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Entropy Function for Heterotic Black Holes

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Abstract

We use the entropy function formalism to study the effect of the Gauss-Bonnet term on the entropy of spherically symmetric extremal black holes in heterotic string theory in four dimensions. Surprisingly the resulting entropy and the near horizon metric, gauge field strengths and the axion-dilaton field are identical to those obtained by Cardoso et. al. for a supersymmetric version of the theory that contains Weyl tensor squared term instead of the Gauss-Bonnet term. We also study the effect of holomorphic anomaly on the entropy using our formalism. Again the resulting attractor equations for the axion-dilaton field and the black hole entropy agree with the corresponding equations for the supersymmetric version of the theory. These results suggest that there might be a simpler description of supergravity with curvature squared terms in which we supersymmetrize the Gauss-Bonnet term instead of the Weyl tensor squared term.

Contents

1 Introduction and Summary

In a previous paper[1] we developed a simple method for computing the entropy of a spherically symmetric extremal black hole in a theory of gravity coupled to abelian gauge fields, neutral scalar fields (and possibly other anti-symmetric tensor fields in dimension > 4) with arbitary higher derivative interactions. In particular we gave an algorithm for constructing an 'entropy function', – a function of the parameters labelling the near horizon background, – such that extremization of this function with respect to the parameters determines the correct values of these parameters for a given set of charges carried by the black hole. Furthermore the value of this function at the extremum gives the entropy of the black hole. Related (but complementary) results have been obtained in [2, 3].

In this paper we apply this method to study the effect of higher derivative terms on the entropy of various extremal black holes in four dimensional heterotic string theory. More specifically we add to the usual supergravity action a Gauss-Bonnet term that is known to arise in tree level heterotic string theory[4, 5], and analyze the black hole entropy in the resulting theory. This problem was studied earlier in [6] in an approximation where the modification of the near horizon geometry due to the Gauss-Bonnet term was ignored, and only the additional contribution of the Gauss-Bonnet term to the expression for the entropy was taken into account. A somewhat different scheme, where again we do not explicitly take into account the effect of backreaction of the Gauss-bonnet term on the near horizon geometry, has been suggested in [7]. The entropy function formalism allows us to go beyond these approximations.

During this study we find some surprises. Refs.[8, 9, 10, 11, 12, 13, 14, 15] studied a

closely related theory, where instead of the Gauss-Bonnet term we add to the action a term proportional to the square of the Weyl tensor and infinite number of other terms required for supersymmetric completion of the action. The coefficient function of the Weyl tensor squared term is adjusted so that the term involving the square of the Riemann tensor has the same coefficient in both theories. Black hole entropy in these supersymmetric theories was computed using a completely different method. In the appropriate approximation the results of these computations turned out to agree with those of [6, 7]. We find that after taking into account the effect of backreaction our results for not only the black hole entropy, but also the near horizon metric, gauge field strengths and the axion-dilaton field, agree with those of [8, 9, 10, 11, 12, 13, 14, 15]. This exact agreement between the two sets of results is surprising considering that we do not even have a fully supersymmetric action. This perhaps suggests that there is a simpler way to supersymmetrize curvature squared terms in the action based on the Gauss-Bonnet combination rather than the Weyl tensor squared term. It will be interesting to explore this possibility.

Heterotic string theory on T^6 and more general CHL compactifications discussed in [16, 17, 18, 19, 20, 21] have S-duality symmetry group SL(2, $\mathbb{Z}$) (as in the case of toroidal compactification[22]) or a subgroup of SL(2, $\mathbb{Z}$) (as in the case of CHL models[19]). The tree level curvature squared terms are not invariant under this S-duality group, and additional terms are needed to restore the S-duality invariance of the action. This amounts to changing the coefficient of the curvature squared term to an S-duality invariant function of the axion-dilaton field. If this function can be regarded as the imaginary part of a holomorphic function, then the action based on the Weyl tensor squared term can be supersymmetrized and the resulting correction to the black hole entropy can be computed[8, 9, 10, 11, 12, 13, 14, 15]. Generically however this holomorphicity requirement is not compatible with the requirement of S-duality and the coefficient of the curvature squared term contains a part that cannot be regarded as the imaginary part of a holomorphic function[23, 24, 25]. In such cases it is not known how to supersymmetrize this term, and hence the method of [8, 9, 10, 11, 12, 13, 14, 15] is not directly applicable. Nevertheless a form of the modified entropy and attractor equations was guessed in [11, 13] by requiring S-duality invariance of the final result. On the other hand since in our analysis we do not supersymmetrize the action and just work with the Gauss-Bonnet term, we do not have any difficulty in extending our analysis to include the non-holomorphic terms, and find an expression for the modified entropy and the attractor equation. Surprisingly,

the results again match with the equations guessed in [11, 13]. Thus in a sense our analysis gives a derivation of the equations conjectured in [11, 13], although in the context of a different (but related) theory.

The rest of the paper is organized as follows. In section 2 we review the construction of the black hole entropy function for an extremal black hole in four dimensions with near horizon geometry $AdS_2 \times S^2$ and use this formalism to demonstrate that the black hole entropy is independent of the asymptotic values of the moduli scalar fields. We also show that the entropy computed using this formalism is unchanged under a field redefinition of the metric and scalar fields and also under an electric-magnetic duality transformation. In section 3 we analyze extremal black hole solutions in heterotic string theory compactified on $\mathcal{M} \times T^2$ where $\mathcal{M}$ stands for a suitable compact manifold (*e.g.* T^4 or $K3$ or some orbifolds of these.) We analyze black hole solutions carrying momentum and winding charges, as well as Kaluza-Klein monopole and H-monopole charges associated with the two circles of T^2, – first in the supergravity approximation and then including the Gauss-Bonnet term that arises at the heterotic string tree level. We find that the final result for the entropy as well as the near horizon values of the metric, gauge field strengths and the axion-dilaton field are identical to the ones found in [8, 9, 10, 11, 12, 13, 14, 15] based on supersymmetrized Weyl tensor squared terms. In section 4 we focus on the special case of $\mathcal{N} = 4$ supersymmetric heterotic string compactification, – either by taking heterotic string theory on T^6 or by considering more general class of CHL compactifications[16, 17, 18, 19, 20, 21], – but go beyond the tree approximation and include higher order corrections so as to restore S-duality of the effective theory. We analyze the black hole entropy in the modified effective field theory using the entropy function, and find an S-duality covariant expression for the black hole entropy and the near horizon value of the axion-dilaton field. These equations agree with the form of the answer guessed in [11, 13] using the requirement of S-duality invariance.

Appendix A is devoted to fixing the normalization of various electric and magnetic charges which arise in this theory.

