|\omega_{j,m}\rangle &= \frac{1}{2}|\mathcal{O}^L_{j-1,m}\rangle \times |\tau^R_{j-1,m}\rangle - \frac{1}{2}|\tau^L_{j-1,m}\rangle \times |\mathcal{O}^R_{j-1,m}\rangle \\ &= \frac{1}{2\sqrt{2}}\left(|\mathcal{O}^L_{j-1,m}\rangle \times \sum_{n=1}^{\infty}\bar{c}_{-n}\bar{\alpha}_n|Y^R_{j,m}\rangle - \sum_{n=1}^{\infty}c_{-n}\alpha_n|Y^L_{j,m}\rangle \times |\mathcal{O}^R_{j-1,m}\rangle\right). \quad (3.29)\end{aligned}$$

In going from the first to the second line of (3.29) we have used eq.(A.12). The first term in (4.25) is built on the primary $|j,m\rangle_X$, whereas the second term $|\omega_{j,m}\rangle$ is built on the primary $|j-1,m\rangle_X$. $|\omega_{j,m}\rangle$ are BRST invariant ghost number two closed string states, carrying $SU(2)_L \times SU(2)_R$ quantum numbers $(j-1,m;j-1,m)$, which are not of the form $|Y^L_{j-1,m}\rangle \times |Y^R_{j-1,m}\rangle$. They represent elements of the relative BRST cohomology which cannot be factored into a ghost number 1 state on the left and a ghost number 1 state on the right. For example $|\omega_{1,0}\rangle \propto (c_{-1}c_1 - \bar{c}_{-1}\bar{c}_1)|0\rangle$.[10] In general $|\omega_{j,m}\rangle$ will be proportional to the BRST invariant state $\sum_n n(c_{-n}c_n - \bar{c}_{-n}\bar{c}_n)|\mathcal{O}^L_{j-1,m}\rangle \times |\mathcal{O}^R_{j-1,m}\rangle$[61] up to addition of BRST trivial terms.

Eqs.(2.26)-(2.27) now tell us that for a D-brane in this two dimensional string theory described by a boundary state $|\mathcal{B}\rangle$,

$$\partial_x\left(e^{2imx}F_{j,m}(x)\right) = 0\,, \quad (3.30)$$

where

$$F_{j,m}(x) = \int \frac{dp}{2\pi}\, e^{-ipx}\langle\mathcal{B}|(c_0-\bar{c}_0)|\phi_{j,m}(p)\rangle\,. \quad (3.31)$$

Since (3.30) shows that $F_{j,m}(x) \propto e^{-2imx}$, the contribution to (3.31) must come from a term proportional to $\delta(p-2m)$ in the integrand. Thus computation of the actual value of $F_{j,m}(x)$ should only involve $|\phi_{j,m}(p)\rangle$ evaluated at $p=2m$, despite the fact that its definition in terms of the boundary state involves $|\phi_{j,m}(p)\rangle$ for general p.

The Minkowski continuation $x \to ix^0$ then gives $e^{-2mx^0}F_{j,m}(ix^0)$ as the conserved charge. We define

$$Q_{j,m}(x^0) = -\frac{1}{2\sqrt{\pi}}F_{j,m}(ix^0)\,, \quad (3.32)$$

where the overall normalization constant $-\frac{1}{2\sqrt{\pi}}$ has been fixed arbitrarily in order to simplify some of the formulæ which will appear later. $e^{-2mx^0}Q_{j,m}(x^0)$ gives the conserved charge in the Minkowski theory associated with the global symmetry generated by $|\Lambda_{j,m}\rangle$.

[10] As pointed out in footnote 6, this state is BRST trivial for $\mu=0$, but not for $\mu\neq 0$. Physically this reflects that a constant dilaton, which was a pure gauge deformation for $\mu=0$ since it could be absorbed by translating φ, is no longer a pure gauge for $\mu\neq 0$ since translation of φ now will also change μ. In fact one can explicitly check that the vertex operator associated with the state $(c_0-\bar{c}_0)|\omega_{1,0}\rangle$ has a non-vanishing inner product with the boundary state describing a D0-brane.

We shall loosely refer to $Q_{j,m}$ as the conserved charge with the understanding that it is $e^{-2mx^0}Q_{j,m}(x^0)$ that is really conserved. We could have absorbed the factor e^{-2mx^0} in the definition of $Q_{j,m}$ so that $Q_{j,m}$ itself would be conserved. However the definition (3.32) has the advantage that it does not involve any explicit time dependent factors. This guarantees that when we relate $Q_{j,m}$ to some quantity in the matrix model, then it must be formed out of matrix model variables without any explicit time dependent factors.

Before concluding this section we shall illustrate our result with a simple example. For $(j,m) = (1,0)$ the symmetry generator $|\Lambda_{1,0}\rangle$ takes the form

$$|\Lambda_{1,0}\rangle = (c_1\alpha_{-1} - \bar{c}_1\bar{\alpha}_{-1})|0\rangle \tag{3.33}$$

up to an overall normalization constant. This represents translation along x. We define

$$|\Lambda_{1,0}(p)\rangle = (c_1\alpha_{-1} - \bar{c}_1\bar{\alpha}_{-1})|p\rangle\,, \tag{3.34}$$

where $|p\rangle = e^{ipX(0)}|0\rangle$. Using the definition (3.25) and the form of Q_B given in (A.5) we get

$$|\phi_{1,0}(p)\rangle = \frac{1}{\sqrt{2}}\left[\frac{p}{2\sqrt{2}}(c_0+\bar{c}_0)\,(c_1\alpha_{-1} - \bar{c}_1\bar{\alpha}_{-1}) - 2c_1\bar{c}_1\alpha_{-1}\bar{\alpha}_{-1} + (c_{-1}c_1 - \bar{c}_{-1}\bar{c}_1)\right]|p\rangle\,. \tag{3.35}$$

We can use this to define $F_{1,0}(x)$. However as argued earlier, the actual value of $F_{1,0}$ can only depend on $|\phi_{1,0}(p=0)\rangle$, and comes from a term in $|\mathcal{B}\rangle$ built over the $|p=0\rangle$ state:

$$(c_0+\bar{c}_0)\left[Ac_1\bar{c}_1\alpha_{-1}\bar{\alpha}_{-1} + B(c_{-1}c_1 - \bar{c}_{-1}\bar{c}_1) + \ldots\right]|0\rangle\,, \tag{3.36}$$

where $\ldots$ involves higher level oscillators. (3.31), (3.35) now gives:

$$F_{1,0} \propto (A+B)\,. \tag{3.37}$$

Upon continuation to Minkowski space this agrees precisely with the expression for the energy of a D-brane as given in [25].

4 Charges Carried by the Rolling Tachyon Background

Two dimensional string theory has a D0-brane, described by the boundary state

$$|\mathcal{B}_0\rangle = |\mathcal{B}_0\rangle_X \otimes |\mathcal{B}\rangle_{liouville} \otimes |\mathcal{B}\rangle_{ghost}\,. \tag{4.1}$$

Here $|\mathcal{B}\rangle_{ghost}$ denotes the standard boundary state in the ghost sector that accompanies any D-brane

$$|\mathcal{B}\rangle_{ghost} = \exp\left(-\sum_{n=1}^{\infty}(\bar{b}_{-n}c_{-n} + b_{-n}\bar{c}_{-n})\right)(c_0+\bar{c}_0)c_1\bar{c}_1|0\rangle_{ghost}\,. \tag{4.2}$$

$|\mathcal{B}\rangle_{liouville}$ denotes the boundary state of the D0-brane CFT in the Liouville sector[23, 2]:

$$|\mathcal{B}\rangle_{liouville} = -\frac{i}{g_s}\frac{1}{2\sqrt{\pi}}\int \frac{dP}{2\pi}\,\sinh(\pi P)\,\frac{\Gamma(-iP)}{\Gamma(iP)}\,|P\rangle\rangle_{liouville}\,, \tag{4.3}$$

where $|P\rangle\rangle_{liouville}$ is the Virasoro Ishibashi state[63] built on the primary $|P\rangle_{liouville} \equiv V_{2+iP}(0)|0\rangle_{liouville}$.[11] Finally $|\mathcal{B}_0\rangle_X$ is given by:

$$|\mathcal{B}_0\rangle_X = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\bar{\alpha}_{-n}\right)|0\rangle_X\,. \tag{4.4}$$

Besides an overall factor of g_s^{-1}, the boundary state $|\mathcal{B}_0\rangle$ apparently differs from the corresponding boundary state given in [44] by a factor of $\frac{1}{2}$. This is a reflection of the difference in the choice of convention for V_{2+iP} as mentioned below eq.(3.11). Alternatively we could multiply the right hand side of (4.3) by a factor of 2, and restrict the integration over P to positive real axis only.

The open string spectrum on this D0-brane has a tachyon. We can construct a family of classical solutions describing the rolling of this tachyon away from the maximum of the tachyon potential[24, 25]. In the euclidean version the family of boundary CFT's describing these solutions is obtained by deforming the original CFT of the D0-brane by the boundary term

$$\lambda\int dt\,\cos(X(t))\,, \tag{4.5}$$

where t denotes a parameter labelling the boundary of the world-sheet and λ is the parameter labelling the rolling tachyon solution. The boundary state $|\mathcal{B}\rangle$ associated with this classical solution has the form:

$$|\mathcal{B}\rangle = |\mathcal{B}_1\rangle + |\mathcal{B}_2\rangle \tag{4.6}$$

where

$$\begin{aligned} |\mathcal{B}_1\rangle &= |\mathcal{B}_1\rangle_X \otimes |\mathcal{B}\rangle_{liouville} \otimes |\mathcal{B}\rangle_{ghost} \\ |\mathcal{B}_2\rangle &= |\mathcal{B}_2\rangle_X \otimes |\mathcal{B}\rangle_{liouville} \otimes |\mathcal{B}\rangle_{ghost}\,. \end{aligned} \tag{4.7}$$

$|\mathcal{B}\rangle_{ghost}$ and $|\mathcal{B}\rangle_{liouville}$ are as given in (4.2), (4.3). Computation of $|\mathcal{B}_1\rangle_X + |\mathcal{B}_2\rangle_X$ has been given in refs.[64, 65, 66, 67, 68, 44, 69, 70]. Here we shall follow the approach of ref.[44] and express the total boundary state as a sum of two parts. $|\mathcal{B}_1\rangle_X$ is given by

$$|\mathcal{B}_1\rangle_X = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\bar{\alpha}_{-n}\right)\,f(X(0))|0\rangle_X\,, \tag{4.8}$$

[11] Note that $|P=0\rangle_{liouville} \neq |0\rangle_{liouville}$. We apologize for this inconsistency in notation.

where

$$f(x) = \frac{1}{1+\sin(\pi\lambda)e^{ix}} + \frac{1}{1+\sin(\pi\lambda)e^{-ix}} - 1 = \sum_{2m\in Z} (-1)^{2m} \sin^{2|m|}(\pi\lambda)\, e^{2imx} \,. \quad (4.9)$$

On the other hand $|\mathcal{B}_2\rangle_X$ is given by:

$$|\mathcal{B}_2\rangle_X = \sum_{j\geq 1} \sum_{m=1-j}^{j-1} f_{j,m}(\lambda) |j,m\rangle\rangle_X \quad (4.10)$$

where $|j,m\rangle\rangle_X$ denotes the Virasoro Ishibashi state[63] built over the primary state $|j,m\rangle_X$, and

$$f_{j,m}(\lambda) = D^j_{m,-m}(2\pi\lambda) \frac{(-1)^{2m}}{D^j_{m,-m}(\pi)} - (-1)^{2m} \sin^{2|m|}(\pi\lambda) \,. \quad (4.11)$$

Here $D^j_{m,n}(\theta)$ denotes the representation of the SU(2) group element $e^{i\theta\sigma_1/2}$ in the spin j representation. Since $f_{j,m}(\lambda)$ involves only the ratio $D^j_{m,-m}(2\pi\lambda)/D^j_{m,-m}(\pi)$, it is independent of the choice of the phase of the basis states used to define $D^j_{m,n}$. With some particular choice of the phases of the basis states we have (see *e.g.* [65])

$$\begin{aligned} D^j_{m,n}(2\pi\lambda) \;\; = & \sum_{\mu=max(0,n-m)}^{min(j-m,j+n)} \frac{[(j+m)!(j-m)!(j+n)!(j-n)!]^{\frac{1}{2}}}{(j-m-\mu)!(j+n-\mu)!\mu!(m-n+\mu)!} \\ & (\cos(\pi\lambda))^{2(j-\mu)+n-m} (i\sin(\pi\lambda))^{m-n+2\mu} \,. \end{aligned} \quad (4.12)$$

Using (4.12) and some algebra, eq.(4.11) may be rewritten as

$$f_{j,m}(\lambda) = (-1)^{2|m|} \cos^2(\pi\lambda) \sin^{2|m|}(\pi\lambda) \sum_{s=0}^{j-|m|-1} \alpha^s_{j,m} \sin^{2s}(\pi\lambda) \,, \quad (4.13)$$

where

$$\begin{aligned} \alpha^s_{j,m} \;\; \equiv & \sum_{\mu=0}^{s} \frac{(j+|m|)!(j-|m|)!(j-|m|-1-\mu)!(-1)^{s-j+|m|}}{((j-|m|-\mu)!)^2 \mu!(2|m|+\mu)!(s-\mu)!(j-|m|-1-s)!} - 1 \\ & \qquad \text{for } s \leq j-|m|-1 \,, \\ \equiv & \; 0 \quad \text{for } s > j-|m|-1 \,. \end{aligned} \quad (4.14)$$

Thus $f_{j,m}(\lambda)$ depends on the magnitude of m but not its sign.