2 Black Hole Entropy Function

Since we shall be analyzing extremal black hole solutions in four dimensional string theory, we begin by briefly reviewing the results of [1] in four dimensions. Let us consider a theory

of gravity coupled to a set of abelian gauge fields $A_\mu^{(i)}$ and a set of neutral scalar fields $\{\phi_s\}$, described by a gauge and general coordinate invariant Lagrangian density $\mathcal{L}$. In this theory we consider an extremal black hole solution with near horizon geometry $AdS_2 \times S^2$. The most general near horizon field configuration consistent with the $SO(2,1) \times SO(3)$ symmetry of $AdS_2 \times S^2$ is of the form:

$$\begin{aligned} ds^2 &= v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \\ \phi_s &= u_s \\ F^{(i)}_{rt} &= e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi} \sin\theta \,, \end{aligned} \tag{2.1}$$

where $F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu$ and v_1, v_2, $\{u_s\}$, $\{e_i\}$ and $\{p_i\}$ are constants labelling the background. We now define:

$$f(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta \, d\phi \, \sqrt{-\det g} \, \mathcal{L} \tag{2.2}$$

evaluated for the background (2.1). Furthermore we define

$$q_i \equiv \frac{\partial f}{\partial e_i}, \qquad F(\vec{u}, \vec{v}, \vec{q}, \vec{p}) \equiv 2\,\pi (e_i\, q_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{p})) \tag{2.3}$$

so that $F/2\pi$ is the Legendre transform of the function f with respect to the variables $\{e_i\}$. Then it follows as a consequence of the equations of motion that for a black hole carrying electric charge $\vec{q}$ and magnetic charge $\vec{p}$, the constants $\vec{v}$, $\vec{u}$ and $\vec{e}$ are given by:

$$\frac{\partial F}{\partial u_s} = 0, \qquad \frac{\partial F}{\partial v_1} = 0 \,, \qquad \frac{\partial F}{\partial v_2} = 0 \,. \tag{2.4}$$

$$e_i = \frac{1}{2\pi} \frac{\partial F(\vec{u}, \vec{v}, \vec{q}, \vec{p})}{\partial q_i} \,. \tag{2.5}$$

Furthermore, using the results of [26, 27, 28, 29] one can prove that the entropy associated with the black hole is given by:

$$S_{BH} = F(\vec{u}, \vec{v}, \vec{q}, \vec{p}) \tag{2.6}$$

evaluated at the extremum (2.4).

An alternative but equivalent formulation is to treat $\vec{e}$ and $\vec{q}$ as independent variables, and define:

$$F(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p}) \equiv 2\,\pi (e_i\, q_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{p})) \tag{2.7}$$

The equations determining $\vec{u}$, $\vec{v}$ and $\vec{e}$ are then given by:

$$\frac{\partial F}{\partial u_s} = 0, \quad \frac{\partial F}{\partial v_1} = 0\,, \quad \frac{\partial F}{\partial v_2} = 0\,, \quad \frac{\partial F}{\partial e_i} = 0\,. \tag{2.8}$$

The entropy associated with the black hole is given by:

$$S_{BH} = F(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p})\,, \tag{2.9}$$

at the extremum (2.8).

It is worth emphasizing again that these results follow as a consequence of the equations of motion and Wald's formula for entropy[26, 27, 28, 29] in the presence of higher derivative terms[1]. Supersymmetry was not used in this analysis. However in the special case of $\mathcal{N} = 2$ supersymmetric theories, these results reproduce the observation of [30] that the Legendre transform of the entropy with respect to the electric charges gives the prepotential of the theory[1].

Before concluding this section we shall discuss some important consequences of this result:

1. Since the construction of the function F involves knowledge of only the Lagrangian density, the functional form of F is independent of asymptotic values of the moduli scalar fields. Thus if the extremization equations (2.4) determine all the parameters $\vec{u}$, $\vec{v}$, then the value of F at the extremum and hence the entropy S_{BH} is completely independent of the asymptotic values of the moduli fields. If on the other hand the function F has flat directions then only some combinations of the parameters $\vec{u}$, $\vec{v}$ are determined by extremizing F, and the rest may depend on the asymptotic values of the moduli fields. However since F is independent of the flat directions, it depends only on the combination of parameters which are fixed by the extremization equations. As a result the value of F at the extremum is still independent of the asymptotic moduli. Thus the entropy of the black hole is independent of the asymptotic values of the moduli fields irrespective of whether or not F has flat directions. This is a generalization of the usual attractor mechanism for supersymmetric black holes in supergravity theories[31, 32, 33].

2. An arbitrary field redefinition of the metric and the scalar fields will induce a redefinition of the parameters $\vec{u}$, $\vec{v}$, and hence the functional form of F will change. However, since the value of F at the extremum is invariant under non-singular field

redefinition, the entropy is unchanged under a redefinition of the metric and other scalar fields. To see this more explicitly, let us consider a reparametrization of $\vec{u}$ and $\vec{v}$ of the form:

$$\widehat{u}_s = g_s(\vec{u}, \vec{v}, \vec{e}, \vec{p}), \quad \widehat{v}_i = h_i(\vec{u}, \vec{v}, \vec{e}, \vec{p})\,, \tag{2.10}$$

for some functions $\{g_s\}, \{h_i\}$. Then it follows from eqs.(2.2), (2.7) that the new entropy function $\widehat{F}(\vec{\widehat{u}}, \vec{\widehat{v}}, \vec{e}, \vec{q}, \vec{p})$ is given by:

$$\widehat{F}(\vec{\widehat{u}}, \vec{\widehat{v}}, \vec{e}, \vec{q}, \vec{p}) = F(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p})\,. \tag{2.11}$$

It is easy to see that eqs.(2.8) are equivalent to:

$$\frac{\partial \widehat{F}}{\partial \widehat{u}_s} = 0, \quad \frac{\partial \widehat{F}}{\partial \widehat{v}_1} = 0\,, \quad \frac{\partial \widehat{F}}{\partial \widehat{v}_2} = 0\,, \quad \frac{\partial \widehat{F}}{\partial e_i} = 0\,. \tag{2.12}$$

Thus the value of $\widehat{F}$ evaluated at this extremum is equal to the value of F evaluated at the extremum (2.8). This result of course is a consequence of the field redefinition invariance of Wald's entropy formula as discussed in [27].

3. As is well known, Lagrangian density is not invariant under an electric-magnetic duality transformation. However, the function F, being Legendre transformation of the Lagrangian density with respect to the electric field variables, is invariant under an electric-magnetic duality transformation. In other words, if instead of the original Lagrangian density $\mathcal{L}$, we use an equivalent dual Lagrangian density $\widetilde{\mathcal{L}}$ where some of the gauge fields have been dualized, and construct a new entropy function $\widetilde{F}(\vec{u}, \vec{v}, \vec{q}, \vec{p})$ from this new Lagrangian density, then F and $\widetilde{F}$ are related to each other by exchange of the appropriate q_i's and p_i's. In the context of two derivative theories this point has been noted in [34].