$|\mathcal{B}_1\rangle_X$ given in (4.8) can also be expressed in terms of the Ishibashi states $|j,m\rangle\rangle_X$ as follows. We rewrite (4.8) as

$$\begin{aligned} |\mathcal{B}_1\rangle_X \;\; = & \sum_{2m\in Z} \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n}\bar{\alpha}_{-n}\right) (-1)^{2m} \sin^{2|m|}(\pi\lambda) e^{2imX(0)} |0\rangle \\ = & \sum_{2m\in Z} (-1)^{2m} \sin^{2|m|}(\pi\lambda) \sum_N |N,m\rangle_L \otimes |N,m\rangle_R \,, \end{aligned} \quad (4.15)$$

where $|N,m\rangle_L$ ($|N,m\rangle_R$) denote a complete basis of orthonormal states constructed out of products of α_{-n}'s ($\bar\alpha_{-n}$'s) acting on $e^{2imX_L(0)}|0\rangle$ ($e^{2imX_R(0)}|0\rangle$). In going from the first to the second line of (4.15) we have expanded $\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\bar\alpha_{-n}\right)$ in a power series expansion. If $|\widetilde{N,m}\rangle_L$ ($|\widetilde{N,m}\rangle_R$) denote another basis of orthonormal states, related to the previous basis by a rotation:

$$|\widetilde{N,m}\rangle_L = R_{NN'}|N',m\rangle_L, \quad |\widetilde{N,m}\rangle_R = R_{NN'}|N',m\rangle_R\,, \quad R_{NM}R_{NM'} = \delta_{MM'}\,, \tag{4.16}$$

then (4.15) may be reexpressed as

$$|\mathcal{B}_1\rangle_X = \sum_{2m\in Z}(-1)^{2m}\sin^{2|m|}(\pi\lambda)\sum_N|\widetilde{N,m}\rangle_L\otimes|\widetilde{N,m}\rangle_R\,. \tag{4.17}$$

Choosing $|\widetilde{N,m}\rangle_L$ and $|\widetilde{N,m}\rangle_R$ to be the orthonormal basis of states formed out of $|j,m\rangle_L$, $|j,m\rangle_R$ and their Virasoro descendants, we arrive at the equation:

$$|\mathcal{B}_1\rangle_X = \sum_{j\geq 0}\sum_{m=-j}^{j}(-1)^{2m}\sin^{2|m|}(\pi\lambda)\,|j,m\rangle\rangle_X\,. \tag{4.18}$$

We now turn to the computation of (3.31) for the boundary state $|\mathcal{B}\rangle$ described above. The first important point to note is that the contribution to (3.31) from the boundary state $|\mathcal{B}_1\rangle$ vanishes. To see this we note that the state:

$$|\mathcal{B}_1(p)\rangle \equiv \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\bar\alpha_{-n}\right)e^{ip.X(0)}|0\rangle_X\otimes|\mathcal{B}\rangle_{liouville}\otimes|\mathcal{B}\rangle_{ghost} \tag{4.19}$$

is BRST invariant for every p[44]:

$$(Q_B+\bar Q_B)|\mathcal{B}_1(p)\rangle = 0\,. \tag{4.20}$$

As a result, using (3.25) we get

$$\langle\mathcal{B}_1(p')|(c_0-\bar c_0)|\phi_{j,m}(p)\rangle = (p-2m)^{-1}\langle\mathcal{B}_1(p')|\{(c_0-\bar c_0),(Q_B+\bar Q_B)\}|\Lambda_{j,m}(p)\rangle\,. \tag{4.21}$$

Now $\{(c_0-\bar c_0),(Q_B+\bar Q_B)\}$ does not have any c or $\bar c$ zero mode. Using (3.24) we also see that $|\Lambda_{j,m}(p)\rangle$ does not have any $(c_0-\bar c_0)$ zero mode. On the other hand the $|\mathcal{B}\rangle_{ghost}$ appearing in the expression for $|\mathcal{B}_1\rangle$ contains only a zero mode of $(c_0+\bar c_0)$ but not of $(c_0-\bar c_0)$. As a result the operators appearing in the matrix element (4.21) does not have any $(c_0-\bar c_0)$ zero mode and hence the matrix element vanishes. This gives:

$$\langle\mathcal{B}_1(p')|(c_0-\bar c_0)|\phi_{j,m}(p)\rangle = 0\,, \tag{4.22}$$

for every $p \neq 2m$. By analytic continuation[12] this result must hold also for $p = 2m$. Since $|\mathcal{B}_1\rangle$ defined in (4.7), (4.8) is a linear combination of $|\mathcal{B}_1(p)\rangle$ defined in (4.19), we see that

$$\langle \mathcal{B}_1|(c_0 - \bar{c}_0)|\phi_{j,m}(p)\rangle = 0\,. \tag{4.23}$$

This agrees with the conclusion of [44] that the $|\mathcal{B}_1\rangle$ part of the boundary state does not carry any information about conserved charges.

Thus we have

$$F_{j,m}(x) = \int \frac{dp}{2\pi}\, e^{-ipx} \langle \mathcal{B}_2|(c_0 - \bar{c}_0)|\phi_{j,m}(p)\rangle\,. \tag{4.24}$$

As pointed out below (3.31), the contribution to $F_{j,m}(x)$ only involves $|\phi_{j,m}(p = 2m)\rangle$. Furthermore, since $|\mathcal{B}_2\rangle_X$ contains contribution from states built on primaries of the form $|j,m\rangle_L \times |j,m\rangle_R$, we can throw away from $|\phi_{j,m}(p = 2m)\rangle$ all terms which are built on primaries of the form $|j',m\rangle_L \times |j'',m\rangle_R$ with $j' \neq j''$. The part of $|\phi_{j,m}(p = 2m)\rangle$ that contributes to (4.24) is given by the terms in the first line on the right hand side of (3.28):

$$|Y^L_{j,m}\rangle \times |Y^R_{j,m}\rangle + |\omega_{j,m}\rangle\,, \tag{4.25}$$

since $|\widehat{\eta}^L_{(j),m}\rangle$ ($|\widehat{\eta}^R_{(j),m}\rangle$) is a linear combination of states with left (right) SU(2) quantum numbers $(j-1,m)$ and $(j-2,m)$. The contribution to (4.24) from the first term of (4.25) is easy to evaluate. The relevant part of the boundary state $|\mathcal{B}_2\rangle$ that contributes to this matrix element is

$$f_{j,-m}(\lambda)|j,-m\rangle_X \otimes |\mathcal{B}\rangle_{liouville} \otimes (c_0 + \bar{c}_0)c_1\bar{c}_1|0\rangle_{ghost} \tag{4.26}$$

Using eq.(3.8), (3.13) and the normalization condition

$$\langle 0|c_{-1}\bar{c}_{-1}c_0\bar{c}_0c_1\bar{c}_1|0\rangle_{ghost} = 1\,, \tag{4.27}$$

we get the contribution to (4.24) from this term to be

$$2\, f_{j,-m}(\lambda)\, e^{-2imx} \;\; {}_{liouville}\langle \mathcal{B}|V_{2(1-j)}(0)|0\rangle_{liouville}\,. \tag{4.28}$$

Since $|\omega_{j,m}\rangle$ carries left and right SU(2) quantum numbers $(j-1,m)$, the contribution to (4.24) from the second term on the right hand side of (4.25) will be proportional to $2f_{j-1,-m}(\lambda)\, e^{-2imx} \;\; {}_{liouville}\langle \mathcal{B}|V_{2(1-j)}(0)|0\rangle_{liouville}$. If we denote the constant of proportionality by $a_{j,m}$, then this contribution is

$$2\, a_{j,m}\, f_{j-1,-m}(\lambda)\, e^{-2imx} \;\; {}_{liouville}\langle \mathcal{B}|V_{2(1-j)}(0)|0\rangle_{liouville}\,. \tag{4.29}$$

[12] The inner product in the left hand side of (4.22) carries a factor of $2\pi\delta(p+p')$ which is clearly not an analytic function of p. Eq.(4.22) implies that the coefficient of this δ-function, regarded as a function of p after setting $p' = -p$, vanishes. It is this coefficient that is expected to be an analytic function of p.

Combining (4.28) and (4.29) we now get

$$F_{j,m}(x) = 2\left(f_{j,-m}(\lambda) + a_{j,m}\, f_{j-1,-m}(\lambda)\right) e^{-2imx}\ \ {}_{liouville}\langle \mathcal{B}|V_{2(1-j)}(0)|0\rangle_{liouville}\,. \tag{4.30}$$

While a direct computation of $a_{j,m}$ is somewhat involved we can calculate it indirectly by using (4.23) and the form of $|\mathcal{B}_1\rangle_X$ given in (4.18). Using the same line of argument as used for $|\mathcal{B}_2\rangle$, we get the contribution to $F_{j,m}(x)$ from $|\mathcal{B}_1\rangle$ to be

$$2\,(-1)^{2m}\, \sin^{2|m|}(\pi\lambda)\,(1+a_{j,m})\ e^{-2imx}\ \ {}_{liouville}\langle \mathcal{B}|V_{2(1-j)}(0)|0\rangle_{liouville}\,. \tag{4.31}$$

On the other hand from the general argument given earlier we know that the contribution to $F_{j,m}(x)$ from $|\mathcal{B}_1\rangle$ must vanish. This gives:

$$a_{j,m} = -1\,. \tag{4.32}$$

Hence (4.30) takes the form:

$$F_{j,m}(x) = 2\left(f_{j,-m}(\lambda) - f_{j-1,-m}(\lambda)\right) e^{-2imx}\ \ {}_{liouville}\langle \mathcal{B}|V_{2(1-j)}(0)|0\rangle_{liouville}\,. \tag{4.33}$$

Using eqs.(3.10), (4.3) we get

$${}_{liouville}\langle \mathcal{B}|V_{2+iP}(0)|0\rangle_{liouville} = \frac{1}{g_s}\,\frac{1}{\sqrt{\pi}}\, i\sinh(\pi P)\,\frac{\Gamma(iP)}{\Gamma(-iP)}\,. \tag{4.34}$$

Taking $P\to 2ij$ limit in the above formula we get

$${}_{liouville}\langle \mathcal{B}|V_{2(1-j)}(0)|0\rangle_{liouville} = \frac{1}{g_s}\,\frac{\sqrt{\pi}}{(2j)!(2j-1)!}\,. \tag{4.35}$$

Thus

$$F_{j,m}(x) = \frac{2\sqrt{\pi}}{(2j)!(2j-1)!}\,\frac{1}{g_s}\left(f_{j,-m}(\lambda) - f_{j-1,-m}(\lambda)\right) e^{-2imx}\,. \tag{4.36}$$

Under the replacement $x\to ix^0$ this gives

$$Q_{j,m}(x^0) \equiv -\frac{1}{2\sqrt{\pi}}\, F_{j,m}(ix^0) = -\frac{1}{g_s}\,\frac{1}{(2j)!(2j-1)!}\left(f_{j,-m}(\lambda) - f_{j-1,-m}(\lambda)\right) e^{2mx^0}\,, \tag{4.37}$$

Using (4.13) we can rewrite this as

$$Q_{j,m}(x^0) = \frac{1}{g_s}\,(-1)^{2m}\, e^{2mx^0}\ \cos^2(\pi\lambda)\sum_{l=|m|}^{j-1} b_j^{l,m}\sin^{2l}(\pi\lambda)\,, \tag{4.38}$$

where

$$
\begin{aligned}
b_j^{l,m} &= -\frac{1}{(2j)!(2j-1)!}\left[\alpha_{j,-m}^{l-|m|} - \alpha_{j-1,-m}^{l-|m|}\right] \quad \text{for } |m| \le l \le j-1\,, \\
&= 0 \quad \text{otherwise}\,.
\end{aligned} \tag{4.39}
$$

The coefficients $\alpha_{j,m}^s$ have been defined in (4.14). It is understood that the sum over l in (4.38) will always be in integer steps but l itself can be integer or half integer depending on whether m is integer or half integer.

From (4.38), (4.39), (4.14) we get

$$
Q_{1,0} = \frac{1}{g_s}\cos^2(\pi\lambda)\, b_1^{0,0} = \frac{1}{g_s}\cos^2(\pi\lambda)\,. \tag{4.40}
$$

Thus $Q_{1,0}$ measures the total energy of the D0-brane[24, 25].

Although the above analysis defines infinite number of conserved charges, for a single D0-brane information about these conserved charges is highly superfluous. Since the classical trajectories of a D0-brane are labelled by a single parameter λ, knowing one of these charges (say $Q_{1,0}$) determines all the other charges. However, the information contained in these conserved charges becomes useful if we have a system of multiple (say n) D0-branes. In that case the tachyon on each D0-brane can roll independently with parameter values $\lambda_1, \ldots, \lambda_n$, and furthermore, there may be arbitrary time delay between the motion of these tachyons, reflected in the freedom of shifting x^0 for the rolling tachyon backgrounds by arbitrary constants $c_1, \ldots, c_n$. The conserved charges $Q_{j,m}$ for such a system are given by:

$$
Q_{j,m}(x^0) = \frac{1}{g_s}(-1)^{2m}\, e^{2mx^0} \sum_{l=|m|}^{j-1} b_j^{l,m} \sum_{r=1}^{n} e^{2mc_r}\, \cos^2(\pi\lambda_r)\sin^{2l}(\pi\lambda_r)\,. \tag{4.41}
$$

In principle, knowing the infinite number of $Q_{j,m}$'s we can determine the parameters λ_r and c_r for a given configuration.

5 Asymptotic Fields Produced by the Rolling Tachyon

Since the boundary state acts as source for closed string fields, it produces closed string background $|\Phi\rangle$. In the normalization convention we are using, the equation of motion determining $|\Phi\rangle$ takes the form[44]:

$$
2(Q_B + \bar{Q}_B)|\Phi\rangle = g_s^2|\mathcal{B}\rangle\,. \tag{5.1}
$$

The solution to this equation in Siegel gauge is given by:

$$|\Phi\rangle = g_s^2\,(b_0+\bar{b}_0)\,\Big(2(L_0+\bar{L}_0)\Big)^{-1}|\mathcal{B}\rangle\,. \tag{5.2}$$

The right hand side of (5.2) is ambiguous due to the existence of zero eigenvalue of $(L_0+\bar{L}_0)$ in the Minkowski space. We use the Hartle-Hawking prescription in which we compute the result in the Euclidean theory and then analytically continue the result to Minkowski theory[26, 27]. As was discussed in [44], $|\Phi\rangle$ produced by the $|\mathcal{B}_1\rangle$ part of the boundary state represents the closed string radiation produced by the rolling tachyon solution whereas the closed string background produced by $|\mathcal{B}_2\rangle$ carries information about the conserved charges carried by this system. We shall focus on the field produced by the $|\mathcal{B}_2\rangle$ part of the boundary state and discuss some subtleties which were overlooked in the analysis of [44].

We begin with the general expression for $|\mathcal{B}_2\rangle$:

$$|\mathcal{B}_2\rangle = -\frac{i}{g_s}\frac{1}{2\sqrt{\pi}}\sum_{j\ge 1}\sum_{m=1-j}^{j-1} f_{j,m}(\lambda)\int\frac{dP}{2\pi}\,\sinh(\pi P)\,\frac{\Gamma(-iP)}{\Gamma(iP)}\,|j,m\rangle\rangle_X\otimes|P\rangle\rangle_{liouville}\otimes|\mathcal{B}\rangle_{ghost}\,. \tag{5.3}$$

The part of $|\mathcal{B}_2\rangle$ which involves the ground state in the ghost sector and primary states in the matter and the Liouville sector is given by:

$$-\frac{i}{g_s}\frac{1}{2\sqrt{\pi}}\sum_{j\ge 1}\sum_{m=1-j}^{j-1} f_{j,m}(\lambda)\int\frac{dP}{2\pi}\,\sinh(\pi P)\,\frac{\Gamma(-iP)}{\Gamma(iP)}\,|j,m\rangle_X\otimes|P\rangle_{liouville}\otimes(c_0+\bar{c}_0)\,c_1\bar{c}_1|0\rangle_{ghost}\,. \tag{5.4}$$

In the region of large negative φ, the liouville primary state $|P\rangle_{liouville}$ behaves as

$$|P\rangle_{liouville} \simeq e^{(2+iP)\varphi(0)}|0\rangle_{liouville} - \left(\frac{\Gamma(iP)}{\Gamma(-iP)}\right)^2 e^{(2-iP)\varphi(0)}|0\rangle_{liouville}\,. \tag{5.5}$$

Furthermore, acting on the state $|j,m\rangle_X\otimes|P\rangle_{liouville}\otimes(c_0+\bar{c}_0)c_1\bar{c}_1|0\rangle_{ghost}$, the operator $2(L_0+\bar{L}_0)$ gives (P^2+4j^2). Thus for large negative φ the closed string field configuration produced by the component of the boundary state given in (5.4) is given by:

$$\begin{aligned}|\check{\Phi}^{(2)}\rangle \;=\; & -\frac{ig_s}{\sqrt{\pi}}\sum_{j\ge 1}\sum_{m=1-j}^{j-1} f_{j,m}(\lambda)\int\frac{dP}{2\pi}\,\frac{1}{P^2+4j^2}\,\sinh(\pi P)\,\frac{\Gamma(-iP)}{\Gamma(iP)}\,|j,m\rangle_X \\ & \otimes\left(e^{(2+iP)\varphi(0)}|0\rangle_{liouville} - \left(\frac{\Gamma(iP)}{\Gamma(-iP)}\right)^2 e^{(2-iP)\varphi(0)}|0\rangle_{liouville}\right)\otimes c_1\bar{c}_1|0\rangle_{ghost}\,.\end{aligned} \tag{5.6}$$

We can evaluate the first term by closing the contour in the lower half plane and the second term by closing the contour in the upper half plane. The two contributions are identical and the sum of them gives:[13]

$$|\check{\Phi}^{(2)}\rangle = \sqrt{\pi}\, g_s \sum_{j\geq 1} \sum_{m=-(j-1)}^{j-1} \frac{1}{((2j)!)^2} f_{j,m}(\lambda)|j,m\rangle_X \otimes e^{2(1+j)\varphi(0)}|0\rangle_{liouville} \otimes c_1\bar{c}_1|0\rangle_{ghost} \,. \tag{5.7}$$

This expression agrees with the corresponding expression in [44] up to factors of g_s which was set to 1 in [44].[14]

Now from eq.(4.37) we have

$$f_{j,-m}(\lambda) = -g_s \sum_{\substack{|m|+1\leq j'\leq j \\ j-j'\in Z}} (2j')!(2j'-1)! Q_{j',m} e^{-2mx^0} \,. \tag{5.8}$$

Eqs.(5.7), (5.8) relate the asymptotic form of $|\Phi\rangle$ produced by the rolling tachyon background to the conserved charges $Q_{j,m}$ carried by the boundary state. In section 9 we shall derive a general formula relating asymptotic field configurations to conserved charges.