3 Dyonic Black Holes in Heterotic String Theory

We shall now apply the results described in section 2 to heterotic string theory on $\mathcal{M} \times S^1 \times \widetilde{S}^1$, where $\mathcal{M}$ is some four manifold (possibly accompanied by background gauge fields and anti-symmetric tensor fields) suitable for heterotic string compactification. Examples of $\mathcal{M}$ are $K3$ or T^4, but more general orbifold compactifications are also possible. We use $\alpha' = 16$ unit and denote by x^8 and x^9 the coordinates along $\widetilde{S}^1$ and S^1 respectively. We

also normalize the coordinates x^8 and x^9 such that they have periodicity $2\pi\sqrt{\alpha'} = 8\pi$. We denote by n and w the number of units of momentum and winding along S^1, by $\tilde{n}$ and $\tilde{w}$ the number of units of momentum and winding along $\widetilde{S}^1$, by N and W the number of units of Kaluza-Klein monopole charge[35, 36] and H-monopole charge[37] associated with S^1 and by $\widetilde{N}$ and $\widetilde{W}$ the number of units of Kaluza-Klein monopole charge and H-monopole charge associated with $\widetilde{S}^1$.[1]

Although eventually we shall be interested in studying a general black hole solution carrying all the eight charges, we shall first consider black hole solution with non-zero n, w and $\widetilde{N}$, $\widetilde{W}$, setting $\tilde{n} = \tilde{w} = N = W = 0$. In the supergravity approximation the four dimensional fields relevant for the construction of this black hole solution are related to the ten dimensional string metric $G^{(10)}_{MN}$, anti-symmetric tensor field $B^{(10)}_{MN}$ and the dilaton $\Phi^{(10)}$ via the relations:[2]

$$
\begin{aligned}
&\Phi = \Phi^{(10)} - \frac{1}{4}\,\ln(G^{(10)}_{99}) - \frac{1}{4}\,\ln(G^{(10)}_{88}) - \frac{1}{2}\ln V_{\mathcal{M}}\,,\\
&S = e^{-2\Phi}\,, \qquad R = \sqrt{G^{(10)}_{99}}\,, \qquad \widetilde{R} = \sqrt{G^{(10)}_{88}}\,,\\
&G_{\mu\nu} = G^{(10)}_{\mu\nu} - (G^{(10)}_{99})^{-1}\, G^{(10)}_{9\mu}\, G^{(10)}_{9\nu} - (G^{(10)}_{88})^{-1}\, G^{(10)}_{8\mu}\, G^{(10)}_{8\nu}\,,\\
&A^{(1)}_\mu = \frac{1}{2}(G^{(10)}_{99})^{-1}\, G^{(10)}_{9\mu}\,, \qquad A^{(2)}_\mu = \frac{1}{2}(G^{(10)}_{88})^{-1}\, G^{(10)}_{8\mu}\,,\\
&A^{(3)}_\mu = \frac{1}{2}B^{(10)}_{9\mu}\,, \qquad A^{(4)}_\mu = \frac{1}{2}B^{(10)}_{8\mu}\,,
\end{aligned}
\tag{3.1}
$$

where $V_{\mathcal{M}}$ denotes the volume of $\mathcal{M}$ measured in the string metric. The effective action involving these fields is given by

$$
\begin{aligned}
\mathcal{S} = \frac{1}{32\pi}\int d^4x\,\sqrt{-\det G}\,S\,\Big[&R_G + S^{-2}\,G^{\mu\nu}\,\partial_\mu S\partial_\nu S\\
&-R^{-2}\,G^{\mu\nu}\,\partial_\mu R\partial_\nu R - \widetilde{R}^{-2}\,G^{\mu\nu}\,\partial_\mu\widetilde{R}\partial_\nu\widetilde{R}\\
&-R^2\,G^{\mu\nu}\,G^{\mu'\nu'}\,F^{(1)}_{\mu\mu'}F^{(1)}_{\nu\nu'} - \widetilde{R}^2\,G^{\mu\nu}\,G^{\mu'\nu'}\,F^{(2)}_{\mu\mu'}F^{(2)}_{\nu\nu'}\\
&-R^{-2}\,G^{\mu\nu}\,G^{\mu'\nu'}\,F^{(3)}_{\mu\mu'}F^{(3)}_{\nu\nu'} - \widetilde{R}^{-2}\,G^{\mu\nu}\,G^{\mu'\nu'}\,F^{(4)}_{\mu\mu'}F^{(4)}_{\nu\nu'}\Big]\\
&+\text{ higher derivative terms } + \text{ string loop corrections}
\end{aligned}
\tag{3.2}
$$

[1] A Kaluza-Klein monopole associated with S^1 represents a background where the circle S^1 is non-trivially fibered over the two sphere labelled by θ, ϕ. An H-monopole associated with S^1 represents a five-brane wrapped on $\mathcal{M}\times\widetilde{S}^1$.

[2] We use the symmetry $x^8 \to -x^8$ together with a parity transformation of the non-compact directions to set $G_{89} = 0$, $B_{89} = 0$ and $B_{\mu\nu} = 0$.

From the definition of $A^{(i)}_\mu$ given in (3.1) it follows that the fields $A^{(1)}_\mu$ and $A^{(3)}_\mu$ couple to the momentum and winding numbers along the x^9 direction, whereas the fields $A^{(2)}_\mu$ and $A^{(4)}_\mu$ couple to the momentum and winding numbers along the x^8 direction. Thus the electric charges q_1 and q_3 associated with the fields $A^{(1)}_\mu$ and $A^{(3)}_\mu$ are proportional to n and w respectively, while the magnetic charges p_2 and p_4 associated with the fields $A^{(2)}_\mu$ and $A^{(4)}_\mu$ are proportional to the quantum numbers $\widetilde{N}$ and $\widetilde{W}$ respectively. The constants of proportionality have been determined in appendix A and the result is as follows:

$$q_1 = \frac{1}{2}n, \quad q_3 = \frac{1}{2}w, \quad p_2 = 4\pi\widetilde{N}, \quad p_4 = 4\pi\widetilde{W}\,. \tag{3.3}$$

We now consider an extremal black hole solution in this theory with near horizon geometry:

$$\begin{aligned} ds^2 &= v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2(d\theta^2 + \sin^2\theta d\phi^2)\,, \\ S &= u_S, \qquad R = u_R, \qquad \widetilde{R} = u_{\widetilde{R}} \\ F^{(1)}_{rt} &= e_1, \quad F^{(3)}_{rt} = e_3, \quad F^{(2)}_{\theta\phi} = \frac{p_2}{4\pi}, \quad F^{(4)}_{\theta\phi} = \frac{p_4}{4\pi} \end{aligned} \tag{3.4}$$

For this background geometry, we shall first compute the function f defined in (2.2) by ignoring the higher derivative terms and string loop corrections in the action (3.2). A straightforward calculation gives:

$$\begin{aligned} & f(u_S, u_R, \widetilde{u}_R, v_1, v_2, e_1, e_3, p_2, p_4) \equiv \int d\theta d\phi\, \sqrt{-\det G}\, \mathcal{L} \\ = \;& \frac{1}{8}\, v_1\, v_2\, u_S \left[-\frac{2}{v_1} + \frac{2}{v_2} + \frac{2\, u_R^2\, e_1^2}{v_1^2} + \frac{2\, e_3^2}{u_R^2\, v_1^2} - 2\, u_{\widetilde{R}}^2\, \frac{p_2^2}{16\pi^2 v_2^2} - 2\, u_{\widetilde{R}}^{-2}\, \frac{p_4^2}{16\pi^2 v_2^2}\right] \end{aligned} \tag{3.5}$$

Eqs.(2.3) now give:

$$q_1 = \frac{1}{2}\, u_S\, \frac{v_2}{v_1}\, u_R^2\, e_1, \quad q_3 = \frac{1}{2}\, u_S\, \frac{v_2}{v_1}\, u_R^{-2}\, e_3\,, \tag{3.6}$$

and

$$\begin{aligned} & F(u_S, u_R, \widetilde{u}_R, v_1, v_2, q_1, q_3, p_2, p_4) \\ = \;& \frac{\pi}{4}\, v_1\, v_2\, u_S \left[\frac{2}{v_1} - \frac{2}{v_2} + \frac{8\, q_1^2}{u_R^2\, v_2^2\, u_S^2} + \frac{8\, u_R^2\, q_3^2}{v_2^2\, u_S^2} + 2\, u_{\widetilde{R}}^2\, \frac{p_2^2}{16\pi^2 v_2^2} + 2\, u_{\widetilde{R}}^{-2}\, \frac{p_4^2}{16\pi^2 v_2^2}\right] \end{aligned} \tag{3.7}$$

It is now straightforward to solve eqs.(2.4). The result is:[3]

$$
\begin{aligned}
& v_1 = v_2 = \frac{1}{4\pi^2} p_2 p_4 = 4 \widetilde{N}\, \widetilde{W} \\
& u_S = 8\pi \sqrt{\frac{q_1 q_3}{p_2 p_4}} = \sqrt{\frac{n\, w}{\widetilde{N}\, \widetilde{W}}}\ , \quad u_R = \sqrt{\frac{q_1}{q_3}} = \sqrt{\frac{n}{w}}, \quad u_{\widetilde{R}} = \sqrt{\frac{p_4}{p_2}} = \sqrt{\frac{\widetilde{W}}{\widetilde{N}}}\ .
\end{aligned}
\tag{3.8}
$$

Eq.(2.6) now gives

$$
S_{BH} = \sqrt{q_1 q_3 p_2 p_4} = 2\pi \sqrt{n w \widetilde{N} \widetilde{W}}. \tag{3.9}
$$

This agrees with the standard result for the entropy of four charge black holes in (3+1) dimensions.

By examining the background (3.8) we see that the effective string coupling square at the horizon is given by $u_S^{-1} \sim \sqrt{\widetilde{N}\widetilde{W}/nw}$, whereas the sizes of AdS_2 and S^2, measured in string metric, are of order $\sqrt{v_1}$, $\sqrt{v_2} \sim \sqrt{\widetilde{N}\widetilde{W}}$. Finally the squares of various field strengths appearing in the action are of order $1/\widetilde{N}\widetilde{W}$. This shows that in this background the string loop expansion is controlled by the combination $\sqrt{\widetilde{N}\widetilde{W}/nw}$ and the α' expansion is controlled by the parameter $1/\widetilde{N}\widetilde{W}$. From now on we shall work in the limit $nw >> \widetilde{N}\widetilde{W}$ so that higher loop correction terms are negligible, and work with the tree level heterotic string theory. In this case we expect that the α' corrections to the solution and the entropy will generate a power series expansion in $1/\widetilde{N}\widetilde{W}$.

We shall now consider a specific higher derivative correction to the action, namely the Gauss-Bonnet term.[4] At tree level in heterotic string theory this corresponds to an additional term in the Lagrangian density of the form[4][5,6]

$$
\Delta\mathcal{L} = \frac{S}{16\pi} \left\{ R_{G\mu\nu\rho\sigma} R_G^{\mu\nu\rho\sigma} - 4 R_{G\mu\nu} R_G^{\mu\nu} + R_G^2 \right\}\ , \tag{3.10}
$$

[3]We are implicitly assuming that q_i and p_i are all positive. Otherwise q_i, p_i must be replaced by $|q_i|$, $|p_i|$ in the final formulæ.

[4]There is no *a priori* reason to believe that this calculation would reproduce the correct entropy for heterotic black holes even for large $nw/\widetilde{N}\widetilde{W}$, since the full tree level effective action contains other terms. However we shall see that the final result for the entropy agrees with that of Cardoso *et.al.*, which in turn is known to reproduce correctly the microscopic entropy of the black hole.

[5]The Lagrangian density also contains a term involving the product of the axion field a and the Pontryagin density[5], but this term vanishes identically in $AdS_2 \times S^2$ background, and we do not need to include this term in our analysis.

[6]Some recent discussion on the effect of Gauss-Bonnet and other higher derivative terms on black hole solutions can be found in [38, 39, 40, 41, 42].

where $R_{G\mu\nu\rho\sigma}$ denotes the Riemann tensor computed using the string metric $G_{\mu\nu}$. This induces the following change in the functions f and F

$$\Delta f = -2\, u_S \quad \rightarrow \quad \Delta F = 4\,\pi\, u_S \tag{3.11}$$

and does not change the relation (3.6) between q_i and e_i, since the correction term is independent of e_i. Eqs.(2.4), (2.6) with the new function $F + \Delta F$ now give:

$$\begin{aligned} &v_1 = v_2 = 4\,\widetilde{N}\,\widetilde{W} + 8 \\ &u_S = \sqrt{\frac{n\,w}{\widetilde{N}\,\widetilde{W}+4}}\,, \quad u_R = \sqrt{\frac{n}{w}}, \quad u_{\widetilde{R}} = \sqrt{\frac{\widetilde{W}}{\widetilde{N}}}\,. \end{aligned} \tag{3.12}$$

and

$$S_{BH} = 2\pi\,\sqrt{nw}\sqrt{\widetilde{N}\widetilde{W}+4}\,. \tag{3.13}$$

For $\widetilde{N} = \widetilde{W} = 0$ in (3.13) we get $S_{BH} = 4\pi\sqrt{nw}$. This agrees with the results of [43, 44, 45, 46, 47] (which in turn agrees with the microscopic counting based on degeneracy of elementary string states) even though the action used in the analysis of these papers was quite different. This surprising agreement was noted in [34] and will be extended and discussed in more detail later in this section. We also note that if we set $\widetilde{N} = \widetilde{W} = 0$ and take nw to be negative, we get $S_{BH} = 4\pi\sqrt{|nw|}$ (see footnote 3). On the other hand these configurations correspond to non-BPS fundamental heterotic string states with only right-moving excitations on the world-sheet. The expected microscopic entropy for this system is $2\sqrt{2}\pi\sqrt{nw}$. Thus the Gauss-Bonnet correction to the low energy effective action is not able to reproduce the microscopic entropy of these non-BPS states.