In ref.[44] it was argued that the rest of the asymptotic field configuration produced by $|\mathcal{B}_2\rangle$, involving states with oscillator excitations, could be gauged away. This was done by showing that the boundary state $|\mathcal{B}_2\rangle$ does not produce any source term in the region of large negative φ. Eq.(5.1) then shows that the string field $|\Phi\rangle$ produced by $|\mathcal{B}_2\rangle$ must be BRST invariant in this region. (Here we regard the BRST operator as a differential operator in the (x,φ) space.) Since the states appearing in expression (5.7) are the only non-trivial elements of the BRST cohomology, it then follows that any other contribution to $|\Phi\rangle$ in this region must be BRST trivial and hence can be gauged away.

There is however a subtlety in this argument which we wish to discuss here. For this let us briefly recall the argument of [44] for the vanishing of the sources for large negarive φ. For a D0-brane the Liouville part of the boundary state is proportional to[15]

$$\int dP \,\sinh(\pi P)\, \frac{\Gamma(-iP)}{\Gamma(iP)} |P\rangle\rangle_{liouville} \,. \tag{5.9}$$

If we focus on string field components which do not involve any oscillator excitation in the Liouville sector, then for large negative φ (5.9) gives a source term for these fields

[13] Note that the classical background produced by the D0-brane corresponds to non-normalizable states in the notation of ref.[71]. The same result is true for string theories based on $c<1$ minimal models coupled to gravity[72].

[14] For a recent discussion of why such backgrounds may be needed to describe a D0-brane from the viewpoint of the matrix model, see [73].

[15] In (5.9) φ denotes the zero mode of φ labelling the space coordinate. We shall use the same symbol φ to label the full Liouville field φ and its zero mode, but it should be clear from the context which interpretation is appropriate.

proportional to

$$\int dP\, \sinh(\pi P)\, \frac{\Gamma(-iP)}{\Gamma(iP)}\, e^{(2+iP)\varphi}\ . \tag{5.10}$$

This can be evaluated by closing the contour in the lower half plane. Since $\sinh(\pi P)\, \frac{\Gamma(-iP)}{\Gamma(iP)}$ does not have a pole in the lower half plane, the result vanishes. Thus $|\mathcal{B}_2\rangle$ does not contribute to any source term for these fields for large negative φ.

A higher level state in the Liouville sector, appearing from the higher level states in $|P\rangle\rangle_{liouville}$, will typically have a P dependent coefficient $f(P)$. Thus the source term associated with such a string field component will be proportional to

$$\int dP\, \sinh(\pi P)\, f(P)\, \frac{\Gamma(-iP)}{\Gamma(iP)}\, e^{(2+iP)\varphi}\ . \tag{5.11}$$

As long as $f(P)$ does not have any pole in the lower half P plane, our argument for the vanishing of this integral for large negative φ still goes through. However typically $f(P)$ does have poles, and the residues at the poles correspond precisely to the null states of the Virasoro algebra. To see an example of this consider the expansion of $|P\rangle\rangle_{liouville}$ to states of level (1,1):

$$|P\rangle\rangle_{liouville} = \left(1 + \frac{2}{P^2+4} L^{\varphi}_{-1} \bar L^{\varphi}_{-1} + \ldots\right) |P\rangle_{liouville}\,, \tag{5.12}$$

where L^{φ}_n, $\bar L^{\varphi}_n$ denote the Liouville Virasoro generators, $|P\rangle_{liouville} = V_{2+iP}(0)|0\rangle_{liouville}$ and $\ldots$ stand for higher level terms. The pole at $P=-2i$ gives a contribution to (5.9) proportional to

$$L^{\varphi}_{-1}\bar L^{\varphi}_{-1} e^{4\varphi(0)}|0\rangle\,, \tag{5.13}$$

for large negative φ. This is a null state of the Liouville Virasoro algebra which does not vanish when we express the L^{φ}_n, $\bar L^{\varphi}_n$ in terms of the oscillators of φ.

The net conclusion from this is that the source term produced by $|\mathcal{B}_2\rangle$ vanishes for large negative φ *only if we set the null states to zero.* This in turn shows that the string field background $|\Phi\rangle$ produced by the boundary state in the region of large negative φ is BRST invariant only if we set the null states to zero.[16] However, in this case there are additional elements of the BRST cohomology in the ghost number two sector, which for large negative φ take the form[61]:

$$g_s \sum_n n(c_{-n}c_n - \bar c_{-n}\bar c_n)\, \mathcal{R}^L_{j-1}\, \mathcal{R}^R_{j-1}\, e^{2imX(0)}\, e^{2(1+j)\varphi(0)}\, c_1\bar c_1|0\rangle\,, \tag{5.14}$$

[16] Operationally this means that we take a complete basis of states (not necessarily orthonormal) in our conformal field theory to be the one generated by the action of X and ghost oscillators and the Liouville Virasoro generators on primaries of the form $e^{ikX(0)}|0\rangle_X \otimes |P\rangle_{liouville} \otimes |0\rangle_{ghost}$, and specify a state in the CFT by specifying its inner product with all members of this basis. Since a null state is orthogonal to any member of the basis, it can be regarded as zero.

where $\mathcal{R}^L_{j-1}$, $\mathcal{R}^R_{j-1}$ are the same combination of oscillators which appear in the construction of $|\mathcal{O}^L_{j-1,m}\rangle$, $|\mathcal{O}^R_{j-1,m}\rangle$ in eq.(3.15). Thus the background string field $|\Phi\rangle$ produced by the rolling tachyon boundary state will also contain linear combination of these states. We can in principle calculate the coefficients of these terms using the analog of (5.6) for higher level states, but we shall not do this here. For later use we note however that the state given in (5.14) carries SU(2) quantum numbers $(j-1, m)$ both on the right and the left sector of the world-sheet. Thus it must come from terms in $|\mathcal{B}_2\rangle$ carrying SU(2) quantum numbers $(j-1, m)$, and as a result the coefficient of this term must be proportional to $f_{j-1,m}(\lambda)$.

6 Relation to Conserved Charges in the Matrix Model

The two dimensional string theory has an equivalent description as a matrix model[17, 18, 19]. This matrix model in turn can be mapped to a theory of free fermions, each of them moving under an inverted harmonic oscillator Hamiltonian:

$$h(q,p) = \frac{1}{2}(p^2 - q^2) + \frac{1}{g_s}\,, \tag{6.1}$$

where g_s is a parameter which corresponds to the coupling constant of the closed string theory in the continuum description. The fermi level is at zero energy; thus all negative energy states are filled and positive energy states are empty. A single D0-brane of energy E corresponds to a single fermion excited from the fermi level to an energy level E[2]. Thus the dynamics of a single D0-brane is described by the single particle Hamiltonian (6.1) with an additional constraint:

$$h(q,p) \geq 0\,, \tag{6.2}$$

due to Pauli exclusion principle. (6.1) together with the constraint (6.2) can be regarded as the open string field theory of a single D0-brane in two dimensional string theory. The configuration $(q = 0, p = 0)$ represents the D0-brane with all open string fields set to zero.

The rolling tachyon solution of the continuum theory parametrized by λ corresponds to a classical trajectory in this open string field theory with energy[1, 2, 3]

$$E = \frac{1}{g_s}\cos^2(\pi\lambda)\,. \tag{6.3}$$

This gives

$$q = -\sqrt{\frac{2}{g_s}}\,\sin(\pi\lambda)\cosh x^0, \qquad p = -\sqrt{\frac{2}{g_s}}\,\sin(\pi\lambda)\sinh x^0\,, \tag{6.4}$$

where for definiteness we have taken the trajectory to be in the negative q region. At the classical level and for energy $E < g_s^{-1}$ there is no tunneling from the negative q to the positive q region. (6.4) may be rewritten as

$$(q \pm p) = -\sqrt{\frac{2}{g_s}}\,\sin(\pi\lambda)e^{\pm x^0}\,. \tag{6.5}$$

We can now express the conserved charges $Q_{j,m}(x^0)$ of the continuum theory in terms of matrix model variables[44] by reexpressing (4.38) in terms of q and p using (6.5). This gives

$$Q_{j,m} = \sum_{l=|m|}^{j-1} \left(\frac{g_s}{2}\right)^l b_j^{l,m}\, W_{l,m}\,, \tag{6.6}$$

where

$$W_{l,m} = h(q,p)\,(q+p)^{l+m}(q-p)^{l-m}\,. \tag{6.7}$$

The charges $e^{-2mx^0}W_{l,m}$ generate symmetries of the action corresponding to the Hamiltonian (6.1), and leave the condition (6.2) invariant. Thus (6.6) expresses the charges in the continuum theory as a linear combination of the charges in the matrix model description.

Eq.(6.6) can be inverted[44] to give

$$W_{l,m} = \left(\frac{g_s}{2}\right)^{-l} \sum_{j=|m|+1}^{l+1} c_j^{l,m} Q_{j,m} \tag{6.8}$$

where the coefficients $c_j^{l,m}$ are determined from the equations:

$$c_{j'}^{l,m} = 0 \quad \text{for} \quad j' < |m|+1 \quad \text{or} \quad j' > l+1 \quad \text{or} \quad l < |m|\,, \tag{6.9}$$

$$\sum_{l=|m|}^{j-1} b_j^{l,m} c_{j'}^{l,m} = \delta_{jj'} \quad \text{for} \quad |m|+1 \le j' \le j\ . \tag{6.10}$$

A systematic procedure for solving these equations is to first use eq.(6.10) for $(m,j') = (j-1,j)$ to determine $c_j^{j-1,j-1}$ for all j. Next we use these equations for $(m,j') = (j-2,j)$ and $(j-2,j-1)$ to determine $c_j^{j-1,j-2}$ and $c_{j-1}^{j-1,j-2}$. Proceeding this way we can eventually determine all the coefficients $c_{j'}^{l,m}$ for $|m|+1 \le j' \le l+1$.

Note that the relations (6.6) between the conserved charges $Q_{j,m}$ in the continuum theory and those in the matrix model have been derived by working with the single fermion excitations in the range $0 \le E \le \frac{1}{g_s}$. However, once these relations have been derived they hold for all ranges of the energy. For example, we could extend our expressions for $Q_{j,m}$ is to fermions carrying energy larger than $\frac{1}{g_s}$. The trajectory associated with a configuration of this type is given by:

$$q = -\sqrt{\frac{2}{g_s}}\,\sinh(\pi\lambda)\,\sinh(x^0)\,, \qquad p = -\sqrt{\frac{2}{g_s}}\,\sinh(\pi\lambda)\,\cosh(x^0)\,. \tag{6.11}$$

The associated $Q_{j,m}$, as computed from (6.6), (6.7) are given by:

$$Q_{j,m}(x^0) = \frac{1}{g_s}\,e^{2mx^0}\,\cosh^2(\pi\lambda)\sum_{l=|m|}^{j-1}(-1)^{l+m}b_j^{l,m}\sinh^{2l}(\pi\lambda)\,. \tag{6.12}$$

These results can also be derived directly from the rolling tachyon boundary state with energy $> \frac{1}{g_s}$[24].

In section 7 we shall discuss how (6.6) can be used to derive constraints on the description of the hole states in the continuum theory.

7 Comments on Hole States

As discussed in section 6, the D0-brane in the continuum description of two dimensional string theory can be identified with single fermion excitations in the matrix theory. The matrix theory contains another set of states which are closely related to the single fermion excitations, – namely the single hole excitations. This involves taking a fermion with energy below the fermi level, and exciting it to an energy just at the fermi level. The question that arises naturally is: what is the description of these states in the continuum description of the two dimensional string theory?

Analysis of the conserved charges carried by a hole state provides a clue. Since a hole associated with the trajectory $(q(x^0), p(x^0))$ corresponds to absence of the fermion associated with this trajectory, the charges $W_{l,m}$ associated with a hole are given by the negative of the expression given in (6.7):

$$W^h_{l,m} = -h(q,p)\,(q+p)^{l+m}(q-p)^{l-m}\,. \tag{7.1}$$

Hence from (6.6) the charges $Q_{j,m}$ carried by a hole state are given by

$$Q^h_{j,m} = -h(q,p)\sum_{l=|m|}^{j-1}\left(\frac{g_s}{2}\right)^l b_j^{l,m}(q+p)^{l+m}(q-p)^{l-m}\,. \tag{7.2}$$

In particular, if we take the following negative energy trajectory

$$q = -\sqrt{\frac{2}{g_s}}\,\cosh(\pi\alpha)\,\cosh(x^0)\,, \qquad p = -\sqrt{\frac{2}{g_s}}\,\cosh(\pi\alpha)\,\sinh(x^0)\,, \tag{7.3}$$

then the hole state associated with this trajectory carries

$$Q^h_{j,m}(x^0) = \frac{1}{g_s}\, e^{2mx^0}(-1)^{2m}\, \sinh^2(\pi\alpha) \sum_{l=|m|}^{j-1} b_j^{l,m} \cosh^{2l}(\pi\alpha)\,. \tag{7.4}$$

Note that $Q^h_{j,m}$ given in (7.4) are related to $Q_{j,m}$ carried by a fermion, as given in (4.38), by a formal replacement of λ by $\frac{1}{2}+i\alpha$ and an overall change in sign.

Now suppose the hole state corresponds to some D-brane system in the continuum theory. Since according to the analysis of section 3 certain terms of the boundary state carry information about these conserved charges, the charges $Q_{j,m}$ computed from this boundary state using eqs.(3.31), (3.32) must agree with (7.4). In particular, these terms must be given by simply continuing the corresponding terms in the boundary state of a D0-brane to $\lambda = \frac{1}{2}+i\alpha$, and then changing the sign of these terms. This is in accordance with the proposal put forward in [5, 29] where it was suggested that the complete boundary state of the hole state is obtained by analytic continuation of λ to $\frac{1}{2}+i\alpha$ followed by a overall change in sign of the boundary state of a D0-brane. Unfortunately this prescription does not seem to work for the $|\mathcal{B}_1\rangle$ component of the boundary state. To see this we note that analytically continuing $-|\mathcal{B}_1\rangle$ to $\lambda = \frac{1}{2}+i\alpha$ amounts to replacing $f(x)$ defined in (4.9) by

$$\begin{aligned}\widetilde{f}(x) &= -\left(\frac{1}{1+\cosh(\pi\alpha)e^{ix}} + \frac{1}{1+\cosh(\pi\alpha)e^{-ix}} - 1\right)\\ &= \frac{1}{1+\mathrm{sech}(\pi\alpha)e^{ix}} + \frac{1}{1+\mathrm{sech}(\pi\alpha)e^{-ix}} - 1\,.\end{aligned} \tag{7.5}$$

Thus the net effect of this is to replace $\sin(\pi\lambda)$ in the original expression by a $\mathrm{sech}(\pi\alpha)$. Since both $\sin(\pi\lambda)$ and $\mathrm{sech}(\pi\alpha)$ are less than 1, there is no qualitative difference between the closed string field configuration produced by $|\mathcal{B}_1\rangle$ and the new boundary state $|\widetilde{\mathcal{B}}_1\rangle$. In particular it is known that the closed string field configuration induced by $|\mathcal{B}_1\rangle$ corresponds to a configuration of the closed string tachyon that describes precisely a single fermion excitation of the matrix model with a time delay given by $-\ln\sin(\pi\lambda)$[2, 44]. Thus the closed string field configuration induced by $|\widetilde{\mathcal{B}}_1\rangle$ also describes a single fermion excitation of the matrix model with a time delay proportional to $-\ln \mathrm{sech}(\pi\alpha)$ if we use the same Hartle-Hawking prescription to compute the closed string radiation from the brane.[17] In contrast a hole state should be described by a closed string field configuration of

[17] If however we first compute the closed string radiation from a D0-brane and then analytically continue the result to $\lambda = \frac{1}{2}+i\alpha$ followed by a change of sign, we get the answer expected of a hole state[5, 29]. This is due to the fact that the poles of $f(x)$ crosses the Hartle-Hawking contour as λ passes through $\frac{1}{2}$, and hence the analytic continuation of λ to $\frac{1}{2}+i\alpha$ does not commute with integration along the Hartle-Hawking contour.

opposite sign, and by examining the classical trajectory of a hole one can argue that the corresponding time delay should be given by $-\ln\cosh(\pi\alpha)$ instead of $-\ln\mathrm{sech}(\pi\alpha)$.