We shall now extend our analysis to a general black hole solution carrying all eight charges n, w, $\widetilde{n}$, $\widetilde{w}$, N, W, $\widetilde{N}$, $\widetilde{W}$. In order to describe a black hole configuration of this type, we need to include in our analysis a more general set of fields. These include the metric $G_{\mu\nu}$, four gauge fields $A^{(i)}_\mu$ $(1 \le i \le 4)$, the axion field a, the dilaton field S, and a real 4×4 matrix valued scalar field M satisfying:

$$MLM^T = M, \quad M^T = M, \quad L \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{3.14}$$

In the supergravity approximation these fields are related to the ten dimensional fields $G^{(10)}_{MN}$, $B^{(10)}_{MN}$ and $\Phi^{(10)}$ as follows (see *e.g.* [22]):

$$
\begin{aligned}
&\widehat{G}_{mn} = G^{(10)}_{mn}\,, \quad \widehat{B}_{mn} = B^{(10)}_{mn}\,, \qquad m,n = 8,9\,,\\
&\widehat{G}^{mn} = (\widehat{G}^{-1})^{mn}\,,\\
&M = \begin{pmatrix} \widehat{G}^{-1} & \widehat{G}^{-1}B \\ -\widehat{B}\widehat{G}^{-1} & \widehat{G} - \widehat{B}\widehat{G}^{-1}\widehat{B} \end{pmatrix}\\
&A^{(10-m)}_{\mu} = \frac{1}{2}\widehat{G}^{mn}G^{(10)}_{m\mu}, \quad A^{(12-m)}_{\mu} = \frac{1}{2}B^{(10)}_{m\mu} - \widehat{B}_{mn}A^{(10-m)}_{\mu}, \qquad 0 \le \mu,\nu \le 3\,,\\
&G_{\mu\nu} = G^{(10)}_{\mu\nu} - \widehat{G}^{mn}G^{(10)}_{m\mu}G^{(10)}_{n\nu}\,,\\
&B_{\mu\nu} = B^{(10)}_{\mu\nu} - 4\widehat{B}_{mn}A^{(10-m)}_{\mu}A^{(10-n)}_{\nu} - 2(A^{(10-m)}_{\mu}A^{(12-m)}_{\nu} - A^{(10-m)}_{\nu}A^{(12-m)}_{\mu})\,,\\
&\Phi = \Phi^{(10)} - \frac{1}{4}\ln\det\widehat{G} - \frac{1}{2}\ln V_{\mathcal{M}}\,, \qquad S = e^{-2\Phi}\,.
\end{aligned}
\tag{3.15}
$$

In interpreting $\widehat{G}$ and $\widehat{B}$ as matrices we must take $m = 9$ ($n = 9$) as a label of the first row (column) and $m = 8$ ($n = 8$) as a label of the second row (column). Thus for example $\widehat{G}_{99}$ appears in the top left hand corner of the matrix $\widehat{G}$. Finally a is defined through the relation

$$
\begin{aligned}
H_{\mu\nu\rho} &\equiv (\partial_\mu B_{\nu\rho} + 2A^{(i)}_{\mu}L_{ij}F^{(j)}_{\nu\rho}) + \text{cyclic permutations of } \mu,\,\nu,\,\rho\,, \qquad 1 \le i,j \le 4\,,\\
&= G_{\mu\mu'}G_{\nu\nu'}G_{\rho\rho'}S^{-1}\,(\sqrt{-\det G})^{-1}\,\epsilon^{\mu'\nu'\rho'\sigma'}\,\partial_{\sigma'}a\,.
\end{aligned}
\tag{3.16}
$$

In terms of these four dimensional fields, the effective action in the supergravity approximation is given by:

$$
\begin{aligned}
\mathcal{S} &= \frac{1}{32\pi}\int d^4x\,\sqrt{-\det G}\,S\,\Big[R_G + \frac{1}{S^2}\,G^{\mu\nu}(\partial_\mu S\partial_\nu S - \frac{1}{2}\partial_\mu a\partial_\nu a) + \frac{1}{8}G^{\mu\nu}Tr(\partial_\mu ML\partial_\nu ML)\\
&\quad -G^{\mu\mu'}G^{\nu\nu'}\,F^{(i)}_{\mu\nu}(LML)_{ij}F^{(j)}_{\mu'\nu'} - \frac{a}{S}G^{\mu\mu'}G^{\nu\nu'}\,F^{(i)}_{\mu\nu}L_{ij}\widetilde{F}^{(j)}_{\mu'\nu'}\Big]\,.
\end{aligned}
\tag{3.17}
$$

We now look for a near horizon field configuration of the form:

$$
\begin{aligned}
ds^2 &= v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2(d\theta^2 + \sin^2\theta\,d\phi^2)\,,\\
S &= u_S, \qquad a = u_a, \qquad M_{ij} = u_{Mij}\\
F^{(i)}_{rt} &= e_i, \quad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi}\,.
\end{aligned}
\tag{3.18}
$$

Substituting (3.18) into (3.17) and using (2.2) we get

$$
f(u_S, u_a, u_M, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta d\phi\,\sqrt{-\det G}\,\mathcal{L}
$$

$$= \frac{1}{8}\, v_1\, v_2\, u_S \left[-\frac{2}{v_1} + \frac{2}{v_2} + \frac{2}{v_1^2} e_i (L u_M L)_{ij} e_j - \frac{1}{8\pi^2 v_2^2} p_i (L u_M L)_{ij} p_j + \frac{u_a}{\pi u_S v_1 v_2} e_i L_{ij} p_j \right] \tag{3.19}$$

Its Legendre transform with respect to the variables e_i gives the entropy function F:

$$q_i \equiv \frac{\partial f}{\partial e_i} = \frac{v_2 u_S}{2 v_1} (L u_M L)_{ij} e_j + \frac{u_a}{8\pi} L_{ij} p_j \,, \tag{3.20}$$

$$\begin{aligned} F(u_S, u_a, u_M, \vec{v}, \vec{q}, \vec{p}) &\equiv 2\pi \left(e_i q_i - f(u_S, u_a, u_M, \vec{v}, \vec{e}, \vec{p}) \right) \\ &= 2\pi \Big[\frac{u_S}{4} (v_2 - v_1) + \frac{v_1}{v_2 u_S}\, q^T u_M q + \frac{v_1}{64\pi^2 v_2 u_S} (u_S^2 + u_a^2) p^T L u_M L p \\ &\quad - \frac{v_1}{4\pi v_2 u_S}\, u_a\, q^T u_M L p \Big] \,. \end{aligned} \tag{3.21}$$