As a result of these discrepancies, finding the correct expression for the boundary state describing the hole remains an open problem. It is possible that the expression for the conserved charges together with the requirement of world-sheet conformal invariance determines the form of the boundary state completely. Unfortunately for a time dependent background the full implication of world-sheet conformal invariance (the Cardy conditions[74]) is not clear. Trying to understand this CFT as an analytic continuation from an Euclidean theory might shed some light on this issue.

Another possible guideline we could try to follow is to try to use solutions which have a target space interpretation for large negative φ and hence are likely to exist in the full string theory. In this context we note that if we are in the region of large negative φ where the potential $e^{2\varphi}$ for the Liouville field is negligible, we can treat φ as an ordinary spatial direction with a linear dilaton background $\Phi_D = 2\varphi$. In this case we can construct a family of solutions describing the motion of an ordinary D0-brane in this linear dilaton background[75].[18] Such branes experience a potential proportional to their mass $e^{-\Phi_D}$ and tend to move towards the strong coupling region so as to minimize their effective mass. Thus these branes are never stationary. This is indeed one of the characteristics of the hole states since they involve orbits below the fermi level, and unlike the $\lambda = 0$ D0-brane configuration describing a fermion sitting at the maximum of the potential, there is no stationary orbit below the fermi level. The full time dependent classical solution involves a D0-brane travelling from the strong coupling to the weak coupling region, reaching a turning point depending on the total energy it carries, and then going back towards the strong coupling region. The more the energy the deeper the D0-brane probes into the weak coupling region. Thus there is no upper limit to how much energy such a D0-brane can carry. This again matches the characteristic of the hole states which do not have any limit on how much energy they can carry due to the inverted harmonic oscillator potential being unbounded from below. The fact that the effective action describing the dynamics of these holes[75] looks very similar to the one used for describing the dynamics of rolling tachyon[76] provides additional support to this proposed identification.

Although we do not have a complete calculation of the charges $Q_{j,m}$ carried by these time dependent D0-brane configurations, for high energy D0-branes which probe well into the weak coupling region we can make an estimate of $Q_{j,m}$ by computing the conserved charges at the instant $x^0 = 0$ when the D0-brane is at the lowest value of φ. For this suppose the lowest value of φ that a D0-brane reaches is $-K$ for some constant K. Since

[18] I wish to thank A. Strominger and J. Karczmarek for discussion on the issue of interpretation of ordinary D0-branes in two dimensional string theory.

the computation of $Q_{j,m}$ involves computing the disk one point function of $\phi_{j,m}$ which has a φ dependence of the form $e^{2(1-j)\varphi}$ for large negative φ, we expect that at $x^0 = 0$, $Q_{j,m}$ for this configuration will be of order

$$\frac{1}{g_s}\, e^{-2\varphi}\, e^{2(1-j)\varphi}\Big|_{\varphi=-K} \sim \frac{1}{g_s}\, e^{2jK}\,. \tag{7.6}$$

In (7.6) the first factor of $g_s^{-1}\, e^{-2\varphi}$ reflects the effect of the overall factor of $g_s^{-1}\, e^{-\Phi_D} = g_s^{-1}\, e^{-2\varphi}$ in the D0-brane world-volume action. On the other hand for a high energy hole state (7.4) for large α gives

$$Q_{j,m}(x^0 = 0) \sim \frac{1}{g_s}\, e^{2j\alpha}\,. \tag{7.7}$$

Thus (7.6) and (7.7) agree under the identification $\alpha = K$.

The boundary state for these hole solutions have recently been proposed for $\mu = 0$[77]. In the euclidean version these branes correspond to the hairpin brane solutions[78, 79], which for large negative φ take the form of a pair of parallel D-branes with Neumann boundary condition on φ and Dirichlet boundary condition on X. Thus we would expect that in the $\mu \neq 0$ theory, for large negative φ these branes will look like a pair of FZZT branes[80, 81] with Dirichlet boundary condition on X. This suggests that the CFT description of these branes might come from beginning with FZZT branes with Dirichlet boundary condition on X(which has $T_{xx} = 0$ and hence must represent a zero energy hole) and then deforming the world-sheet theory by an appropriate boundary term. It will be interesting to carry out this construction and see if the corresponding conserved charges agree with the expected charges carried by the hole.

8 $\mu \to 0$ Limit

So far in our analysis we have set $\mu = 1$ by a shift of the Liouville field φ. In this section we shall bring in the factors of μ and then take the limit $\mu \to 0$ in order to compare our results with those given in [36, 37, 40, 42].[19] For this we need to shift $\varphi \to \varphi + \frac{1}{2}\ln\mu$ so that the cosmological constant term $e^{2\varphi}$ gets transformed to $\mu e^{2\varphi}$. From eqs.(3.23), (3.13) and (3.16) we see that in the weak coupling region of large negative φ the gauge transformation parameter $\Lambda_{j,m}$ has exponential dependence on φ of the form $e^{2(1-j)\varphi}$. Thus it will pick up a multiplicative factor of μ^{1-j} under this shift of φ. This suggests

[19] Alternatively, we could keep μ fixed at 1 and take the large imaginary λ limit so that the total energy $\frac{1}{g_s}\cos^2(\pi\lambda)$ becomes large compared to $\frac{1}{g_s}$.. In the matrix model language this will correspond to taking the limit $|p|, |q| >> \frac{1}{\sqrt{g_s}}$.

that we should now use new gauge transformation parameters $\widehat{\Lambda}_{j,m}$ which are related to the old parameters by the relation:

$$\widehat{\Lambda}_{j,m} = \mu^{j-1}\,\Lambda_{j,m}\,, \tag{8.1}$$

so that $\widehat{\Lambda}_{j,m}$ do not have any explicit μ-dependence in the weak coupling region. The corresponding charges $\widehat{Q}_{j,m}$ are related to $Q_{j,m}$ by a multiplicative factor of μ^{j-1}:

$$\widehat{Q}_{j,m} = \mu^{j-1}\,Q_{j,m}\,. \tag{8.2}$$

The shift in φ also induces a redefinition of the closed string coupling constant g_s. To see this note that before the shift the kinetic terms for closed string fields are multiplied by an overall factor of $g_s^{-2}e^{-2\Phi_D} = g_s^{-2}e^{-4\varphi}$. After shifting φ by $\frac{1}{2}\ln\mu$ this factor becomes $g_s^{-2}\mu^{-2}e^{-4\varphi}$. This suggests that we define a new closed string coupling constant

$$\widehat{g}_s = g_s\,\mu\,. \tag{8.3}$$

We shall now express $\widehat{Q}_{j,m}$ carried by a single D0-brane in terms of the matrix model variables. Replacing the $\frac{1}{g_s}$ factor in the matrix model by $\frac{\mu}{\widehat{g}_s}$ in eqs.(6.6), (6.7) gives[20]

$$Q_{j,m} = h(q,p)\sum_{l=|m|}^{j-1}\left(\frac{\widehat{g}_s}{2\mu}\right)^{l} b_j^{l,m}(q+p)^{l+m}(q-p)^{l-m} \tag{8.4}$$

with

$$h(q,p) = \frac{p^2}{2} - \frac{q^2}{2} + \frac{\mu}{\widehat{g}_s}\,. \tag{8.5}$$

Hence from (8.2)

$$\widehat{Q}_{j,m} = \mu^{j-1}\left(\frac{p^2}{2} - \frac{q^2}{2} + \frac{\mu}{\widehat{g}_s}\right)\sum_{l=|m|}^{j-1}\left(\frac{\widehat{g}_s}{2\mu}\right)^{l} b_j^{l,m}(q+p)^{l+m}(q-p)^{l-m}\,. \tag{8.6}$$

Let us now consider the $\mu\to 0$ limit of (8.6) keeping $\widehat{g}_s$, p and q fixed. In this limit only the $l=(j-1)$ term contributes to the sum, and we get

$$\widehat{Q}_{j,m} = -\frac{1}{2}\left(\frac{\widehat{g}_s}{2}\right)^{j-1} b_j^{j-1,m}(q+p)^{j+m}(q-p)^{j-m}\,. \tag{8.7}$$

[20]Since in the $\mu\to 0$ limit the potential barrier at the fermi level between the negative q side and the positive q side disappears, we expect that this limit gives a sensible theory only when negative energy levels on both sides of the barrier are filled. Thus strictly speaking the analysis of this section and of section 10 will be sensible only when we repeat this in the type 0B string theory[4, 5]. However since our analysis is based on the study of conserved charges and does not involve the details of the dynamics, we expect that the results obtained from our analysis will survive in the full theory where negative energy states on both sides of the barrier are filled, even though the system that we are studying has negative energy states filled only on one side of the barrier.

From (4.39) and the fact that $\alpha^{j-|m|-1}_{j-1,-m}$ vanishes since $\alpha^s_{j',m}$ vanishes for $s > j' - |m| - 1$, we get:

$$b_j^{j-1,m} = -\frac{1}{(2j)!(2j-1)!}\,\alpha^{j-|m|-1}_{j,-m}\,. \tag{8.8}$$

An explicit expression for $\alpha^s_{j,m}$ has been given in eq.(4.14). From this one can show that

$$\alpha^{j-|m|-1}_{j,m} = -\frac{(2j)!}{(j+m)!(j-m)!}\,. \tag{8.9}$$

The identity required for proving this is obtained by comparing the coefficients of $x^{j-m}y^{j+m}$ in the equation:

$$(x+y)^{2j} = (x+y)^{j+m}(x+y)^{j-m}\,, \tag{8.10}$$

by expressing each of the three factors in a binomial expansion. Using eqs.(8.8) and (8.9), eq.(8.7) may be expressed as

$$\begin{aligned}\widehat{Q}_{j,m} &= \frac{1}{2}\left(\frac{\widehat{g}_s}{2}\right)^{j-1}\frac{1}{(2j)!(2j-1)!}\,\alpha^{j-|m|-1}_{j,-m}\,(q+p)^{j+m}(q-p)^{j-m} \\ &= -\frac{1}{2}\,\frac{1}{(2j-1)!(j+m)!(j-m)!}\left(\frac{\widehat{g}_s}{2}\right)^{j-1}(q+p)^{j+m}(q-p)^{j-m}\,. \end{aligned}\tag{8.11}$$

Up to an overall normalization constant this expression for $\widehat{Q}_{j,m}$ agrees with the results of [36, 37, 40, 42] based on the analysis of the algebra of symmetries generated by these transformations. Some issues regarding normalization factors have been discussed in appendix B.

It is also instructive to consider the $\mu \to 0$ limit of the discrete state closed string field configuration produced by the $|\mathcal{B}_2\rangle$ component of the boundary state. After the shift of φ by $\frac{1}{2}\ln\mu$, (5.7) takes the form:

$$|\check{\Phi}^{(2)}\rangle = \widehat{g}_s\sum_j\sum_{m=-(j-1)}^{j-1}\frac{\sqrt{\pi}}{((2j)!)^2}\mu^j\, f_{j,m}(\lambda)|j,m\rangle_X \otimes e^{2(1+j)\varphi(0)}|0\rangle_{liouville}\otimes c_1\bar{c}_1|0\rangle_{ghost}\,. \tag{8.12}$$

The relation between $f_{j,m}$ and $Q_{j',m'}$ has been given in (5.8). Using (8.2) and (8.3) we can rewrite this as

$$f_{j,-m}(\lambda) = -\widehat{g}_s\sum_{\substack{1\le j'\le j\\ j-j'\in Z}}(2j')!(2j'-1)!\mu^{-j'}\left(\widehat{Q}_{j',m}e^{-2mx^0}\right)\,. \tag{8.13}$$

Substituting this into (8.12) we see that only the $j' = j$ term contributes in the $\mu \to 0$ limit, and gives:

$$|\check{\Phi}^{(2)}\rangle = -(\widehat{g}_s)^2\sum_j\sum_{m=-(j-1)}^{j-1}\frac{\sqrt{\pi}}{2j}\left(e^{2mx^0}\,\widehat{Q}_{j,-m}\right)|j,m\rangle_X \otimes e^{2(1+j)\varphi(0)}|0\rangle_{liouville}\otimes c_1\bar{c}_1|0\rangle_{ghost}\,. \tag{8.14}$$

Thus $|\check{\Phi}^{(2)}\rangle$ has a finite non-zero $\mu \to 0$ limit.

As argued in section 5, $|\mathcal{B}_2\rangle$ also produces additional closed string background involving states of the form given in (5.14), with coefficient proportional to $f_{j-1,m}(\lambda)$. When we express $f_{j-1,m}(\lambda)$ in terms of $\widehat{Q}_{j',m'}$ by replacing j by $j-1$ in (8.13), the leading contribution is of order $\mu^{-(j-1)}$. On the other hand the shift of φ by $\frac{1}{2}\ln\mu$ in (5.14) and expressing the overall factor of g_s in terms of $\widehat{g}_s$ produces the same factor of μ^j as in (8.12). Thus in the $\mu \to 0$ limit the contribution from this term vanishes. This shows that (8.14) is the only contribution left in this limit. This is consistent with the fact that for $\mu = 0$ we can choose to use the oscillator description of the field φ, and in that case the additional states appearing in (5.14) are not BRST invariant as $(Q_B + \bar{Q}_B)$ acting on these states do not vanish when we express all states in terms of φ oscillators. Thus had the coefficients of the additional states not vanished in the $\mu \to 0$ limit, we shall be left with a field configuration that is not on-shell asymptotically.

For $\mu = 0$ the string coupling $\widehat{g}_s$ is an irrelevant constant. This is obvious in the matrix model since $h(q,p)$ given in (8.5) is independent of $\widehat{g}_s$ in this limit. In the continuum description we see this by noting that a shift $\varphi \to \varphi - \frac{1}{2}\ln\widehat{g}_s$ removes the $\widehat{g}_s$ from the pre-factor $\widehat{g}_s^{-2}e^{-4\varphi}$ in the closed string action. A consistency check of eq.(8.14) will be that after this shift of φ the relation between $|\check{\Phi}^{(2)}\rangle$ and the matrix model variables q, p should be independent of $\widehat{g}_s$. Is this true? Using (8.11) we see that this shift in φ changes (8.14) to

$$\begin{aligned} |\check{\Phi}^{(2)}\rangle \;&=\; \sqrt{\pi}\sum_j \sum_{m=-(j-1)}^{j-1} 2^{-j}\,\frac{1}{(2j)!(j+m)!(j-m)!}\, e^{2mx^0}\,(q+p)^{j-m}\,(q-p)^{j+m} \\ &\quad |j,m\rangle_X \otimes e^{2(1+j)\varphi(0)}|0\rangle_{liouville} \otimes c_1\bar{c}_1|0\rangle_{ghost}\,. \end{aligned} \tag{8.15}$$

As required, this equation has no $\widehat{g}_s$ dependence.