As shown in appendix A, the charges $\vec{q}$, $\vec{p}$ are related to the quantum numbers n, w, $\widetilde{n}$, $\widetilde{w}$, N, W, $\widetilde{N}$, $\widetilde{W}$ as

$$\begin{aligned} &q_1 = \frac{1}{2}\, n, \quad q_2 = \frac{1}{2}\, \widetilde{n}, \quad q_3 = \frac{1}{2}\, w, \quad q_4 = \frac{1}{2}\, \widetilde{w} \,, \\ &p_1 = 4\pi N, \quad p_2 = 4\pi \widetilde{N}, \quad p_3 = 4\pi W, \quad p_4 = 4\pi \widetilde{W} \,. \end{aligned} \tag{3.22}$$

This suggests that we define new charge vectors:

$$Q_i = 2 q_i, \qquad P_i = \frac{1}{4\pi}\, L_{ij} p_j, \tag{3.23}$$

so that P_i and Q_i are integers. In terms of $\vec{Q}$ and $\vec{P}$ the entropy function is given by:[7]

$$F = \frac{\pi}{2} \Big[u_S (v_2 - v_1) + \frac{v_1}{v_2 u_S} \left(Q^T u_M Q + (u_S^2 + u_a^2)\, P^T u_M P - 2\, u_a\, Q^T u_M P \right) \Big] \,. \tag{3.24}$$

We now need to find the extremum of F with respect to u_S, u_a, u_{Mij}, v_1 and v_2. In general this leads to a complicated set of equations. However we can simplify the analysis by noting that the action (3.17) has an $SO(2,2)$ symmetry acting on M and $F^{(i)}_{\mu\nu}$:

$$M \to \Omega M \Omega^T, \quad F^{(i)}_{\mu\nu} \to \Omega_{ij} F^{(j)}_{\mu\nu} \,, \tag{3.25}$$

where Ω is a matrix satisfying

$$\Omega^T L \Omega = L \,. \tag{3.26}$$

[7] As expected, F is invariant under the S-duality transformation $\begin{pmatrix} Q' \\ P' \end{pmatrix} = \begin{pmatrix} m & n \\ r & s \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}$, $u'_a + i u'_S = \{ m(u_a + i u_S) + n \} / \{ r(u_a + i u_S) + s \}$, $v'_i = v_i u_S / u'_S$.

(3.25) induces the following transformation on the various parameters:

$$\begin{aligned} & e_i \to \Omega_{ij} e_j, \qquad p_i \to \Omega_{ij} p_j, \qquad u_M \to \Omega u_M \Omega^T\,, \\ & q_i \to (L\Omega L)_{ij} q_j\,, \qquad Q_i \to (L\Omega L)_{ij} Q_j, \qquad P_i \to (L\Omega L)_{ij} P_j\,. \end{aligned} \tag{3.27}$$

The entropy function (3.24) is invariant under these transformations. Since at its extremum with respect to u_{Mij} the entropy function depends only on $\vec{P}$, $\vec{Q}$, v_1, v_2, u_S and u_a it must be a function of the $SO(2,2)$ invariant combinations:

$$Q^2 = Q_i L_{ij} Q_j, \quad P^2 = P_i L_{ij} P_j, \quad Q\cdot P = Q_i L_{ij} P_j\,, \tag{3.28}$$

besides v_1, v_2, u_S and u_a. Let us for definiteness take $Q^2 > 0$, $P^2 > 0$, and $(Q\cdot P)^2 < Q^2 P^2$. In that case with the help of an $SO(2,2)$ transformation we can make

$$(I-L)_{ij} Q_j = 0, \quad (I-L)_{ij} P_j = 0\,, \tag{3.29}$$

where I denotes the 4×4 identity matrix. It can be easily seen that for $\vec{P}$ and $\vec{Q}$ satisfying this condition, every term in (3.24) is extremized with respect to u_M for[8]

$$u_M = I\,. \tag{3.30}$$

Substituting (3.30) into (3.24) and using (3.28), (3.29), we get:

$$F = \frac{\pi}{2}\left[u_S(v_2-v_1) + \frac{v_1}{v_2}\left(\frac{Q^2}{u_S} + \frac{P^2}{u_S}(u_S^2+u_a^2) - 2\,\frac{u_a}{u_S}\,Q\cdot P\right)\right]\,. \tag{3.31}$$

Written in this $SO(2,2)$ invariant manner, eq.(3.31) is valid for general $\vec{P}$, $\vec{Q}$ satisfying $P^2 > 0$, $Q^2 > 0$ and $(Q\cdot P)^2 < Q^2 P^2$.

It remains to extremize F with respect to v_1, v_2, u_S and u_a. This gives

$$v_1 = v_2 = 2P^2\,, \qquad u_S = \frac{\sqrt{Q^2P^2 - (Q\cdot P)^2}}{P^2}\,, \qquad u_a = \frac{Q\cdot P}{P^2}\,. \tag{3.32}$$

The black hole entropy, given by the value of F for this configuration, is

$$S_{BH} = \pi\sqrt{Q^2P^2 - (Q\cdot P)^2}\,. \tag{3.33}$$

[8]This is most easily seen by diagonalizing L to the form $\begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}$ where I_2 is a 2×2 identity matrix. In this case Q and P satisfying (3.29) will have third and fourth components zero, and hence a variation δu_{Mij} with either i or $j = 3$, 4 will give vanishing contribution to each term in δF computed from (3.24). On the other hand due to the constraint (3.14) on M, any variation δM_{ij} (and hence δu_{Mij}) with $i, j = 1, 2$ must vanish. Thus each term in δF vanishes under all the allowed variations of u_M.

Finally let us discuss the effect of adding the Gauss-Bonnet term given in (3.10). Since this does not affect the relation between $\vec{q}$ and $\vec{e}$, we get, as in (3.11),

$$\Delta f = -2\, u_S \quad \rightarrow \quad \Delta F = 4\,\pi\, u_S\,. \tag{3.34}$$

For F given in (3.24) and Q, P satisfying (3.29) we have an extremum of $F + \Delta F$ at

$$u_M = I, \tag{3.35}$$

$$\begin{aligned} u_S &= \sqrt{\frac{Q^2P^2 - (Q\cdot P)^2}{P^2(P^2+8)}}\,, \qquad u_a = \frac{Q\cdot P}{P^2}\,, \\ v_1 &= v_2 = 2P^2 + 8\,, \end{aligned} \tag{3.36}$$

and

$$S_{BH} = F = \pi\sqrt{\frac{(P^2+8)(Q^2P^2 - (Q\cdot P)^2)}{P^2}}\,. \tag{3.37}$$

Since eqs.(3.36), (3.37) are written in an $SO(2,2)$ covariant form, they are valid for general $\vec{Q}$, $\vec{P}$ with $Q^2 > 0$, $P^2 > 0$, $(Q\cdot P)^2 < Q^2P^2$.