9 Conserved Charges from Asymptotic String Field Configurations

So far in our analysis we have discussed how to compute the conserved charges $\widehat{Q}_{j,m}$ for a D-brane system in terms of its boundary state. As discussed in section 5, since the boundary state of a D-brane acts as sources for various closed string fields, it produces closed string field configuration. We should expect that the asymptotic field configuration also contains information about the conserved charges of the system, just as long range electric field and gravitational field contain information about the charge and mass of a configuration in Maxwell-Einstein theory. It will be useful to find an expression for the conserved charges in terms of the asymptotic field configuration, since such a formula will have a more general validity even for configurations which are not D-branes. This is what we shall do now. Our strategy will be to first find the relation between the conserved charges and asymptotic field configurations for a D-brane system, and then treat this as a universal relation that holds for any configuration in this theory. During this analysis we shall work with general μ from the beginning and call the string coupling constant $\widehat{g}_s$, so that after inverse Wick rotation the conserved charges obtained from this analysis directly gives the charges $\widehat{Q}_{j,m}$.

Our strategy will be to formally manipulate the expression for the conserved charge carried by a D-brane to write it as an integral of a total derivative with respect to the Liouville coordinate φ, so that we can express it as a boundary term evaluated at large negative φ. We begin with the expression (3.31) of the conserved charge $F_{j,m}(x)$. In this expression $|\phi_{j,m}(p)\rangle$ is built on a fixed Liouville primary $V_{2(1-j)}(0)|0\rangle_{liouville}$. Let us consider a general family of states $|\phi_{j,m}(p,q)\rangle$ built on the Liouville primary $V_{2+iq}(0)|0\rangle_{liouville}$ such that

$$|\phi_{j,m}(p,q=2ij)\rangle = |\phi_{j,m}(p)\rangle\,, \tag{9.1}$$

and define:

$$F_{j,m}(x,q) = \int \frac{dp}{2\pi}\, e^{-ipx} \langle\mathcal{B}|(c_0-\bar{c}_0)|\phi_{j,m}(p,q)\rangle\,. \tag{9.2}$$

Clearly $F_{j,m}(x, q = 2ij)$ gives us the conserved charge $F_{j,m}(x)$ defined in (3.31). Our goal will be to express $F_{j,m}(x,q)$ in terms of the closed string field $|\Phi\rangle$ produced by the boundary state. For this we note that since $|\phi_{j,m}(p)\rangle$ is BRST invariant, $(Q_B + \bar{Q}_B)|\phi_{j,m}(p,q)\rangle$ must vanish at $q = 2ij$. This allows us to define a new family of states $|\Omega_{j,m}(p,q)\rangle$ through the relations:

$$(Q_B+\bar{Q}_B)|\phi_{j,m}(p,q)\rangle = (q-2ij)|\Omega_{j,m}(p,q)\rangle\,. \tag{9.3}$$

Next we note that the linearized closed string field equation (5.1) with g_s replaced be $\widehat{g}_s$ takes the form:

$$-2\langle\Phi|(Q_B+\bar{Q}_B) = \widehat{g}_s^{\,2}\,\langle\mathcal{B}|\,. \tag{9.4}$$

Using (9.3), (9.4) and the fact that neither $|\phi_{j,m}(p,q)\rangle$ nor the closed string field $|\Phi\rangle$ carries a $(c_0-\bar{c}_0)$ zero mode, we can now express (9.2) as

$$F_{j,m}(x,q) = (q-2ij)G_{j,m}(x,q)\,, \tag{9.5}$$

where

$$G_{j,m}(x,q) = 2\,\widehat{g}_s^{\,-2} \int \frac{dp}{2\pi}\, e^{-ipx}\langle\Phi|(c_0-\bar{c}_0)|\Omega_{j,m}(p,q)\rangle\,. \tag{9.6}$$

If we define:

$$\widetilde{G}_{j,m}(x,\varphi) = \int \frac{dq}{2\pi}\, e^{(2-iq)\varphi} G_{j,m}(x,q)\,, \tag{9.7}$$

then (9.5) gives

$$F_{j,m}(x,q) = i\int d\varphi e^{i(q-2ij)\varphi}\,\partial_\varphi\left(e^{-2(1+j)\varphi}\widetilde{G}_{j,m}(x,\varphi)\right)\,. \tag{9.8}$$

Thus

$$F_{j,m}(x) = F_{j,m}(x, q=2ij) = i\int d\varphi\,\partial_\varphi\left(e^{-2(1+j)\varphi}\widetilde{G}_{j,m}(x,\varphi)\right)\,. \tag{9.9}$$

The right hand side of (9.9) is a total derivative in the Liouville coordinate. Thus we can identify the conserved charge $F_{j,m}(x)$ as the boundary value of $-ie^{-2(1+j)\varphi}\widetilde{G}_{j,m}(x,\varphi)$ as $\varphi\to-\infty$:

$$\begin{aligned} F_{j,m}(x) &= -i\lim_{\varphi\to-\infty}\left(e^{-2(1+j)\varphi}\widetilde{G}_{j,m}(x,\varphi)\right) \\ &= -2i\,\widehat{g}_s^{-2}\lim_{\varphi\to-\infty}\left(e^{-2(1+j)\varphi}\int\frac{dp}{2\pi}\,e^{-ipx}\left[\int\frac{dq}{2\pi}\,e^{(2-iq)\varphi}\,\langle\Phi|(c_0-\bar{c}_0)|\Omega_{j,m}(p,q)\rangle\right]\right)\,. \end{aligned} \tag{9.10}$$

In this expression the term inside the square bracket may be interpreted as the φ-space wave-function of an appropriate component of the state $|\Phi\rangle$ at large φ. Thus (9.10) after inverse Wick rotation gives the desired expression for the charges $\widehat{Q}_{j,m}$ in terms of the asymptotic closed string field configuration. Since this relation expresses $\widehat{Q}_{j,m}$ in terms of field configurations in the weak coupling region, we can use this to define $\widehat{Q}_{j,m}$ even in the $\mu\to 0$ limit.

There are clearly infinite number of ways of defining the family of states $|\phi_{j,m}(p,q)\rangle$ satisfying the requirement that $|\phi_{j,m}(p,q=2ij)\rangle = |\phi_{j,m}(p)\rangle$. This induces an ambiguity in the definition of $|\Omega_{j,m}(p,q)\rangle$ in the form of the freedom of adding a BRST exact contribution. However as long as the asymptotic closed string field configuration is on-shell, the freedom of adding a BRST exact contribution to $|\Omega_{j,m}\rangle$ does not affect the expression (9.10) for the conserved charge since a BRST exact state has vanishing inner product with an on-shell state.

We shall now illustrate this construction for the $\mu=0$ case that will be of interest to us in section 10. In this case we can regard φ as a scalar field with background charge and use the oscillators of φ to label states. Let us suppose $|\Omega_{j,m}(p,q)\rangle$ has the form

$$|\Omega_{j,m}(p,q)\rangle = \widehat{\Omega}_{j,m}(p,q)e^{ipX(0)}|0\rangle_X\otimes e^{(2+iq)\varphi(0)}|0\rangle_{liouville}\otimes|0\rangle_{ghost}\,, \tag{9.11}$$

for some combination $\widehat{\Omega}_{j,m}(p,q)$ of ghost oscillators and non-zero mode X and φ oscillators. Then if the asymptotic closed string field $|\Phi\rangle$ has a term of the form:

$$\widehat{\Phi}\, e^{ip'X(0)}|0\rangle_X\otimes e^{(2+iq')\varphi(0)}|0\rangle_{liouville}\otimes|0\rangle_{ghost}\,, \tag{9.12}$$

for some combination $\widehat{\Phi}$ of ghost oscillators and non-zero mode X and φ oscillators, then the contribution to the right hand side of (9.10) from this term in $|\Phi\rangle$ is given by:

$$-2i\widehat{g}_s^{-2} \lim_{\varphi\to-\infty} \left(e^{-2(1+j)\varphi} \langle\langle 0|\widehat{\Phi}^T \, (c_0 - \bar{c}_0) \, \widehat{\Omega}_{j,m}(-p', -q')|0\rangle\rangle e^{ip'x} e^{(2+iq')\varphi} \right) . \tag{9.13}$$

Here $\widehat{\Phi}^T$ denotes the BPZ conjugate of $\widehat{\Phi}$, and $\langle\langle 0|\cdot|0\rangle\rangle$ denotes matrix element involving only the ghost oscillators and the non-zero mode oscillators of the X and φ fields.

Since conservation laws imply that $F_{j,m}(x)$ must have the form e^{-2imx} we see from (9.13) that the relevant component of $|\Phi\rangle$ that contributes to $F_{j,m}(x)$ must have $p' = -2m$. As a result in (9.13) we can replace $-p'$ in the argument of $\widehat{\Omega}_{j,m}$ by $2m$. Thus the computation of the value of the conserved charge requires $|\Omega_{j,m}(p, q)\rangle$ for $p = 2m$. This computation can be simplified by the following observation. Since for $\mu = 0$ the only non-trivial elements of BRST cohomology at ghost number two and Liouville momentum $q = 2ij$ are the states $|Y^L_{j,m}\rangle \times |Y^R_{j,m}\rangle$, the rest of the contributions in (3.28) are BRST trivial and can be ignored. This allows us to choose

$$|\phi_{j,m}(p = 2m, q)\rangle = |j, m\rangle_X \otimes e^{(2+iq)\varphi(0)}|0\rangle_{liouville} \otimes c_1\bar{c}_1|0\rangle_{ghost} \tag{9.14}$$

and hence from (9.3)

$$|\Omega_{j,m}(p = 2m, q)\rangle = \frac{1}{4}\,(q+2ij)|j, m\rangle_X \otimes e^{(2+iq)\varphi(0)}|0\rangle_{liouville} \otimes (c_0 + \bar{c}_0)\, c_1\bar{c}_1|0\rangle_{ghost} \, . \tag{9.15}$$

Comparing this with (9.11) and using (3.6) we see that

$$\widehat{\Omega}_{j,m}(p = 2m, q) = \frac{1}{4}(q + 2\,i\,j)\, \mathcal{P}^L_{j,m}\, \mathcal{P}^R_{j,m}\, (c_0 + \bar{c}_0)\, c_1\bar{c}_1 \, . \tag{9.16}$$

(9.13) together with the orthonormality relations (3.8) now shows that the contribution to $F_{j,m}$ from a $|\Phi\rangle$ that is on-shell asymptotically comes from the coefficient of the BRST cohomology element:

$$|j, -m\rangle_X \otimes e^{2(1+j)\varphi(0)}|0\rangle_{liouville} \otimes c_1\bar{c}_1|0\rangle_{ghost} \, , \tag{9.17}$$

in $|\Phi\rangle$.

We shall now test (9.13) by computing the charges from the asymptotic field configuration given in (8.14) (after making the replacement $x^0 \to -ix$). The only term in $|\Phi\rangle$ that contributes to $F_{j,m}(x)$ is the term:

$$-(\widehat{g}_s)^2 \,\frac{\sqrt{\pi}}{2j}\, \left(e^{2imx}\, \widehat{Q}_{j,m}\right)\, |j, -m\rangle_X \otimes e^{2(1+j)\varphi(0)}|0\rangle_{liouville} \otimes c_1\bar{c}_1|0\rangle_{ghost} \, . \tag{9.18}$$

Comparison with (9.12) shows that for this term

$$\widehat{\Phi} = -(\widehat{g}_s)^2 \, \frac{\sqrt{\pi}}{2j} \, \left(e^{2imx} \, \widehat{Q}_{j,m}\right) \, \mathcal{P}^L_{j,-m} \, \mathcal{P}^R_{j,-m} \, c_1 \bar{c}_1 \, , \qquad p' = -2m, \qquad q' = -2ij \, . \quad (9.19)$$

Substituting (9.16) and (9.19) into (9.13), and using the normalization condition (3.8) we get

$$\begin{aligned} F_{j,m}(x) &= -2i\widehat{g}_s^{-2} \lim_{\varphi\to-\infty} \left(e^{-2(1+j)\varphi} (-(\widehat{g}_s)^2) \, \frac{\sqrt{\pi}}{2j} \, \left(e^{2imx} \, \widehat{Q}_{j,m}\right) \, (ij) \, (2) \, e^{-2imx} \, e^{2(1+j)\varphi} \right) \\ &= -2\sqrt{\pi} \, \widehat{Q}_{j,m} \, . \end{aligned} \quad (9.20)$$

This agrees with the relation between $F_{j,m}$ and $Q_{j,m}$ introduced in (3.32). One factor of (2) in (9.20) comes from the ghost correlator.

Of course in this case this agreement is not a surprise since we have defined $Q_{j,m}$ through eq.(3.32). However we can take this agreement as a test of our equations (9.10), (9.13) relating the charges $\widehat{Q}_{j,m}$ to the asymptotic field configuration. We shall use these relations in section 10 to compute the charges $\widehat{Q}_{j,m}$ carried by the black hole.

10 Two Dimensional Black Holes

We can use the results of section 9 to calculate the charges $\widehat{Q}_{j,m}$ for any string field configuration in two dimensional string theory, and then try to identify a corresponding matrix model state that carries the same set of conserved charges. One particular configuration that is of interest is the two dimensional black hole solution[30, 31] in the $\mu \to 0$ limit. In this section we shall try to compute the conserved charges $\widehat{Q}_{j,m}$ carried by the black hole and then try to identify an appropriate configuration in the matrix model that carries such charges.[21] Since for $\mu = 0$ we can remove the string coupling constant $\widehat{g}_s$ by a shift of φ, we shall set $\widehat{g}_s = 1$ in this analysis.

In order to calculate the conserved charges carried by the black hole we need to find the closed string field configuration associated with the black hole for large negative φ. Thus we need to be able to represent the black hole as a classical solution in string field theory. We begin by writing down the solution in the effective field theory in the Schwarzschild like coordinates[30, 31]:

$$G_{00} = -(1 - ae^{4\varphi}), \quad G_{\varphi\varphi} = (1 - ae^{4\varphi})^{-1}, \quad \Phi_D = 2\varphi \, , \quad (10.1)$$

[21]For previous attempts at identifying the black hole in the matrix model see [32, 82, 83, 84, 85, 86, 87]. It is not completely clear how to compare our results for $\mu = 0$ Lorentzian black hole with the results of these papers.

where $G_{\mu\nu}$ denotes the string metric, Φ_D denotes the dilaton, and a is a parameter related to the mass of the black hole. This satisfies the equations of motion derived from the action of two dimensional dilaton gravity:

$$\mathcal{S} = \int dx^0\, d\varphi\, e^{-2\Phi_D}\, \sqrt{-\det G}\, (R_G + 4G^{\mu\nu}\partial_\mu\Phi_D\partial_\nu\Phi_D + 16)\,. \tag{10.2}$$

Defining $h_{\mu\nu}$ and ϕ_D as the deviation of the metric and the dilaton from the flat Minkowski metric and linear dilaton background:

$$h_{\mu\nu} = G_{\mu\nu} - \eta_{\mu\nu}\,, \quad \phi_D = \Phi_D - 2\varphi\,, \tag{10.3}$$

we see that for large negative φ, (10.1) takes the form:

$$h_{00} = ae^{4\varphi}, \quad h_{\varphi\varphi} \simeq ae^{4\varphi}, \quad \phi_D = 0\,. \tag{10.4}$$

These give a solution of the linearized equations of motion derived from the action (10.2) around the linear dilaton background. The complete solution (10.1) can now be recovered as a power series expansion in $e^{4\varphi}$ by beginning with (10.4) and then iteratively solving the full equations of motion derived from (10.2). In particular r iterations will be needed to recover the term of order $e^{4r\varphi}$ in the expansion of the solution (10.1) in powers of $e^{4\varphi}$.