Another quantity of interest is the apparent entropy of the black hole that we would get had we used the original Bekenstein-Hawking formula for our computation. Although this is not the real entropy, this gives a convention independent measure of the near horizon metric. For the normalization of the action used in (3.17) the Newton's constant $G_N = 2$. Since the canonical metric is given by $g_{\mu\nu} = S\, G_{\mu\nu}$, the area of the horizon measured in the canonical metric is $4\pi\, u_S\, v_2$. Thus the apparent entropy is given by:

$$S_{app} = \frac{4\pi u_S v_2}{4G_N} = \pi\,(P^2+4)\sqrt{\frac{Q^2P^2 - (Q\cdot P)^2}{P^2(P^2+8)}}\,. \tag{3.38}$$

Thus

$$\frac{S_{app}}{S_{BH}} = \frac{P^2+4}{P^2+8}\,. \tag{3.39}$$

We now compare these results with the computations of [8, 9, 10, 11, 12, 13, 14, 15]. In these papers a general method for analyzing supersymmetric black holes in the presence of curvature squared terms was developed. The analysis uses a fully supersymmetric action[48, 49], and the curvature squared term takes the form of Weyl tensor squared rather than the Gauss-Bonnet combination. The general results derived in these papers

can be easily applied to supersymmetric black holes in heterotic string theory on T^6, $K3 \times T^2$ or related orbifold models. Surprisingly, the result agrees exactly with the formula (3.37) for the black hole entropy (see *e.g.* eq.(6.64) of [13]). In fact not only the entropy but also the values of the dilaton-axion field given in (3.36) agree with eq.(6.63) of [13]. Recalling that $S_{app} = \pi|Z|^2$ in the convention of [13] one can easily check that the ratio S_{app}/S_{BH} given in (3.39) also agrees with the results of [13]. This in turn shows that the near horizon metric is also identical in the two formalisms. Finally, since $S_{BH}(\vec{q}, \vec{p})$ in the two formalisms are identical, the near horizon electric fields $e_i = (2\pi)^{-1}\partial S_{BH}/\partial q_i$ are also identical.[9] Given that the starting points are quite different, – we have added a Gauss-Bonnet term to the action without supersymmetrizing it, whereas ref.[13] uses a fully supersymmetrized version of the Weyl-tensor squared term, – this agreement is quite surprising. Perhaps this indicates that there is an alternative formulation of the fully supersymmetric action based on the Gauss-Bonnet combination that is simpler than the one based on the Weyl tensor squared term.

4 $N = 4$ Supersymmetric Theories and Holomorphic Anomaly

Toroidally compactified heterotic string theory is S-duality invariant under the transformation

$$\tau \to \frac{m\tau + n}{r\tau + s}, \qquad \tau \equiv a + iS, \qquad m, n, r, s \in \mathbb{Z}, \quad ms - nr = 1\,, \tag{4.1}$$

together with appropriate transformation on other fields[22] (see footnote 7 for the action of S-duality transformation on various parameters). The supergravity theory described by the action (3.17) reflects this symmetry.[10] However the additional term (3.10) does not respect this symmetry and hence the effective action must receive additional corrections. In particular we expect the full effective Lagrangian density to contain a term of the

[9] The near horizon magnetic field is directly given by $p_i/4\pi$ in both formalisms.

[10] Although both for the toroidal and the CHL compactifications the ranks of the matrices M and L are larger, using the continuous T-duality symmetry of the action we can align the charges of the black hole so that only the fields which appear in the action (3.17) are excited.

form:[11,12]

$$\Delta\mathcal{L} = \phi(a,S)\,\left\{R_{G\mu\nu\rho\sigma}R_G^{\mu\nu\rho\sigma} - 4R_{G\mu\nu}R_G^{\mu\nu} + R_G^2\right\}\,, \tag{4.2}$$

where $\phi(a,S)$ is invariant under the S-duality transformation (4.1) and for weak coupling, i.e. for large S, ϕ approaches $S/16\pi$. For heterotic string theory on T^6 the correct choice of $\phi(a,S)$ is[13]:

$$\phi(a,S) = -\frac{3}{16\pi^2}\ln\left(2\,S\,|\eta(a+iS)|^4\right)\,, \qquad \eta(\tau) \equiv e^{2\pi i\tau/24}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})\,. \tag{4.3}$$

The form of the action (3.17), (4.2) is valid also for a general four dimensional heterotic string compactification with $\mathcal{N}=4$ supersymmetry[50, 51], *e.g.* in CHL models[16, 17, 18, 19, 20, 21], with a different choice of $\phi(a,S)$. In general $\phi(a,S)$ is of the form[50, 51]:

$$\phi(a,S) = -\frac{3}{16\pi^2}\left[\frac{r-4}{24}\,\ln\left(2\,S\,|\eta(a+iS)|^4\right) + \psi(a+iS) + \psi(a+iS)^*\right]\,, \tag{4.4}$$

where r is the total number of U(1) gauge fields in the theory and $\psi(\tau)$ is a complex analytic function of τ in the upper half plane such that $\partial_\tau\psi(\tau)$ is a modular form of weight two under the S-duality group of the theory. * denotes complex conjugation. $\psi(\tau)$ itself gets shifted by constants under various modular transformations, and grows linearly with τ for large S:

$$\psi(\tau) \simeq \frac{i\pi}{144}\,(28-r)\,\tau\,, \tag{4.5}$$

so that $\phi(a,S) \simeq S/16\pi$ for large S. For the Z_N orbifold models discussed in [51]:

$$\psi(\tau) = \frac{1}{N-1}\frac{28-r}{12}\,(\ln\eta(N\tau) - \ln\eta(\tau))\,. \tag{4.6}$$

It is now easy to study the effect of the term (4.2) on the entropy function. It gives an additional contribution to f and F of the form

$$\Delta f = -32\pi\phi(u_a,u_S) \quad \to \quad \Delta F = 64\pi^2\phi(u_a,u_S)\,. \tag{4.7}$$

[11]The metric that remains invariant under an S-duality transformation is the Einstein metric $g_{\mu\nu} = SG_{\mu\nu}$ and not the string metric. Thus in order to get a fully S-duality invaraint combination we need to replace the curvature tensor $R_{G\mu\nu\rho\sigma}$ in (4.2) by the curvature tensor $R_{g\mu\nu\rho\sigma}$ computed using the Einstein metric, and also the $\sqrt{-\det G}$ multiplying this term in the expression for the action by $\sqrt{-\det g}$. However this difference is irrelevant for a constant dilaton background considered in (3.18).

[12]Besides the Gauss-Bonnet combination the action is also expected to contain a term proportional to the Pontryagin density[5]. However as discussed in footnote 5, contribution from this term vanishes for the near horizon $AdS_2\times S^2$ geometry, and hence we shall not need to include this term in our analysis.