We shall now use this construction as a guideline to construct the black hole solution in closed string field theory[49]. We begin with the observation that up to gauge transformation, the solution (10.4) corresponds to switching on a closed string background proportional to

$$a\,|j=1, m=0\rangle_X \otimes e^{4\varphi(0)}|0\rangle_{liouville} \otimes c_1\bar{c}_1|0\rangle_{ghost}\,. \tag{10.5}$$

This, being a BRST invariant state, is clearly a solution of the linearized equations of motion of closed string field theory. We can now generate the full solution of the closed string field equations by beginning with (10.5) and then solving the closed string field equations iteratively.[22] Our analysis is facilitated by the fact that the initial configuration (10.5) is independent of the time coordinate x^0. Thus we can work in a restricted subsector of closed string field theory involving time independent fields (*i.e.* fields carrying zero X^0 momentum). This restricted string field theory action has an $SU(2)_L \times SU(2)_R$ symmetry inherited from the $SU(2)_L \times SU(2)_R$ symmetry of the euclidean theory at the self-dual

[22]For Euclidean black hole it has been suggested that the solution is obtained by deforming a flat linear dilaton background by a non-normalizable operator carrying non-zero winding number along the X direction[32, 88]. The relationship between our approach for generating the solution in the Minkowski theory and that of [32, 88] is not clear. We note however that even a conventional D-brane wrapping a compact direction carries winding charge along that direction. But once we go to the universal cover of the circle describing a non-compact direction, the winding charge disappears from the boundary state describing the D-brane.

radius. In particular the string field interaction terms can couple a field of $SU(2)_L$ (or $SU(2)_R$) quantum number j to a set of fields carrying SU(2) quantum numbers $j_1, \ldots j_r$ only if $j \le j_1 + \ldots j_r$.

It is now clear that the order $e^{4r\varphi}$ term, obtained by iteration of the string field equations with (10.5) as the starting point, must be a linear combination of terms of the form:

$$\mathcal{K}_r \, (|j', m=0\rangle_L \times |j'', m=0\rangle_R) \otimes e^{4r\varphi(0)}|0\rangle_{liouville} \otimes c_1\bar{c}_1|0\rangle_{ghost}\,, \quad j', j'' \le r\,, \tag{10.6}$$

where $\mathcal{K}_r$ is some combination of X and Liouville Virasoro generators and ghost oscillators. Comparing this with (9.17) we see that in order that it contributes to $F_{j,m}(x)$ we must have $j = 2r-1$ so as to match the power of e^{φ} in (9.17) and (10.6). On the other hand comparing the SU(2) quantum numbers we see that in order that (10.6) contributes to $F_{j,m}(x)$, j' and j'' must be equal to j. This gives $j \le r$. This gives the restriction:

$$2r - 1 \le r, \qquad i.e. \qquad r \le 1\,. \tag{10.7}$$

Thus we see that all contributions to $F_{j,m}(x)$ vanish for $j > 1$. After inverse Wick rotation this gives:

$$\widehat{Q}_{j,m}/\widehat{Q}_{1,0} = 0 \quad \text{for } j > 1\,. \tag{10.8}$$

We can now look for a configuration in the matrix model that carries the same quantum numbers. We focus on the quantum numbers $\widehat{Q}_{j,0}$. (8.11) shows that any single fermion state of energy E will carry non-zero $\widehat{Q}_{j,0}$ for all j, and hence cannot represent a black hole. The situation does not improve much by superposing a finite number of fermion / hole states of different energies.[23] To see this note that if we have fermions of energy E_1, E_2, … and holes of energies ϵ_1, ϵ_2, …, so that the total energy of the system is given by

$$\sum_k E_k + \sum_r \epsilon_r\,, \tag{10.9}$$

then $\widehat{Q}_{j,0}$ associated with this state, as computed from (8.11), is proportional to

$$\left(\sum_r (\epsilon_r)^j - \sum_k (-E_k)^j\right)\,. \tag{10.10}$$

For odd j all the contributions add and hence it seems impossible to adjust the E_k's and the ϵ_r's so that the total $\widehat{Q}_{1,0}$ is finite or large while all other $\widehat{Q}_{j,0}/\widehat{Q}_{1,0}$ ratios vanish. The

[23] We are working under the assumption that the Lorentzian black hole can be described within the free fermion description of the matrix model, and does not, for example, involve U(N) non-singlet sector of the matrix model[32].

only way to avoid this situation will be to take a superposition of many single fermion and hole states, each with energy E_k or ϵ_r close to zero, so that the total energy adds up to the mass M of the black hole.[24] If there are N fermions, and we take each of E_k and ϵ_r to be of order M/N, then the $\widehat{Q}_{j,m}$ associated with this configuration is of order

$$N(M/N)^j \sim M^j N^{1-j} \,. \tag{10.11}$$

Thus in the $N \to \infty$ limit this vanishes for all $j > 1$, as is required for a black hole.

This analysis suggests that the black hole of two dimensional string theory can be thought of as a collection of a large number of fermions and holes, each with a very small energy, so that the total energy adds up to the mass M of the black hole. This is consistent with the fact that classically for $\mu = 0$ only a pulse of very small height can stay in the strong coupling region near the top of the potential for a long period.

If this is really the correct description of the black hole then it gives rise to an apparent puzzle. Since a black hole consists of a large number of fermions and holes close to zero energy, a finite energy fermion or hole will not feel the effect of the black hole in the classical limit. If we identify these finite energy excitations with D0-branes as in the $\mu \neq 0$ case, then this will imply that a classical D0-brane in this string theory should not be able to distinguish between the black hole geometry and the usual linear dilaton background. Can this be true? D0-branes in two dimensional black hole geometry have been analyzed from various viewpoints in [91, 78, 92, 93], and we can borrow these results to make this comparison. First let us note that the usual linear dilaton background of two dimensional string theory is given by

$$ds^2 \equiv G_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + d\varphi^2, \qquad \Phi_D = 2\varphi \,, \tag{10.12}$$

where $G_{\mu\nu}$ denotes the string metric, Φ_D is the dilaton, and x^0 and $x^1 \equiv \varphi$ are the coordinates of the two dimensional space. In this background the world-line theory of a D0-brane is given by the action:[25,26]

$$-\int d\tau e^{-\Phi_D} \sqrt{-G_{\mu\nu}\partial_\tau X^\mu \partial_\tau X^\nu} = -\int d\tau \sqrt{-e^{-2\Phi_D} G_{\mu\nu}\partial_\tau X^\mu \partial_\tau X^\nu} \,, \tag{10.13}$$

[24] Since these results were first reported in the Strings 2004 conference in Paris[89], ref.[90] has made similar observations.

[25] Note that according to the proposal made at the end of section 7 these D0-branes describe hole states rather than fermion states, but this does not affect our argument.

[26] Naively we would expect that in this case the D0-brane, besides containing the mode that allows it to move along the Liouville direction, will also have a tachyonic mode. From the description of the D0-brane as a single fermion in the matrix model, we do not have an obvious identification of this mode. We note however that in the presence of a linear dilaton background the D0-brane accelerates, and hence it is not completely clear if the tachyonic mode is really present even in the continuum description. On the other hand, due to this acceleration it is also not clear if the Dirac-Born-Infeld action provides a reliable description of the D0-brane dynamics. We shall nevertheless proceed with this action in the absence of a better alternative.

where τ is a parameter labelling the D0-brane world-line. Thus effectively the motion of the D0-brane is described by a particle moving under the metric[91]

$$ds_{D0}^2 \equiv e^{-2\Phi_D} G_{\mu\nu} dx^\mu dx^\nu = e^{-4\varphi}(-(dx^0)^2 + d\varphi^2)\,. \tag{10.14}$$

On the other hand the black hole background in the Schwarzschild-like coordinate system is given by[30, 31]:

$$\widetilde{ds}^2 = -(1 - ae^{4\varphi})(dx^0)^2 + (1 - ae^{4\varphi})^{-1} d\varphi^2, \qquad \Phi_D = 2\varphi\,, \tag{10.15}$$

where $\widetilde{ds}$ denotes the line element in the black hole background measured in the string metric. The D0-brane moving in this background sees an effective metric

$$\widetilde{ds}_{D0}^2 \equiv e^{-2\Phi_D}\, \widetilde{ds}^2 = e^{-4\varphi} \left[-(1 - ae^{4\varphi})(dx^0)^2 + (1 - ae^{4\varphi})^{-1} d\varphi^2\right]. \tag{10.16}$$

If we define

$$w = -\frac{1}{4} \ln\left(e^{-4\varphi} - a\right)\,, \tag{10.17}$$

then in this new coordinate system (10.16) takes the form:

$$\widetilde{ds}_{D0}^2 = e^{-4w}(-(dx^0)^2 + dw^2)\,. \tag{10.18}$$

This is identical to (10.14) up to a relabelling of the coordinate φ as w. Thus we see that the classical dynamics of a D0-brane in the black hole and the usual linear dilaton background are indeed identical. It is important to note that the coordinate transformation that relates the two metrics involves only a redefinition of the Liouville coordinate and does not involve x^0. This is consistent with the fact that for both the black hole and the linear dilaton background, the asymptotic time coordinate x^0 is identified as the time coordinate of the matrix model.

It is in fact easy to show that both (10.14) and (10.18) describe a flat two dimensional metric[91]. The easiest way to see this is to begin with the black hole background in the conformal gauge[30, 31]:

$$\widetilde{ds}^2 = -e^{2\Phi_D} du dv, \qquad e^{-2\Phi_D} = a - 4uv\,. \tag{10.19}$$

This gives:

$$\widetilde{ds}_{D0}^2 = e^{-2\Phi_D} \widetilde{ds}^2 = -du dv\,, \tag{10.20}$$

which is manifestly a flat metric. Thus in this coordinate system the classical solutions correspond to D0-branes moving with uniform velocity. Given such a solution we can convert the answer to any other coordinate system. Note however that unlike the previous transformation that related the black hole background to the linear dilaton background, in this case the coordinate transformation involves x^0 in a non-trivial fashion. Thus time in the coordinate system in which the D0-brane sees a flat metric does not coincide with the time coordinate of the matrix model.

Clearly there are many questions which remain to be answered, but we hope that our results will provide a useful starting point for a complete understanding of the two dimensional black hole in the matrix model. Understanding this and other questions will invariably lead to a deeper understanding of two dimensional string theory and also possibly critical string theories.

11 Lessons for Critical String Theory

Although we have made heavy use of the closed string gauge transformations, it should be noted that the final conclusions about the existence of global symmetries and associated conserved charges are for the open string field theory. For the D0-brane of the two dimensional string theory the action of this open string field theory can be taken to be the usual cubic action[50] evaluated for the specific boundary CFT describing the D0-brane. The open string spectrum in this theory is given by the product of an arbitrary state in the X and ghost CFT, and a state of the Liouville theory built by the action of the Liouville Virasoro generators on the SL(2,R) invariant vacuum[23]. In particular there is no open string state constructed over a non-trivial primary of the Liouville sector. As a result the relevant correlation functions needed for the construction of this open string field theory involves arbitrary combination of X and ghost operators and the Liouville stress tensor. These can be computed without any knowledge of the Liouville theory except its total central charge which is 25.

Now consider a different theory where we have replaced the Liouville theory by an arbitrary $c = 25$ CFT. Let us take an arbitrary D-brane in this theory and denote the corresponding boundary CFT by $\mathcal{F}$. The open string spectrum on this D-brane will of course involve excitations over non-trivial primaries of $\mathcal{F}$. However irrespective of what $c = 25$ theory we have, or what D-brane we are considering, there will always be a universal subsector of open string states obtained by restricting ourselves to only the $c = 25$ Virasoro descendants of the SL(2,R) invariant vacuum of $\mathcal{F}$ and arbitrary excitations in the ghost and X sector.[27] The open string field theory of this universal sector will be identical to the open string field theory of the D0-brane of the two dimensional string theory since all the relevant correlation functions are identical. Thus understanding the classical dynamics

[27] This is in fact the same universal theory that was used in [94, 95] for the construction of the tachyon lump solution and the rolling tachyon solution in open string field theory.

of the D0-brane in the two dimensional string theory gives us direct information about the classical dynamics of this universal subsector of the open string field theory on any D-brane in any background. In particular open string field configurations in this universal subsector will be characterized by the same conserved charges as in the two dimensional string theory.

The above discussion has been given using the open string viewpoint, but we could also try to directly analyze these conservation laws for a general D-brane system in a generic string theory following the procedure given in sections 3 and 4. For this we need to construct the analogs of the states $|\phi_{j,m}(p)\rangle$ in this more general string theory. Since in section 3 we have given an algorithm for constructing $|\phi_{j,m}(p)\rangle$ beginning with two sets of chiral states, $|Y^L_{j,m}(p)\rangle$ and $|\mathcal{O}^L_{j,m}(p)\rangle$ (and their right-moving analogs), all we need to do is to give the generalization of these states. This can be done as follows. Let us assume that the $c = 25$ theory has one free scalar field Z, and that the z-direction is orthogonal to the D-brane under consideration so that we have Dirichlet boundary condition on Z. We now replace in eqs.(3.13), (3.16) (and analogous equations for right-handed states) $V^L_{2(1-j)}$ by

$$e^{2\sqrt{j^2-1}\,Z_L}\,, \tag{11.1}$$

and the Virasoro generators of the Liouville field by the total Virasoro generators of the $c = 25$ theory. Formally $e^{2\sqrt{j^2-1}\,Z_L}$ is a primary of dimension $1 - j^2$, – same as that of $V^L_{2(1-j)}$ in the Liouville theory. This defines the analogs of $|\mathcal{O}^L_{j-1,m}(p)\rangle$ and $|Y^L_{j,m}(p)\rangle$ in this more general string theory. We can now define $|\eta^L_{(j),m}(p)\rangle$, $|\psi^L_{(j),m}(p)\rangle$ as in eq.(3.19) and (3.20), and $|\Lambda_{j,m}(p)\rangle$, $|\phi_{j,m}(p)\rangle$ through eqs. (3.23), (3.26). This in turn can be used to define the conserved charges through eq. (3.31). Note that due to the presence of the operator $e^{2\sqrt{j^2-1}\,Z}$, the gauge transformation parameter $|\Lambda_{j,m}(p)\rangle$ grows exponentially for large Z. However since the D-brane is localized at a finite value of Z, the action of the gauge transformation on the degrees of freedom on the D-brane is expected to be finite and give rise to finite conserved charges.

There is however a subtlety in this analysis. In general the states $|\mathcal{O}^L_{j-1,m}\rangle$ defined this way may not be annihilated by Q_B when we express the state in terms of Z oscillators. What is guaranteed however is that the result is a linear combination of null states of the $c = 25$ Virasoro algebra. As a result the right hand sides of eq.(3.19), (3.20), (3.25) now could contain additional terms which are linear combinations of these null states. The conservation law (3.30) still holds if the inner product of the boundary state $\langle\mathcal{B}|$ with these null states vanish. Typically null states are orthogonal to regular primary and secondary states, but have non-zero inner product with states which are neither primary states nor can be regarded as a secondary state over another primary. Thus as long as the boundary state $|\mathcal{B}\rangle$ does not have such states, the inner product of $\langle\mathcal{B}|$ with these null states will vanish and the conservation laws will hold.

In particular if the D-brane is such that there is separate conservation of the world-sheet energy momentum tensor associated with the $c = 25$ theory and the X CFT at the boundary of the world-sheet, then we expect that the null states of the $c = 25$ Virasoro algebra will have vanishing one point function on the disk and hence its inner product with the boundary state will vanish. The rolling tachyon solution on a D0-brane of the critical string theory of course satisfies this condition, and hence possesses conserved charges. Computation of these conserved charges can be done following the analysis of section 4. The final expression will be given by eq.(4.33) with ${}_{liouville}\langle\mathcal{B}|V_{2(1-j)}(0)|0\rangle_{liouville}$ replaced by the one point function of $e^{2\sqrt{j^2-1}Z(0)}$ on the unit disk. Let us denote this by $\langle e^{2\sqrt{j^2-1}Z(0)}\rangle_D$. This in turn is obtained by first computing $\langle e^{ikZ(0)}\rangle_D$, and then analytically continuing the result to $k = -2i\sqrt{j^2-1}$. Thus for example if the D-brane is localized at $z = a$, then we have $\langle e^{ikZ(0)}\rangle_D = Ce^{ika}$ for some known constant C, and hence $\langle e^{2\sqrt{j^2-1}Z(0)}\rangle_D = Ce^{2\sqrt{j^2-1}\,a}$.

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A Properties of $|\psi^L_{(j),m}\rangle$ and $|\eta^L_{(j),m}\rangle$

In section 3 we defined $|\eta^L_{(j),m}(p)\rangle$ and $|\psi^L_{(j),m}(p)\rangle$ through the relations:

$$Q_B|\mathcal{O}^L_{j-1,m}(p)\rangle = (p-2m)|\eta^L_{(j),m}(p)\rangle\,, \tag{A.1}$$

and

$$Q_B|Y^L_{j,m}(p)\rangle = (p-2m)|\psi^L_{(j),m}(p)\rangle\,, \tag{A.2}$$

where $|\mathcal{O}^L_{j-1,m}(p)\rangle$ and $|Y^L_{j,m}(p)\rangle$ are chiral states of conformal weight $(\frac{p^2}{4}-m^2)$:

$$|Y^L_{j,m}(p)\rangle = \mathcal{P}^L_{j,m}\, e^{ipX_L(0)}|0\rangle_X \otimes V^L_{2(1-j)}(0)|0\rangle_{liouville} \otimes c_1|0\rangle_{ghost}\,, \tag{A.3}$$

and

$$|\mathcal{O}^L_{j-1,m}(p)\rangle = \mathcal{R}^L_{j-1,m} e^{ipX_L(0)}|0\rangle_X \otimes V^L_{2(1-j)}(0)|0\rangle_{liouville} \otimes c_1|0\rangle_{ghost} \,. \tag{A.4}$$

In this appendix we shall determine the SU(2) quantum numbers and other properties of $|\eta^L_{(j),m}(p=2m)\rangle$ and $|\psi^L_{(j),m}(p=2m)\rangle$. We begin with the observation that in a subspace with matter momentum p, Q_B takes the form:

$$Q_B = \frac{1}{4}\, p^2\, c_0 + \frac{1}{\sqrt{2}}\, p \sum_{n\neq 0} c_{-n}\alpha_n + \widehat{Q}_B \,, \tag{A.5}$$

where $\widehat{Q}_B$ does not depend on p. Knowing that Q_B annihilates $|Y^L_{j,m}(p)\rangle$ and $|\mathcal{O}^L_{j-1,m}(p)\rangle$ for $p = 2m$, and that the oscillator parts $\mathcal{P}^L_{j,m}$, $\mathcal{R}^L_{j-1,m}$ involved in the construction of $|Y^L_{j,m}(p)\rangle$, $|\mathcal{O}^L_{j-1,m}(p)\rangle$ do not depend on p, we can write:

$$\begin{aligned} Q_B|Y^L_{j,m}(p)\rangle &= \left(\frac{1}{4}(p^2-4m^2)c_0 + \frac{1}{\sqrt{2}}(p-2m)\sum_{n\neq 0} c_{-n}\alpha_n\right)|Y^L_{j,m}(p)\rangle \\ Q_B|\mathcal{O}^L_{j-1,m}(p)\rangle &= \left(\frac{1}{4}(p^2-4m^2)c_0 + \frac{1}{\sqrt{2}}(p-2m)\sum_{n\neq 0} c_{-n}\alpha_n\right)|\mathcal{O}^L_{j-1,m}(p)\rangle \,. \end{aligned} \tag{A.6}$$

Comparing this with (A.1), (A.2) we get

$$\begin{aligned} |\psi^L_{(j),m}(p)\rangle &= \frac{1}{4}(p+2m)c_0|Y^L_{j,m}(p)\rangle + \frac{1}{\sqrt{2}}\sum_{n\geq 1} c_{-n}\alpha_n|Y^L_{j,m}(p)\rangle \\ |\eta^L_{(j),m}(p)\rangle &= \frac{1}{4}(p+2m)c_0|\mathcal{O}^L_{j-1,m}(p)\rangle + \frac{1}{\sqrt{2}}\sum_{n\neq 0} c_{-n}\alpha_n|\mathcal{O}^L_{j-1,m}(p)\rangle \,. \end{aligned} \tag{A.7}$$

Note that the sum over n in the first equation is restricted to $n \geq 1$ since $|Y^L_{j,m}(p)\rangle$ is annihilated by c_n for $n \geq 1$.

Let us now focus on the case $p = 2m$. From (A.7) it follows that $|\eta^L_{(j),m}\rangle \equiv |\eta^L_{(j),m}(p = 2m)\rangle$ and $|\psi^L_{(j),m}\rangle \equiv |\psi^L_{(j),m}(p = 2m)\rangle$ are given by

$$\begin{aligned} |\psi^L_{(j),m}\rangle &= mc_0|Y^L_{j,m}\rangle + \frac{1}{\sqrt{2}}\sum_{n\geq 1} c_{-n}\alpha_n|Y^L_{j,m}\rangle \\ |\eta^L_{(j),m}\rangle &= mc_0|\mathcal{O}^L_{j-1,m}\rangle + \frac{1}{\sqrt{2}}\sum_{n\neq 0} c_{-n}\alpha_n|\mathcal{O}^L_{j-1,m}\rangle \,. \end{aligned} \tag{A.8}$$

Clearly $|\eta^L_{(j),m}\rangle$ and $|\psi^L_{(j),m}\rangle$ must be linear combination of states which are obtained by the action of negative moded matter and Liouville Virasoro generators and negative or zero moded ghost oscillators on states of the form:

$$|j', m\rangle_L \otimes V^L_{2(1-j)}(0)|0\rangle_{liouville} \otimes c_1|0\rangle_{ghost} \,, \tag{A.9}$$

for some j'. We want to analyze the possible values of j' which could arise. The first terms in both equations on the right hand side of (A.8) clearly have $j' = j$ and $j-1$ respectively since c_0 is an SU(2) singlet. On the other hand, in the language of SU(2) current algebra, α_{-n} is simply the SU(2) current J^3_{-n}. Thus the action of α_{-n} on $|\mathcal{O}^L_{j-1,m}\rangle$ must be a linear combination of states of spin lying between $j-2$ and j, and that on $|Y^L_{j,m}\rangle$ must be a linear combination of states of spin j and $j \pm 1$. This leads us to the conclusion that $|\eta^L_{(j),m}\rangle$ is built on states of the form (A.9) with $(j-2) \leq j' \leq j$, and $|\psi^L_{(j),m}\rangle$ is built on states of the form (A.9) with $(j-1) \leq j' \leq (j+1)$.

We can further restrict the structure of $|\psi^L_{(j),m}\rangle$ and $|\eta^L_{(j),m}\rangle$ by analyzing their conformal weights. For this we note that both $|\psi^L_{(j),m}\rangle$ and $|\eta^L_{(j),m}\rangle$ have conformal weight 0. On the other hand the state given in (A.9) has conformal weight $(j')^2 - j^2$. Thus $|\eta^L_{(j),m}\rangle$ and $|\psi^L_{(j),m}\rangle$ can only be linear combination of states built on states of the form (A.9) with $j' \leq j$. Furthermore, for $j' = j$, there cannot be any non-zero mode oscillator acting on (A.9). Since $|\eta^L_{(j),m}\rangle$ and $|\psi^L_{(j),m}\rangle$ have ghost numbers 1 and 2 respectively, it follows that the $j' = j$ state appearing in the expression for $|\eta^L_{(j),m}\rangle$ and $|\psi^L_{(j),m}\rangle$ must be proportional to $|Y^L_{j,m}\rangle$ and $c_0|Y^L_{j,m}\rangle$ respectively. It is known that the coefficient of $|Y^L_{j,m}\rangle$ appearing in the expression for $|\eta^L_{(j),m}\rangle$ is non-zero[36]. We shall normalize $|\mathcal{O}^L_{j-1,m}\rangle$ such that this coefficient is one. Thus we get

$$|\eta^L_{(j),m}\rangle = |Y^L_{j,m}\rangle + |\widehat{\eta}^L_{(j),m}\rangle\,, \tag{A.10}$$

where $|\widehat{\eta}^L_{(j),m}\rangle$ is built on states $|j', m\rangle$ with $j' = (j-1)$ or $(j-2)$.

For $|\psi^L_{(j),m}\rangle$, eq.(A.8), together with the constraints based on SU(2) symmetry, ghost number and conformal weight, give

$$|\psi^L_{(j),m}\rangle = m\, c_0\, |Y^L_{j,m}\rangle + |\tau^L_{j-1,m}\rangle \tag{A.11}$$

where

$$|\tau^L_{j-1,m}\rangle = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} c_{-n}\alpha_n |Y^L_{j,m}\rangle\,, \tag{A.12}$$

must be built by the action of X and Liouville Virasoro and ghost oscillators on a state of the form (A.9) with $j' = (j-1)$.

B Normalization of $\widehat{Q}_{j,m}$

In this appendix we shall compare the expression (8.11) for the charges $\widehat{Q}_{j,m}$ with those found in [36, 37, 40, 62] by comparing the symmetry algebras in the continuum theory and the matrix model. In particular [40] constructed a set of symmetry generators $\overline{Q}_{j,m}$ in the continuum theory at $\mu = 0$ with Poisson bracket:

$$\{\overline{Q}_{j,m}, \overline{Q}_{j',m'}\} = 2\,(jm' - j'm)\ \overline{Q}_{j+j'-1,m+m'}\,. \tag{B.1}$$

Comparing this with the Poisson brackets computed using (8.11) we see that it is consistent to make the assignment:

$$\overline{Q}_{j,m} = -\left[(2j-1)!(j+m)!(j-m)!\right]\ \widehat{Q}_{j,m}\,. \tag{B.2}$$

The charges $\overline{Q}_{j,m}$ found in [40] were related to a set of symmetry transformation parameters $|\overline{\Lambda}_{j,m}\rangle$ given by expressions similar to (3.23) with $Y^{L,R}_{j,m}$, $\mathcal{O}^{L,R}_{j,m}$ replaced by $\overline{Y}^{L,R}_{j,m}$ and $\overline{\mathcal{O}}^{L,R}_{j,m}$ respectively:

$$|\overline{\Lambda}_{j,m}\rangle \sim \frac{1}{2}\left[|\overline{\mathcal{O}}^{L}_{j-1,m}\rangle \times |\overline{Y}^{R}_{j,m}\rangle - |\overline{Y}^{L}_{j,m}\rangle \times |\overline{\mathcal{O}}^{R}_{j-1,m}\rangle\right]\,. \tag{B.3}$$

Here $\sim$ denotes equality up to an overall j and m independent numerical factor. $\overline{Y}^{L,R}_{j,m}$ and $\overline{\mathcal{O}}^{L,R}_{j,m}$ differ from $Y^{L,R}_{j,m}$ and $\mathcal{O}^{L,R}_{j,m}$ used in this paper by normalization factors. The relative normalization between $Y^{L,R}_{j,m}$ and $\overline{Y}^{L,R}_{j,m}$ is given by (see eq.(2.6) of [40]):

$$\overline{Y}^{L,R}_{j,m} = \left[(2j)!(j+m)!(j-m)!\right]^{\frac{1}{2}}\ Y^{L,R}_{j,m}\,. \tag{B.4}$$

We shall now try to find the relative normalization between $\mathcal{O}^{L,R}_{j,m}$ and $\overline{\mathcal{O}}^{L,R}_{j,m}$ so that we can find the relation between $|\overline{\Lambda}_{j,m}\rangle$ and[28]

$$|\widehat{\Lambda}_{j,m}\rangle = \frac{1}{2}\left[|\mathcal{O}^{L}_{j-1,m}\rangle \times |Y^{R}_{j,m}\rangle - |Y^{L}_{j,m}\rangle \times |\mathcal{O}^{R}_{j-1,m}\rangle\right] \tag{B.5}$$

and compare this with (B.2).

$\overline{\mathcal{O}}^{L,R}_{j,m}$ are normalized so that they satisfy the operator product algebra[40]:

$$\overline{\mathcal{O}}^{L,R}_{j-1,m}\,\overline{\mathcal{O}}^{L,R}_{j'-1,m'} = \overline{\mathcal{O}}^{L,R}_{j+j'-2,m+m'}\,. \tag{B.6}$$

On the other hand according to (A.8), (A.10) $\mathcal{O}^{L,R}_{j-1,m}$ is normalized so that

$$\frac{1}{\sqrt{2}}\sum_{n\neq 0} c_{-n}\alpha_n|\mathcal{O}^{L}_{j-1,m}\rangle = |Y^{L}_{j,m}\rangle + \cdots\,, \qquad \frac{1}{\sqrt{2}}\sum_{n\neq 0} \bar{c}_{-n}\bar{\alpha}_n|\mathcal{O}^{R}_{j-1,m}\rangle = |Y^{R}_{j,m}\rangle + \cdots\,, \tag{B.7}$$

[28]As in section 9 we shall assume that we are working with the tachyon potential $\mu e^{2\varphi}$ from the beginning, and hence we can replace Λ in (3.23) by $\widehat{\Lambda}$. This way, while taking the $\mu \to 0$ limit we do not have to include any additional μ dependent multiplicative factor in the definition of the charges.

where $\cdots$ denote terms with SU(2) quantum number $(j-1,m)$ and $(j-2,m)$. By examining eq.(B.6) we see that the action of an SU(2) rotation matrix on $\overline{\mathcal{O}}_{j,m}$ cannot be given by the standard unitary spin j representation, since in that case the right hand side of the equation will contain a factor of the Clebsch-Gordon coefficient

$$\begin{aligned} C^{j-1,j'-1,j+j'-2}_{m,m',m+m'} &= \Big[\frac{(2j-2)!}{(j-1+m)!(j-1-m)!}\,\frac{(2j'-2)!}{(j'-1+m')!(j'-1-m')!} \\ &\quad \frac{(j+j'+m+m'-2)!(j+j'-2-m-m')!}{(2(j+j'-2))!}\Big]^{\frac{1}{2}} . \end{aligned} \tag{B.8}$$

Thus if we want the SU(2) transformations to be represented by unitary matrices, we need to choose the basis of operators to be:

$$\left[\frac{(2j-2)!}{(j-1+m)!(j-1-m)!}\right]^{\frac{1}{2}} \overline{\mathcal{O}}^{L,R}_{j-1,m} . \tag{B.9}$$

On the other hand by examining (B.7) we see that the operators $\mathcal{O}^{L,R}_{j-1,m}$ also do not transform by unitary matrices under an SU(2) rotation, since in that case the right hand side of this equation will contain a factor of[29]

$$C^{j-1,1,j}_{m,0,m} = \left[\frac{(j+m)(j-m)}{j(2j-1)}\right]^{\frac{1}{2}} . \tag{B.10}$$

The operators which do transform in a unitary representation of SU(2) are of the form:

$$\left[\frac{(j+m)(j-m)}{j(2j-1)}\right]^{\frac{1}{2}} \mathcal{O}^{L,R}_{j-1,m} . \tag{B.11}$$

Since the operators given in (B.9) and (B.11) transform in identical representations of the SU(2) group, they must be related by an m-independent multiplicative constant. This gives:

$$\begin{aligned} \overline{\mathcal{O}}^{L}_{j-1,m} &= F(j)\left[\frac{(j-1+m)!(j-1-m)!}{(2j-2)!}\,\frac{(j+m)(j-m)}{j(2j-1)}\right]^{\frac{1}{2}} \mathcal{O}^{L}_{j-1,m} \\ &= \sqrt{2}\,F(j)\left[\frac{(j+m)!(j-m)!}{(2j)!}\right]^{\frac{1}{2}} \mathcal{O}^{L}_{j-1,m} , \end{aligned} \tag{B.12}$$

for some function $F(j)$ of j.