Together with (3.24) this gives

$$\begin{aligned} F &= \frac{\pi}{2}\Big[u_S(v_2-v_1)+\frac{v_1}{v_2u_S}\left(Q^T u_M Q+(u_S^2+u_a^2)\,P^T u_M P\right. \\ &\quad \left.-2\,u_a\,Q^T u_M P\right)+128\,\pi\,\phi(u_a,u_S)\Big]\,. \end{aligned} \tag{4.8}$$

As in section 3, we can use the SO(2,2) symmetry of the action to align the vectors Q and P to be annihilated by $(I-L)$. In this case each term in the expression for F is extremized for $u_M = I$. Substituting this back into the expression for F we get

$$F=\frac{\pi}{2}\left[u_S(v_2-v_1)+\frac{v_1}{v_2}\left(\frac{Q^2}{u_S}+\frac{P^2}{u_S}(u_S^2+u_a^2)-2\,\frac{u_a}{u_S}\,Q\cdot P\right)+128\pi\phi(u_a,u_S)\right]\,. \tag{4.9}$$

Extremization with respect to v_1 and v_2 give:

$$v_1=v_2=u_S^{-2}\,\left(Q^2+P^2(u_S^2+u_a^2)-2u_a\,Q\cdot P\right)\,. \tag{4.10}$$

Substituting this into (4.9) gives:

$$F=\frac{\pi}{2}\Big[\frac{1}{u_S}\,\left(Q^2-2\,u_a\,Q\cdot P+P^2(u_S^2+u_a^2)\right)+128\pi\phi(u_a,u_S)\Big]\,. \tag{4.11}$$

The values of u_a and u_S at the horizon are determined by extremizing F with respect to u_a and u_S. This gives:

$$\begin{aligned} P^2u_a-Q\cdot P+64\,\pi\,u_S\,\frac{\partial\phi}{\partial u_a}&=0\,, \\ -\frac{1}{u_S^2}\left(Q^2-2\,u_a\,Q\cdot P+P^2u_a^2\right)+P^2+128\,\pi\,\frac{\partial\phi}{\partial u_S}&=0\,. \end{aligned} \tag{4.12}$$

We can now try to compare eqs.(4.11), (4.12) with the corresponding results in the supersymmetric version of the theory. Since $\phi(a,S)$ given in (4.3) (or more generally (4.4)) is not a sum of a holomorphic and an anti-holomorphic term, it is in general difficult to supersymmetrize the corresponding Weyl tensor squared term. Due to this reason the analysis of Cardoso et. al. cannot be directly extended to this case. Nevertheless [11] guessed a form of the corrected equations based on the requirement of S-duality invariance of the theory. It can easily be shown that the expression (4.11) for the black hole entropy as well as the attractor equations (4.12) agree with the results guessed by Cardoso et. al. (see *e.g.* eqs.(1) and (2) of [52] for toroidal compactification). This again is a surprising agreement whose significance needs to be explored.

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A Normalization of the Charges

In this appendix we shall fix the normalization of the electric charges q_i and magnetic charges p_i associated with the gauge fields $A_\mu^{(i)}$. For this we start by noting that according to our convention, the presence of an electric charge q_i induces a coupling

$$q_i \int dx^0 A_0^{(i)} \,, \tag{A.1}$$

to the constant mode of $A_0^{(i)}$. On the other hand the magnetic charge p_i is defined in terms of the asymptotic form of $F_{\theta\phi}^{(i)}$:

$$F_{\theta\phi}^{(i)} = \frac{p_i}{4\pi} \sin\theta \,. \tag{A.2}$$

We begin by considering an elementary string wrapped once along S^1. In the presence of such a string there is a coupling

$$\frac{1}{4\pi\alpha'} \int d\xi^0 \int d\xi^1 \, \varepsilon^{\alpha\beta} \, B_{MN} \, \partial_\alpha X^M \partial_\beta X^N \,, \tag{A.3}$$

where ξ^0 and ξ^1 are the world-sheet coordinates and $\varepsilon^{\alpha\beta}$ is the totally anti-symmetric tensor on the world-sheet with $\varepsilon^{10} = 1$. In the static gauge $X^0 = \xi^0$, $X^9 = \xi^1$, this gives a coupling:

$$\frac{1}{2\pi\alpha'} B_{90} \int dx^0 \int dx^9 = \frac{1}{4} \int dx^0 B_{90} \,, \tag{A.4}$$

where we have used $\alpha' = 16$ and the fact that x^9 has periodicity $2\pi\sqrt{\alpha'} = 8\pi$. Using the identification $B_{9\mu} = 2\, A_\mu^{(3)}$ given in (3.1) we can rewrite (A.4) as

$$\frac{1}{2} \int dx^0 \, A_0^{(3)} \,. \tag{A.5}$$

Comparing with (A.1) we see that an elementary string wound once along S^1 carry half a unit of q_3 charge. Thus for an elementary string wound w times along S^1, we have

$$q_3 = \frac{1}{2} \, w \,. \tag{A.6}$$

Since T-duality along the circle S^1 interchanges the momentum and winding along S^1 and also the gauge fields $A^{(1)}$ and $A^{(3)}$, we get

$$q_1 = \frac{1}{2} \, n \,. \tag{A.7}$$

Finally, symmetry under the exchange $S^1 \leftrightarrow \widetilde{S}^1$ gives

$$q_2 = \frac{1}{2}\,\widetilde{n}, \qquad q_4 = \frac{1}{2}\,\widetilde{w}\,. \tag{A.8}$$

We now turn to the normalization of the magnetic charges. Asymptotically a Kaluza-Klein monopole solution associated with the circle $\widetilde{S}^1$ corresponds to a (9+1) dimensional background:

$$ds^2_{10} = -dt^2 + a\,(dx^8 + 2(1-\cos\theta)d\phi)^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + b\,(dx^9)^2 + d\vec{y}^2\,, \tag{A.9}$$

where $d\vec{y}^2$ denotes the metric on $\mathcal{M}$ and a, b are constants. The coefficient in front of $(1-\cos\theta)$ has been fixed so that at $\theta = \pi$, the $\phi \to \phi + 2\pi$ transformation can be compensated by translating x^8 by its periodicity 8π. Otherwise the ϕ circle will not shrink to a point at $\theta = \pi$. Using (3.1) we see that for large r this corresponds to

$$A^{(2)}_\phi = (1-\cos\theta)\,, \tag{A.10}$$

and hence

$$F^{(2)}_{\theta\phi} = \sin\theta\,. \tag{A.11}$$

From (2.1) it follows that the above configuration has $p_2 = 4\pi$. Thus $\widetilde{N}$ Kaluza-Klein monopoles associated with the circle $\widetilde{S}^1$ will have

$$p_2 = 4\pi\,\widetilde{N}\,. \tag{A.12}$$

Since T-duality associated with $\widetilde{S}^1$ exchanges $\widetilde{N}$ and $\widetilde{W}$ and also $A^{(2)}_\mu$ and $A^{(4)}_\mu$, it follows that

$$p_4 = 4\pi\,\widetilde{W}\,. \tag{A.13}$$

Finally, using the $S^1 \leftrightarrow \widetilde{S}^1$ symmetry we get

$$p_1 = 4\pi\,N, \qquad p_3 = 4\pi\,W\,. \tag{A.14}$$

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