[29]Note that since $Y^{L,R}_{j,m}$ are normalized, they transform by unitary matrices under an SU(2) rotation.

Substituting (B.4), (B.12) and their right handed counterpart into (B.3) we see that

$$|\overline{\Lambda}_{j,m}\rangle \sim \sqrt{2}\,F(j)\,(j+m)!\,(j-m)!\;|\Lambda_{j,m}\rangle\,. \tag{B.13}$$

Comparing (B.13) with (B.2) we see that at least the m dependence of the relative normalization between $\overline{Q}_{j,m}$ and $\widehat{Q}_{j,m}$ is compatible with that between $|\overline{\Lambda}_{j,m}\rangle$ and $|\widehat{\Lambda}_{j,m}\rangle$.

In order to compare the j dependence of the relative normalization of $Q_{j,m}$ and $\overline{Q}_{j,m}$ with that of $\Lambda_{j,m}$ and $\overline{\Lambda}_{j,m}$, we can choose a convenient value of m. We shall choose $m = j-1$. In this case from (B.2) we get

$$\overline{Q}_{j,j-1} = -\left[(2j-1)!\right]^2\,\widehat{Q}_{j,j-1}\,. \tag{B.14}$$

Also (B.4) gives

$$\overline{Y}^{L,R}_{j,j-1} = \left[(2j)!(2j-1)!\right]^{\frac{1}{2}}\;Y^{L,R}_{j,j-1}\,. \tag{B.15}$$

To find the relation between $\overline{\mathcal{O}}_{j-1,j-1}$ and $\mathcal{O}_{j-1,j-1}$ we note that (B.7) for $m = j-1$ gives

$$\frac{1}{\sqrt{2}}\sum_{n\neq 0} c_{-n}\alpha_n|\mathcal{O}^L_{j-1,j-1}\rangle = |Y^L_{j,j-1}\rangle + \cdots\,, \qquad \frac{1}{\sqrt{2}}\sum_{n\neq 0} \bar{c}_{-n}\bar{\alpha}_n|\mathcal{O}^R_{j-1,j-1}\rangle = |Y^R_{j,j-1}\rangle + \cdots\,. \tag{B.16}$$

On the other hand an expression for $\overline{\mathcal{O}}^{L,R}_{j,m}$ satisfying (B.6) was given in [62] in terms of Schur polynomials. Using this expression and using the properties of the Schur polynomial one finds

$$\frac{1}{\sqrt{2}}\sum_{n\neq 0} c_{-n}\alpha_n|\overline{\mathcal{O}}^L_{j-1,j-1}\rangle = \frac{1}{\sqrt{2}}\,\left[(2j)!(2j-1)!\right]^{\frac{1}{2}}\;|Y^L_{j,j-1}\rangle + \cdots\,, \tag{B.17}$$

and a similar expression relating $|\overline{\mathcal{O}}^R_{j-1,j-1}\rangle$ to $|Y^R_{j,j-1}\rangle$. Comparing (B.16) and (B.17) we see that

$$|\overline{\mathcal{O}}^{L,R}_{j-1,j-1}\rangle = \frac{1}{\sqrt{2}}\,\left[(2j)!(2j-1)!\right]^{\frac{1}{2}}\;|\mathcal{O}^{L,R}_{j-1,j-1}\rangle\,. \tag{B.18}$$

Substituting (B.15) and (B.18) into eq.(B.3) we now get:

$$|\overline{\Lambda}_{j,j-1}\rangle \sim \left[(2j)!(2j-1)!\right]\;|\widehat{\Lambda}_{j,j-1}\rangle\,. \tag{B.19}$$

Comparing (B.14) with (B.19) we see that the proportionality factor between $\overline{Q}_{j,m}$ and $\widehat{Q}_{j,m}$ differs from that between $\overline{\Lambda}_{j,m}$ and $\widehat{\Lambda}_{j,m}$ by a factor proportional to j. The precise meaning of this mismatch is not entirely clear to us. It appears as if the conserved charges constructed following the procedure of sections 2 and 3 are related to the charges which generate the corresponding symmetries by a j-dependent normalization factor. Repeating a similar exercise for other states of this theory, *e.g.* the hole states might throw some light on this issue.

References

[1] J. McGreevy and H. Verlinde, arXiv:hep-th/0304224.

[2] I. R. Klebanov, J. Maldacena and N. Seiberg, JHEP **0307**, 045 (2003) [arXiv:hep-th/0305159].

[3] J. McGreevy, J. Teschner and H. Verlinde, arXiv:hep-th/0305194.

[4] T. Takayanagi and N. Toumbas, JHEP **0307**, 064 (2003) [arXiv:hep-th/0307083].

[5] M. R. Douglas, I. R. Klebanov, D. Kutasov, J. Maldacena, E. Martinec and N. Seiberg, arXiv:hep-th/0307195.

[6] A. Sen, Mod. Phys. Lett. A **19**, 841 (2004) [arXiv:hep-th/0308068].

[7] A. Sen, Phys. Rev. D **68**, 106003 (2003) [arXiv:hep-th/0305011].

[8] A. Sen, Phys. Rev. Lett. **91**, 181601 (2003) [arXiv:hep-th/0306137].

[9] J. de Boer, A. Sinkovics, E. Verlinde and J. T. Yee, JHEP **0403**, 023 (2004) [arXiv:hep-th/0312135].

[10] J. Ambjorn and R. A. Janik, Phys. Lett. B **584**, 155 (2004) [arXiv:hep-th/0312163].

[11] G. Mandal and S. R. Wadia, arXiv:hep-th/0312192.

[12] D. Gaiotto and L. Rastelli, arXiv:hep-th/0312196.

[13] D. Ghoshal, S. Mukhi and S. Murthy, arXiv:hep-th/0406106.

[14] F. David, Mod. Phys. Lett. A **3**, 1651 (1988).

[15] J. Distler and H. Kawai, Nucl. Phys. B **321**, 509 (1989).

[16] V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. A **3**, 819 (1988).

[17] D. J. Gross and N. Miljkovic, Phys. Lett. B **238**, 217 (1990).

[18] E. Brezin, V. A. Kazakov and A. B. Zamolodchikov, Nucl. Phys. B **338**, 673 (1990).

[19] P. Ginsparg and J. Zinn-Justin, Phys. Lett. B **240**, 333 (1990).

[20] S. R. Das and A. Jevicki, Mod. Phys. Lett. A **5**, 1639 (1990).

[21] A. M. Sengupta and S. R. Wadia, Int. J. Mod. Phys. A **6**, 1961 (1991).

[22] D. J. Gross and I. R. Klebanov, Nucl. Phys. B **352**, 671 (1991).

[23] A. B. Zamolodchikov and A. B. Zamolodchikov, arXiv:hep-th/0101152.

[24] A. Sen, JHEP **0204**, 048 (2002) [arXiv:hep-th/0203211].

[25] A. Sen, JHEP **0207**, 065 (2002) [arXiv:hep-th/0203265].

[26] N. Lambert, H. Liu and J. Maldacena, arXiv:hep-th/0303139.

[27] D. Gaiotto, N. Itzhaki and L. Rastelli, Nucl. Phys. B **688**, 70 (2004) [arXiv:hep-th/0304192].

[28] J. L. Karczmarek, H. Liu, J. Maldacena and A. Strominger, JHEP **0311**, 042 (2003) [arXiv:hep-th/0306132].

[29] D. Gaiotto, N. Itzhaki and L. Rastelli, Phys. Lett. B **575**, 111 (2003) [arXiv:hep-th/0307221].

[30] G. Mandal, A. M. Sengupta and S. R. Wadia, Mod. Phys. Lett. A **6**, 1685 (1991).

[31] E. Witten, Phys. Rev. D **44**, 314 (1991).

[32] V. Kazakov, I. K. Kostov and D. Kutasov, Nucl. Phys. B **622**, 141 (2002) [arXiv:hep-th/0101011].

[33] S. Dasgupta and T. Dasgupta, arXiv:hep-th/0311177.

[34] G. W. Moore and N. Seiberg, Int. J. Mod. Phys. A **7** (1992) 2601.

[35] D. Minic, J. Polchinski and Z. Yang, Nucl. Phys. B **369**, 324 (1992).

[36] E. Witten, Nucl. Phys. B **373**, 187 (1992) [arXiv:hep-th/9108004].

[37] I. R. Klebanov and A. M. Polyakov, Mod. Phys. Lett. A **6**, 3273 (1991) [arXiv:hep-th/9109032].

[38] S. R. Das, A. Dhar, G. Mandal and S. R. Wadia, Int. J. Mod. Phys. A **7**, 5165 (1992) [arXiv:hep-th/9110021].

[39] J. Avan and A. Jevicki, Nucl. Phys. B **397**, 672 (1993) [arXiv:hep-th/9209036].

[40] I. R. Klebanov and A. Pasquinucci, arXiv:hep-th/9210105.

[41] A. Jevicki, arXiv:hep-th/9302106.

[42] E. Witten and B. Zwiebach, Nucl. Phys. B **377**, 55 (1992) [arXiv:hep-th/9201056].

[43] A. Dhar, G. Mandal and S. R. Wadia, Nucl. Phys. B **454**, 541 (1995) [arXiv:hep-th/9507041].

[44] A. Sen, arXiv:hep-th/0402157.

[45] B. Zwiebach, Annals Phys. **267**, 193 (1998) [arXiv:hep-th/9705241], and references therein.

[46] B. H. Lian and G. J. Zuckerman, Phys. Lett. B **254**, 417 (1991); Phys. Lett. B **266**, 21 (1991).

[47] S. Mukherji, S. Mukhi and A. Sen, Phys. Lett. B **266**, 337 (1991).

[48] P. Bouwknegt, J. G. McCarthy and K. Pilch, Commun. Math. Phys. **145**, 541 (1992).

[49] S. Mukherji, S. Mukhi and A. Sen, Phys. Lett. B **275**, 39 (1992).

[50] E. Witten, Nucl. Phys. B **268**, 253 (1986).

[51] A. Sen, arXiv:hep-th/0403200.

[52] T. Asakawa, S. Kobayashi and S. Matsuura, JHEP **0310**, 023 (2003) [arXiv:hep-th/0309074].

[53] J. Polchinski, Nucl. Phys. B **346**, 253 (1990).

[54] S. R. Das, A. Dhar and S. R. Wadia, Mod. Phys. Lett. A **5**, 799 (1990).

[55] R. Dijkgraaf, E. Verlinde and H. Verlinde, Commun. Math. Phys. **115**, 649 (1988).

[56] M. Goulian and M. Li, Phys. Rev. Lett. **66**, 2051 (1991).

[57] H. Dorn and H. J. Otto, Phys. Lett. B **291**, 39 (1992) [arXiv:hep-th/9206053].

[58] H. Dorn and H. J. Otto, Nucl. Phys. B **429**, 375 (1994) [arXiv:hep-th/9403141].

[59] A. B. Zamolodchikov and A. B. Zamolodchikov, Nucl. Phys. B **477**, 577 (1996) [arXiv:hep-th/9506136].

[60] J. Teschner, Class. Quant. Grav. **18**, R153 (2001) [arXiv:hep-th/0104158].

[61] C. Imbimbo, S. Mahapatra and S. Mukhi, Nucl. Phys. B **375**, 399 (1992).

[62] Y. S. Wu and C. J. Zhu, Nucl. Phys. B **404**, 245 (1993) [arXiv:hep-th/9209011].

[63] N. Ishibashi, Mod. Phys. Lett. A **4**, 251 (1989).

[64] C. G. . Callan, I. R. Klebanov, A. W. W. Ludwig and J. M. Maldacena, Nucl. Phys. B **422**, 417 (1994) [arXiv:hep-th/9402113].

[65] A. Recknagel and V. Schomerus, Nucl. Phys. B **545**, 233 (1999) [arXiv:hep-th/9811237].

[66] T. Okuda and S. Sugimoto, Nucl. Phys. B **647**, 101 (2002) [arXiv:hep-th/0208196].

[67] F. Larsen, A. Naqvi and S. Terashima, JHEP **0302**, 039 (2003) [arXiv:hep-th/0212248].

[68] N. R. Constable and F. Larsen, JHEP **0306**, 017 (2003) [arXiv:hep-th/0305177].

[69] M. Gutperle and A. Strominger, Phys. Rev. D **67**, 126002 (2003) [arXiv:hep-th/0301038].

[70] K. Okuyama, JHEP **0309**, 053 (2003) [arXiv:hep-th/0308172].

[71] N. Seiberg, Prog. Theor. Phys. Suppl. **102**, 319 (1990).

[72] D. Kutasov, K. Okuyama, J. Park, N. Seiberg, D. Shih [arXiv:hep-th/0406030].

[73] S. R. Das, arXiv:hep-th/0401067.

[74] J. L. Cardy, Nucl. Phys. B **324**, 581 (1989).

[75] D. Kutasov, arXiv:hep-th/0405058.

[76] A. Sen, Mod. Phys. Lett. A **17**, 1797 (2002) [arXiv:hep-th/0204143].

[77] Y. Nakayama, Y. Sugawara and H. Takayanagi, arXiv:hep-th/0406173.

[78] S. Ribault and V. Schomerus, JHEP **0402**, 019 (2004) [arXiv:hep-th/0310024].

[79] S. L. Lukyanov, E. S. Vitchev and A. B. Zamolodchikov, Nucl. Phys. B **683**, 423 (2004) [arXiv:hep-th/0312168].

[80] V. Fateev, A. B. Zamolodchikov and A. B. Zamolodchikov, arXiv:hep-th/0001012.

[81] J. Teschner, arXiv:hep-th/0009138.

[82] S. R. Das, Mod. Phys. Lett. A **8**, 69 (1993) [arXiv:hep-th/9210107].

[83] E. J. Martinec and S. L. Shatashvili, Nucl. Phys. B **368**, 338 (1992).

[84] A. Dhar, G. Mandal and S. R. Wadia, Mod. Phys. Lett. A **7**, 3703 (1992) [arXiv:hep-th/9210120].

[85] S. R. Das, Mod. Phys. Lett. A **8**, 1331 (1993) [arXiv:hep-th/9303116].

[86] A. Dhar, G. Mandal and S. R. Wadia, Mod. Phys. Lett. A **8**, 1701 (1993) [arXiv:hep-th/9304072].

[87] A. Jevicki and T. Yoneya, Nucl. Phys. B **411**, 64 (1994) [arXiv:hep-th/9305109].

[88] J. Maldacena, Talk at Strings 2004.

[89] A. Sen, Talk at Strings 2004, Paris (http://strings04.lpthe.jussieu.fr/talks/Sen.pdf).

[90] E. Martinec and K. Okuyama, arXiv:hep-th/0407136.

[91] J. M. Maldacena, G. W. Moore and N. Seiberg, JHEP **0107**, 046 (2001) [arXiv:hep-th/0105038].

[92] A. Fotopoulos, V. Niarchos and N. Prezas, arXiv:hep-th/0406017.

[93] K. P. Yogendran, to appear.

[94] N. Moeller, A. Sen and B. Zwiebach, JHEP **0008**, 039 (2000) [arXiv:hep-th/0005036].

[95] N. Moeller and B. Zwiebach, JHEP **0210**, 034 (2002) [arXiv:hep-th/0207107].